Chapter 2
Approximation Theorems

Summary. In this chapter we embed the important work of Gräter (cf. [Gr], [Gr_1] and [Gr_2]) on approximation theorems in the book. Approximation theorems are a well-known and important topic in classical valuation theory of fields (cf. [E] and [Rib]). The question is to decide for given valuations $v_1, \ldots, v_n$ of a field, elements $a_1, \ldots, a_n$ in the field and $\alpha_1, \ldots, \alpha_n$ in the value groups whether there is an element x in the field such that

$$v_i(x - a_i) \geq \alpha_i \text{ resp. } v_i(x - a_i) = \alpha_i$$

for all i; i.e. if the elements a_i can be approximated by some x up to a certain degree. The approximation theorems were then generalized to certain classes of rings as "rings of Krull type" (cf. [G_3]).

Gräter elaborated various approximation theorems in our general setting of R-Prüfer rings and has found deep connections, to be reflected below.

We consider three types of approximation theorems: the approximation theorem in the neighbourhood of zero, the general approximation theorem and the reinforced approximation theorem. The first concerns the condition $v_i(x) = \alpha_i$, the second $v_i(x - a_i) \geq \alpha_i$ and the last one $v_i(x - a_i) = \alpha_i$. The reinforced approximation theorem was formulated by Gräter. He perceived the important connection with the intersection ring of the v_i to be Prüfer. He did this also in the case of families with finite avoidance (in his terminology "with finite character", cf. also [G_4]). The approximation theorem in the neighbourhood of zero (going already back to Manis [M], cf. also [Gr_1]) was stated in [Al], but only for finitely many valuations. Our conception of the general approximation theorem appears to be more natural then the formulation of Gräter. We give it also in the case of families having finite avoidance.

The approximation theorems are treated in Sect. 5–7. After the basic Sect. 1 we give in Sect. 2–4 the important notions of dependence, inverse property and essential valuation which will be used widely for the approximation theorems.

In this chapter R, A denote commutative rings with 1.

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1 Coarsening of Valuations

Already in Volume I coarsenings of a given valuation $v : R \to \Gamma \cup \{\infty\}$ (cf. [Vol. I, Definition 9 in I §1]) played a major role at many places, and then again here in Chap. 1. In the present section we collect basic facts about coarsenings, mostly in the case that v is Manis. Section 1 should be regarded as a tool box for the approximation theorems on families of valuations, to be studied later.

All these facts are explicitly or virtually contained in the previous chapters, or very easy consequences of results there. We recommend that the reader does not bother much about the (hints of) proofs in Sect. 1. Instead, whenever he has doubts about an assertion in Sect. 1, he should first try to prove it as exercise.

Definition 1 (cf. [Vol. I, Definition 9 in I §1]). Let v, w be valuations on R.

a) Then w is called *coarser* than v, or v is called *finer* than w, if there exists a homomorphism of ordered monoids $f : \Gamma_v \cup \{\infty\} \to \Gamma_w \cup \{\infty\}$[1] such that $w(x) = f(v(x))$ for all $x \in R$.[2] We write $v \leq w$.
b) If $v \leq w$ or $w \leq v$, the valuations v and w are called *comparable*. Otherwise they are called *incomparable*.

It is obvious that $A_v \subset A_w$, $\mathfrak{p}_v \supset \mathfrak{p}_w$ and $\operatorname{supp} v = \operatorname{supp} w$ if $v \leq w$. The latter shows that in the case of PM-valuations the above relation is not the same as the relation in Definition 2 of Chap. 1, Sect.3. (In the case of non-trivial valuations they are the same as we will see later.) Note also that we follow the definition of [Gr$_1$] and not the definition of [Al-M] where the opposite ordering is used.

Remark 1.1. The homomorphism in Definition 1 is uniquely determined and an epimorphism.

Proof. We work in the situation of Definition 1.

a) Let f and g be homomorphisms of ordered monoids fulfilling the condition. Let $\gamma \in \Gamma_v$. There are $x, y \in R \setminus \operatorname{supp} v$ such that $\gamma = v(x) - v(y)$. We obtain

$$\begin{aligned} f(\gamma) &= f(v(x) - v(y)) = f(v(x)) - f(v(y)) = w(x) - w(y) \\ &= g(v(x)) - g(v(y)) = g(v(x) - v(y)) = g(\gamma). \end{aligned}$$

b) Given $\delta \in \Gamma_w$ we have to find some $\gamma \in \Gamma_v$ such that $f(\gamma) = \delta$. There are $x, y \in R \setminus \operatorname{supp} w$ such that $\delta = w(x) - w(y)$. We obtain with $\gamma := v(x) - v(y) \in \Gamma_v$

$$f(\gamma) = f(v(x) - v(y)) = f(v(x)) - f(v(y)) = w(x) - w(y) = \delta.$$

□

[1] This means that $f(\alpha) \geq f(\beta)$ if $\alpha \geq \beta$ (cf. [Vol. I, p. 17]). Note that necessarily $f(\Gamma_v) \subset \Gamma_w$ and that $f|\Gamma_v : \Gamma_v \to \Gamma_w$ is a homomorphism of ordered groups.

[2] Note that then necessarily $f(\infty) = \infty$.

Proposition 1.2. *The set of valuations on R is partially ordered by the coarsening relation $\leq$ up to equivalence.*

Proof. It is clear that the relation $\leq$ is reflexive and transitive. For antisymmetry we have to show the following. Let v, w be valuations on R such that $v \leq w$ and $w \leq v$. Then v and w are equivalent. Let $f : \Gamma_v \cup \{\infty\} \to \Gamma_w \cup \{\infty\}$ and $g : \Gamma_w \cup \{\infty\} \to \Gamma_v \cup \{\infty\}$ be the homomorphisms of ordered monoids such that $w = f \circ v$ and $v = g \circ w$. We show that f is an isomorphism and then are done. By Remark 1 f is surjective. For the injectivity of f let $\gamma \in \Gamma_v$ be given such that $f(\gamma) = 0$. Let $x, y \in R \setminus \operatorname{supp} v$ such that $\gamma = v(x) - v(y)$. Then by the above

$$\gamma = v(x) - v(y) = g(w(x)) - g(w(y))$$

and therefore

$$0 = f(\gamma) = f(g(w(x))) - f(g(w(y))) = w(x) - w(y).$$

Hence

$$\gamma = g(w(x)) - g(w(y)) = g(w(x) - w(y)) = g(0) = 0$$

and we are done. □

Before looking at characterizations of coarsening we collect facts about the influence of coarsening on various properties and constructions introduced in Volume I.

Remarks 1.3. Let v, w be valuations on R such that $v \leq w$. Then the following holds.

(1) If v is trivial then w is trivial.
(2) If v is special then w is special.
(3) If v is Manis then w is Manis.
(4) If w is local then v is local.
(5) v is Manis and local iff w is Manis and local.
(6) v has maximal support iff w has maximal support.
(7) If v is Prüfer–Manis then w is Prüfer–Manis.
(8) If v is principal then w is principal.

Proof. (1), (2), (3) and (6) are clear by the definitions. To prove (4) let $x \in A_v \setminus \mathfrak{p}_v$. We have to show that x is a unit of A_v. Since $v(x) = 0$ also $w(x) = 0$, hence $x \in A_w \setminus \mathfrak{p}_w$. Therefore $x \in A_w^*$ by the assumption and we obtain some $y \in A_w$ such that $xy = 1$. Consequently $v(y) = 0$ and so $y \in A_v$ and $x \in A_v^*$. (5) follows from [Vol. I, Proposition I.1.3.ii]. (7) follows from (3) and [Vol. I, Corollary I.5.3]. (8) can be seen by [Vol. I, Proposition III.8.1.a] and Remark 1. □

Proposition 1.4. *Let v, w be valuations on R such that $v \leq w$. Let B be a subring of R. Then $v|_B \leq w|_B$ holds for the special restrictions of v resp. w to B.*

Proof. Let $u_1 := v|B : B \to \Gamma_v \cup \{\infty\}$ and $u_2 := w|B : B \to \Gamma_w \cup \{\infty\}$. Let $\Delta_1 := c_{u_1}(\Gamma_v)$ and $\Delta_2 := c_{u_2}(\Gamma_w)$. Let $f : \Gamma_v \cup \{\infty\} \to \Gamma_w \cup \{\infty\}$ be the homomorphism of ordered monoids such that $w = f \circ v$. We show that $f(\Delta_1) \subset \Delta_2$ and $f(\Gamma_v \setminus \Delta_1) \subset \Gamma_w \setminus \Delta_2$. For the first assertion let $\gamma \in \Delta_1$. Then there is some $x \in B$ with $v(x) \leq 0$ such that $v(x) \leq \gamma \leq -v(x)$. We obtain $w(x) = f(v(x)) \leq 0$ and

$$w(x) = f(v(x)) \leq f(\gamma) \leq f(-v(x)) = -f(v(x)) = -w(x).$$

Hence $f(\gamma) \in \Delta_2$. For the second assertion let $\delta \in \Gamma_v \setminus \Delta_1$. Clearly $\delta \neq 0$. We may assume that $\delta > 0$. Then $\delta > -v(x)$ for all $x \in B$. Assume that $f(\delta) \in \Delta_2$. Then there is some $x_0 \in B$ with $w(x_0) < 0$ such that $f(\delta) \leq -w(x_0)$. We obtain $\delta > -v(x_0^2)$ and $f(\delta) < -w(x_0^2)$. This contradicts the fact that f is order preserving. From this two observations and the definition of $v|_B = u_1|\Delta_1$ resp. $w|_B = u_2|\Delta_2$ we see that f induces a well-defined homomorphism of ordered monoids $g : \Delta_1 \cup \{\infty\} \to \Delta_2 \cup \{\infty\}$ such that $g \circ v|_B = w|_B$. □

Proposition 1.5. *Let v, w be valuations on R with* $\operatorname{supp} v = \operatorname{supp} w =: \mathfrak{q}$. *Let S be a multiplicative subset of R with $S \cap \mathfrak{q} = \emptyset$. We consider the valuations $v_S : S^{-1}R \to \Gamma_v \cup \{\infty\}, w_S : S^{-1}R \to \Gamma_w \cup \{\infty\}$ (cf. [Vol. I, Chap. I §1]). The following are equivalent:*

(1) $v \leq w$.
(2) $v_S \leq w_S$.

Proof. (1) $\Rightarrow$ (2): Let $f : \Gamma_v \cup \{\infty\} \to \Gamma_w \cup \{\infty\}$ be the homomorphism of ordered monoids such that $f \circ v = w$. Let $\tilde{v} := v_S$ and $\tilde{w} := w_S$. We have $\Gamma_{\tilde{v}} = \Gamma_v$ and $\Gamma_{\tilde{w}} = \Gamma_w$. Hence f is a homomorphism from $\Gamma_{\tilde{v}} \cup \{\infty\}$ to $\Gamma_{\tilde{w}} \cup \{\infty\}$. For $a/s \in S^{-1}R$ we have

$$f(\tilde{v}(\frac{a}{s})) = f(v(a) - v(s)) = f(v(a)) - f(v(s)) = w(a) - w(s) = \tilde{w}(\frac{a}{s}).$$

(2) $\Rightarrow$ (1): Let $f : \Gamma_v \cup \{\infty\} \to \Gamma_w \cup \{\infty\}$ be the homomorphism of ordered monoids such that $f \circ v_S = w_S$. Then obviously $f \circ v = w$. We are done. □

Proposition 1.6. *Let v, w be valuations on R with* $\operatorname{supp} v = \operatorname{supp} w := \mathfrak{q}$. *We consider the valuations $\overline{v} : R/\mathfrak{q} \to \Gamma_v \cup \{\infty\}, \overline{w} : R/\mathfrak{q} \to \Gamma_w \cup \{\infty\}$ and $\hat{v} : k(\mathfrak{q}) \to \Gamma_w \cup \{\infty\}, \hat{w} : k(\mathfrak{q}) \to \Gamma_w \cup \{\infty\}$ (cf. [Vol. I, Chap. I §1]). The following are equivalent:*

(1) $v \leq w$.
(2) $\overline{v} \leq \overline{w}$.
(3) $\hat{v} \leq \hat{w}$.

Proof. The equivalence of (1) and (2) is clear. The equivalence of (2) and (3) follows from Proposition 5. □

Note that in the situation above $\hat{v} \le \hat{w}$ iff $\mathfrak{o}_v \subset \mathfrak{o}_w$ by classical valuation theory (cf. [Vol. I, Theorem I.2.6]).

Scholium 1.7 (cf. [Vol. I, Remarks I.1.12]). Let Γ be a totally ordered abelian group. Given a convex subgroup H of Γ, the quotient Γ/H can be made naturally into an ordered abelian group such that the canonical map $\Gamma \to \Gamma/H$ is an order preserving homomorphism. If $v : R \to \Gamma \cup \{\infty\}$ is a valuation we obtain a coarsening $w : R \to \Gamma/H \cup \{\infty\}$ by setting $w(x) := v(x) + H$ for $x \in R$. This latter valuation is denoted by v/H.

Assume that $\Gamma = \Gamma_v$ (for the given valuation v on R). The coarsenings w of v correspond, up to equivalence, uniquely with the convex subgroups H of Γ via $w = v/H$.

Assume that v is Manis. Let H be a convex subgroup of Γ. Then the following holds for the coarsening $w := v/H$ of v (cf. [Vol. I, Scholium I.1.18]):

$$A_w = A_H := \{x \in R \mid v(x) \ge h \text{ for some } h \in H\}$$

$$\mathfrak{p}_w = \mathfrak{p}_H := \{x \in R \mid v(x) > h \text{ for all } h \in H\}.$$

In the case of Manis valuations we establish more criteria for coarsening.

Proposition 1.8. *Let v, w be non-trivial Manis valuations on R. The following are equivalent:*

(1) $v \le w$.
(2) $\mathfrak{p}_w \subset \mathfrak{p}_v \subset A_v \subset A_w$,
(3) $\mathfrak{p}_w$ *is an ideal of* A_v *contained in* $\mathfrak{p}_v$.
(4) $\mathfrak{p}_w$ *is a proper* v*-convex ideal of* A_v.

Proof. The equivalence of (1)–(3) was established in [Vol. I, Theorem I.2.6.i].

(3) $\Rightarrow$ (4): Note that $\mathfrak{p}_w$ is a prime ideal of A_v since $A_v \subset A_w$ by (2) and $\mathfrak{p}_w$ is a prime ideal of A_w. We show that $\operatorname{supp} v \subset \mathfrak{p}_w$ and then will be done by [Vol. I, Proposition I.1.10]. Since v and w are both Manis and non-trivial and since $A_v \subset A_w$ we obtain $\operatorname{supp} v = [A_v : R] \subset [A_w : R] = \operatorname{supp} w \subset \mathfrak{p}_w$.

(4) $\Rightarrow$ (3): Assume that $\mathfrak{p}_w \not\subset \mathfrak{p}_v$. Then there is some $x \in \mathfrak{p}_w$ with $v(x) = 0$. Since $\mathfrak{p}_w$ is v-convex we obtain $A_v \subset \mathfrak{p}_w$, contradiction. □

Remark 1.9. In the situation of Proposition 8 we have that $\mathfrak{p}_w$ is a prime ideal of A_v.

Scholium 1.10 (cf. [Vol. I, Corollary I.2.7]). Let v be a Manis valuation on R. The coarsenings w of v correspond uniquely, up to equivalence, with the prime ideals $\mathfrak{p}$ of $A := A_v$ between $\operatorname{supp} v$ and $\mathfrak{p}_v$ via $\mathfrak{p} = \mathfrak{p}_w$. Also $A_w = A_{[\mathfrak{p}]}$.

Remark 1.11. Let v be a Manis valuation on R and let $\mathfrak{p}$ be a prime ideal of $A := A_v$ with $\operatorname{supp} v \subset \mathfrak{p} \subset \mathfrak{p}_v$. Then the following holds.

i) $(A_{[\mathfrak{p}]}, \mathfrak{p})$ is a Manis pair in R.
ii) $A_{[\mathfrak{p}]} = [\mathfrak{p} : \mathfrak{p}] = \{x \in R \mid x\mathfrak{p} \subset \mathfrak{p}\}$.
iii) $(A_{[\mathfrak{p}]})_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$ and $\mathfrak{p}_{[\mathfrak{p}]} = \mathfrak{p}$.

Proof. By Scholium 10 there is a coarsening w of v such that $A_w = A_{[\mathfrak{p}]}$ and $\mathfrak{p}_w = \mathfrak{p}$.

i): This follows since w is Manis by (3) of Remarks 3.
ii): This has been proved in [Vol. I, Theorem I.2.6.ii].
iii): This follows from [Vol. I, Lemma III.1.0]. □

Definition 2. Let v be a Manis valuation on R and let $\mathfrak{p}$ be a prime ideal of A_v with $\operatorname{supp} v \subset \mathfrak{p} \subset \mathfrak{p}_v$. Then the corresponding coarsening of v is denoted by $v^{\mathfrak{p}}$.

Proposition 1.12. *Let v be a Manis valuation on R and let $\mathfrak{p}$ be a prime ideal of A_v with $\operatorname{supp} v \subset \mathfrak{p} \subset \mathfrak{p}_v$. Then the following are equivalent:*

(1) $v^{\mathfrak{p}}$ *is non-trivial.*
(2) $\operatorname{supp} v \subsetneqq \mathfrak{p}$.

Proof. (1) $\Rightarrow$ (2): If $\operatorname{supp} v = \mathfrak{p}$ we have $A_{[\mathfrak{p}]} = [\mathfrak{p} : \mathfrak{p}] = R$ by Remark 11(ii). Hence $v^{\mathfrak{p}}$ is trivial.

(2) $\Rightarrow$ (1): Let $\operatorname{supp} v \subsetneqq \mathfrak{p}$. Then there is some $x \in \mathfrak{p}$ with $v(x) \neq \infty$. Since v is Manis there is some $y \in R$ with $v(y) = -v(x)$. By Remark 11(ii) we get $A_{v^{\mathfrak{p}}} = [\mathfrak{p} : \mathfrak{p}] \subset [\mathfrak{p}_v : \mathfrak{p}]$. But $y \notin [\mathfrak{p}_v : \mathfrak{p}]$ since $yx \in A_v \setminus \mathfrak{p}_v$. So $A_{v^{\mathfrak{p}}} \neq R$. □

Combining Scholium 7 and Scholium 10 we obtain

Remark 1.13. Let v be a Manis valuation on R.

a) Let H be a convex subgroup of Γ_v. Then $v/H = v^{\mathfrak{p}}$ where

$$\mathfrak{p} = \mathfrak{p}_H = \{x \in R \mid v(x) > h \text{ for all } h \in H\}.$$

b) Let $\mathfrak{p}$ be a prime ideal of A_v with $\operatorname{supp} v \subset \mathfrak{p} \subset \mathfrak{p}_v$. Then $v^{\mathfrak{p}} = v/H$ where

$$H = \{\gamma \in \Gamma_v \mid v(z) > \gamma > -v(z) \text{ for all } z \in \mathfrak{p}\} = \{\pm v(x) \mid x \in A_v \setminus \mathfrak{p}\}.$$

Theorem 1.14. *Let v and w be non-trivial Manis valuations on R. Assume that v is Prüfer–Manis. Then the following are equivalent.*

(1) $v \leq w$,
(2) $A_v \subset A_w$,

Proof. (1) $\Rightarrow$ (2): This follows from Proposition 8.

(2) $\Rightarrow$ (1): Since v is PM we have that A_v is Prüfer in R. By [Vol. I, Theorem III.1.2] there is an R-regular prime ideal $\mathfrak{p}$ of $A := A_v$ such that $A_w = A_{[\mathfrak{p}]}$, namely $\mathfrak{p} = \mathfrak{p}_{A_w} \cap A$ where $\mathfrak{p}_{A_w} = \{x \in A_w \mid \exists s \in R \setminus A_w \text{ such that } sx \in A_w\}$ (see [Vol. I, Definition 2 in I §2]). Note that $\mathfrak{p}_{A_w} = \mathfrak{p}_w$ by [Vol. I, Proposition I.2.3]. By [Vol. I,Theorem III.1.3] we get $\mathfrak{p} \subset \mathfrak{p}_v$ and $\mathfrak{p} = \mathfrak{p}_{[\mathfrak{p}]}$. But $\mathfrak{p}_{[\mathfrak{p}]} = \mathfrak{p}_w$ by the proof of [Vol. I, Theorem III.1.2]. Hence $\mathfrak{p}_w \subset \mathfrak{p}_v$. By Proposition 8 we obtain $v \leq w$. □

Theorem 1.15. *Let v be a non-trivial Prüfer–Manis valuation on R. Let B be a proper R-overring of A. Then there is up to equivalence a unique valuation w on R such that $B = A_w$. Moreover, $v \leq w$.*

Proof. For the existence we can copy the proof of Theorem 14 (2) $\Rightarrow$ (1): Since v is PM we have by [Vol. I, Corollary III.3.2] that B is Prüfer–Manis in R. Let w be a Manis valuation such that $A_w = B$. By Theorem 14 we have $v \leq w$ for the corresponding Manis valuation.

For the uniqueness let w_1, w_2 be Manis valuations on R such that $A_{w_1} = A_{w_2} = B$. Applying Theorem 14 we get $w_1 \leq w_2$ and $w_2 \leq w_1$. By Proposition 2 we get that w_1 and w_2 are equivalent. $\square$

Scholium 1.16. Let v be a Prüfer–Manis valuation on R.

a) The coarsenings of v correspond uniquely, up to equivalence, with the R-overrings of A_v.
b) The non-trivial coarsenings of v correspond uniquely, up to equivalence, with the R-regular prime ideals of A_v.

Proof. a): This is a consequence of Theorem 15 (the unique trivial valuation being coarser than v corresponds of course with R itself).
b): This is a consequence of Scholium 10, Proposition 12 and [Vol. I, Theorem III.2.5]. $\square$

Corollary 1.17. *Let v and w be non-trivial Manis valuations in R. Let A be a Prüfer subring of R such that $A \subset A_v \cap A_w$. The following are equivalent.*

(1) $v \leq w$,
(2) $A_v \subset A_w$,
(3) $\mathfrak{p}_w \subset \mathfrak{p}_v$.
(4) $\mathfrak{p}_w \cap A \subset \mathfrak{p}_v \cap A$.

Proof. Since v is PM by [Vol. I, Corollary I.5.3] we can apply Theorem 14. This gives the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (3) follows from Proposition 8.
(3) $\Rightarrow$ (4): This is trivial.
(4) $\Rightarrow$ (2): Let $\mathfrak{p} := \mathfrak{p}_v \cap A$ and $\mathfrak{p}' := \mathfrak{p}_w \cap A$. We have $A_v = A_{[\mathfrak{p}]}$ and $A_w = A_{[\mathfrak{p}']}$ by [Vol. I, Theorem III.1.2]. This gives $A_v \subset A_w$. $\square$

2 Dependent Families of Manis Valuations

In this section the classical notion of dependence resp. independence (cf. [E]) is formulated in the setting of Manis valuations (cf. [Gr_1], [Gr_2]). Special attention is paid to the case of valuations over a Prüfer subring.

Definition 1. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. If there is a non-trivial Manis valuation u on R with $v_i \leq u$ for $i \in I$ the family is called *dependent*; otherwise it is called *independent*.

We then often say that *the valuations $v_i, i \in I$, are dependent resp. independent.*

N.B. If $(v_i \mid i \in I)$ is a family of dependent Manis valuations on R then v_i is non-trivial for $i \in I$ and $\operatorname{supp} v_i = \operatorname{supp} v_j$ for $i, j \in I$.

Proposition 2.1. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. The following are equivalent.*

(1) $(v_i \mid i \in I)$ is dependent.
(2) There is a subset $\mathfrak{p}$ of R such that $\mathfrak{p}$ is a prime ideal of A_{v_i} with $\operatorname{supp} v_i \subsetneqq \mathfrak{p} \subset \mathfrak{p}_{v_i}$ *for all $i \in I$.*

Proof. (1) $\Rightarrow$ (2): Let u be a non-trivial Manis valuation on R with $v_i \leq u$ for $i \in I$. Let $\mathfrak{p} := \mathfrak{p}_u$. Then $\operatorname{supp} v_i = \operatorname{supp} u$ and $\operatorname{supp} u \subsetneqq \mathfrak{p}_u$ since u is non-trivial. By Proposition 1.8 (and Remark 1.9) we see that $\mathfrak{p} \subset \mathfrak{p}_{v_i}$ and that $\mathfrak{p}$ is a prime ideal of A_{v_i} for all $i \in I$.
(2) $\Rightarrow$ (1): By Scholium 1.10 the valuation u corresponding to $\mathfrak{p}$ is a coarsening of v_i for $i \in I$. Since $\operatorname{supp} u = \operatorname{supp} v_i \subsetneqq \mathfrak{p}$ for some (resp. all) $i \in I$ we get that u is non-trivial. Hence $(v_i \mid i \in I)$ is dependent. $\square$

Definition 2. Assume that $(B_i \mid i \in I)$ is a family of subrings of R. Then we denote the set of subsets $\mathfrak{p}$ of R such that $\mathfrak{p}$ is a prime ideal of every B_i by $\bigcap_{i \in I} \operatorname{Spec} B_i$.

Remark 2.2. Let $\mathfrak{V} = (v_i \mid i \in I)$ be a family of Manis valuations on R. By Proposition 1 we know that $\mathfrak{V}$ is dependent iff

$$X_{\mathfrak{V}} := \{\mathfrak{p} \in \bigcap_{i \in I} \operatorname{Spec} A_{v_i} \mid \operatorname{supp} v_i \subsetneqq \mathfrak{p} \subset \mathfrak{p}_{v_i} \text{ for } i \in I\}$$

is non-empty. By [Vol. I, Proposition I.1.10] every $\mathfrak{p} \in X_{\mathfrak{V}}$ is v_i-convex for all $i \in I$. Hence the elements of $X_{\mathfrak{V}}$ are ordered by inclusion, and $X_{\mathfrak{V}}$ has the largest element

$$\mathfrak{p}_{\mathfrak{V}} := \bigcup_{\mathfrak{p} \in X_{\mathfrak{V}}} \mathfrak{p}$$

if $\mathfrak{V}$ is dependent.

Corollary 2.3. *Let $\mathfrak{V} = (v_i \mid i \in I)$ be a dependent family of Manis valuations on R. Then there is a finest (non-trivial) Manis valuation which is coarser than v_i for every $i \in I$. It is (up to equivalence) the unique Manis valuation u on R with $\mathfrak{p}_u = \mathfrak{p}_{\mathfrak{V}}$.*

Definition 3. Let $(v_i \mid i \in I)$ be a family of dependent Manis valuations on R. The finest Manis valuation which is coarser than v_i for all $i \in I$ is denoted by $\bigvee_{i \in I} v_i$.

We use the results of Sect. 1 to establish criteria for dependence and to describe $\bigvee v_i$.

Remark 2.4. Let $\mathfrak{V} = (v_i \mid i \in I)$ be a dependent family of Manis valuations on R.

i) We have $\mathfrak{p}_{\bigvee v_i} = \mathfrak{p}_{\mathfrak{V}}$.
ii) Let u be a Manis valuation such that $v_i \leq u$ for $i \in I$. Then $\bigvee_{i \in I} v_i \leq u$.

Definition 4. Assume that $\mathfrak{V} = (v_i \mid i \in I)$ is a family of Manis valuations on R. If $\mathfrak{V}$ is dependent and $i \in I$, let $H^i_{\mathfrak{V}}$ denote the convex subgroup of Γ_{v_i} generated by $v_i(A_{v_i} \setminus \mathfrak{p}_{\bigvee v_j})$ with j running through I. If $\mathfrak{V}$ is independent, we set $H^i_{\mathfrak{V}} = \Gamma_{v_i}$ for $i \in I$.

Remark 2.5 (cf. Remarks 1.13). If $\mathfrak{V} = (v_i \mid i \in I)$ is dependent then for $i \in I$

$$\Gamma_{\bigvee v_j} \cong \Gamma_{v_i}/H^i_{\mathfrak{V}} \text{ and } \bigvee v_j = v_i/H^i_{\mathfrak{V}}.$$

Remark 2.6. Let $(v_i \mid i \in I)$ be a family of dependent Manis valuations on R. Then $A_{\bigvee v_i} \supset \prod_{i \in I} A_{v_i}$ and $\mathfrak{p}_{\bigvee v_i} \subset [\bigcap_{i \in I} \mathfrak{p}_{v_i} : \prod_{i \in I} A_{v_i}]$.

Proposition 2.7. *Let $(v_i \mid i \in I)$ be a family of PM-valuations on R. The following are equivalent.*

(1) $(v_i \mid i \in I)$ is dependent.
(2) There is a proper subring of R containing A_{v_i} for all $i \in I$.
(3) $\prod_{i \in I} A_{v_i} \neq R$.

Proof. The equivalence of (1) and (2) is a consequence of Scholium 1.16(a). The equivalence of (2) and (3) is obvious. □

Corollary 2.8. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R such that $\bigcap_{i \in I} A_{v_i}$ is Prüfer in R. The following are equivalent.*

(1) $(v_i \mid i \in I)$ is dependent.
(2) There is a proper subring of R containing A_{v_i} for all $i \in I$.
(3) $\prod_{i \in I} A_{v_i} \neq R$.

Proposition 2.9. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. Let A be a Prüfer subring of R with $A \subset \bigcap_{i \in I} A_{v_i}$. The following are equivalent.*

(1) $(v_i \mid i \in I)$ is dependent.
(2) There is an R-regular prime ideal $\mathfrak{p}$ of A contained in $\mathfrak{p}_{v_i} \cap A$ for all $i \in I$.

Proof. (1) $\Rightarrow$ (2): Let $u := \bigvee v_i$ and $\mathfrak{p} := \mathfrak{p}_u \cap A$. Then $\mathfrak{p} \subset \mathfrak{p}_{v_i} \cap A$ for $i \in I$ by Corollary 1.17. Since u is non-trivial we have $A_u = A_{[\mathfrak{p}]}$ by [Vol. I, Theorem III.1.2] and [Vol. I, Proposition I.2.3]. By [Vol. I, Lemma III.1.1], $\mathfrak{p}$ is an R-regular prime ideal of A.

(2) $\Rightarrow$ (1): Let $\mathfrak{p}$ be an R-regular prime ideal $\mathfrak{p}$ of A contained in $\mathfrak{p}_{v_i} \cap A$ for $i \in I$. Then again by [Vol. I, Theorem III.1.2] $A_{[\mathfrak{p}]} \neq R$ and $A_{[\mathfrak{p}]}$ is an R-overring of $A_{[\mathfrak{p}_{v_i} \cap A]} = A_{v_i}$ for $i \in I$. We get the claim by Proposition 7 or Corollary 8. □

Proposition 2.10. *Let $(v_i \mid i \in I)$ be a dependent family of PM-valuations on R. Then $A_{\bigvee v_i} = \prod_{i \in I} A_{v_i}$ and $\mathfrak{p}_{\bigvee v_i} = [\bigcap_{i \in I} \mathfrak{p}_{v_i} : \prod_{i \in I} A_{v_i}]$.*

Proof. Let $u := \bigvee_{i \in I} v_i$.

a) We have $A_u \supset \prod_{i \in I} A_{v_i}$ by Remark 6. On the other hand there is by Scholium 1.16(a) a Manis valuation u' on R such that $A_{u'} = \prod_{i \in I} A_{v_i}$ and $v_i \leq u'$ for $i \in I$. Since $u \leq u'$ we get $A_u \subset A_{u'}$.
b) Let $\mathfrak{p} := [\bigcap_{i \in I} \mathfrak{p}_{v_i} : \prod_{i \in I} A_{v_i}]$. Then $\mathfrak{p}$ is a proper ideal of A_u containing $\mathfrak{p}_u$ by Remark 6. Since u is non-trivial, the ideal $\mathfrak{p}_u$ is R-regular by [Vol. I, Theorem III.2.5]. Since $\mathfrak{p}$ contains $\mathfrak{p}_u$ it is clearly also R-regular. By [Vol. I, Theorem III.3.10] we get that $\mathfrak{p}$ is contained in $\mathfrak{p}_u$. □

Corollary 2.11. *Let $(v_i \mid i \in I)$ be a dependent family of Manis valuations on R. If $\bigcap_{i \in I} A_{v_i}$ is Prüfer in R, then $A_{\bigvee v_i} = \prod_{i \in I} A_{v_i}$ and $\mathfrak{p}_{\bigvee v_i} = [\bigcap_{i \in I} \mathfrak{p}_{v_i} : \prod_{i \in I} A_{v_i}]$.*

We collect basic facts about dependent and independent families.

Remark 2.12. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R and let $J \subset I$. If $(v_i \mid i \in I)$ is dependent then $(v_i \mid i \in J)$ is dependent and $\bigvee_{i \in J} v_i \leq \bigvee_{i \in I} v_i$.

Proposition 2.13. *Let $(v_i \mid i \in I), (w_i \mid i \in I)$ be families of Manis valuations on R such that $v_i \leq w_i$ for all $i \in I$. If $(w_i \mid i \in I)$ is dependent then $(v_i \mid i \in I)$ is dependent and $\bigvee_{i \in I} v_i \leq \bigvee_{i \in I} w_i$.*

Proof. By the transitivity of $\leq$ we get $v_i \leq \bigvee_{j \in I} w_j$ for all $i \in I$. Hence $(v_i \mid i \in I)$ is dependent and $\bigvee_{i \in I} v_i \leq \bigvee_{i \in I} w_j$. □

Definition 5. Let $(v_i \mid i \in I)$ be an independent family of Manis valuations on R. If $\operatorname{supp} v_i = \operatorname{supp} v_j$ for all $i, j \in I$ we denote by $\bigvee_{i \in I} v_i$ the trivial valuation with $\operatorname{supp} \bigvee_{i \in I} v_i = \operatorname{supp} v_j$ for $j \in I$. Otherwise let $\bigvee_{i \in I} v_i$ denote the map $R \to \{0\}$. Notice that in this case $\bigvee_{i \in I} v_i$ is not a valuation.

Remark 2.14. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R with $\operatorname{supp} v_i = \operatorname{supp} v_j$ for $i, j \in I$. Then $\bigvee_{j \in I} v_j$ is a coarsening of v_i for $i \in I$. We have

$$A_{\bigvee v_j} = (A_{v_i})_{[\mathfrak{p}_{\bigvee v_j}]}$$

for $i \in I$ (see Scholium 1.10).

Proposition 2.15. *Let $\mathfrak{V} = (v_i \mid i \in I)$ be a family of Manis valuations on R with $\operatorname{supp} v_i = \operatorname{supp} v_j$ for all $i, j \in I$. Let $w_i := v_i|_{A_{\bigvee v_j}}$ for $i \in I$. Then w_i is a Manis valuation on $A_{\bigvee v_j}$ with $\Gamma_{w_i} = H^i_{\mathfrak{V}}$ and $\operatorname{supp} w_i = \mathfrak{p}_{\bigvee v_j}$ for $i \in I$ (cf. Definitions 4 and 5 above).*

Proof. a) If $\mathfrak{V}$ is independent then $A_{\bigvee v_j} = R$ and nothing is to show. So we assume that $\mathfrak{V}$ is dependent. Let $i \in I$. By [Vol. I, Proposition I.1.17] and Remark 6 w_i is a Manis valuation on A_u where $u := \bigvee v_j$.

b) We show that $\Gamma_{w_i} = H^i_{\mathfrak{V}}$. Let $\gamma \in \Gamma_{w_i}$. Then there is some $x \in A_u$ with $w_i(x) = \gamma$. Since $w_i(x) < \infty$ we have $w_i(x) = v_i(x)$ and find some $y \in A_u$ with $v_i(y) \leq 0$ and $v_i(y) \leq v_i(x) \leq -v_i(y)$ by the definition of the special restriction. Since $v_i \leq u$ and $y \in A_u$ we get $u(y) = u(x) = 0$ and therefore $\gamma \in H^i_{\mathfrak{V}}$. Let $\delta \in H^i_{\mathfrak{V}} \subset \Gamma_{v_i}$. Let $x \in R$ with $v_i(x) = \delta$. Then $u(x) = 0$ and therefore $x \in A_u$. Let $x' \in R$ with $v_i(x') = -\delta$. By the same argument we get $x' \in A_u$. By the definition of the special restriction we get $w_i(x) = \delta$ and therefore $\delta \in \Gamma_{w_i}$.

c) We show that $\operatorname{supp} w_i = \mathfrak{p}_u$. Let $x \in \operatorname{supp} w_i \subset A_u$. Assume that $x \notin \mathfrak{p}_u$. Then $u(x) = 0$ and therefore $v(x) \in H^i_{\mathfrak{V}}$ and $w_i(x) = v_i(x) \neq \infty$ by the argument in b), contradiction. Let $x \in \mathfrak{p}_u \subset A_u$. Assume that $w_i(x) \neq \infty$. Then by the definition of the special restriction there is some $y \in A_u$ with $v_i(y) \leq 0$ and $v_i(y) \leq v_i(x) \leq -v_i(y)$. Since $v_i \leq u$ we obtain $u(y) \leq u(x) \leq -u(y)$. This gives $u(y) < 0$, contradiction to $y \in A_u$. □

3 The Inverse Property

The inverse property is a substitute for the inverse element in the case of fields (cf. [M]). We prove various inequalities to be used later and investigate the connection between dependence and the inverse property (see also [Gr$_1$], [Al] and [Al$_1$]).

Definition 1. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R.

i) The family has the *inverse property* if for every $x \in R$, there is some $x' \in R$ such that $v_i(xx') = 0$ for all $i \in I$ with $v_i(x) \neq \infty$.
ii) The family has the *finite inverse property* if every finite subfamily has the inverse property.

We then often say that the valuations $v_i, i \in I$, have the inverse (resp. finite inverse) property.

Remarks 3.1. a) Let v be a Manis valuation on R. Then v has the inverse property.
b) If a family of Manis valuations on R has the inverse property then it has also the finite inverse property.
c) Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having the inverse (resp. finite inverse) property. Then for $J \subset I$, the subfamily $(v_i \mid i \in J)$ has the inverse (resp. finite inverse) property.
d) Any family of valuations on a field has the inverse property.

Proof. a): The inverse property for a single valuation is equivalent with the Manis property.
b), c): This is obvious.
d): Let R be a field. Given $x \in R^*$ take x^{-1}. □

Remark 3.2. A family $(v_i \mid i \in I)$ of Manis valuations on R has the inverse property iff for every $x \in R$ there is some $x' \in R$ such that $v_i(x^2x') = v_i(x)$ for all $i \in I$.

Remark 3.3. Let $(v_i \mid i \in I_1)$ be a family of Manis valuations on R with the (finite) inverse property. Let $(v_i \mid i \in I_2)$ be a family of trivial Manis valuations on R such that for every $i \in I_2$ there is an $i' \in I_1$ with $\operatorname{supp} v_{i'} = \operatorname{supp} v_i$. Then the family $(v_i \mid i \in I_1 \cup I_2)$ has the (finite) inverse property.

Remark 3.4. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. Let I be an ideal of R such that $I \subset \operatorname{supp} v_i$ for all $i \in I$. For $i \in I$ we denote by $\overline{v}_i$ the corresponding Manis valuation of v_i on R/I. The following are equivalent.

(1) $(v_i \mid i \in I)$ has the inverse (resp. finite inverse) property.
(2) $(\overline{v}_i \mid i \in I)$ has the inverse (resp. finite inverse) property.

Before exploiting the inverse property we prove useful inequalities for a finite set of Manis valuations.

Lemma 3.5 (cf. [Vol. I, Lemma I.6.9]). *Let k be a subring of R and let $v_1, \dots, v_n$ be valuations on R with $A_{v_i} \supset k$ for all $1 \le i \le n$. Let $m \in \mathbb{N}$. Given an element x of R, there exists a monic polynomial $F(T) \in k[T]$ with $F(0) = 0$ and the following property:*

If $G(T) \in k[T]$ is any monic polynomial of degree ≥ 1 with absolute term $G(0) \in k^*$, then $v_i(G(F(x))) = 0$ if $v_i(x) \ge 0$ and $v_i(G(F(x))) \le mv_i(x)$ if $v_i(x) < 0$ for $1 \le i \le n$.

Proof. We take $F(T) := T^m F_1(t) \dots F_n(T)$ in the proof of [Vol. I, Lemma I.6.9]. □

Proposition 3.6. *Let $v_1, \dots, v_n$ be valuations on R and let $x \in R$. Let $m \in \mathbb{N}$. Then there exists $y \in R$ such that $v_i(y) = 0$ if $v_i(x) \ge 0$ and $v_i(y) \le mv_i(x)$ if $v_i(x) < 0$ for all $1 \le i \le n$.*

Proof. In the previous Lemma 5 we take $k := \mathbb{Z} \cdot 1_R$. Let $F(T) \in k[T]$ be a monic polynomial with $F(0) = 0$ and the above property. We take $G(T) := 1 + T$. Then $y := 1 + F(x)$ fulfills the requirements. □

Lemma 3.7. *Let $v_1, \dots, v_n, w_1, \dots, w_m$ be valuations on R. Let $x_1, \dots, x_k \in R$ such that for every $i \in \{1, \dots, n\}$ there is at most one $l \in \{1, \dots, k\}$ with $v_i(x_l) = 0$. Then there are $y_1, \dots, y_k \in R$ such that the following properties hold.*

i) If $u \in \{v_1, \dots, v_n, w_1, \dots, w_m\}$, $l \in \{1, \dots, k\}$ and $u(x_l) < \infty$, then $u(y_l) < \infty$.
ii) If $u \in \{v_1, \dots, v_n, w_1, \dots, w_m\}$, $l \in \{1, \dots, k\}$ and $u(x_l)\sigma 0$, then $u(y_l)\sigma 0$ where $\sigma \in \{<, =, >\}$.
iii) If $i \in \{1, \dots, n\}$, $l_1, l_2 \in \{1, \dots, k\}, l_1 \neq l_2$, and $v_i(x_{l_1}) < \infty$ or $v_i(x_{l_2}) < \infty$, then $v_i(y_{l_1}) \neq v_i(y_{l_2})$.

Proof. We do induction on k.
$k = 1$: There is nothing to show, we can take $y_1 := x_1$.
$k \to k+1$: By the inductive hypothesis applied to $x_1, \ldots, x_k$ we find $y_1, \ldots, y_k \in R$ with the above properties.

Claim: Let $1 \leq i \leq n$. Let $1 \leq l \leq k$ such that $v_i(x_{k+1}) < \infty$ or $v_i(y_l) < \infty$. Then there is at most one $m_{i,l} \in \mathbb{N}$ such that $v_i(x_{k+1}^{m_{i,l}}) = v_i(y_l)$.

Proof of the Claim: Let $1 \leq l \leq k$. By the assumption of the lemma and the inductive hypotheses we have $v_i(x_{k+1}) \neq 0$ or $v_i(y_l) \neq 0$. This gives the claim.

We choose now $m \in \mathbb{N}$ such that $m > m_{i,l}$ for all such $m_{i,l}$ above. Then we take $y_{k+1} := x_{k+1}^m$. □

Proposition 3.8. *Let $v_1, \ldots, v_n, w$ be valuations on R such that $A_w \not\subset A_{v_i}$ for all $1 \leq i \leq n$. Then there is some $x \in R$ such that $w(x) = 0$ and $v_i(x) < 0$ for all $1 \leq i \leq n$.*

Proof. We do induction on n.
$n = 1$: Since $A_w \not\subset A_{v_1}$ there is some $x' \in R$ such that $w(x') \geq 0$ and $v_1(x') < 0$. By Proposition 6 there is some $x \in R$ such that $w(x) = 0$ and $v_1(x) < 0$.
$n \to n+1$: By the inductive hypothesis and the case $n = 1$ there are $x', x'' \in R$ such that

$$\begin{aligned} w(x') &= 0, v_1(x') < 0, \ldots, v_n(x') < 0, \\ w(x'') &= 0, v_{n+1}(x'') < 0. \end{aligned}$$

If $v_{n+1}(x') < 0$ we take $x := x'$. So we assume that $v_{n+1}(x') \geq 0$. Applying Lemma 7 we can assume that $v_i(x') \neq v_i(x'')$ for all $1 \leq i \leq n+1$. Let $\hat{x} := x' + x''$. Then $w(\hat{x}) \geq 0$ and $v_i(\hat{x}) < 0$ for all $1 \leq i \leq n+1$. By Proposition 6 there is some $x \in R$ with $w(x) = 0$ and $v_i(x) < 0$ for all $1 \leq i \leq n$. □

Corollary 3.9. *Let $v_1, \ldots, v_n, w_1, \ldots, w_m$ be valuations on R such that the following properties hold.*

a) $A_{w_1} \not\subset A_{v_i}$ *for all* $1 \leq i \leq n$.
b) $A_{w_1} \subset A_{w_j}$ *for all* $1 \leq j \leq m$.

Then there is some $x \in R$ such that $w_j(x) = 0$ for all $1 \leq j \leq m$ and $v_i(x) < 0$ for all $1 \leq i \leq n$.

Proof. We do induction on m.

$m = 1$: This is covered by Proposition 8.

$m \to m+1$: By the inductive hypothesis there is some $x' \in R$, such that $w_j(x') = 0$ for all $1 \leq j \leq m$, and $v_i(x') < 0$ for all $1 \leq i \leq n$. By assumption b) we have $w_{m+1}(x') \geq 0$. By Proposition 6 we find some $x \in R$ such that $w_j(x) = 0$ for all $1 \leq j \leq m+1$ and $v_i(x) \leq v_i(x') < 0$ for $1 \leq i \leq n$. □

Remark 3.10. Assume that v is a non-trivial special valuation on R. Then for every $\alpha \in \Gamma_v$ there exists some $x \in R$ with $v(x) < \alpha$.

Proof. Let $\alpha \in \Gamma_v$. Then there are $y, z \in R \setminus \operatorname{supp} v$ with $\alpha = v(y) - v(z)$. Since v is special and non-trivial there is some $z' \in R$ with $v(z') < -v(z)$. We take $x := yz'$. $\square$

Proposition 3.11. *Let $v_1, \dots, v_n$ be non-trivial special valuations on R. Let $\alpha_i \in \Gamma_{v_i}$ for $1 \le i \le n$. Then there is some x in R such that $v_i(x) < \alpha_i$ for all $1 \le i \le n$.*

Proof. We do induction on n.

$n = 1$: This is clear by Remark 10.

$n \to n+1$: Since the valuations $v_1, \dots, v_{n+1}$ are non-trivial we may assume that $\alpha_i < 0$ for all $1 \le i \le n+1$. By the inductive hypothesis there are $x', x'' \in R$ such that $v_i(x') < \alpha_i$ for $1 \le i \le n$ and $v_i(x'') < \alpha_i$ for $2 \le i \le n+1$. By Lemma 7 we can assume that $v_i(x') \neq v_i(x'')$ for all $1 \le i \le n+1$. If $v_{n+1}(x') < \alpha_{n+1}$ or $v_1(x'') < \alpha_1$ we are done. Otherwise let $x := x' + x''$. For $2 \le i \le n$ we have $v_i(x) = \min\{v_i(x'), v_i(x'')\} < \alpha_i$. Also, $v_1(x) = v_1(x') < \alpha_1$ and $v_{n+1}(x) = v_{n+1}(x'') < \alpha_{n+1}$. $\square$

Proposition 3.12. *Let $v_1, \dots, v_n$ be non-trivial special valuations on R and let $w_1, \dots, w_m$ be trivial valuations on R. Let $\alpha_i \in \Gamma_{v_i}$ for $1 \le i \le n$. Then there is some $x \in R$ such that $v_i(x) < \alpha_i$ for $1 \le i \le n$ and $w_j(x) = 0$ for $1 \le j \le m$.*

Proof. We do induction on m.

$m = 1$: We may assume that $\alpha_i < 0$ for $1 \le i \le n$. By Proposition 11 there is some $x' \in R$ such that $v_i(x') < \alpha_i$ for $1 \le i \le n$. If $w_1(x') = 0$ we take $x := x'$. If $w_1(x') = \infty$ we take $x = 1 + x'$. Then $v_i(x) = v_i(x') < \alpha_i$ for $1 \le i \le n$ and $w_1(x) = 0$.

$m \to m+1$: Again we may assume that $\alpha_i < 0$ for $1 \le i \le n$. By the inductive hypothesis there is for $1 \le j \le m+1$ some $x_j \in R$ such that $v_i(x_j) < \alpha_i$ for $1 \le i \le n$ and $w_k(x_j) = 0$ for $k \in \{1, \dots, m+1\} \setminus \{j\}$. If there is some $j \in \{1, \dots, m+1\}$ such that $w_j(x_j) = 0$ we are done. Otherwise let $y_j := \prod_{k \neq j} x_k$ for $1 \le j \le m+1$. Then

$$v_i(y_j) = \sum_{k \neq j} v_i(x_k) < m\alpha_i < \alpha_i$$

for $1 \le i \le n$. Moreover, $w_j(y_j) = 0$ and $w_k(y_j) = \infty$ for $k \in \{1, \dots, m+1\} \setminus \{j\}$. By Lemma 7 we may assume that $v_i(y_{j_1}) \neq v_i(y_{j_2})$ for all $1 \le i \le n$ and $j_1 \neq j_2$. We set $x := y_1 + \dots + y_{m+1}$. Then $v_i(x) = \min\{v_i(y_1), \dots, v_i(y_{m+1})\} < \alpha_i$ for $1 \le i \le n$ and $w_j(x) = 0$ for $1 \le j \le m+1$. $\square$

Corollary 3.13. *Let $v_1, \dots, v_n$ be special valuations on R. Let $\alpha_i \in \Gamma_{v_i}$ for $1 \le i \le n$. Then there is some $x \in R$ such that $v_i(x) \le \alpha_i$ for all $1 \le i \le n$.*

Proof. Without restriction we may assume that there is some $k \in \{0, \dots, n\}$ such that v_i is non-trivial for $1 \leq i \leq k$ and trivial for $k+1 \leq i \leq n$. Then $\alpha_i = 0$ for $k+1 \leq i \leq n$. We get the claim by Proposition 12. □

Now the inverse property comes into the game. We start by exploiting the results above.

Corollary 3.14. *Let $v_1, \dots, v_n, w$ be Manis valuations on R having the inverse property such that $A_w \not\subset A_{v_i}$ for all $1 \leq i \leq n$. Then there is some $x \in R$ such that $w(x) = 0$ and $0 < v_i(x) < \infty$ for all $1 \leq i \leq n$.*

Proof. By Proposition 8 there is some $x' \in R$ such that $w(x') = 0$ and $v_i(x') < 0$ for all $1 \leq i \leq n$. Since $v_1, \dots, v_n, w$ have the inverse property there is some $x \in R$ such that $w(x) = w(x') = 0$ and $v_i(x) = -v_i(x') > 0$ for all $1 \leq i \leq n$. □

Corollary 3.15. *Let $v_1, \dots, v_n, w_1, \dots, w_m$ be Manis valuations on R having the inverse property such that the following hold.*

a) $A_{w_1} \not\subset A_{v_i}$ *for all* $1 \leq i \leq n$.
b) $A_{w_1} \subset A_{w_j}$ *for all* $1 \leq j \leq m$.

Then there is some $x \in R$ such that $w_j(x) = 0$ for all $1 \leq j \leq m$ and $0 < v_i(x) < \infty$ for all $1 \leq i \leq n$.

Proof. This follows from Corollary 9 and the inverse property. □

Corollary 3.16. *Let $v_1, \dots, v_n$ be non-trivial Manis valuations on R having the inverse property. Let $\alpha_i \in \Gamma_{v_i}$ for $1 \leq i \leq n$. Then there is some x in R such that $\alpha_i < v_i(x) < \infty$ for all $1 \leq i \leq n$.*

Proof. By Proposition 11 there is some $x' \in R$ such that $v_i(x') < -\alpha_i$ for all $1 \leq i \leq n$. The statement follows from the inverse property. □

Corollary 3.17. *Let $v_1, \dots, v_n, w_1, \dots, w_m$ be Manis valuations on R having the inverse property such that $v_1, \dots, v_n$ are non-trivial and $w_1, \dots, w_m$ are trivial. Let $\alpha_i \in \Gamma_{v_i}$ for $1 \leq i \leq n$. Then there is some $x \in R$ such that $\alpha_i < v_i(x) < \infty$ for $1 \leq i \leq n$ and $w_j(x) = 0$ for $1 \leq j \leq m$.*

Proof. This follows from Proposition 12 and the inverse property. □

Corollary 3.18. *Let $v_1, \dots, v_n$ be Manis valuations on R having the inverse property. Let $\alpha_i \in \Gamma_{v_i}$ for $1 \leq i \leq n$. Then there is some $x \in R$ such that $\alpha_i \leq v_i(x) < \infty$ for all $1 \leq i \leq n$.*

Proof. This follows from Corollary 13 and the inverse property. □

Proposition 3.19. *Let $v_1, \dots, v_n$ be Manis valuations on R. The following are equivalent.*

(1) $v_1, \dots, v_n$ *have the inverse property.*
(2) For any $x \in R$ there is an element $y \in R$ such that, for all $1 \leq i \leq n$, $v_i(y) = v_i(x)$ if $v_i(x) \geq 0$, and $-v_i(x) \leq v_i(y) < \infty$ if $v_i(x) < 0$.

(3) *For any* $x \in R$ *there is an element* $y \in R$ *such that, for all* $1 \le i \le n$, $v_i(y) = v_i(x)$ *if* $v_i(x) \ge 0$, *and* $-v_i(x) \le v_i(y)$ *if* $v_i(x) < 0$.
(4) *If* $1 \le i \le n$ *and* $x \in R$ *with* $v_i(x) < 0$ *and* $v_j(x) = 0$ *for* $j \ne i$, *then there is an element* $y \in R$ *such that* $0 \le v_i(xy) < \infty$ *and* $v_j(y) = 0$ *for* $j \ne i$.
(5) *If* $1 \le i \le n$ *and* $x \in R$ *with* $v_i(x) < 0$ *and* $v_j(x) = 0$ *for* $j \ne i$, *then there is an element* $y \in R$ *such that* $0 \le v_i(xy)$ *and* $v_j(y) = 0$ *for* $j \ne i$.

Proof. (1) $\Rightarrow$ (2): By Proposition 6 we find some $y' \in R$ such that for all $1 \le i \le n$ $v_i(y') = 0$ if $v_i(x) \ge 0$ and $v_i(y') \le 2v_i(x)$ if $v_i(x) < 0$. Since $v_1, \dots, v_n$ have the inverse property there is some $y'' \in R$ such that for all $1 \le i \le n$ $v_i(y'') = 0$ if $v_i(x) \ge 0$ and $-2v_i(x) \le v_i(y'') < \infty$ if $v_i(x) < 0$. We set $y := xy''$. Let $1 \le i \le n$. If $v_i(x) \ge 0$ then $v_i(y) = v_i(x) + v_i(y'') = v_i(x)$. If $v_i(x) < 0$ then $v_i(y) = v_i(x) + v_i(y'') \ge v_i(x) - 2v_i(x) = -v_i(x)$ and $v_i(y) \ne \infty$.
(2) $\Rightarrow$ (3): This is obvious.
(2) $\Rightarrow$ (4): Take y from (2).
(3) $\Rightarrow$ (5): Take y from (3).
(4) $\Rightarrow$ (5): This is obvious.
(5) $\Rightarrow$ (1): We do induction on n.
$n = 1$: $\{v_1\}$ has the inverse property since v_1 is Manis (cf. Remarks 1(a)).
$\le n \to n + 1$: Let $x \in R$. By the inductive hypothesis we may assume that $v_i(x) \ne \infty$ for $1 \le i \le n+1$. Also by the inductive hypothesis there is some $y_1 \in R$ such that $v_i(xy_1) = 0$ for $2 \le i \le n + 1$. By (5) we may assume that $v_1(xy_1) \ge 0$. (Otherwise, $v_1(xy_1) < 0$. By (5) there is some $z_1 \in R$ with $v_i(xy_1z_1) = 0$ for $2 \le i \le n + 1$, and $v_1(xy_1z_1) \ge 0$. Replacing y_1 by y_1z_1, we are done.) In the same way there are $y_2, \dots, y_{n+1} \in R$ such that for $1 \le i \le n + 1$ $v_j(xy_i) = 0$ for $j \ne i$ and $v_i(xy_i) \ge 0$. If there is some $1 \le i \le n + 1$ such that $v_i(xy_i) = 0$ we take $x' := y_i$ and are done. Otherwise, $v_i(xy_i) > 0$ for all $1 \le i \le n + 1$. We set $x'_i := x^{n-1} \prod_{j \ne i} y_j$. Then

$$v_i(xx'_i) = v_i(x^n \prod_{j \ne i} y_j) = \sum_{j \ne i} v_i(xy_j) = 0$$

for $1 \le i \le n + 1$ and

$$v_j(xx'_i) = v_j(x^n \prod_{k \ne i} y_k) = \sum_{k \ne i} v_j(xy_k) = v_j(xy_j) > 0$$

for all $j \ne i$. Let $x' := x'_1 + \dots + x'_{n+1}$. Then $v_i(xx') = v_i(xx'_i) = 0$ for all $1 \le i \le n + 1$.

□

Corollary 3.20. *Let* $v_1, \dots, v_n$ *be Manis valuations on* R *having the inverse property. Let* B *be an* R*-overring of* $\prod_{i=1}^n A_{v_i}$. *Then the special restrictions* $v_1|_B, \dots, v_n|_B$ *of* $v_1, \dots, v_n$ *to* B *have the inverse property.*

Proof. By [Vol. I, Proposition I.1.17], $v_i|_B$ is a Manis valuation for $1 \le i \le n$. We show that condition (5) of Proposition 19 holds. Without restriction we show it for $i = 1$. Let $x \in B$ with $v_1|_B(x) < 0$ and $v_j|_B(x) = 0$ for $2 \le j \le n$. Then $v_i|_B(x) = v_i(x)$ for all $1 \le i \le n$. By Proposition 19(5) applied to $v_1, \dots, v_n$ there is some $y \in R$ such that $v_1(xy) \ge 0$ and $v_j(y) = 0$ for $2 \le j \le n$. We see that $y \in \bigcap_{i=1}^n A_{v_i} \subset B$. We get $v_1|_B(xy) \ge 0$ and $v_j|_B(y) = 0$ for $2 \le j \le n$. $\square$

We show in Proposition 4.19 below that the statement of Corollary 20 holds even for overrings of $\bigcap_{1 \le i \le n} A_{v_i}$.

Now we investigate the connection of the inverse property with coarsening and independence.

Proposition 3.21. *Let $(v_i \mid i \in I), (w_j \mid j \in J)$ be families of Manis valuations on R such that the following holds.*

a) $(v_i \mid i \in I)$ has the inverse (resp. finite inverse) property.
b) For every $j \in J$ there is some $i \in I$ such that $v_i \le w_j$.

Then $(w_j \mid j \in J)$ has the inverse (resp. finite inverse) property.

Proof. We may concentrate on the inverse property. Let $x \in R$. Since $(v_i \mid i \in I)$ has the inverse property there is by Remark 2 some $x' \in R$ such that $v_i(x^2x') = v_i(x)$ for all $i \in I$. For $j \in J$ let $i_j \in I$ with $v_{i_j} \le w_j$. Let $f_{i_j} : \Gamma_{v_{i_j}} \cup \{\infty\} \to \Gamma_w \cup \{\infty\}$ be the homomorphism of ordered monoids such that $w_j = f_{i_j} \circ v_{i_j}$. We obtain

$$w_j(x^2x') = f_{i_j}(v_{i_j}(x^2x')) = f_{i_j}(v_{i_j}(x)) = w_j(x)$$

for all $j \in J$. By Remark 2 we get the claim. $\square$

Proposition 3.22. *Let v, w be two Manis valuations on R with w non-trivial. The following are equivalent.*

(1) $v \le w$.
(2) v, w have the inverse property and $A_v \subset A_w$.
(3) v, w have the inverse property and $\operatorname{supp} v \subsetneq \mathfrak{p}_w \subset \mathfrak{p}_v$.

Proof. (1) $\Rightarrow$ (2): The valuations v, w have the inverse property by Remarks 1(a) and Proposition 21. The valuation v is non-trivial by Remarks 1.3(1). We get by Proposition 1.8 that $A_v \subset A_w$.

(2) $\Rightarrow$ (3): We show that $\mathfrak{p}_w \subset \mathfrak{p}_v$. Assume that $\mathfrak{p}_w \not\subset \mathfrak{p}_v$. Let $x \in \mathfrak{p}_w \setminus \mathfrak{p}_v$. Then $w(x) > 0$ and $v(x) \le 0$.

Case 1: $w(x) < \infty$. Since v, w have the inverse property there is some $x' \in R$ such that $v(x') = -v(x) \ge 0$ and $w(x') = -w(x) < 0$, i.e. $x' \in A_v \setminus A_w$, contradiction.

Case 2: $w(x) = \infty$. Since w is non-trivial there is some $y \in R$ such that $0 < w(y) < \infty$. By Case 1 we get $y \in \mathfrak{p}_v$, i.e. $v(y) > 0$. Let $x' := x + y$. Then $v(x') = v(x) \le 0$ and $0 < w(x') = w(y) < \infty$, contradiction to Case 1.

It remains to show that $\operatorname{supp} v \subsetneq \mathfrak{p}_w$. Since w is non-trivial $A_w \ne R$. Hence $A_v \ne R$ and therefore v is also non-trivial. We obtain

$$\operatorname{supp} v = [A_v : R] \subset [A_w : R] = \operatorname{supp} w \subsetneqq \mathfrak{p}_w.$$

(3) $\Rightarrow$ (1): Since $\operatorname{supp} v \subsetneqq \mathfrak{p}_v$ the valuation v is non-trivial. By Proposition 1.8 it suffices to show that $A_v \subset A_w$. Assume that this does not hold. By Corollary 14 we find some $x \in R$ with $v(x) = 0$ and $w(x) > 0$, hence $x \in \mathfrak{p}_w \setminus \mathfrak{p}_v$, contradiction to (3). $\square$

Corollary 3.23. *Let $v_1, \dots, v_n, w$ be Manis valuations on R having the inverse property such that $v_1, \dots, v_n$ are non-trivial and $w \not\leq v_i$ for all $1 \leq i \leq n$. Then there is some $x \in R$ such that $w(x) = 0$ and $0 < v_i(x) < \infty$ for all $1 \leq i \leq n$.*

Proof. By Proposition 22 we see that $A_w \not\subset A_{v_i}$ for all $1 \leq i \leq n$. We get the claim by Corollary 14. $\square$

Corollary 3.24. *Let $v_1, \dots, v_n, w_1, \dots, w_m$ be Manis valuations on R having the inverse property such that $v_1, \dots, v_n$ are non-trivial and the following hold.*

a) $w_1 \not\leq v_i$ *for all* $1 \leq i \leq n$.
b) $w_1 \leq w_j$ *for all* $1 \leq j \leq m$.

Then there is some $x \in R$ such that $w_j(x) = 0$ for all $1 \leq j \leq m$ and $0 < v_i(x) < \infty$ for all $1 \leq i \leq n$.

Proof. By Proposition 22 we get that $A_{w_j} \not\subset A_{v_i}$ for all $1 \leq j \leq m$ and $1 \leq i \leq n$. Clearly, $A_{w_1} \subset A_{w_j}$ for all $2 \leq j \leq m$. We get the claim by Corollary 15. $\square$

Let v, w be Manis valuations with $\operatorname{supp} v = \operatorname{supp} w$ having the inverse property. Our next goal is to describe $v \vee w$ in this situation.

Lemma 3.25. *Let v be a valuation on R and let B be a subring of R. Then $\mathfrak{p} := [\mathfrak{p}_v : B]$ is a v-convex prime ideal of A_v with $\operatorname{supp} v \subset \mathfrak{p} \subset \mathfrak{p}_v$.*

Proof. Since $\mathfrak{p}_v$ is an ideal of A_v, clearly $\mathfrak{p}$ is an ideal of A_v. Since $1 \in B$ we have $\mathfrak{p} \subset \mathfrak{p}_v$. Obviously $\mathfrak{p}$ is a v-convex ideal with $\operatorname{supp} v \subset \mathfrak{p}$. It remains to show that $\mathfrak{p}$ is prime. Let $a, b \in A_v \setminus \mathfrak{p}$. There are $x, y \in B$ such that $v(ax) \leq 0$ and $v(by) \leq 0$. Hence $v(abxy) \leq 0$ and therefore $ab \notin \mathfrak{p}$ since $xy \in B$. $\square$

Remark 3.26. Let v be a valuation on R and let B be a subring of R. Then $[\mathfrak{p}_v : B] = [\mathfrak{p}_v : \operatorname{conv}(B)]$, with $\operatorname{conv}(B)$ denoting the v-convex hull of B.

Proposition 3.27. *Let v be a Manis valuation on R and let B_1, B_2 be subrings of R. The following are equivalent.*

(1) $[\mathfrak{p}_v : B_1] = [\mathfrak{p}_v : B_2]$,
(2) $\operatorname{conv}(B_1) = \operatorname{conv}(B_2)$.

Proof. (2) $\Rightarrow$ (1): This follows with Remark 26.

(1) $\Rightarrow$ (2): By Remark 26 we can assume that B_1 and B_2 are v-convex and have to show that $B_1 = B_2$. Assume that there is some $x \in B_1 \setminus B_2$. Then $v(x) < v(y)$

for all $y \in B_2$. Let $z \in R$ with $v(z) = -v(x)$. Then $v(zy) > 0$ for all $y \in B_2$, hence $z \in [\mathfrak{p}_v : B_2]$. But $v(zx) = 0$, hence $z \notin [\mathfrak{p}_v : B_1]$, contradiction. $\square$

Proposition 3.28. *Let v, w be Manis valuations on R having the inverse property such that* $\operatorname{supp} v = \operatorname{supp} w$. *Let $\mathfrak{p} := [\mathfrak{p}_v : A_w]$. Then $\mathfrak{p}$ is a w-convex prime ideal of A_w with* $\operatorname{supp} w \subset \mathfrak{p} \subset \mathfrak{p}_w$.

Proof. a) Since $\operatorname{supp} w = \operatorname{supp} v$ we get that $\operatorname{supp} w \subset \mathfrak{p}$ by Lemma 25.

b) We show that $\mathfrak{p} \subset \mathfrak{p}_w$. Suppose that $\mathfrak{p} \not\subset \mathfrak{p}_w$. Let $x \in \mathfrak{p} \setminus \mathfrak{p}_w$. Then $x \notin \operatorname{supp} w = \operatorname{supp} v$. Since v, w have the inverse property there is some $x' \in R$ with $v(xx') = w(xx') = 0$. Since $x \notin \mathfrak{p}_w$ we get $x' \in A_w$. From $x \in \mathfrak{p} = [\mathfrak{p}_v\colon A_w]$ we see that $xx' \in \mathfrak{p}_v$, contradiction to $v(xx') = 0$.

c) We show that $\mathfrak{p}$ is an ideal of A_w. Given $x \in \mathfrak{p}$ and $a \in A_w$ we get $xaA_w \subset xA_w \subset \mathfrak{p}_v$, hence $xa \in \mathfrak{p}$.

d) Finally we show that $\mathfrak{p}$ is a prime ideal of A_w. Let $x, y \in A_w$ with $xy \in \mathfrak{p}$ and $y \notin \mathfrak{p}$. Then there is some $a \in A_w$ with $ya \notin \mathfrak{p}_v$ and therefore $v(ya) \leq 0$. For any $b \in A_w$ we get

$$v(xb) \geq v(xb) + v(ya) = v(xyab).$$

Since $xy \in \mathfrak{p}, ab \in A_w$ and $\mathfrak{p}A_w \subset \mathfrak{p}_v$ by the definition of $\mathfrak{p}$, we have $v(xyab) > 0$. So $v(xb) > 0$ for all $b \in B$ and hence $x \in \mathfrak{p}$.

By [Vol. I, Proposition I.1.10] $\mathfrak{p}$ is w-convex. $\square$

Proposition 3.29. *Let v, w be Manis valuations on R having the inverse property such that* $\operatorname{supp} v = \operatorname{supp} w$. *Then $\mathfrak{p}_{v \vee w} = [\mathfrak{p}_v : A_w]$.*

Proof. Let $\mathfrak{p} := [\mathfrak{p}_v : A_w]$. By Lemma 25 and Proposition 28, $\mathfrak{p}$ is a prime ideal of A_v and A_w such that $\operatorname{supp} v, \operatorname{supp} w \subset \mathfrak{p} \subset \mathfrak{p}_v \cap \mathfrak{p}_w$.

Case 1: v and w are dependent. By Remark 2.2 and Corollary 2.3 we have to show the following. Let $\mathfrak{r}$ be a prime ideal of A_v and A_w with $\mathfrak{r} \subset \mathfrak{p}_v \cap \mathfrak{p}_w$. Then $\mathfrak{r} \subset \mathfrak{p}$. Let $x \in \mathfrak{r}$ and let $a \in A_w$ be arbitrary. Then $xa \in \mathfrak{r} \subset \mathfrak{p}_v$ and therefore $x \in \mathfrak{p}$ by the definition of $\mathfrak{p}$.

Case 2: v and w are independent. Again by Remark 2.2 we get that $\mathfrak{p} = \operatorname{supp} v = \operatorname{supp} w$ and therefore $\mathfrak{p} = \mathfrak{p}_{v \vee w} = \operatorname{supp}(v \vee w)$ according to Definition 5 in Sect. 2. $\square$

Corollary 3.30. *Let v, w be Manis valuations on R having the inverse property such that* $\operatorname{supp} v = \operatorname{supp} w$. *Then $[\mathfrak{p}_v : A_w] = [\mathfrak{p}_w : A_v]$.*

Proof. By Proposition 29 we get

$$[\mathfrak{p}_v : A_w] = \mathfrak{p}_{v \vee w} = \mathfrak{p}_{w \vee v} = [\mathfrak{p}_w : A_v].$$

$\square$

Proposition 3.31. *Let v, w be Manis valuations on R having the inverse property such that* $\operatorname{supp} v = \operatorname{supp} w$*. Then* $[\mathfrak{p}_v : A_w] = [\mathfrak{p}_v \cap \mathfrak{p}_w : A_v A_w] = [\mathfrak{p}_w : A_v]$.

Proof. Let $\mathfrak{p} := [\mathfrak{p}_v : A_w]$. We show that $[\mathfrak{p}_v \cap \mathfrak{p}_w : A_v A_w] = \mathfrak{p}$ and then are done by Corollary 30. By Lemma 25 and Proposition 28, $\mathfrak{p}$ is a prime ideal of A_v and A_w with $\mathfrak{p} \subset \mathfrak{p}_v \cap \mathfrak{p}_w$. Hence $\mathfrak{p} \subset [\mathfrak{p}_v \cap \mathfrak{p}_w : A_v A_w]$. The other inclusion is trivial. □

Corollary 3.32. *Let v, w be Manis valuations on R having the inverse property such that* $\operatorname{supp} v = \operatorname{supp} w$*. Then* $\mathfrak{p}_{v \vee w} = [\mathfrak{p}_v \cap \mathfrak{p}_w : A_v A_w]$.

Proof. This follows from Propositions 29 and 31. □

We extend the above results to finitely many Manis valuations. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. For $J \subset I$ we set $v_J := \bigvee_{i \in J} v_i$, $A_J := \prod_{i \in J} A_{v_i}$ and $\mathfrak{p}_J := \mathfrak{p}_{v_J}$. For J finite we often omit the brackets in the index, writing $A_{1,2}$ instead of $A_{\{1,2\}}$ etc.

Proposition 3.33. *Let I be a finite set and let $(v_i \mid i \in I)$ be a family of Manis valuations on R having the inverse property such that* $\operatorname{supp} v_i = \operatorname{supp} v_j$ *for all $i, j \in I$. Given a non-empty subset J of I we have* $\mathfrak{p}_I = [\mathfrak{p}_J : A_K]$ *for every non-empty subset K of I with $J \cup K = I$.*

Proof. Let $\emptyset \neq J \subset I$ and let $\emptyset \neq K \subset I$ with $J \cup K = I$.

Case 1: $J = I$. Clearly $\mathfrak{p}_I = [\mathfrak{p}_I : A_{v_I}]$. By Remark 2.6 and Definition 5 in Sect. 2 $A_{v_I} \supset A_K$. Hence $\mathfrak{p}_I \subset [\mathfrak{p}_I : A_K]$. Since $1 \in A_K$, we have equality.

Case 2: $J \subsetneqq I$. Without restriction we assume that $I = \{1, \dots, n\}$ for some $n \in \mathbb{N}$. We do induction on n.

$n = 1$: Nothing is to show.

$n \to n+1$: Without restriction we may assume that $n+1 \notin J$. We have $v_{1,\dots,n+1} = v_{1,\dots,n} \vee v_{n+1}$. By Proposition 21 the valuations $v_{1,\dots,n}, v_{n+1}$ have the inverse property. By Proposition 29 we obtain $\mathfrak{p}_{1,\dots,n+1} = [\mathfrak{p}_{1,\dots,n} : A_{n+1}]$. By the inductive hypothesis we get $\mathfrak{p}_{1,\dots,n+1} = [[\mathfrak{p}_J : A_{K \setminus \{n+1\}}] : A_{n+1}]$. But the latter set clearly coincides with $[\mathfrak{p}_J : A_K]$. □

Proposition 3.34. *Let I be a finite set and let $(v_i \mid i \in I)$ be a family of Manis valuations on R having the inverse property such that* $\operatorname{supp} v_i = \operatorname{supp} v_j$ *for all $i, j \in I$. For every non-empty subset J of I we have* $\mathfrak{p}_I = [\bigcap_{i \in J} \mathfrak{p}_i : A_I]$.

Proof. We may assume that $I = \{1, \dots, n\}$ for some $n \in \mathbb{N}$.

Special Case: $J = I$. We do induction on n.

$n = 1$: Clearly $\mathfrak{p}_v = [\mathfrak{p}_v : A_v]$ for any valuation v on R.

$n \to n+1$: We have $v_{1,\dots,n+1} = v_{1,\dots,n} \vee v_{n+1}$. By Proposition 21 the valuations $v_{1,\dots,n}, v_{n+1}$ have the inverse property. By Corollary 32 we obtain

$$\mathfrak{p}_I = [\mathfrak{p}_{1,\dots,n} \cap \mathfrak{p}_{n+1} : A_{1,\dots,n} A_{n+1}] = [\mathfrak{p}_{1,\dots,n} \cap \mathfrak{p}_{n+1} : A_I].$$

By the inductive hypothesis we get

$$\mathfrak{p}_I = [[\bigcap_{i=1}^{n} \mathfrak{p}_i : A_{1,\ldots,n}] \cap \mathfrak{p}_{n+1} : A_I].$$

But the latter set clearly coincides with $[\bigcap_{i \in I} \mathfrak{p}_i : A_I]$.

General Case. By Proposition 33 we have $\mathfrak{p}_I = [\mathfrak{p}_J : A_I]$. By the special case we obtain

$$\mathfrak{p}_I = [[\bigcap_{i \in J} \mathfrak{p}_i : A_J] : A_I].$$

But the latter set clearly coincides with $[\bigcap_{i \in J} \mathfrak{p}_i : A_I]$. □

Proposition 3.35. *Let $v_1, \ldots, v_n$ be Manis valuations on R having the inverse property such that* $\operatorname{supp} v_i = \operatorname{supp} v_j$ *for all* $1 \le i, j \le n$. *Then* $v_1|_{A_{\bigvee v_j}}, \ldots, v_n|_{A_{\bigvee v_j}}$ *are independent Manis valuations on* $A_{\bigvee v_j}$ *having the inverse property.*

Proof. If $v_1, \ldots, v_n$ are independent then $A_{\bigvee v_j} = R$ and nothing is to show. So we assume that $v_1, \ldots, v_n$ are dependent. Let $u := \bigvee_{1 \le j \le n} v_j$. For $1 \le i \le n$ let $w_i := v_i|_{A_u}$. Suppose that $w_1, \ldots, w_n$ are dependent. Let $u' := \bigvee_{1 \le i \le n} w_i$. By Remark 2.6, $A_u \supset \prod_{i=1}^{n} A_{v_i}$. By Corollary 20 the valuations $w_1, \ldots, w_n$ have the inverse property. By Remark 2.2 we find some $x \in \mathfrak{p}_{u'} \setminus \operatorname{supp} w_1$. Then $x \notin \mathfrak{p}_u$ by Proposition 2.15. By Proposition 33 we get $x \notin [\mathfrak{p}_{v_1} : B]$ where $B := \prod_{i=2}^{n} A_{v_i}$. Hence we find some $a \in B$ such that $xa \notin \mathfrak{p}_{v_1}$ resp. $v_1(ax) \le 0$. By the above $B \subset A_u$. Hence $ax \in A_u$ and therefore $w_1(ax) = v_1(ax) \le 0$. But $ax \in \mathfrak{p}_{u'} \subset \mathfrak{p}_{w_1}$ by Remark 2.2. So $w_1(ax) > 0$, contradiction. □

4 Essential Valuations

The notion of essentiality in the field case (cf. [E]) and for rings of Krull type (cf. [G_4], [Al-O]) is generalized to arbitrary Manis valuations: A valuation over a subring A is called A-essential if v can be recovered from the trace of v on A. If the intersection ring A of a family of valuations is Prüfer then every valuation in this family is A-essential. We want to establish criteria for the converse. For this we introduce the important notion of a family of valuations having finite avoidance (compare with Definition 2 in Chap. 1, Sect. 1). (Compare the notion of "endlichem Typ" in the work of Gräter and of "finite character" in the work of Griffin and Alajbegović.) For finitely many valuations the inverse property is equivalent with essentiality. We establish this result for families having finite avoidance by defining the finite avoidance inverse property.

Definition 1. Let v be a valuation on R and let A be a subring of R.

a) If $A \subset A_v$ we say that v *is a valuation over* A.
b) Let v be a valuation over A. Then the prime ideal $\mathfrak{p}_v \cap A$ of A is called the *center* of v on A and is denoted by $\mathrm{cent}_A(v)$.
c) Let v be a valuation over A. We say that v is A*-essential* if $A_v = A_{[\mathrm{cent}_A(v)]}$.

Example 4.1. Let v be any valuation on R. Then v is A_v-essential by [Vol. I, Lemma III.1.0].

Remarks 4.2. Let v be any valuation on R.

i) $\mathfrak{p}_v$ is the center of v on A_v (compare with [Vol. I, p. 11]).
ii) Let A be a subring of R such that v is a valuation over A. Then $A_{[\mathrm{cent}_A(v)]} \subset A_v$.

Proposition 4.3. *Let A be a subring of R and let v be an A-essential valuation on R. Let B be an R-overring of A such that v is a valuation over B. Then v is B-essential.*

Proof. Let $\mathfrak{r}_1 := \mathrm{cent}_A(v)$ and $\mathfrak{r}_2 := \mathrm{cent}_B(v)$. By Remark 2(ii) it is enough to show $A_v \subset B_{[\mathfrak{r}_2]}$. Let $x \in A_v$. Since v is A-essential there is some $y \in A \setminus \mathfrak{r}_1$ such that $xy \in A$. Then clearly $y \in B \setminus \mathfrak{r}_2$ and $xy \in B$. Hence $x \in B_{[\mathfrak{r}_2]}$. □

Proposition 4.4. *Let A be a subring of R and let v be an A-essential valuation on R. Then $\mathfrak{p}_v = \mathrm{cent}_A(v)_{[\mathrm{cent}_A(v)]}$.*

Proof. Let $\mathfrak{r} := \mathrm{cent}_A(v)$. Obviously $\mathfrak{p}_v \supset \mathfrak{r}_{[\mathfrak{r}]}$. Let $x \in \mathfrak{p}_v \subset A_v$. Then since v is A-essential there is some $s \in A \setminus \mathfrak{r}$ with $xs \in A$. We have $v(s) = 0$ and therefore $v(xs) = v(x) > 0$. So $xs \in \mathfrak{p}_v \cap A = \mathfrak{r}$ and therefore $x \in \mathfrak{r}_{[\mathfrak{r}]}$. □

Proposition 4.5. *Let v, w be Manis valuations on R with $v \leq w$. Let A be a subring of R. If v is A-essential then w is A-essential.*

Proof. We have $\mathfrak{p}_w \subset \mathfrak{p}_v \subset A_v \subset A_w$ by Sect. 1. In particular w is a valuation over A. Let $\mathfrak{r}_v := \mathrm{cent}_A(v)$ and $\mathfrak{r}_w := \mathrm{cent}_A(w)$. Then $\mathfrak{r}_w \subset \mathfrak{r}_v$. By Remark 2(ii) it is enough to show $A_w \subset A_{[\mathfrak{r}_w]}$. Let $x \in A_w$.

Case 1: $x \in A_v$. Since $A_v = A_{[\mathfrak{r}_v]}$ by assumption there is some $s \in A \setminus \mathfrak{r}_v$ such that $xs \in A$. Since $A \setminus \mathfrak{r}_v \subset A \setminus \mathfrak{r}_w$ we are done.

Case 2: $x \in A_w \setminus A_v$. Then $x \in A_w \setminus \mathfrak{p}_w$ and therefore $w(x) = 0$. Since $x \notin A_v$ there is some $y \in \mathfrak{p}_v$ with $xy \in A_v \setminus \mathfrak{p}_v$. Therefore $xy \in A_w \setminus \mathfrak{p}_w$ and hence $w(xy) = 0$. We obtain $w(y) = 0$ and so $y \in \mathfrak{p}_v \setminus \mathfrak{p}_w$. From $A_v = A_{[\mathfrak{r}_v]}$ we get $s, t \in A \setminus \mathfrak{r}_v$ such that $xys \in A$ and $yt \in A$. Then $xyst \in A$. We have $s \in A \setminus \mathfrak{r}_w$, so we show that $yt \in A \setminus \mathfrak{r}_w$ and are done. But $y \in \mathfrak{p}_v \setminus \mathfrak{p}_w \subset A_w \setminus \mathfrak{p}_w$ and $t \in A \setminus \mathfrak{r}_w$, hence $yt \notin \mathfrak{p}_w$. □

Corollary 4.6. *Let v, w be a Manis valuations on R such that $v \leq w$. Then w is B-essential for every ring B with $A_v \subset B \subset A_w$.*

Proof. This follows immediately from Example 1, Propositions 3 and 5. □

In particular we obtain in the above situation $A_w = (A_v)_{[\mathrm{cent}_{A_v}(w)]}$ and $\mathfrak{p}_w = \mathrm{cent}_{A_v}(w)$ (cf. Scholium 1.10).

Theorem 4.7. *Let $A \subset B$ be subrings of R and let v be an A-essential Manis valuation on R. Then the special restriction $v|_B$ is a Manis valuations on B that is A-essential.*

Proof. Let $\mathfrak{r} := \mathrm{cent}_A(v)$ and $w := v|_B$.

a) We show that w is A-essential. Since v is A-essential we have

$$A_v = A^R_{[\mathfrak{r}]} = \{x \in R \mid \exists\, s \in A \setminus \mathfrak{r} \text{ such that } xs \in A\}$$

and

$$\mathfrak{p}_v = \mathfrak{r}^R_{[\mathfrak{r}]} = \{x \in R \mid \exists\, s \in A \setminus \mathfrak{r} \text{ such that } xs \in \mathfrak{r}\}.$$

Since $A_w = A_v \cap B$ and $\mathfrak{p}_w = \mathfrak{p}_v \cap B$ we get that

$$A_w = A^B_{[\mathfrak{r}]} = \{x \in B \mid \exists\, s \in A \setminus \mathfrak{r} \text{ such that } xs \in A\}$$

and

$$\mathfrak{p}_w = \mathfrak{r}^B_{[\mathfrak{r}]} = \{x \in B \mid \exists\, s \in A \setminus \mathfrak{r} \text{ such that } xs \in \mathfrak{r}\}.$$

This shows that w is A-essential.

We show that w is Manis. Let $x \in B$ with $w(x) < \infty$. We need to find some $x' \in B$ with $w(xx') = 0$.

Case 1: $w(x) \leq 0$. Then $w(x) = v(x) < \infty$. Since v is Manis there is some $y \in R$ such that $v(y) = -v(x) \geq 0$. By the above we have some $z \in A$ and $s \in A \setminus \mathfrak{r}$ with $ys = z$, hence $v(z) = v(y) = -v(x)$. Then $x' := z \in B$ and $w(xx') = 0$.

Case 2: $w(x) > 0$. Since w is special, there exists some $y \in B$ with $w(y) \leq -w(x)$, hence $w(xy) \leq 0$. Then by Case 1 we obtain some $z \in B$ with $w(xyz) = 0$. So $x' := yz \in B$ does the job. □

Theorem 4.8. *Let A be Prüfer in R and let v be a Manis valuation over A. Then v is A-essential.*

Proof. Let $\mathfrak{r} := \mathrm{cent}_A(v)$. Since A is Prüfer the pair $(A_{[\mathfrak{r}]}, \mathfrak{r}_{[\mathfrak{r}]})$ is a Manis pair. By Remarks 2(ii) we know that $A_{[\mathfrak{r}]} \subset A_v$. We show that $\mathfrak{p}_v \cap A_{[\mathfrak{r}]} = \mathfrak{r}_{[\mathfrak{r}]}$ and obtain by [Vol. I, Theorem I.2.4 i) ⇒ ii)] that $A_{[\mathfrak{r}]} = A_v$.

Let $x \in \mathfrak{p}_v \cap A_{[\mathfrak{r}]}$. Then $v(x) > 0$ and there is some $s \in A \setminus \mathfrak{r}$ such that $xs \in A$. Since $v(s) = 0$ we get $v(xs) > 0$ and hence $xs \in \mathfrak{r}$. This shows that $x \in \mathfrak{r}_{[\mathfrak{r}]}$. Let

$x' \in \mathfrak{r}_{[\mathfrak{r}]}$. Then there is some $s' \in A \setminus \mathfrak{r}$ with $x's' \in \mathfrak{r}$. So $v(x') = v(x's') > 0$ and therefore $x' \in \mathfrak{p}_v \cap A_{[\mathfrak{r}]}$. □

Corollary 4.9. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. If $A := \bigcap_{i \in I} A_{v_i}$ is Prüfer in R then every v_i is A-essential.*

Corollary 4.10. *Let A be Prüfer in R and let v be a Manis valuation over A. Let B be an R-overring of A. Then $v|_B$ is a PM-valuation on B.*

Proof. Let $w := v|_B$. By Theorem 7 w is a Manis valuation on B. By the definition of a special restriction (cf. [Vol. I, Definition 11 in I §1]) we have $A_w = B \cap A_v \supset A$. By [Vol. I, Corollary I.5.3] A_w is Prüfer in B. Hence w is a PM-valuation. □

Corollary 4.11. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. Then the following are equivalent.*

(1) $\bigcap_{i \in I} A_{v_i}$ is Prüfer in R.
(2) For all $i \in I$ we have that $\bigcap_{j \neq i} A_{v_j}$ is Prüfer in R and $v_i|_{\bigcap_{j \neq i} A_{v_j}}$ is a PM-valuation on $\bigcap_{j \neq i} A_{v_j}$.
(3) There is some $i \in I$ such that $\bigcap_{j \neq i} A_{v_j}$ is Prüfer in R and $v_i|_{\bigcap_{j \neq i} A_{v_j}}$ is a PM-valuation on $\bigcap_{j \neq i} A_{v_j}$.

Proof. Let $A := \bigcap_{i \in I} A_{v_i}$. For $i \in I$ let $B_i := \bigcap_{j \neq i} A_{v_i}$ and $w_i := v_i|_{B_i}$. By the very definition of special restriction we see that $A_{w_i} = A_{v_i} \cap B_i = A$.

(1) ⇒ (2): For $i \in I$ v_i is a valuation over A. By Corollary 10 we obtain that w_i is a PM-valuation on B_i.

(2) ⇒ (3): This is trivial.

(3) ⇒ (1): Let $i \in I$ such that the property in (3) holds. Since w_i is a PM-valuation by assumption we get that $A = A_{w_i}$ is Prüfer in B_i. Since B_i is Prüfer in R by assumption we get that A is Prüfer in R by [Vol. I, Theorem I.5.6]. □

Corollary 4.12. *Let v, w be PM-valuations on R. The following are equivalent.*

(1) $A_v \cap A_w$ is Prüfer in R.
(2) $v|_{A_w}$ is a PM-valuation on A_w.
(3) $w|_{A_v}$ is a PM-valuation on A_v.

Proposition 4.13. *Let $v_1, \ldots, v_n$ be Manis valuations on R and let $A \subset A_{v_1} \cap \ldots \cap A_{v_n}$ be Prüfer in R. Let B be an R-overring of A. If $v_1|_B, \ldots, v_n|_B$ are dependent then $v_1, \ldots, v_n$ are dependent and then $\bigvee v_i|_B = (\bigvee v_i)|_B$.*

Proof. Let $w_i := v_i|_B$ for $1 \leq i \leq n$. By Corollary 10 these valuations are Manis on B. We have

$$\bigcap_{i \leq i \leq n} A_{w_i} = \bigcap_{1 \leq i \leq n} A_{v_i} \cap B \supset A.$$

By [Vol. I, Corollary I.5.3] $\bigcap_{1\le i\le n} A_{w_i}$ is Prüfer in B. We assume that $w_1,\dots,w_n$ are dependent.

a) We show that $v_1,\dots,v_n$ are dependent. We have $\prod_{i=1}^n A_{w_i} \neq B$ by Corollary 2.8 and $A_{\bigvee w_i} = \prod_{i=1}^n A_{w_i}$ by Corollary 2.11. Again by Corollary 2.8 we have to show that $\prod_{i=1}^n A_{v_i} \neq R$. Assume this does not hold. Since $\prod_{i=1}^n A_{w_i} \neq B$ we find some $a \in B$ with $(\bigvee w_i)(a) < 0$. Clearly $a \in R = \prod_{i=1}^n A_{v_i}$. Hence there is some $q \in \mathbb{N}$ and $b_{i,1},\dots,b_{i,q} \in A_{v_i}$ for $1 \le i \le n$ such that $a = \sum_{p=1}^q \prod_{i=1}^n b_{i,p}$. The valuations $v_1,\dots,v_n$ are A-essential by Theorem 8. Hence there are $s_{i,p} \in A\backslash \mathrm{cent}_A(v_i)$ such that $b_{i,p}s_{i,p} \in A$ for $1 \le i \le n$ and $1 \le p \le q$. By Proposition 1.8 we have for $1 \le i \le n$

$$\mathfrak{p}_{\bigvee w_j} \subset \mathfrak{p}_{w_i} = \mathrm{cent}_B(v_i) \subset \mathfrak{p}_{v_i}.$$

Hence $(\bigvee w_j)(s_{i,p}) = 0$ for all $1 \le i \le n$ and $1 \le p \le q$. Let $x := \prod_{i=1}^n \prod_{p=1}^q s_{i,p}$. For $1 \le p \le q$ we get

$$(\bigvee w_j)\left(\prod_{i=1}^n b_{i,p}x\right) = (\bigvee w_j)\left(\prod_{i=1}^n b_{i,p}s_{i,p}\right) \ge 0$$

since $b_{i,p}s_{i,p} \in A \subset A_{\bigvee w_j}$. Hence

$$\begin{aligned}(\bigvee w_j)(ax) &= (\bigvee w_j)\left(\sum_{p=1}^q (\prod_{i=1}^n b_{i,p}x)\right)\\ &\ge \min\{(\bigvee w_j)\left(\prod_{i=1}^n b_{i,p}x\right) \mid 1 \le p \le q\} \ge 0.\end{aligned}$$

But $(\bigvee w_j)(ax) = (\bigvee w_j)(a) < 0$, contradiction.

b) We have

$$A_{w_1} \cap \ldots \cap A_{w_n} = A_{v_1} \cap \ldots \cap A_{v_n} \cap B = A.$$

By [Vol. I, Corollary I.5.3] A is Prüfer in B. By the definition of special restriction and Scholium 1.16 we have to show that $A_{\bigvee w_i} = A_{\bigvee v_i} \cap B$. By a) $v_1,\dots,v_n$ are dependent. The proof has shown that $(\prod_{i=1}^n A_{v_i}) \cap B \subset \prod_{i=1}^n A_{w_i}$. Since

$$\prod_{i=1}^n A_{w_i} = \prod_{i=1}^n (A_{v_i} \cap B) \subset (\prod_{i=1}^n A_{v_i}) \cap B$$

we get that $\prod_{i=1}^n A_{w_i} = (\prod_{i=1}^n A_{v_i}) \cap B$. Hence $A_{\bigvee w_i} = A_{\bigvee v_i} \cap B$ by Corollary 2.11. $\square$

We introduce the important notion of a family of Manis valuations with finite avoidance. Many theorems for the finite case can be extended to this more general situation.

Definition 2. A family of Manis valuations $(v_i \mid i \in I)$ on R has *finite avoidance* if the family $(A_{v_i} \mid i \in I)$ has finite avoidance in the sense of Definition 2 in Chap. 1, Sect. 1; i.e. for every $x \in R$ the set of indices $i \in I$ such that $v_i(x) < 0$ is finite.

The following will be useful for later.

Remark 4.14. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having finite avoidance. Let $i \in I$ such that v_i is non-trivial. Then the set $\{j \in I \mid v_j \leq v_i\}$ is finite.

Proof. Let J be the above set. Since v_i is non-trivial there is some $x \in R$ such that $v_i(x) < 0$. By the finite avoidance property there is a finite set $\tilde{J} \subset I$ such that $v_k(x) \geq 0$ for all $k \in I \setminus \tilde{J}$. Hence $J \subset \tilde{J}$ and we are done. $\square$

Proposition 4.15. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having finite avoidance. Let $A := \bigcap_{i \in I} A_{v_i}$. The following are equivalent.*

(1) A is Prüfer in R.
(2) $A \subset R$ is a PF-extension (cf. Definition 1 in Chap. 1, Sect. 4).

Proof. (1) ⇒ (2): We can assume that all v_i are non-trivial. We can also assume that the valuations v_i are pairwise incomparable (by taking only the minimal elements with respect to $\leq$, using Remark 14). By Theorem 1.4.1 we get that A is a PF-extension.

(2) ⇒ (1): This is obvious by the definition of a PF-extension. $\square$

Theorem 4.16. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having finite avoidance such that $A := \bigcap_{i \in I} A_{v_i}$ is Prüfer in R. Let $\mathfrak{m}$ be a maximal ideal of A such that $\mathfrak{m} \neq \mathrm{cent}_A(v_i)$ for all $i \in I$ and let w be a non-trivial Manis valuation on R such that $v_i \leq w$ for some $i \in I$. Then $\mathrm{cent}_A(w) \not\subset \mathfrak{m}$.*

Proof. It clearly suffices to do the proof in the case that all v_i are non-trivial. Moreover, we can assume that the valuations are pairwise incomparable. Then $A_{v_i} \not\subset A_{v_j}$ for $i \neq j$ by Corollary 1.17. Let $\mathfrak{r}_i := \mathrm{cent}_A(v_i)$ for $i \in I$. Then $A_{v_i} = A_{[\mathfrak{r}_i]}$ by Theorem 8. Hence $\mathfrak{r}_i \not\subset \mathfrak{r}_j$ for $i \neq j$. By [Vol. I, Lemma III.1.1] we have that $\mathfrak{r}_i$ is an R-regular prime ideal of A for all $i \in I$. Similarly to Proposition 1.1.7 we see that $\{\mathfrak{r}_i \mid i \in I\}$ is the set of all maximal R-regular prime ideals of A. So if $\mathfrak{m}$ is a maximal ideal of A with $\mathfrak{m} \neq \mathfrak{r}_i$ for all $i \in I$ then $\mathfrak{m}$ is not R-regular.

Let now w be a non-trivial Manis valuation on R such that $v_i \leq w$ for some $i \in I$. By Proposition 5 we get that w is A-essential, so $A_{[\mathrm{cent}_A(w)]} = A_w \neq R$. Therefore again by [Vol. I, Lemma III.1.1] the ideal $\mathrm{cent}_A(w)$ of A is R-regular and so $\mathrm{cent}_A(w) \not\subset \mathfrak{m}$, simply by [Vol. I, Definition 1 in II §1]. $\square$

We show a result converse to Corollary 9 (in the case of finite avoidance) and Theorem 16 for PM-valuations.

Theorem 4.17. *Let $(v_i \mid i \in I)$ be a family of PM-valuations on R having finite avoidance. Let $A := \bigcap_{i \in I} A_{v_i}$. Assume that the following properties hold.*

i) Every v_i is A-essential.
ii) If $\mathfrak{m}$ is a maximal ideal of A such that $\mathfrak{m} \neq \operatorname{cent}_A(v_i)$ for all $i \in I$ and if w is a non-trivial Manis valuation on R such that $v_i \leq w$ for some $i \in I$, then $\operatorname{cent}_A(w) \not\subset \mathfrak{m}$.

Then A is Prüfer in R.

Proof. Let $\mathfrak{m}$ be a maximal ideal of A. We show that $(A_{[\mathfrak{m}]}, \mathfrak{m}_{[\mathfrak{m}]})$ is a Manis pair and are done.

Case 1: $\mathfrak{m} = \operatorname{cent}_A(v_i)$ for some $i \in I$. Since v_i is A-essential by (i) we have $A_{[\mathfrak{m}]} = A_{v_i}$. By Proposition 4 we get $\mathfrak{m}_{[\mathfrak{m}]} = \mathfrak{p}_v$. Since v_i is Manis by assumption we are done.

Case 2: $\mathfrak{m} \neq \operatorname{cent}_A(v_i)$ for all $i \in I$. It is enough to show $A_{[\mathfrak{m}]} = R$. Suppose that $A_{[\mathfrak{m}]} \neq R$. Let $x \in R \setminus A_{[\mathfrak{m}]}$. Then

$$(A :_A x) = \{a \in A \mid xa \in A\} \subset \mathfrak{m}.$$

We have $(A :_A x) = \bigcap_{i \in I} (A_{v_i} :_A x)$. Since the family $(v_i \mid i \in I)$ has finite avoidance there are $i_1, \ldots, i_n$ in I such that $v_i(x) \geq 0$ for all $i \notin J := \{i_1, \ldots, i_n\}$. Hence $A = (A_{v_i} :_A x)$ for all $i \notin J$. This gives $\bigcap_{1 \leq l \leq n} (A_{v_{i_l}} :_A x) \subset \mathfrak{m}$ and therefore $(A_{v_{i_k}} :_A x) \subset \mathfrak{m}$ for some $1 \leq k \leq n$. We write $j := i_k$.

Case 2.1: v_j is trivial. Then $(A_{v_j} :_{A_{v_j}} x) = R$ and hence

$$(A_{v_j} :_A x) = (A_{v_j} :_{A_{v_j}} x) \cap A = A.$$

We get $\mathfrak{m} = A$, contradiction.

Case 2.2: v_j is non-trivial. From $(A_{v_j} :_A x) \subset \mathfrak{m}$ we conclude $\sqrt{(A_{v_j} :_A x)} \subset \mathfrak{m}$. Obviously the ideals $(A_{v_j} :_{A_{v_j}} x)$ and $\sqrt{(A_{v_j} :_{A_{v_j}} x)}$ of A_{v_j} are v_j-convex. Therefore $\sqrt{(A_{v_j} :_{A_{v_j}} x)}$ is a prime ideal of A_{v_j}. Since v_j is Manis we have $(A_{v_j} :_{A_{v_j}} x) \supsetneqq \operatorname{supp} v_j$. With [Vol. I, Theorem III.2.5] we get that $\mathfrak{n} := \sqrt{(A_{v_j} :_{A_{v_j}} x)}$ is an R-regular prime ideal of A_{v_i} contained in $\mathfrak{p}_{v_j}$. By Scholium 1.10 and Proposition 1.12 we obtain a non-trivial coarsening w of v_j with $A_w = (A_{v_j})_{[\mathfrak{n}]}$ and $\mathfrak{p}_w = \mathfrak{n}$. Since $\sqrt{(A_{v_j} :_{A_{v_j}} x)} \cap A = \sqrt{(A_{v_j} :_A x)}$ we get $\mathfrak{p}_w \cap A \subset \mathfrak{m}$, contradiction to (ii). $\square$

The next results establish the connection between the inverse property and essential valuations.

Theorem 4.18. *Let $v_1, \ldots, v_n$ be Manis valuations on R and let $A := A_{v_1} \cap \ldots \cap A_{v_n}$. The following are equivalent.*

(1) $v_1, \ldots, v_n$ have the inverse property.
(2) Every v_i is A-essential.

Proof. Let $\mathfrak{r}_i := \mathrm{cent}_A(v_i)$ for $1 \le i \le n$.

(1) $\Rightarrow$ (2): Let $i \in \{1, \ldots, n\}$. Clearly v_i is a valuation over A. By Remarks 2(ii) it is enough to show $A_{v_i} \subset A_{[\mathfrak{r}_i]}$. Let $x \in A_{v_i}$. By Proposition 3.6 there is some $y \in R$ such that for $1 \le j \le n$ $v_j(y) = 0$ if $v_j(x) \ge 0$ and $v_j(y) \le v_j(x)$ if $v_j(x) \le 0$. Since $v_1, \ldots, v_n$ have the inverse property we get some $s \in R$ such that $v_j(s) = -v_j(y)$ for all $1 \le j \le n$. Then $v_j(s) \ge 0$ for all $1 \le j \le n$ and $v_i(s) = 0$. This gives $s \in A \setminus \mathfrak{r}_i$. We have $v_j(xs) = v_j(x) - v_j(y) \ge 0$ for all $1 \le j \le n$. Hence $xs \in A$. This shows that $x \in A_{[\mathfrak{r}_i]}$.

(2) $\Rightarrow$ (1): By Proposition 3.21 we can assume that v_i, v_j are incomparable for all $i \neq j$.

Special Case: We assume that every v_i is non-trivial. We do induction on n.

$n = 1$: $\{v_1\}$ has the inverse property since v_1 is Manis (cf. Remark 3.1(a)).

$n \to n+1$: Let $x \in R$. By the inductive hypothesis and Proposition 3 we may assume that $v_i(x) \neq \infty$ for all $1 \le j \le n+1$ (otherwise we can omit those v_i with $v_i(x) = \infty$). We have to find some $x' \in R$ with $v_i(x') = -v_i(x)$ for all $1 \le i \le n+1$. Let $B := \bigcap_{i=1}^{n} A_{v_i}$. By Proposition 3 $v_1, \ldots, v_n$ are B-essential. By the inductive hypothesis the valuations $v_1, \ldots, v_n$ have the inverse property.

Claim 1: There is some $y \in R$ such that

$$v_1(y) = 0, v_2(y) > 0, \ldots, v_{n+1}(y) > 0.$$

Proof of Claim 1: By Corollary 3.23 there is some $y' \in R$ with $v_1(y') = 0$ and $0 < v_i(y') < \infty$ for all $2 \le i \le n$. We may assume that $v_{n+1}(y') \ge 0$. Otherwise, since $A_{v_1} = A_{[\mathfrak{r}_1]}$ there is some $a \in A \setminus \mathfrak{r}_1$ with $y'a \in A$. Then we replace y' by $y'a$. Applied the same arguments to the valuations $v_1, v_3, \ldots, v_{n+1}$ we find some $y'' \in R$ with $v_1(y'') = 0, v_2(y'') \ge 0$ and $v_i(y'') > 0$ for all $3 \le i \le n+1$. With $y := y'y''$ we get $v_1(y) = 0$ and $v_i(y) > 0$ for all $2 \le i \le n+1$.

By above $v_1, \ldots, v_n$ have the inverse property. Hence there is some $z_1 \in R$ with $v_1(xz_1) = 0$ and $v_i(xz_1) \ge 0$ for all $2 \le i \le n$. As in the proof of Claim 1 we can assume that $v_{n+1}(xz_1) \ge 0$. Let $y_1 \in R$ be as in Claim 1. Then $v_1(xy_1z_1) = 0$ and $v_i(xy_1z_1) > 0$ for all $2 \le i \le n+1$. In the same way we obtain $y_2, \ldots, y_{n+1} \in R$ and $z_2, \ldots, z_{n+1} \in R$ such that $v_i(xy_iz_i) = 0$ and $v_j(xy_iz_i) > 0$ for all $j \neq i$. Let $x' := y_1z_1 + \ldots + y_nz_n$. Then $v(xx') = 0$ for all $1 \le i \le n$.

General Case: We may assume that $v_1, \ldots, v_k$ are trivial and $v_{k+1}, \ldots, v_{n+1}$ are non-trivial for some $0 \le k \le n+1$. We do induction on k.

$k = 0$: This is covered by the special case.

$k \to k+1$: We may assume that $\operatorname{supp} v_1$ is minimal in $\{\operatorname{supp} v_i \mid 1 \le i \le k+1\}$ with respect to inclusion.

Claim 2: Let $\varepsilon_i \in \Gamma_{v_i}$ for $k+2 \leq i \leq n$. Then there is some $z \in R$ such that

$$v_1(z) = 0, v_2(z) = \infty, \ldots, v_{k+1}(z) = \infty, v_{k+2}(z) > \varepsilon_{k+2}, \ldots, v_n(z) > \varepsilon_n.$$

Proof of Claim 2: We can assume that $\varepsilon_i > 0$ for $k+2 \leq i \leq n$. By Proposition 3.11 there is some $z' \in R$ with $v_i(z') < -\varepsilon_i$ for all $k+2 \leq i \leq n$. We have $A_{v_1} = R = A_{[\mathfrak{r}_1]}$ by assumption. Hence there is some $s \in A \setminus \mathfrak{r}_1$ with $z's \in A$, We get that $v_1(s) = 0$ and $v_i(s) > \varepsilon_i$ for $k+1 \leq i \leq n$. By assumption $\operatorname{supp} v_1$ is minimal in $\{\operatorname{supp} v_i \mid 1 \leq i \leq k+1\}$. Since v_i and v_j are incomparable for $i \neq j$ we have $\operatorname{supp} v_1 \neq \operatorname{supp} v_i$ for $2 \leq i \leq k+1$. Hence there is for $2 \leq i \leq k+1$ some $b_i \in \operatorname{supp} v_i \setminus \operatorname{supp} v_1$. Let $b := sb_2 \cdot \ldots \cdot b_{k+1}$. Then $v_1(b) = 0$ and $v_i(b) = \infty$ for $2 \leq i \leq k+1$. Since $R = A_{[\mathfrak{r}_1]}$ there is some $t \in A \setminus \mathfrak{r}_1$ such that $c := bt \in A$. We obtain

$$v_1(c) = 0, v_2(c) = \infty, \ldots, v_{k+1}(c) = \infty, v_{k+2}(c) \geq 0, \ldots, v_n(c) \geq 0.$$

Let $z := sc$. Then z fulfills the requirements of the claim.

Let now $x \in R$. We have to find some $x' \in R$ with $v_i(x') = -v_i(x)$ for all $1 \leq i \leq n$ with $v_i(x) < \infty$. By the inductive hypothesis there is some $y \in R$ such that $v_i(y) = -v_i(x)$ for all $2 \leq i \leq n$ with $v_i(x) < \infty$. If $v_1(x) = \infty$ nothing is to show. So we assume that $v_1(x) = 0$. If $v_1(y) = 0$ we are done. Otherwise $v_1(y) = \infty$. By the claim there is some $z \in R$ such that $v_1(z) = 0$, $v_i(z) = \infty$ for $2 \leq i \leq k+1$ and $v_j(z) > v_j(y)$ for all $k+1 \leq j \leq n$ with $v_j(y) < \infty$. Let $x' := y + z$. Then $v_i(x') = -v_i(x)$ for all $1 \leq i \leq n$ such that $v_i(x) \neq \infty$. □

We want to extend Theorem 18 to the case of finite avoidance. For that reason we introduce the following technical definition.

Definition 3. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. We say that it has the *finite avoidance inverse property* if for all $i_1, \ldots, i_n \in I$ and all R-overrings B of $\bigcap_{i \in I} A_{v_i}$ the special restrictions $v_{i_1}|_B, \ldots, v_{i_n}|_B$ are Manis valuations on B having the inverse property.

Proposition 4.19. *Let $v_1, \ldots, v_n$ be finitely many Manis valuations on R. The following are equivalent.*

(1) $v_1, \ldots, v_n$ have the inverse property.
(2) $v_1, \ldots, v_n$ have the finite avoidance inverse property.

Proof. (1) $\Rightarrow$ (2): Let $A := \bigcap_{1 \leq i \leq n} A_{v_i}$. Let $i_1, \ldots, i_q \in \{1, \ldots, n\}$ and let B be an R-overring of A. We have to show that $v_{i_1}|_B, \ldots, v_{i_q}|_B$ are Manis valuations on B having the inverse property. We write $w_p := v_{i_p}|_B$ for $1 \leq p \leq q$. By Theorems 18 and 7 $w_1, \ldots, w_q$ are Manis valuations on B and A-essential. Let $C := \bigcap_{1 \leq p \leq q} A_{w_i}$. Clearly $A \subset C \subset B$. By Proposition 3 we get that $w_1, \ldots, w_q$ are C-essential. By Theorem 18 we get that they have the inverse property.

(2) $\Rightarrow$ (1): Apply (2) to $1, \ldots, n$ and $B = R$. □

Remark 4.20. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having the finite avoidance inverse property.

a) For $J \subset I$, the subfamily $(v_i \mid i \in J)$ has the finite avoidance inverse property.
b) The family $(v_i \mid i \in I)$ has the finite inverse property.

Proof. a): This is clear from the definition.
b): This follows from (a) and Proposition 19. □

Theorem 4.21. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having finite avoidance and let $A := \bigcap_{i \in I} A_{v_i}$. The following are equivalent.*

(1) $(v_i \mid i \in I)$ has the finite avoidance inverse property.
(2) Every v_i is A-essential.

Proof. Let $\mathfrak{r}_i := \operatorname{cent}_A(v_i)$ for $i \in I$.

(1) ⇒ (2): Let $i_0 \in I$. Clearly v_{i_0} is a valuation over A. By Remark 2(ii) it is enough to show $A_{v_{i_0}} \subset A_{[\mathfrak{r}_{i_0}]}$. Let $x \in A_{v_{i_0}}$. Since $(v_i \mid i \in I)$ has finite avoidance there is a finite subset $J = \{i_0, \dots, i_n\}$ of I such that $x \in A_{v_k}$ for all $k \notin J$. (Enlarging J if necessary we can assume that $i_0 \in J$.) Let $B := \bigcap_{k \notin J} A_{v_k}$. Then $x \in B$. Since $(v_i \mid i \in I)$ has the finite avoidance inverse property the valuations $v_{i_1}|_B, \dots, v_{i_n}|_B$ are Manis with the inverse property. Let $w_k := v_{i_k}|_B$ for $0 \le k \le n$. We have

$$\bigcap_{0 \le k \le n} A_{w_k} = \bigcap_{0 \le k \le n} (A_{v_{i_k}} \cap B) = (\bigcap_{0 \le k \le n} A_{v_{i_k}}) \cap B = A$$

and

$$\operatorname{cent}_A(w_0) = \mathfrak{p}_{w_0} \cap A = \mathfrak{p}_{v_{i_0}} \cap B \cap A = \operatorname{cent}_A(v_{i_0}) = \mathfrak{r}_{i_0}.$$

By Theorem 18, (1) ⇒ (2), we have that w_0 is A essential. Hence there is some $s \in A \setminus \mathfrak{r}_{i_0}$ such that $xs \in A$.

(2) ⇒ (1): Let $i_1, \dots, i_n \in I$ and let B be an R-overring of A. Let $w_k := v_{i_k}|_B$ for $1 \le k \le n$. By Theorem 7 the valuations $w_1, \dots, w_n$ are Manis on B and A-essential. We have

$$\bigcap_{1 \le k \le n} A_{w_k} = \bigcap_{1 \le k \le n} (A_{v_{i_k}} \cap B) = (\bigcap_{1 \le k \le n} A_{v_{i_k}}) \cap B \supset A.$$

By Proposition 3 $w_1, \dots, w_n$ are $\bigcap_{1 \le k \le n} A_{w_k}$-essential. By Theorem 18, (2) ⇒ (1), we get that $w_1, \dots, w_n$ have the inverse property. □.

Corollary 4.22. *Let $(v_i \mid i \in I)$ be a family of Manis valuation on R having finite avoidance. If $\bigcap_{i \in I} A_{v_i}$ is Prüfer in R then the family has the finite avoidance inverse property.*

Proof. Let $A := \bigcap_{i \in I} A_{v_i}$. By Corollary 9 v_i is A-essential for every $i \in I$. By Theorem 21 the family has the finite avoidance inverse property. □

5 The Approximation Theorem in the Neighbourhood of Zero

The approximation theorem in the neighbourhood of zero for finitely many valuations is equivalent to the inverse property (and essentiality). Therefore the approximation theorem holds if the intersection ring of the finitely many valuations is Prüfer. Extending the existing literature (cf. [Gr_1], [Al-O], [Al-M]) we introduce the notion of the approximation theorem in the neighbourhood of zero for arbitrary families. It implies the finite avoidance inverse property and is implied by the so-called strong finite avoidance inverse property and the Prüfer condition.

Proposition 5.1. *Let v, w be Manis valuations on R with the inverse property. Then the following are equivalent.*

(1) For any $\varepsilon \in \Gamma_v$ with $\varepsilon \geq 0$ there is some $x \in R$ with $v(x) \geq \varepsilon$ and $w(x) \leq 0$.
(2) For any $\varepsilon \in \Gamma_v$ with $\varepsilon \geq 0$ there is some $x \in R$ with $v(x) = \varepsilon$ and $w(x) \leq 0$.
(3) For any $\varepsilon \in \Gamma_v$ with $\varepsilon \geq 0$ there is some $x \in R$ with $v(x) \geq \varepsilon$ and $w(x) = 0$.
(4) For any $\varepsilon \in \Gamma_v$ with $\varepsilon \geq 0$ there is some $x \in R$ with $v(x) = \varepsilon$ and $w(x) = 0$.
(5) For any $\varepsilon \in \Gamma_v$ there is some $x \in R$ with $v(x) = \varepsilon$ and $w(x) = 0$.
(6) For any $\varepsilon \in \Gamma_v$ and $\eta \in \Gamma_w$ there is some $x \in R$ with $v(x) = \varepsilon$ and $w(x) = \eta$.

Proof. Clearly condition (6) implies all the other conditions. We verify (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6), and then will be done.

(1) ⇒ (2): By (1), there is some $y \in R$ with $v(y) \geq \varepsilon$ and $w(y) \leq 0$. If $v(y) = \varepsilon$ we are done. We assume that $v(y) > \varepsilon$. Since v is Manis there is some $z \in R$ with $v(z) = \varepsilon$. If $w(z) \leq 0$ we are done. So we may assume that $w(z) > 0$. With $x := y + z$ we obtain $v(x) = v(y + z) = v(z) = \varepsilon$ and $w(x) = w(y) \leq 0$.

(2) ⇒ (3): For $\varepsilon = 0$ we can take $x = 1$. So we assume that $\varepsilon > 0$. By (2) there is some $y \in R$ such that $v(y) \geq \varepsilon$ and $w(y) \leq 0$. If $w(y) = 0$ we are done. Hence we assume that $w(y) < 0$. Then $v(1+y) = v(1) = 0$ and $w(1+y) = w(y) < 0$. Since v, w have the inverse property there is some $z \in R$ with $v(z) = -v(1 + y) = 0$ and $w(z) = -w(1 + y) = -w(y)$. With $x := yz$ we get $v(x) = v(y) \geq \varepsilon$ and $w(x) = w(y) + w(z) = w(y) - w(y) = 0$.

(3) ⇒ (4): For $\varepsilon = 0$ we can take $x = 1$. So we assume that $\varepsilon > 0$. By (3) there is some $y \in R$ with $v(y) \geq \varepsilon$ and $w(y) = 0$. If $v(y) = \varepsilon$ we are done. Hence we assume that $v(y) > \varepsilon$. Since v is Manis there is some $z \in R$ with $v(z) = \varepsilon$.

Case 1: $w(z) = 0$. Then we can take $x := z$.

Case 2: $w(z) > 0$. Let $x := y + z$. Then $v(x) = v(z) = \varepsilon$ and $w(x) = w(y) = 0$.

Case 3: $w(z) < 0$. Then $v(1+z) = v(1) = 0$ and $w(1+z) = w(z) < 0$. Since v, w have the inverse property there is some $z' \in R$ with $v(z') = -v(1+z) = 0$ and $w(z') = -w(1+z) = -w(z)$. With $x := zz'$ we get $v(x) = v(z) + v(z') = \varepsilon$ and $w(x) = w(z) + w(z') = w(z) - w(z) = 0$.

(4) $\Rightarrow$ (5): By (4), we may assume that $\varepsilon < 0$. Again by (4) there is some $y \in R$ with $v(y) = -\varepsilon$ and $w(y) = 0$. Since v, w have the inverse property there is some $x \in R$ such that $v(x) = -v(y) = \varepsilon$ and $w(x) = -w(y) = 0$.

(5) $\Rightarrow$ (6):

Case 1: $\eta = 0$. The claim follows with (5).

Case 2: $\eta < 0$. Since w is Manis there is some $y \in R$ with $w(y) = \eta$. We may assume that $v(y) \neq \infty$. (Otherwise, we replace y by $1 + y$.) By (5) there is some $z \in R$ with $v(z) = \varepsilon - v(y)$ and $w(z) = 0$. Let $x := yz$. Then $v(x) = \varepsilon$ and $w(x) = w(y) = \eta$.

Case 3: $\eta > 0$. By Case 2 there is an element $y \in R$ such that $v(y) = -\varepsilon$ and $w(y) = -\eta$. Since v and w have the inverse property the claim follows. □

Proposition 5.2. *Let $v_1, \ldots, v_n$ be Manis valuations on R with the inverse property. Let $\varepsilon_i \in \Gamma_{v_i}$ for $1 \leq i \leq n$. The following are equivalent.*

(1) There is some $x \in R$ with $v_1(x) = \varepsilon_1, \ldots, v_n(x) = \varepsilon_n$.
(2) For each pair (i, j) there is some $x \in R$ with $v_i(x) = \varepsilon_i$ and $v_j(x) = \varepsilon_j$.

Proof. (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (1):

Claim 1: If $v_1 \geq v_2$ it is enough to show (1) for $v_2, \ldots, v_n$.

Proof of Claim 1: Assume that there is some $x \in R$ such that $v_2(x) = \varepsilon_2, \ldots, v_n(x) = \varepsilon_n$. By (2) there is some $y \in R$ such that $v_1(y) = \varepsilon_1$ and $v_2(y) = \varepsilon_2$. Let $f : \Gamma_{v_2} \cup \{\infty\} \to \Gamma_{v_1} \cup \{\infty\}$ be the homomorphism of ordered monoids such that $v_1 = f \circ v_2$. Then

$$v_1(x) = f(v_2(x)) = f(\varepsilon_2) = f(v_2(y)) = v_1(y) = \varepsilon_1.$$

By Claim 1 we may assume that $v_i \not\leq v_j$ for all $i \neq j$. (Otherwise, if $v_i \leq v_j$ for some i, j, we may omit v_j.)

Special Case: $v_1, \ldots, v_n$ are non-trivial.

Claim 2: There is some $y \in R$ such that $v_1(y) = \varepsilon_1, v_2(y) > \varepsilon_2, \ldots, v_n(y) > \varepsilon_n$.

Proof of Claim 2: We prove Claim 2 by induction on n.

$n = 1$: This is obvious since v_1 is Manis.

$n \to n+1$: By the inductive hypothesis there is some $y' \in R$ such that

$$v_1(y') = \varepsilon_1, v_2(y') > \varepsilon_2, \ldots, v_n(y') > \varepsilon_n.$$

If $v_{n+1}(y') > \varepsilon_{n+1}$ we are done. Hence we assume that $v_{n+1}(y') \leq \varepsilon_{n+1}$. For the pair $(1, n+1)$ there is by (2) some $z \in R$ with $v_1(z) = \varepsilon_1$ and $v_{n+1}(z) = \varepsilon_{n+1}$. Since v_1 and v_{n+1} have the inverse property there is some $z' \in R$ such that $v_1(z') = -\varepsilon_1$ and $v_{n+1}(z') = -\varepsilon_{n+1}$. Hence $v_1(y'z') = 0$ and $\alpha := v_{n+1}(y'z') \leq 0$. By Proposition 3.19, (1) $\Rightarrow$ (3), applied to $y'z'$ there is some $a \in R$ such that

$$v_1(a) = 0, v_2(a) \geq 0, \ldots, v_n(a) \geq 0, v_{n+1}(a) \geq -\alpha.$$

Since $v_1 \not\leq v_{n+1}$ there is by Corollary 3.23 some $b \in R$ with $v_1(b) = 0$ and $v_{n+1}(b) > 0$. By Proposition 3.19, (1) $\Rightarrow$ (3), we may assume that

$$v_1(b) = 0, v_2(b) \geq 0, \ldots, v_n(b) \geq 0, v_{n+1}(b) > 0.$$

Hence

$$v_1(ab) = 0, v_2(ab) \geq 0, \ldots, v_n(ab) \geq 0, v_{n+1}(ab) > -\alpha.$$

We set $y := y'ab$. Then

$$\begin{aligned} v_1(y) &= v_1(y') + v_1(ab) = \varepsilon_1, \\ v_i(y) &= v_i(y') + v_i(ab) \geq v_i(y') > \varepsilon_i \end{aligned}$$

for $2 \leq i \leq n$ and

$$\begin{aligned} v_{n+1}(y) &= v_{n+1}(y') + v_{n+1}(ab) \\ &> v_{n+1}(y') - \alpha = v_{n+1}(y') - v_{n+1}(y'z') \\ &= -v_{n+1}(z') = v_{n+1}(z) = \varepsilon_{n+1}. \end{aligned}$$

This shows Claim 2.

By Claim 2 there is for each $1 \leq i \leq n$ some $x_i \in R$ such that $v_i(x_i) = \varepsilon_i$ and $v_j(x_i) > \varepsilon_j$ for all $j \neq i$ provided all v_i are non-trivial. Let $x := x_1 + \cdots + x_n$. Then $v_i(x) = \varepsilon_i$ for all $1 \leq i \leq n$.

General Case: We may assume that $v_1, \ldots, v_k$ are non-trivial and $v_{k+1}, \ldots, v_n$ are trivial for some $0 \leq k \leq n$. We can also assume that $v_{k+1}, \ldots, v_n$ are all different. Note that $\varepsilon_i = 0$ for all $k+1 \leq i \leq n$. We write $v_{k+1}, \ldots, v_n$ as

$$v_{11}, \ldots, v_{1l_1}, v_{21}, \ldots, v_{2l_2}, \ldots, v_{m_1}, \ldots, v_{ml_m}$$

with $m \in \mathbb{N}_0$, $l_j \in \mathbb{N}$ (where $1 \leq j \leq m$ and $\sum_{j=1}^{m} l_j = n-k$) such that $\operatorname{supp} v_{j1}$ is maximal in $\{\operatorname{supp} v_i \mid k+1 \leq i \leq n\}$ and $\operatorname{supp} v_{j1} \supset \operatorname{supp} v_{jj'}$ for $1 \leq j \leq m$ and $1 \leq j' \leq l_j$. Note that after rewriting $\varepsilon_{jj'} = 0$ for all j, j'.

By the special case there is some $x \in R$ such that $v_i(x) = \varepsilon_i$ for all $1 \le i \le k$. Let

$$n(x) := \#\{1 \le j \le m \mid v_{j1}(x) = \infty\}.$$

Note that if $v_{j1}(x) = 0$ then $v_{jj'}(x) = 0$ for all $1 \le j' \le l_j$. Hence if $n(x) = 0$ we are done. So we may assume that $n(x) \ge 1$ and without restriction we assume that $v_{11}(x) = \infty$. Since $\operatorname{supp} v_{j1}$ is maximal and the trivial valuations are all different there is for $2 \le j \le m$ some $a_j \in R$ with $a_j \in \operatorname{supp} v_{j1} \setminus \operatorname{supp} v_{11}$. Let $a := a_2 \cdot \ldots \cdot a_m$. Then $v_{11}(a) = 0$ and $v_{j1}(a) = \infty$ for $2 \le j \le m$. By Proposition 3.19, (1) $\Rightarrow$ (3), we can assume that $v_1(a) \ge 0, \ldots, v_k(a) \ge 0$. By Corollary 3.17 we find some $b \in R$ with $v_{11}(b) = 0$ and $v_1(b) > \varepsilon_1, \ldots, v_k(b) > \varepsilon_k$. Then $v_{11}(ab) = 0$, $v_{j1}(ab) = \infty$ for $2 \le j \le n$ and $v_i(ab) = v_i(a) + v_i(b) > \varepsilon_i$ for $1 \le i \le k$. Let $x' := x + ab$. Then $v_i(x') = v_i(x) = \varepsilon_i$ for $1 \le i \le k$, $v_{11}(x') = v_{11}(ab) = 0$ and $v_{j1}(x') = v_{j1}(x)$ for $2 \le j \le m$. So $n(x') < n(x)$. Doing induction on $n(x)$ we obtain the claim. □

Proposition 5.3. *Let $v_1, \ldots, v_n$ be Manis valuations on R having the inverse property. Then the following are equivalent.*

(1) For every $i \ne j$ v_i and v_j are independent.
(2) For any $(\varepsilon_1, \ldots, \varepsilon_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ there is some $x \in R$ with $v_i(x) = \varepsilon_i$ for all $1 \le i \le n$.
(3) For every $i \ne j$ and $\varepsilon_i \in \Gamma_{v_i}$ there is some $x \in R$ with $v_i(x_i) = \varepsilon_i$ and $v_j(x) = 0$.

Proof. By Remark 3.1(c) and Proposition 2 it is enough to do the proof in the case $n = 2$. Let $v := v_1, w := v_2, \varepsilon := \varepsilon_1$ and $\eta := \varepsilon_2$

(1) $\Rightarrow$ (2):

Case 1: $\operatorname{supp} v \not\subset \operatorname{supp} w$. Let $y \in \operatorname{supp} v \setminus \operatorname{supp} w$. Since w is Manis there is some $z \in R$ with $w(yz) = 0$. For any $0 \le \varepsilon' \in \Gamma_v$ we have $v(yz) = \infty \ge \varepsilon'$. By Proposition 1, (3) $\Rightarrow$ (6), we get some $x \in R$ such that $v(x) = \varepsilon$ and $w(x) = \eta$.

Case 2: $\operatorname{supp} w \not\subset \operatorname{supp} v$. We can copy the proof of Case 1.

Case 3: $\operatorname{supp} v = \operatorname{supp} w$. Let $\mathfrak{p} := [\mathfrak{p}_w : A_v]$. Then by Proposition 3.29 we have $\mathfrak{p} = \mathfrak{p}_{v \vee w}$. Since v and w are independent by assumption we get $\mathfrak{p} = \operatorname{supp} v$ (cf. Definition 5 in Sect. 2). Let $\varepsilon' \in \Gamma_v$ with $\varepsilon' \ge 0$. Let $y \in A_v$ with $v(y) = \varepsilon'$. Then $y \notin \operatorname{supp} v = \mathfrak{p}$. Hence there is some $a \in A_v$ such that $z := ay \notin \mathfrak{p}_w$. We obtain $v(z) = v(a) + v(y) \ge v(y) = \varepsilon'$ and $w(z) \le 0$. By Proposition 1, (1) $\Rightarrow$ (6), we get some $x \in R$ such that $v(x) = \varepsilon$ and $w(x) = \eta$.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Suppose that v and w are dependent. Then by Remark 2.2 there is a prime ideal $\mathfrak{p}$ of A_v and A_w such that

$$\operatorname{supp} v, \operatorname{supp} w \subsetneqq \mathfrak{p} \subset \mathfrak{p}_v \cap \mathfrak{p}_w.$$

Let $y \in \mathfrak{p} \setminus \operatorname{supp} v$. Let $\varepsilon := v(y) < \infty$. We show that there is no $x \in R$ such that $v(x) = \varepsilon$ and $w(x) = 0$, contradiction. Hence suppose there is some $x \in R$ with $v(x) = \varepsilon$ and $w(x) = 0$. Since v, w have the inverse property and $v(y), w(y) \neq \infty$ there is some $y' \in R$ such that $v(yy') = 0$ and $w(yy') = 0$. We have

$$v(xy') = v(x) + v(y') = v(y) - v(y) = 0.$$

This shows that $xy' \in A_v$. Since $y \in \mathfrak{p}$ and $\mathfrak{p}$ is an ideal of A_v we get $xyy' \in \mathfrak{p}$. Since $\mathfrak{p} \subset \mathfrak{p}_w$ we get $w(xyy') > 0$. But

$$w(xyy') = w(x) + w(yy') = w(x) = 0,$$

contradiction. □

Definition 1. Let v_1, v_2 be Manis valuations on R. We write H_{v_1,v_2} for the subgroup $H^1_{(v_1,v_2)}$ of Γ_{v_1} introduced in Sect. 2, Definition 4. We set $\Gamma_{v_1,v_2} := \Gamma_{v_1}/H_{v_1,v_2}$ and write $f_{v_1,v_2} : \Gamma_{v_1} \to \Gamma_{v_1,v_2}$ for the canonical homomorphism of (ordered) groups.

Note that $\Gamma_{v_1,v_2} = \Gamma_{v_1 \vee v_2}$ if $\operatorname{supp} v_1 = \operatorname{supp} v_2$ (in particular if v_1, v_2 are dependent) and that $\Gamma_{v_1,v_2} = \{0\}$ if v_1, v_2 are independent. Let $\operatorname{supp} v_1 = \operatorname{supp} v_2$. Extending f_{v_1,v_2} by setting $f_{v_1,v_2}(\infty) = \infty$ the map $f_{v_1,v_2} : \Gamma_{v_1} \cup \{\infty\} \to \Gamma_{v_1 \vee v_2} \cup \{\infty\}$ is the homomorphism of ordered monoids such that $v_1 \vee v_2 = f_{v_1,v_2} \circ v_1$.

Corollary 5.4. *Let $v_1, \dots, v_n$ be Manis valuations on R with the inverse property. Let $\varepsilon_i \in \bigcap_{j \neq i} H_{v_i,v_j}$ for $1 \leq i \leq n$. Then there is some $x \in R$ with $v_1(x) = \varepsilon_1, \dots, v_n(x) = \varepsilon_n$.*

Proof. Let $i, j \in \{1, \dots, n\}$ with $i \neq j$. Since $\varepsilon_i \in H_{v_i,v_j}$ and $\varepsilon_j \in H_{v_j,v_i}$ it is enough by Proposition 2 to show the claim for v_i and v_j. Let $v := v_i$ and $w := v_j$. If v and w are independent the claim follows from Proposition 3. So we assume that v and w are dependent. Then $\operatorname{supp} v = \operatorname{supp} w$. By Proposition 3.35 $v' := v|_{A_{v \vee w}}$ and $w' := w|_{A_{v \vee w}}$ are independent Manis valuations on $A_{v \vee w}$ having the inverse property. By Propositions 2.15 and 3 we find some $x \in A_{v \vee w}$ such that $v'(x) = \varepsilon$ and $w'(x) = \eta$. We get $v(x) = v'(x) = \varepsilon$ and $w(x) = w'(x) = \eta$. □

Definition 2. i) Let v, w be Manis valuations on R. Let $(\alpha, \beta) \in \Gamma_v \times \Gamma_w$. Then (α, β) is called *compatible* if $f_{v,w}(\alpha) = f_{w,v}(\beta)$.

ii) Let $v_1, \dots, v_n$ be Manis valuations on R and let $(\alpha_1, \dots, \alpha_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n}$. The tuple $(\alpha_1, \dots, \alpha_n)$ is called *compatible* if (α_i, α_j) is compatible for every $1 \leq i, j \leq n$.

Remark 5.5. Let $v_1, \dots, v_n$ be Manis valuations on R.

a) Let $x \in R \setminus \bigcup_{1 \leq i \leq n} \operatorname{supp} v_i$. The tuple $(v_1(x), \dots, v_n(x)) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n}$ is compatible.
b) Let $(\alpha_1, \dots, \alpha_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n}$ be compatible. Then for every $1 \leq k \leq n$ the tuple $(\alpha_1, \dots, \alpha_k) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_k}$ is compatible.

Remark 5.6. Let v, w be Manis valuation on R with $v \leq w$. Let $(\alpha, \beta) \in R$. Then (α, β) is compatible iff $\beta = f_{v,w}(\alpha)$.

Proposition 5.7. *Let $v_1, \ldots, v_n$ be Manis valuations on R. For $1 \leq i \leq n$ let $\Delta_i := \bigcap_{j \neq i} H_{v_i,v_j}$. Let $(\delta_1, \ldots, \delta_n) \in \Delta_1 \times \ldots \times \Delta_n$. Then $(\delta_1, \ldots, \delta_n)$ is compatible.*

Proof. Let $1 \leq i, j \leq n$. We may assume that $i \neq j$. Since $\delta_i \in H_{v_i,v_j}$ and $\delta_j \in H_{v_j,v_i}$ we have $f_{v_i,v_j}(\delta_i) = f_{v_j,v_i}(\delta_j) = 0$. □

Proposition 5.8. *Let $v_1, \ldots, v_n$ be Manis valuations on R. Then the following are equivalent.*

(1) For every $i \neq j$ v_i, v_j are independent.
(2) Every $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ is compatible.

Proof. It clearly suffices to do the proof in the case $n = 2$. We set $v := v_1, w := v_2, \alpha := \alpha_1$ and $\beta := \alpha_2$.

(1) ⇒ (2): This follows immediately from $\Gamma_{v,w} = \{0\}$ (cf. Definition 4 in Sect. 2).

(2) ⇒ (1): Assume that v, w are dependent. We choose $x \in R \setminus A_{v \vee w}$. Then $(v(x), 2w(x))$ is not compatible since

$$f_{v,w}(v(x)) = (v \vee w)(x) \neq 2(v \vee w)(x) = f_{w,v}(2w(x)),$$

contradiction. □

Let $v_1, \ldots, v_n$ be Manis valuations on R. Let $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$. By Remark 5(a) it is necessary that $(\alpha_1, \ldots, \alpha_n)$ is compatible to find some $x \in R$ with $v_1(x) = \alpha_1, \ldots, v_n(x) = \alpha_n$.

Definition 3. Let $v_1, \ldots, v_n$ be Manis valuations on R. The *approximation theorem in the neighbourhood of zero* holds for $v_1, \ldots, v_n$ if for every compatible $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ there is some $x \in R$ such that $v_1(x) = \alpha_1, \ldots, v_n(x) = \alpha_n$.

Example 5.9. Let $v_1, \ldots, v_n$ be Manis valuations on R having the inverse property such that v_i and v_j are independent for every $i \neq j$. Then by Proposition 3 the approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_n$.

Remark 5.10. Let $v_1, \ldots, v_n, w_1, \ldots, w_m$ be Manis valuations on R such that for every $1 \leq j \leq m$ there is some $1 \leq i \leq n$ with $v_i \leq w_j$. The following are equivalent.

(1) The approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_n$.
(2) The approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_n, w_1, \ldots, w_m$.

Proof. For $1 \leq j \leq m$ we choose $1 \leq i_j \leq n$ such that $v_{i_j} \leq w_j$.

(1) ⇒ (2): Let $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times \Gamma_{w_1} \times \ldots \times \Gamma_{w_m}$ be compatible. Then $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ is compatible. Hence there is some $x \in R$ such that $v_i(x) = \alpha_i$ for all $1 \leq i \leq n$. Let $1 \leq j \leq m$. Then by Remark 6

$$w_j(x) = f_{v_{i_j},w_j}(v_{i_j}(x)) = f_{v_{i_j},w_j}(\alpha_{i_j}) = \beta_j.$$

(2) $\Rightarrow$ (1): Let $(\alpha_1,\ldots,\alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ be compatible. Let $\beta_j := f_{v_{i_j},w_j}(\alpha_{i_j})$ for $1 \le j \le m$. Then $(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times \Gamma_{w_1} \times \ldots \times \Gamma_{w_m}$ is compatible. To see this we distinguish three cases.

Claim 1: (α_k,α_l) is compatible for $1 \le k,l \le n$. This follows from the setting.

Claim 2: (α_k,β_l) is compatible for $1 \le k \le n$ and $1 \le l \le m$. We may assume that v_k and w_l are dependent. Then $v_k \vee v_{i_l} \le v_k \vee w_l$ by Proposition 2.13. Let $g : \Gamma_{v_k \vee v_{i_l}} \cup \{\infty\} \to \Gamma_{v_k \vee w_l} \cup \{\infty\}$ be the homomorphism of ordered monoids such that $v_k \vee w_l = g \circ (v_k \vee v_{i_l})$. We obtain

$$f_{w_l,v_k}(\beta_l) = f_{w_l,v_k}(f_{v_{i_l},w_l}(\alpha_{i_l})) = g(f_{v_{i_l},v_k}(\alpha_{i_l})) = g(f_{v_k,v_{i_l}}(\alpha_k)) = f_{v_k,w_l}(\alpha_k).$$

Claim 3: (β_k,β_l) is compatible for $1 \le k,l \le m$. We may assume that w_k, w_l are dependent. Then $v_{i_k} \vee v_{i_l} \le w_k \vee w_l$ by Proposition 2.13. Let $h : \Gamma_{v_{i_k} \vee v_{i_l}} \cup \{\infty\} \to \Gamma_{w_k \vee w_l} \cup \{\infty\}$ be the homomorphism of ordered monoids such that $w_k \vee w_l = h \circ (v_{i_k} \vee v_{i_l})$. We obtain

$$\begin{aligned} f_{w_k,w_l}(\beta_k) &= f_{w_k,w_l}(f_{v_{i_k},w_k}(\alpha_{i_k})) = h(f_{v_{i_k},v_{i_l}}(\alpha_{i_k})) \\ &= h(f_{v_{i_l},v_{i_k}}(\alpha_{i_l})) = f_{w_l,w_k}(f_{v_{i_l},w_l}(\alpha_{i_l})) \\ &= f_{w_l,w_k}(\beta_l). \end{aligned}$$

Since $(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n)$ is compatible, there is by assumption some $x \in R$ such that $v_i(x) = \alpha_i$ for $1 \le i \le n$ and $w_j(x) = \beta_j$ for $1 \le j \le m$. Hence we are done. □

Theorem 5.11. *Let $v_1,\ldots,v_n$ be Manis valuations on R. The following are equivalent.*

(1) $v_1,\ldots,v_n$ have the inverse property.
(2) The approximation theorem in the neighbourhood of zero holds for $v_1,\ldots,v_n$.

Proof. (1) $\Rightarrow$ (2): By Definitions 3 and 2 and Proposition 2 we can assume that $n = 2$. We write $v := v_1$ and $w := v_2$. By Proposition 3 we can assume that v and w are dependent. Let $(\alpha,\beta) \in \Gamma_v \times \Gamma_w$ be compatible. Then $f_{v,w}(\alpha) = f_{w,v}(\beta)$ in $\Gamma_{v \vee w}$. Hence there is some $y \in R$ with $(v \vee w)(y) = f_{v,w}(\alpha) = f_{w,v}(\beta)$. We get $v(y) - \alpha \in H_{v,w}$ and $w(y) - \beta \in H_{w,v}$. By Corollary 4 there is some $z \in R$ such that $v(z) = -v(y) + \alpha$ and $w(z) = -w(y) + \beta$. With $x := yz$ we get $v(x) = \alpha$ and $w(x) = \beta$.

(2) $\Rightarrow$ (1): Let $x \in R$. For $1 \le i \le n$ we take $\alpha_i := -v_i(x)$ if $v_i(x) < \infty$ and $\alpha_i := 0$ if $v_i(x) = \infty$.

Claim: $(\alpha_1,\ldots,\alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ is compatible.

Proof of the Claim: It clearly suffices to check the following case: Let $k \neq l$ such that $v_k(x) < \infty$ and $v_l(x) = \infty$. Then (α_k, α_l) is compatible. Since $x \in \operatorname{supp} v_l \setminus \operatorname{supp} v_k$ we get that v_k and v_l are independent. We have $\Gamma_{v,w} = \{0\}$ and are done.

By the assumption there is some $x' \in R$ such that $v_i(x') = \alpha_i$ for $1 \leq i \leq n$. By construction $v_i(x') = -v_i(x)$ for all $1 \leq i \leq n$ such that $v_i(x) \neq \infty$. □

Corollary 5.12. *Let $v_1, \ldots, v_n$ be Manis valuations on R. If $A_{v_1} \cap \ldots \cap A_{v_n}$ is Prüfer in R then the approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_n$.*

Proof. By Theorem 4.18 and Corollary 4.9 the Manis valuations $v_1, \ldots, v_n$ have the inverse property. We get the claim by Theorem 11. □

Corollary 5.13. *Assume that the approximation theorem in the neighbourhood of zero holds for Manis valuations $v_1, \ldots, v_n$. Let $1 \leq k \leq n$. Then the approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_k$.*

Proof. By Theorem 11 $v_1, \ldots, v_n$ have the inverse property. By Remark 3.1(c) we know that $v_1, \ldots, v_k$ have the inverse property. Again by Theorem 11 the approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_k$. □

Corollary 5.14. *Let $v_1, \ldots, v_n$ be non-trivial Manis valuations on R such that the approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_n$. Let $1 \leq k \leq n$. Then a compatible tuple $(\alpha_1, \ldots, \alpha_k) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_k}$ can be enlarged to a compatible tuple $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$.*

Proof. Let $(\alpha_1, \ldots, \alpha_k) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_k}$ be compatible. By Corollary 13 the approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_k$. Hence there is some $x \in R$ such that $v_i(x) = \alpha_i$ for all $1 \leq i \leq k$. By Theorem 11 $v_1, \ldots, v_n$ have the inverse property. Let $I := \{1 \leq i \leq n \mid v_i(x) < \infty\}$. Note that $I \supset \{1, \ldots, k\}$. By Corollary 3.16 there is some $y \in R$ such that $v_i(x) < v_i(y) < \infty$ for all $i \in I$ and $v_i(y) < \infty$ for all $i \notin I$. Let $z := x + y$. Then $v_i(z) = v_i(x) = \alpha_i$ for all $1 \leq i \leq k$ and $v_i(z) < \infty$ for all $k+1 \leq i \leq n$. Let $\alpha_i := v_i(z)$ for $k+1 \leq i \leq n$. Then $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ is compatible by Remark 5(a). □

In [Al-M] an example of two incomparable Manis valuations not fulfilling the approximation theorem in the neighbourhood of zero from [Ar] is formulated:

Example 5.15. Let v be a valuation on a field K with value group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ equipped with the lexicographical order. Let $H := \{0\} \oplus \mathbb{Z}$ and let $w := v/H$. Then $\Gamma_w \cong \mathbb{Z}$. The valuations $\overline{v}$ and $\overline{w}$ defined by

$$\overline{v}(a_0 + a_1 X + \ldots + a_n X^n) := \min\{v(a_k) + k \cdot (1, 0) \mid 1 \leq k \leq n\}$$

resp.

$$\overline{w}(a_0 + a_1 X + \ldots + a_n X^n) := \min\{w(a_k) + k \cdot 1 \mid 1 \leq k \leq n\}$$

are Manis valuations on $K[X]$ that do not satisfy the approximation theorem in the neighbourhood of zero.

We want to consider the case of families with finite avoidance. Some notations have to be introduced.

Definition 4. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. Let $i \in I$ and let $\alpha \in \Gamma_{v_i} \cup \{\infty\}$. If $\alpha \notin \{0, \infty\}$ let H_α denote the largest convex subgroup of Γ_{v_i} that does not contain α. We set

$$I(\alpha) := \begin{cases} \{i\} & \alpha \in \{0,\infty\}, \\ & \text{if} \\ \{j \in I \mid v_j \leq v_i/H_\alpha\} & \alpha \notin \{0,\infty\}. \end{cases}$$

Remark 5.16. Let in the above situation $\alpha \notin \{0, \infty\}$. Let $w := v_i/H_\alpha$. Then w is a non-trivial Manis valuation on R with $v_i \leq w$ (cf. Scholium 1.7). In particular $i \in I(\alpha)$. (So $i \in I(\alpha)$ for all $i \in I$.) If $0 < \alpha < \infty$ and $\alpha = v_i(x)$ for some $x \in R$ then $\mathfrak{p}_w$ is the smallest v_i-convex prime ideal of A_{v_i} containing x (cf. Scholium 1.10).

Proposition 5.17. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. Let $i \in I$ and $\alpha \in \Gamma_{v_i} \setminus \{0\}$. Then*

$$I(\alpha) = \{j \in I \mid f_{v_i,v_j}(\alpha) \neq 0\}.$$

Proof. Let $w := v_i/H_\alpha$. We have that w is non-trivial with $v_i \leq w$. Hence we obtain for $j \in I$ $v_j \leq w$ iff $v_i \vee v_j \leq w$ iff $H_{v_i,v_j} \subset H_\alpha$ iff $\alpha \notin H_{v_i,v_j}$ iff $f_{v_i,v_j}(\alpha) \neq 0$ in Γ_{v_i,v_j}. □

Remark 5.18. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having finite avoidance. Let $i \in I$ and let $\alpha \in \Gamma_{v_i} \cup \infty$. Then $I(\alpha)$ is finite.

Proof. If $\alpha \in \{0, \infty\}$ nothing is to show. If $\alpha \notin \{0, \infty\}$ then we are done by the definition and Remark 4.14. □

Definition 5. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R and let $i_1, \ldots, i_n \in I$. Then $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}}$ is called $\{i_1, \ldots, i_n\}$-*complete* if $\bigcup_{1 \leq k \leq n} I(\alpha_k) = \{i_1, \ldots, i_n\}$.

Remark 5.19. a) Note that $\bigcup_{1 \leq k \leq n} I(\alpha_k) \supset \{i_1, \ldots, i_n\}$ for any $i_1, \ldots, i_n \in I$ and $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}}$ by Remark 16.

b) If $(v_i \mid i \in I)$ consists of pairwise independent Manis valuations then every $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}}$ is $\{i_1, \ldots, i_n\}$-complete for all $i_1, \ldots, i_n \in I$ by Proposition 17.

Remark 5.20. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R and let $i_1, \ldots, i_n \in I$. Let $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}}$ be compatible and $\{i_1, \ldots, i_n\}$-complete. Let $j_1, \ldots, j_m \in I \setminus \{i_1, \ldots, i_n\}$. Then $(\alpha_1, \ldots, \alpha_n, 0, \ldots, 0) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times \Gamma_{v_{j_1}} \times \ldots \times \Gamma_{v_{j_m}}$ is compatible.

Proof. Let $1 \leq k \leq n$ and $1 \leq l \leq m$. If $\alpha_k = 0$ then clearly $f_{v_{i_k}, v_{j_l}}(\alpha_k) = 0$. If $\alpha_k \neq 0$ then $f_{v_{i_k}, v_{j_l}}(\alpha_k) = 0$ by Proposition 17 since $j_l \notin I(\alpha_k)$. □

Definition 6. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. We say that the *approximation theorem in the neighbourhood of zero* holds for the family if for each $i_1, \ldots, i_n \in I$ and each compatible and $\{i_1, \ldots, i_n\}$-complete tuple $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}}$ there is some $x \in R$ with $v_{i_p}(x) = \alpha_p$ for $1 \leq p \leq n$ and $v_j(x) \geq 0$ for all $j \in I \setminus \{i_1, \ldots, i_n\}$.

Remark 5.21. If I is finite then Definition 6 coincides with Definition 3.

Proof. Let $I = \{1, \ldots, n\}$.

a) We show that the approximation theorem in the neighbourhood of zero in the sense of Definition 6 implies the one in the sense of Definition 3. To see this let $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ be compatible. By Remark 19(a) it is $\{1, \ldots, n\}$-complete and we are done.
b) We show that the approximation theorem in the neighbourhood of zero in the sense of Definition 3 implies the one in the sense of Definition 6. For this let $i_1 \ldots, i_k \in \{1, \ldots, n\}$. Without restriction we can assume that $i_1 = 1, \ldots, i_k = k$. Let $(\alpha_1, \ldots, \alpha_k) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_k}$ be compatible and $\{1, \ldots, k\}$-complete. Then $(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ is compatible by Remark 20. Hence there is some $x \in R$ such that $v_i(x) = \alpha_i$ for $1 \leq i \leq k$ and $v_i(x) = 0$ for $k + 1 \leq i \leq n$ and we are done. □

The approximation theorem in the neighbourhood of zero implies the finite avoidance inverse property.

Theorem 5.22. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having finite avoidance. If the approximation theorem in the neighbourhood of zero holds for the family then it has the finite avoidance inverse property.*

Proof. By Theorem 4.21 it is enough to show that every v_i is A-essential. Let $i_0 \in I$. We set $\mathfrak{p} := \mathrm{cent}_A(v_{i_0})$. We have to show that $A_{v_{i_0}} \subset A_{[\mathfrak{p}]}$. Let $x \in A_{v_{i_0}}$. Since the given family has finite avoidance we find $i_1, \ldots, i_n \in I$ such that $v_{i_k}(x) < 0$ for $1 \leq k \leq n$ and $v_j(x) \geq 0$ for all $j \in J := I \setminus \{i_1, \ldots, i_n\}$.

Case 1: $v_{i_0}(x) = 0$. Then the tuple $(v_{i_0}(x), v_{i_1}(x), \ldots, v_{i_n}(x)) \in \Gamma_{v_{i_0}} \times \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}}$ is compatible and $\{i_0, \ldots, i_n\}$-complete. The tuple $(-v_{i_0}(x), -v_{i_1}(x), \ldots, -v_{i_n}(x)) \in \Gamma_{v_{i_0}} \times \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}}$ is then also compatible and $\{i_0, \ldots, i_n\}$-complete. By assumption we find some $y \in \bigcap_{j \in J} A_{v_j}$ such that $v_{i_0}(y) = -v_{i_0}(x) = 0$ and $v_{i_k}(y) = -v_{i_k}(x) \geq 0$ for $1 \leq k \leq n$. Hence $y \in A \setminus \mathfrak{p}$ and $xy \in A$.

Case 2: $v_{i_0}(x) > 0$. Then $v_{i_0}(1 + x) = 0$, $v_{i_k}(1 + x) < 0$ for $1 \leq k \leq n$ and $v_j(1 + x) \geq 0$ for all $j \in I \setminus \{i_1, \ldots, i_n\}$. By Case 1 we have $1 + x \in A_{[\mathfrak{p}]}$ and therefore $x \in A_{[\mathfrak{p}]}$. □

We are not able to prove the other implication. We introduce the following notion.

Definition 7. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. We say that it has the *strong finite avoidance inverse property* if it has the finite avoidance inverse property and the following holds for all $i_1, \ldots, i_n \in I$ and all R-overrings B of $\bigcap_{i \in I} A_{v_i}$: If $v_{i_1}|_B, \ldots, v_{i_n}|_B$ are dependent then $v_{i_1}, \ldots, v_{i_n}$ are dependent and $\bigvee v_{i_k}|B = (\bigvee v_{i_k})|_B$.

Remark 5.23. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R such that $\bigcap_{i \in I} A_{v_i}$ is Prüfer in R. The the family has the strong finite avoidance inverse property.

Proof. This follows from Proposition 4.13. □

Theorem 5.24. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R having finite avoidance. If the family has the strong finite avoidance inverse property then the approximation theorem in the neighbourhood of zero holds for it.*

Proof. Let $i_1, \ldots, i_n \in I$ and let $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}}$ be compatible and $\{i_1, \ldots, i_n\}$-complete. By Remark 4.20(b) the valuations $v_{i_1}, \ldots, v_{i_n}$ have the inverse property. By Theorem 11 the approximation theorem in the neighbourhood of zero holds for $v_{i_1}, \ldots, v_{i_n}$. Hence there is some $y \in R$ such that $v_{i_k}(y) = \alpha_k$ for $1 \le k \le n$. Since $(-\alpha_1, \ldots, -\alpha_n)$ is also compatible there is some $y' \in R$ such that $v_{i_k}(y') = -\alpha_k$ for $1 \le k \le n$. Since the family has finite avoidance there is a finite subset J of I containing $i_1, \ldots, i_n$ such that $v_j(y) \ge 0$ and $v_j(y') \ge 0$ for all $j \in I \setminus J$. Let $B := \bigcap_{j \in I \setminus J} A_{v_j}$. Then $y, y' \in B$. We write $J \setminus \{i_1, \ldots, i_n\}$ as $\{i_{n+1}, \ldots, i_m\}$ for some $m \ge n$. For $1 \le k \le m$ we set $w_k := v_{i_k}|_B$. The family $(v_i \mid i \in I)$ has the finite avoidance inverse property by assumption. Therefore the valuations $w_1, \ldots, w_m$ are Manis on B and have the inverse property. By Theorem 11 the approximation theorem in the neighbourhood of zero holds for $w_1, \ldots, w_m$. Since $y, y' \in B$ we have $\alpha_k \in \Gamma_{w_k}$ and $\alpha_k = w_k(y)$ for $1 \le k \le n$. We set $\alpha_k := 0 \in \Gamma_{w_k}$ for $n + 1 \le k \le m$.

Claim: The tuple $(\alpha_1, \ldots, \alpha_m) \in \Gamma_{w_1} \times \ldots \times \Gamma_{w_m}$ is compatible.

Proof of the Claim: Let $1 \le k < l \le m$. We show that (α_k, α_l) is compatible. We distinguish three cases.

Case 1: $1 \le k, l \le n$. Then $w_k(y) = \alpha_k$ and $w_l(y) = \alpha_l$, so (α_k, α_l) is compatible.

Case 2: $1 \le k \le n$ and $n + 1 \le l \le m$. Since $i_l \not\in I(\alpha_k)$ we get $f_{v_{i_k}, v_{i_l}}(\alpha_k) = 0$ in $\Gamma_{v_{i_k} \vee v_{i_l}}$ by Proposition 17. By the strong finite avoidance inverse property and the definition of special restriction we obtain $f_{w_k, w_l}(\alpha_k) = 0$ in Γ_{w_k, w_l}. So (α_k, α_l) is compatible.

Case 3: $n + 1 \le k < l \le m$. This is obvious.

Since the approximation theorem in the neighbourhood of zero holds for $w_1, \ldots, w_m$ there is some $x \in B$ such that $w_k(x) = \alpha_k$ for $1 \le k \le m$. Then $v_{i_k}(x) = \alpha_k$ for $1 \le k \le n$. It remains to show that $v_j(x) \ge 0$ for all $j \in I \setminus \{i_1, \ldots, i_n\}$. If $j = i_k$ for some $n + 1 \le k \le m$ then $v_j(x) = w_k(x) = 0$. If $j \notin J$ then $v_j(x) \ge 0$ since $x \in B$. □

Corollary 5.25. *Let* $(v_i \mid i \in I)$ *be a family of Manis valuations on* R *having finite avoidance. If* $\bigcap_{i\in i} A_{v_i}$ *is Prüfer in* R *then the approximation theorem in the neighbourhood of zero holds for* $(v_i \mid i \in I)$.

Proof. By Remark 23 the family $(v_i, \mid i \in I)$ has the strong finite avoidance inverse property. We get the claim by Theorem 24. □

6 The General Approximation Theorem

We formulate the general approximation theorem both in the finite and in the infinite case. Note that our reasoning in the finite case also contains Gräter's general approximation theorem (cf. [Gr]). Assuming finite avoidance the general approximation theorem implies the approximation theorem in the neighbourhood of zero and is implied by the Prüfer condition if the valuations are additionally pairwise independent. The theory of distributive submodules (cf. Chap. II) is heavily used.

Definition 1. i) Let v, w be Manis valuations on R. Let $(\alpha, \beta) \in \Gamma_v \times \Gamma_w$ and $a, b \in R$. We call the tuple (α, β, a, b) *weakly compatible* if $(v \vee w)(a-b) \geq \min\{f_{v,w}(\alpha), f_{w,v}(\beta)\}$.

ii) Let $v_1, \dots, v_n$ be Manis valuations on R. Let $(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_k} \times R^n$. The tuple $(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n)$ is called *weakly compatible* if $(\alpha_i, \alpha_j, a_i, a_j)$ is weakly compatible for every $1 \leq i, j \leq n$.

Remark 6.1. Let $v_1, \dots, v_n$ be Manis valuations on R.

a) Let $(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n} \times R^n$. Let $x \in R$ such that $v_1(x-a_1) \geq \alpha_1, \dots, v_n(x-a_n) \geq \alpha_n$. Then $(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n)$ is weakly compatible.

b) For $1 \leq i \leq n$ let $\Delta_i := \bigcap_{j\neq i} H_{v_i,v_j}$. We set $w_i := v_i/\Delta_i$. Let $a_i \in A_{w_i}$ and $\varepsilon \in \Delta_i$. Then the tuple
$$(\varepsilon_1, \dots, \varepsilon_n, a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n} \times R^n$$
is weakly compatible.

Proof. Let $1 \leq i, j \leq n$.

a): We set $v := v_i, w := v_j, \alpha := \alpha_i, \beta := \alpha_j, a := a_i$ and $b := a_j$. We have
$$\begin{aligned}(v \vee w)(a-b) &= (v \vee w)(a-x+x-b) \geq \min\{(v \vee w)(a-x), (v \vee w)(b-x)\}\\ &= \min\{f_{v,w}(v(a-x)), f_{w,v}(w(b-x))\} \geq \min\{f_{v,w}(\alpha), f_{w,v}(\beta)\}.\end{aligned}$$

b): We may assume that $i \neq j$ and that v_i and v_j are dependent. We have $(v_i \vee v_j)(a_i - a_j) \geq (v_i \vee v_j)(a_i)$. Since $\Delta_i \subset H_{v_i,v_j}$ we see $w_i \leq v_i \vee v_j$. Since $a_i \in A_{w_i}$ we get $(v_i \vee v_j)(a_i) \geq 0$. Clearly $f_{v_i,v_j}(\varepsilon_i) = f_{v_j,v_i}(\varepsilon_j) = 0$ and we are done. □

Proposition 6.2. *Let $v_1, \dots, v_n$ be Manis valuations on R. Then the following are equivalent.*

(1) For every $i \neq j$ v_i and v_j are independent.
(2) Every $(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n} \times R^n$ is weakly compatible.

Proof. It clearly suffices to do the proof in the case $n = 2$. We set $v := v_1, w := v_2, \alpha := \alpha_1, \beta := \alpha_2, a := a_1$ and $b := b_1$.

(1) $\Rightarrow$ (2): This follows immediately from $\Gamma_{v,w} = \{0\}$ (cf. Definition 1 in Sect. 5).

(2) $\Rightarrow$ (1): Assume that v, w are dependent. We choose $x \in R \setminus A_{v \vee w}$. Then $(0, 0, x, 0) \in \Gamma_v \times \Gamma_w \times R^2$ is not compatible since $(v \vee w)(x) < 0$. ☐

Definition 2. Let $v_1, \dots, v_n$ be Manis valuations on R. The *general approximation theorem* holds for $v_1, \dots, v_n$ if for every weakly compatible $(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_k} \times R^n$ there is some $x \in R$ such that $v_1(x - a_1) \geq \alpha_1, \dots, v_n(x - a_n) \geq \alpha_n$.

Note that by Remark 1(b) our definition of a general approximation theorem contains the one of Gräter in [Gr].

Remark 6.3. Let $v_1, \dots, v_n$ be Manis valuations on R and let $w_1, \dots, w_m$ be trivial Manis valuations on R. The general approximation theorem holds for $v_1, \dots, v_n$ iff it holds for $v_1, \dots, v_n, w_1, \dots, w_m$. ☐

It is clear by the preceding remark that the notion of general approximation theorem is vacuous in the case of trivial valuations. In the case of non-trivial valuations we get that the general approximation theorem implies the approximation theorem in the neighbourhood of zero.

Theorem 6.4. *Let $v_1, \dots, v_n$ be non-trivial Manis valuations on R. If the general approximation theorem holds for $v_1, \dots, v_n$ then also the approximation theorem in the neighbourhood of zero.*

Proof. We can clearly assume that the valuations $v_1, \dots, v_n$ are pairwise non-isomorphic. Without restriction we write the valuations as $v_1, \dots, v_k, w_1, \dots, w_l$ such that the following properties hold.

i) v_i and v_j are incomparable for $i \neq j$.
ii) There is $\varphi : \{1, \dots, l\} \to \{1, \dots, k\}$ such that $v_{\varphi(j)} \leq w_j$ for all $1 \leq j \leq l$.

Let $(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_k} \times \Gamma_{w_1} \times \dots \times \Gamma_{w_l}$ be compatible. For $1 \leq i \leq k$ let $x_i \in R$ such that $v_i(x_i) = \alpha_i$. For $1 \leq j \leq l$ let $y_j := x_{\varphi(j)}$. Then $w_j(y_j) = \beta_j$ for $1 \leq j \leq l$. Since v_i is non-trivial and $v_i \not\geq v_j$ for all $j \neq i$ we get that $H_{v_i, v_j} \neq \{0\}$ for all $j \neq i$. Since v_i is non-trivial and $v_i \not\geq w_j$ for all $1 \leq j \leq l$ we get that $H_{v_i, w_j} \neq \{0\}$ for all $1 \leq j \leq l$. Since these are convex subgroups of Γ_{v_i} we obtain that

$$\Delta_i := \bigcap_{j \neq i} H_{v_i, v_j} \cap \bigcap_{1 \leq j \leq l} H_{v_i, w_j} \neq \{0\}.$$

Let $\delta_i \in \Delta_i$ with $\delta_i > 0$ and let $\varepsilon_i := \alpha_i + \delta_i$.

Claim: The tuple $(\varepsilon_1, \dots, \varepsilon_k, \beta_1, \dots, \beta_l, x_1, \dots, x_k, y_1, \dots, y_l)$ is weakly compatible.

Proof of the Claim: We distinguish three cases.

Subclaim 1: $(\varepsilon_i, \varepsilon_j, x_i, x_j)$ is weakly compatible for $1 \le i, j \le k$. We have

$$\begin{aligned}(v_i \vee v_j)(x_i - x_j) &\ge \min\{(v_i \vee v_j)(x_i), (v_i \vee v_j)(x_j)\}\\ &= \min\{f_{v_i,v_j}(v_i(x_i)), f_{v_j,v_i}(v_j(x_j))\}\\ &= \min\{f_{v_i,v_j}(\alpha_i), f_{v_j,v_i}(\alpha_j)\}\\ &= \min\{f_{v_i,v_j}(\varepsilon_i), f_{v_j,v_i}(\varepsilon_j)\}.\end{aligned}$$

Subclaim 2: $(\varepsilon_i, \beta_j, x_i, y_j)$ is weakly compatible for $1 \le i \le k$ and $1 \le j \le l$. We may assume that v_i and w_j are dependent. Then $v_i \vee v_{\varphi(j)} \le v_i \vee w_j$ by Proposition 2.13. Let $g : \Gamma_{v_i \vee v_{\varphi(j)}} \cup \{\infty\} \to \Gamma_{v_i \vee w_j} \cup \{\infty\}$ be the homomorphism of ordered monoids such that $v_i \vee w_j = g \circ (v_i \vee v_{\varphi(j)})$. We have

$$\begin{aligned}(v_i \vee w_j)(x_i - y_j) &\ge \min\{(v_i \vee w_j)(x_i), (v_i \vee w_j)(y_j)\}\\ &= \min\{f_{v_i,w_j}(v_i(x_i)), f_{w_j,v_i}(w_j(y_j))\}\\ &= \min\{g(f_{v_i,v_{\varphi(j)}}(v_i(x_i))), g(f_{v_{\varphi(j)},v_i}(v_{\varphi(j)}(x_{\varphi(j)})))\}\\ &= g\left(\min\{f_{v_i,v_{\varphi(j)}}(v_i(x_i)), f_{v_{\varphi(j)},v_i}(v_{\varphi(j)}(x_{\varphi(j)}))\}\right)\\ &= g\left(\min\{f_{v_i,v_{\varphi(j)}}(\alpha_i), f_{v_{\varphi(j)},v_i}(\alpha_{\varphi(j)})\}\right)\\ &= g\left(\min\{f_{v_i,v_{\varphi(j)}}(\varepsilon_i), f_{v_{\varphi(j)},v_i}(\alpha_{\varphi(j)})\}\right)\\ &= \min\{g(f_{v_i,v_{\varphi(j)}}(\varepsilon_i)), g(f_{v_{\varphi(j)},v_i}(\alpha_{\varphi(j)}))\}\\ &= \min\{f_{v_i,w_j}(\varepsilon_i), f_{w_j,v_i}(\beta_j)\}.\end{aligned}$$

Subclaim 3: $(\beta_i, \beta_j, y_i, y_j)$ is weakly compatible for $1 \le i, j \le k$. We may assume that w_i and w_j are dependent. Then $v_{\varphi(i)} \vee v_{\varphi(j)} \le w_i \vee w_j$ by Proposition 2.13. Let $h : \Gamma_{v_{\varphi(i)} \vee v_{\varphi(j)}} \cup \{\infty\} \to \Gamma_{w_i \vee w_j} \cup \{\infty\}$ be the homomorphism of ordered monoids such that $v_{w_i \vee w_j} = h \circ (v_{\varphi(i)} \vee v_{\varphi(j)})$. We have

$$\begin{aligned}(w_i \vee w_j)(y_i - y_j) &\ge \min\{(w_i \vee w_j)(y_i), (w_i \vee w_j)(y_j)\}\\ &= \min\{f_{w_i,w_j}(w_i(y_i)), f_{w_j,w_i}(w_j(y_j))\}\\ &= \min\{h(f_{v_{\varphi(i)},v_{\varphi(j)}}(v_{\varphi(i)}(x_{\varphi(i)}))), h(f_{v_{\varphi(j)},v_{\varphi(i)}}(v_{\varphi(j)}(x_{\varphi(j)})))\}\\ &= h\left(\min\{f_{v_{\varphi(i)},v_{\varphi(j)}}(v_{\varphi(i)}(x_{\varphi(i)})), f_{v_{\varphi(j)},v_{\varphi(i)}}(v_{\varphi(j)}(x_{\varphi(j)}))\}\right)\\ &= h\left(\min\{f_{v_{\varphi(i)},v_{\varphi(j)}}(\alpha_{\varphi(i)}), f_{v_{\varphi(j)},v_{\varphi(i)}}(\alpha_{\varphi(j)})\}\right)\end{aligned}$$

$$= \min\{h(f_{v_{\varphi(i)},v_{\varphi(j)}}(\alpha_{\varphi(i)})), h(f_{v_{\varphi(j)},v_{\varphi(i)}}(\alpha_{\varphi(j)}))\}$$
$$= \min\{f_{w_i,w_j}(\beta_i), f_{w_j,w_i}(\beta_j)\}.$$

Since the general approximation theorem holds for $v_1, \ldots, v_n$ we find by the claim some $x \in R$ such that $v_i(x - x_i) \geq \varepsilon_i$ for $1 \leq i \leq k$ (and $w_j(x - y_j) \geq \beta_j$ for $1 \leq j \leq l$). Then

$$v_i(x) = \min\{v_i(x_i), v_i(x - x_i)\} = \alpha_i$$

for $1 \leq i \leq k$ and

$$w_j(x) = f_{v_{\varphi(j)},w_j}(v_{\varphi(j)}(x)) = f_{v_{\varphi(j)},w_j}(\alpha_j) = \beta_j$$

for $1 \leq j \leq l$. □

Corollary 6.5. *Let $v_1, \ldots, v_n$ be non-trivial Manis valuations on R. If the general approximation theorem holds for $v_1, \ldots, v_n$ then $v_1, \ldots, v_n$ have the inverse property.*

Proof. By Theorem 4 the approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_n$. By Theorem 5.11 $v_1, \ldots, v_n$ have the inverse property. □

Next, we show that the general approximation theorem holds for $v_1, \ldots, v_n$ if $v_1, \ldots, v_n$ are pairwise independent and $A_{v_1} \cap \ldots \cap A_{v_n}$ is Prüfer in R. For this, we need some preparation.

Definition 3. Let A be a ring, M an A-module and $N_1, \ldots, N_k$ submodules of M. We say that the *Chinese Remainder Theorem (CRT)* holds for the submodules $N_1, \ldots, N_k$ of M, if for any elements $x_1, \ldots, x_k$ of M with $x_i \equiv x_j \mod (N_i + N_j)$ and for every $1 \leq i, j \leq k$ there exists some $x \in M$ with $x \equiv x_i \mod N_i$ for all $1 \leq i \leq k$.

Remark 6.6. Conversely, if $x \equiv x_i \mod N_i$ for some $x \in R$ then certainly $x_i \equiv x_j \mod (N_i + N_j)$ for every $1 \leq i, j \leq k$.

Lemma 6.7. *Let M be an A-module. Then CRT holds for any two submodules N_1 and N_2 of M.*

Proof. Let $x_1, x_2 \in M$ with $x_1 - x_2 \in N_1 + N_2$. Then there are $a_1 \in N_1$, $a_2 \in N_2$ such that $x_1 - x_2 = a_1 + a_2$. Then $x := x_1 - a_1 = x_2 + a_2$ does the job. □

Proposition 6.8. *Let M be an A-module and let $N_1, \ldots, N_k$ be distributive submodules of M (cf. [Vol. I, Definition 1 in II §5]). Then CRT holds for $N_1, \ldots, N_k$.*

Proof. We do induction on k.

$k = 1$: This is obvious.

$k-1 \to k$: Let $x_1, \ldots, x_k \in M$ with $x_i \equiv x_j \mod N_i + N_j$ for all $1 \leq i, j \leq k$. By the inductive hypothesis there is some $y \in M$ such that $y - x_i \in N_i$ for all

$1 \leq i \leq k-1$. It is enough to find some $x \in R$ with $x - y \in \bigcap_{i=1}^{k-1} N_i$ and $x - x_k \in N_k$. By Lemma 7 it suffices to show that $y - x_k \in \left(\bigcap_{i=1}^{k-1} N_i\right) + N_k$. Since N_k is distributive we get by applying [Vol. I, Proposition II.5.2(2)] iterated that

$$\left(\bigcap_{i=1}^{k-1} N_i\right) + N_k = \bigcap_{i=1}^{k-1}(N_i + N_k).$$

But for each $1 \leq i \leq k-1$ we have

$$y - x_k = y - x_i + x_i - x_k \in N_i + N_i + N_k = N_i + N_k.$$

□

Corollary 6.9. *Let A be Prüfer in R. Then CRT holds for finitely many R-regular A-submodules of R.*

Proof. By [Vol. I, Example II.5.1] every R-regular A-submodule of R is a distributive submodule of R. Now we can apply Proposition 8. □

Proposition 6.10. *Let $v_1, \ldots, v_n$ be Manis valuations on R such that $A_{v_1} \cap \ldots \cap A_{v_n}$ is Prüfer in R. Let $a_1, \ldots, a_n \in R$ and $\varepsilon_1 \in \Gamma_{v_1}, \ldots, \varepsilon_n \in \Gamma_{v_n}$. Then the following are equivalent.*

(1) There is some $x \in R$ such that $v_i(x - a_i) \geq \varepsilon_i$ for all $1 \leq i \leq n$.
(2) For each $1 \leq i, j \leq n$ there is some $x \in R$ with $v_i(x - a_i) \geq \varepsilon_i$ and $v_j(a - a_j) \geq \varepsilon_j$.

Proof. Let $A := A_{v_1} \cap \ldots \cap A_{v_n}$.

(1) ⇒ (2): This is trivial.

(2) ⇒ (1): Let $I_k := \{y \in R \mid v_k(y) \geq \varepsilon_k\}$ for $1 \leq k \leq n$. We have to show that there is some $x \in R$ such that $x \equiv a_k \mod I_k$ for $1 \leq k \leq n$. Note that $I_k = R$ if v_k is trivial. We have that I_k is a v_k-convex A_{v_k}-submodule of R properly containing $\operatorname{supp} v_k$. By [Vol. I, Theorem III.2.2] we get that I_k is an R-regular A_{v_k}-submodule and therefore an R-regular A-module for all $1 \leq k \leq n$. Hence CRT holds for $I_1, \ldots, I_n$ by Corollary 9. Hence it is enough to show that $a_k - a_l \in I_k + I_l$ for all $1 \leq k, l \leq n$. By Remark 6 we are done if we find some $x \in R$ with $x - a_k \in I_k$ and $x - a_l \in I_l$, i.e. $v_k(x - a_k) \geq \varepsilon_k$ and $v_l(x - a_l) \geq \varepsilon_l$. But this holds by (2). □

Theorem 6.11. *Let $v_1, \ldots, v_n$ be pairwise independent Manis valuations on R such that $A_{v_1} \cap \cdots \cap A_{v_n}$ is Prüfer in R. Then the general approximation theorem holds for $v_1, \ldots, v_n$.*

Proof. By Definition 1, Proposition 10 and [Vol. I, Corollary I.5.3] it is enough to show the case $n = 2$. So let $(\alpha_1, \alpha_2, a_1, a_2) \in \Gamma_{v_1} \times \Gamma_{v_2} \times R^2$ be weakly compatible. We set $I_k := \{y \in R \mid v_k(y) \geq \alpha_k\}$ for $1 \leq k \leq 2$. Then I_1, I_2 are A-modules. By Lemma 7 we have to show that $a_1 - a_2 \in I_1 + I_2$.

Case 1: $v_k(a_1 - a_2) \geq \alpha_k$ for $k = 1$ or $k = 2$. Then $a_1 - a_2 \in I_k \subset I_1 + I_2$ and we are done.

Case 2: $v_k(a_1 - a_2) < \alpha_k$ for $1 \leq k \leq 2$. Let $\gamma_k := \alpha_k - v_k(a_1 - a_2) > 0$. Since v_1 and v_2 are independent and since $A := A_{v_1} \cap A_{v_2}$ is Prüfer we find by Proposition 5.8 and Corollary 5.12 $x_1, x_2 \in R$ such that $v_1(x_1) = \gamma_1, v_2(x_1) = 0$ and $v_1(x_2) = 0, v_2(x_2) = \gamma_2$. We have $v_1(x_1(a_1 - a_2)) = \alpha_1$. So

$$x_1(a_1 - a_2) \in I_1 \subset (I_1 + I_2) \subset (I_1 + I_2)A_{v_2}.$$

By [Vol. I, Theorem III.2.2] I_2 is an R-regular A_{v_2}-module. Hence $(I_1 + I_2)A_{v_2}$ is an R-regular A_{v_2}-module. Again by [Vol. I, Theorem III.2.2] we see that $(I_1 + I_2)A_{v_2}$ is v_2-convex. Since $v_2(x_1) = 0$ we get $a_1 - a_2 \in (I_1 + I_2)A_{v_2}$. In the same way we obtain $a_1 - a_2 \in (I_1 + I_2)A_{v_1}$. Hence

$$a_1 - a_2 \in \bigcap_{k=1,2} (I_1 + I_2)A_{v_k} = I_1 + I_2$$

where the latter equality holds by [Vol. I, Theorem II.1.4(4)]. □

For the following, recall Definitions 4 and 5 from Sect. 5.

Definition 4. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R and let $i_1, \ldots, i_n \in I$. Then $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$ is called $\{i_1, \ldots, i_n\}$- *complete* if

$$\bigcup_{k=1}^{n} I(\alpha_k) \cup \bigcup_{k=1}^{n} I(v_k(a_k)) = \{i_1, \ldots, i_n\}.$$

Remark 6.12. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R and let $i_1, \ldots, i_n \in I$. Let $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times R^n$ be $\{i_1, \ldots, i_n\}$-complete. Then $(\alpha_1, \ldots, \alpha_n)$ is $\{i_1, \ldots, i_n\}$-complete.

Proof. This follows immediately from Remark 5.19(a). □

Remark 6.13. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R and let $i_1, \ldots, i_n \in I$. Let $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times R^n$ be weakly compatible and $\{i_1, \ldots, i_n\}$-complete. Let $j_1, \ldots, j_m \in I \setminus \{i_1, \ldots, i_n\}$. Then

$$(\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, a_1, \ldots, a_n, 0, \ldots, 0) \in$$

$$\Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times \Gamma_{v_{j_1}} \times \ldots \times \Gamma_{v_{jm}} \times R^{n+m}$$

is weakly compatible and $\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$-complete.

Proof. Let $1 \leq p \leq n$ and $1 \leq q \leq m$.

a) We show that $f_{v_{i_p},v_{j_q}}(\alpha_p) = 0$. If $\alpha_p = 0$ this is clear. If $\alpha_p \neq 0$ then $f_{v_{i_p},v_{j_q}}(\alpha_p) = 0$ by Proposition 5.17 since $j_q \notin I(\alpha_p)$.
b) We show that $(v_{i_p} \vee v_{j_q})(a_p) \geq 0$. We have $(v_{i_p} \vee v_{j_q})(a_p) = f_{v_{i_p},v_{j_q}}(v_{i_p}(a_p))$. If $v_{i_p}(a_p) = 0$ or $v_{i_p}(a_p) = \infty$ nothing is to show. Therefore we assume the $v_{i_p}(a_p) \in \Gamma_{v_i} \setminus \{0\}$. Since $j_q \notin I(v_p(a_p))$ we get $f_{v_{i_p},v_{j_q}}(v_{i_p}(a_p)) = 0$ be Proposition 5.17.

By (a) and (b) we get that

$$(\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, a_1, \ldots, a_n, 0, \ldots, 0) \in$$

$$\Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times \Gamma_{v_{j_1}} \times \ldots \times \Gamma_{v_{j_m}} \times R^{n+m}$$

is weakly compatible. It is clear by Definition 4 in Sect. 5 that it is $\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$-complete. □

Definition 5. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. We say that the *general approximation theorem* holds for the family if for each $i_1, \ldots, i_n \in I$ and each weakly compatible and $\{i_1, \ldots, i_n\}$-complete tuple $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times R^n$ there is some $x \in R$ with $v_{i_k}(x - a_i) \geq \alpha_k$ for $1 \leq k \leq n$ and $v_j(x) \geq 0$ for all $j \in I \setminus \{i_1, \ldots, i_n\}$.

Remark 6.14. If I is finite then Definition 5 coincides with Definition 2.

Proof. Let $I = \{1, \ldots, n\}$.

a) We show that the general approximation theorem in the sense of Definition 5 implies the one in the sense of Definition 2. To see this let $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$ be weakly compatible. By Remark 5.19(a) it is $\{1, \ldots, n\}$-complete and we are done.
b) We show that the general approximation theorem in the sense of Definition 2 implies the one in the sense of Definition 5. For this let $i_1 \ldots, i_k \in \{1, \ldots, n\}$. Without restriction we can assume that $i_1 = 1, \ldots, i_k = k$. Let $(\alpha_1, \ldots, \alpha_k, a_1, \ldots, a_k) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_k} \times R^k$ be weakly compatible and $\{1, \ldots, k\}$-complete. Then

$$(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0, a_1, \ldots, a_k, 0, \ldots, 0) \in$$

$$\Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$$

is weakly compatible by Remark 13. Hence there is some $x \in R$ such that $v_i(x - a_i) \geq \alpha_i$ for $1 \leq i \leq k$ and $v_i(x) \geq 0$ for $k + 1 \leq i \leq n$ and we are done. □

We can generalize Theorem 4 to families with finite avoidance.

Theorem 6.15. *Let $(v_i \mid i \in I)$ be a family of non-trivial Manis valuations on R with finite avoidance. If the general approximation theorem holds for $(v_i \mid i \in I)$ then also the approximation theorem in the neighbourhood of zero.*

Proof. We can clearly assume that the valuations $v_i, i \in I$, are pairwise non-isomorphic. We write $I = K\dot{\cup}L$ such that the following holds.

i) The valuations in the family $(v_k \mid k \in K)$ are pairwise incomparable.
ii) There is $\varphi : L \to K$ such that $v_{\varphi(l)} \leq v_l$ for all $l \in L$.

(Note that there are no infinite descending chains by Remark 4.14.) Let $i_1, \ldots, i_n \in K$ and $j_1, \ldots, j_m \in L$. Let

$$(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_k}} \times \Gamma_{v_{j_1}} \times \ldots \times \Gamma_{v_{j_l}}$$

be compatible and $\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$-complete. Applying Remark 5.20 we can assume that $\varphi(j_l) \in \{i_1, \ldots, i_n\}$ for all $1 \leq l \leq m$. For $1 \leq k \leq n$ let $x_k \in R$ such that $v_{i_k}(x_k) = \alpha_k$. For $1 \leq l \leq m$ let $y_l := x_{\varphi(j_l)}$. Then $v_{j_l}(y_l) = \beta_l$ for $1 \leq l \leq m$. Clearly the tuple

$$(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m, x_1, \ldots, x_n, y_1, \ldots, y_m) \in$$

$$\Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times \Gamma_{v_{j_1}} \times \ldots \times \Gamma_{v_{jm}} \times R^{n+m}$$

is $\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$-complete.

Claim : Let $\Delta := \bigcap_{p \in I \setminus \{i_1\}} H_{i_1,p}$. Then $\Delta \neq \{0\}$.

Proof of the Claim: Since $v_{i_1} \neq v_p$ for all $p \neq i_1$ and since $i_1 \in K$ we have $H_{i_1,p} \neq \{0\}$ for all $p \neq i_1$. Assume that $\Delta = \{0\}$. Then we find a sequence $p_1, p_2, \ldots$ in $I \setminus \{i_1\}$ such that

$$\Gamma_{v_{i_1}} \supsetneqq H_{i_1,p_1} \supsetneqq H_{i_1,p_2} \supsetneqq H_{i_1,p_3} \supsetneqq \ldots.$$

So

$$v_{i_1} \vee v_{p_1} > v_{i_1} \vee v_{p_2} > v_{i_1} \vee v_{p_3} > \ldots.$$

Since $v_{i_1} \vee v_{p_1}$ is non-trivial we find some $x \in R$ such that $(v_{i_1} \vee v_{p_1})(x) < 0$. Then $(v_{i_1} \vee v_{p_n})(x) < 0$ for all $n \in \mathbb{N}$. We get $v_{p_n}(x) < 0$ for all $p \in \mathbb{N}$. This contradicts the condition that the family has finite avoidance.

For $1 \leq k \leq n$ we find by the claim $\delta_i \in \bigcap_{p \in I \setminus \{i_k\}} H_{i_k,p}$ with $\delta_i > 0$. Let $\varepsilon_k := \alpha_k + \delta_k$ for $1 \leq k \leq n$. We see as in the proof of Theorem 4 that the tuple $(\varepsilon_1, \ldots, \varepsilon_n, \beta_1, \ldots, \beta_m, x_1, \ldots, x_n, y_1, \ldots, y_m)$ is weakly compatible. By construction it is clearly $\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$-complete.

Since the general approximation theorem holds for $(v_i \mid i \in I)$ we find some $x \in R$ such that $v_{i_k}(x - x_k) \geq \varepsilon_k$ for $1 \leq k \leq n$, $v_{j_l}(x - y_l) \geq \beta_l$ for $1 \leq l \leq m$ and $v_p(x) \geq 0$ for $p \in I \setminus \{i_1, \ldots, i_n, j_1, \ldots, j_m\}$. As in the proof of Theorem 4 we see that $v_{i_k}(x) = \alpha_k$ for $1 \leq k \leq n$ and $v_{j_l}(x) = \beta_l$ for $1 \leq l \leq m$ and are done. $\square$

Theorem 6.16. *Let $(v_i \mid i \in I)$ be a family of pairwise independent Manis valuations on R having finite avoidance. If $\bigcap_{i\in I} A_{v_i}$ is Prüfer in R then the general approximation theorem holds for $(v_i \mid i \in I)$.*

Proof. We set $A := \bigcap_{i\in I} A_{v_i}$. Let $i_1,\dots,i_n \in I$. Let $(\alpha_1,\dots,\alpha_n,a_1,\dots,a_n) \in \Gamma_{v_{i_1}} \times \dots \times \Gamma_{v_{i_n}} \times R^n$. (The tuple is weakly compatible by Proposition 2 and $\{i_1,\dots,i_n\}$-complete by Remark 5.19(b)) For $1 \le k \le n$ let $x_k \in R$ with $v_{i_k}(x_k) = \alpha_k$ and $y_k \in R$ with $v_{i_k}(y_k) = -\alpha_k$. Since $(v_i \mid i \in I)$ has finite avoidance there is a finite subset J of I containing $i_1,\dots,i_n$ such that $v_j(x_k) \ge 0, v_j(y_k) \ge 0$ and $v_j(a_k) \ge 0$ for all $1 \le k \le n$ and all $j \in I \setminus J$. Let $B := \bigcap_{j\in I\setminus J} A_{v_j}$. Then $x_k, y_k, a_k \in B$ for all $1 \le k \le n$. We write $J \setminus \{i_1,\dots,i_n\}$ as $\{i_{n+1},\dots,i_m\}$ for some $m \ge n$. For $1 \le k \le m$ we set $w_k := v_{i_k}|_B$. By Corollary 4.10 $w_1,\dots,w_m$ are Manis valuations. By Proposition 4.13 they are pairwise independent. By construction $\alpha_k \in \Gamma_{w_k}$ for $1 \le k \le n$. The tuple

$$(\alpha_1,\dots,\alpha_n,0,\dots,0,a_1,\dots,a_n,0,\dots,0) \in \Gamma_{w_1} \times \dots \times \Gamma_{w_m} \times B^m.$$

is weakly compatible and $\{i_1,\dots,i_m\}$-complete. We have

$$\bigcap_{1\le k\le m} A_{w_k} = \bigcap_{1\le k\le m} A_{v_{i_k}} \cap B = A.$$

Hence $\bigcap_{1\le k\le m} A_{w_k}$ is Prüfer in B by [Vol. I, Corollary I.5.3]. Applying Theorem 11 there is some $x \in B$ such that $w_k(x - a_k) \ge \alpha_k$ for $1 \le k \le n$ and $w_k(x) \ge 0$ for $n+1 \le k \le m$. Therefore $v_{i_k}(x - a_k) \ge \alpha_k$ for $1 \le k \le n$ and $v_j(x) \ge 0$ for $j \in I \setminus \{i_1,\dots,i_n\}$. □

7 The Reinforced Approximation Theorem

We present the remarkable result of Gräter [Gr$_2$] on the reinforced approximation theorem. It implies the approximation theorem in the neighbourhood of zero. Assuming finite avoidance the reinforced approximation theorem for pairwise non-isomorphic PM-valuations is equivalent to the Prüfer condition. The concept of maximally dominant Manis valuations is introduced and used.

Definition 1. i) Let v, w be Manis valuation on R. Let $(\alpha,\beta,a,b) \in \Gamma_v \times \Gamma_w \times R^2$. We call the tuple (α,β,a,b) *compatible* if it is weakly compatible (cf. Sect. 6, Definition 1) and if the tuple $(\alpha,\beta) \in \Gamma_v \times \Gamma_w$ is compatible (cf. Sect. 5, Definition 2).

ii) Let $v_1,\dots,v_n$ be Manis valuations on R. Let $(\alpha_1,\dots,\alpha_n,a_1,\dots,a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_k} \times R^n$. The tuple $(\alpha_1,\dots,\alpha_n,a_1,\dots,a_n)$ is called *compatible* if $(\alpha_i,\alpha_j,a_i,a_j)$ is compatible for every $1 \le i, j \le n$.

Remark 7.1. Let $v_1, \dots, v_n$ be Manis valuations on R. Let $(\alpha_1, \dots, \alpha_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n}$. The tuple $(\alpha_1, \dots, \alpha_n)$ is compatible (in the sense of Definition 2 in Sect. 5) iff $(\alpha_1, \dots, \alpha_n, 0, \dots, 0) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n} \times R^n$ is compatible (in the sense of Definition 1).

Remark 7.2. Let v, w be Manis valuations on R. A tuple $(\alpha, \beta, a, b) \in \Gamma_v \times \Gamma_w \times R^2$ is compatible iff $(\alpha, \beta) \in \Gamma_v \times \Gamma_w$ is compatible and the following holds:

$$v(a-b) < \alpha \Longrightarrow f_{v,w}(\alpha) = (v \vee w)(a-b) \text{ in } \Gamma_{v,w},$$

or

$$w(a-b) < \beta \Longrightarrow f_{w,v}(\beta) = (v \vee w)(a-b) \text{ in } \Gamma_{v,w}.$$

Remark 7.3. Let $v_1, \dots, v_n$ be Manis valuations on R.

i) Let $x \in R \setminus \bigcup_{1 \le i \le n} \operatorname{supp} v_i$ and let $a_i \in R$ for $1 \le i \le n$. If $v_i(a_i) \ge v_i(x)$ for all $1 \le i \le n$ then $(v_1(x), \dots, v_n(x), a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n} \times R^n$ is compatible.
ii) Let $(\alpha_1, \dots, \alpha_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n}$ be compatible and let $x, a_1, \dots, a_n \in R$ such that $v_i(x - a_i) \ge \alpha_i$ for all $1 \le i \le n$. Then $(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n} \times R^n$ is compatible.

Proof. i): The tuple $(v_1(x), \dots, v_n(x)) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n}$ is compatible by Remark 5.5(a). We show that $(v_1(x), \dots, v_n(x), a_1, \dots, a_n)$ is weakly compatible. Let $1 \le i, j \le n$. We set $v := v_i, w := v_j, \alpha := v(x), \beta := w(x), a := a_i$, and $b := a_j$. Then

$$\begin{aligned}
(v \vee w)(a-b)) &= (v \vee w)(a - x + x - b) \ge \min\{(v \vee w)(a-x), (v \vee w)(b-x)\} \\
&= \min\{f_{v,w}(v(a-x)), f_{w,v}(w(b-x))\} \\
&\ge \min\{f_{v,w}(v(x)), f_{v,w}(v(a)), f_{w,v}(w(x)), f_{w,v}(w(b))\} \\
&= f_{v,w}(\alpha) (= f_{w,v}(\beta)).
\end{aligned}$$

ii): This follows from Remark 6.1(a). □

Definition 2. Let $v_1, \dots, v_n$ be Manis valuations on R. The *reinforced approximation theorem* holds for $v_1, \dots, v_n$ if for every compatible $(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n} \times R^n$ there is some $x \in R$ such that $v_1(x - a_1) = \alpha_1, \dots, v_n(x - a_n) = \alpha_n$.

Remark 7.4. (cf. [Al-M, p. 107]) Let $v_1, \dots, v_n$ be Manis valuations and let

$$(\alpha_1, \dots, \alpha_n, a_1, \dots, a_n) \in \Gamma_{v_1} \times \dots \times \Gamma_{v_n} \times R^n.$$

Assume that there is some $x \in R$ such that $v_i(x - a_i) = \alpha_i$ for all $1 \le i \le n$. Then the above tuple is not necessary compatible.

Proof. Let v_1, v_2 be dependent Manis valuations on R. Let $\alpha_1 \in \Gamma_{v_1}$ and $\alpha_2 \in \Gamma_{v_2}$ with $f_{v_1,v_2}(\alpha_1) < f_{v_2,v_1}(\alpha_2)$. Let $a_1, a_2 \in R$ with $v_1(a_1) = \alpha_1$ and $v_2(a_2) = \alpha_2$. Taking $x = 0$ we have $v_i(x - a_i) = \alpha_i$ for $1 \le i \le 2$. But (α_1, α_2) is not compatible. □

Remark 7.5. Let $v_1, \ldots, v_n, w_1, \ldots, w_m$ be Manis valuations on R such that for every $1 \le j \le m$ there is some $1 \le i \le n$ with $v_i \le w_j$. If the reinforced approximation theorem holds for $v_1, \ldots, v_n, w_1, \ldots, w_m$ then also for $v_1, \ldots, v_n$.

Proof. For $1 \le j \le m$ we choose $1 \le i_j \le n$ such that $v_{i_j} \le w_j$. Let

$$(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$$

be compatible. For $1 \le j \le m$ let $\beta_j := f_{v_{i_j}, w_j}(\alpha_{i_j})$ and $b_j := a_{i_j}$. We see as in the proof of Remark 5.10, (2) ⇒ (1), that the tuple

$$(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m, a_1, \ldots, a_n, b_1, \ldots, b_m) \in$$

$$\Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times \Gamma_{w_1} \times \ldots \times \Gamma_{w_m} \times R^{n+m}$$

is compatible. By assumption there is some $x \in R$ such that $v_i(x - a_i) = \alpha_i$ for all $1 \le i \le n$. □

Remark 7.6. Let $v_1, \ldots, v_n$ be non-trivial Manis valuations on R and let $w_1, \ldots, w_m$ be trivial Manis valuations on R. If the reinforced approximation theorem holds for $v_1, \ldots, v_n, w_1, \ldots, w_m$ then also for $v_1, \ldots, v_n$.

Proof. Let $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$ be compatible. Then

$$(\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, a_1, \ldots, a_n, 0, \ldots, 0) \in$$
$$\Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times \Gamma_{w_1} \times \ldots \times \Gamma_{w_m} \times R^{n+m}$$

is compatible since v_i, w_j are independent and w_j, w_l are independent for $1 \le i \le n$ and $1 \le j \ne l \le m$. By assumption there is some $x \in R$ such that $v_i(x - a_i) = \alpha_i$ for all $1 \le i \le n$. □

Proposition 7.7. *Let $v_1, \ldots, v_n$ be Manis valuations on R. If the reinforced approximation theorem holds for $v_1, \ldots, v_n$ then also the approximation theorem in the neighbourhood of zero.*

Proof. Let $(\alpha_1, \ldots, \alpha_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ be compatible. By Remark 1 the tuple $(\alpha_1, \ldots, \alpha_n, 0, \ldots, 0) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$ is compatible. Hence there is some $x \in R$ such that $v_i(x) = \alpha_i$ for all $1 \le i \le n$. □

Corollary 7.8. *Let $v_1, \ldots, v_n$ be Manis valuations on R. If the reinforced approximation theorem holds for $v_1, \ldots, v_n$ then $v_1, \ldots, v_n$ have the inverse property.*

Proof. By Proposition 7 the approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_n$. By Theorem 5.11 $v_1, \ldots, v_n$ have the inverse property. □

Proposition 7.9. *Let $v_1, \ldots, v_n$ be Manis valuations on R such that $A_{v_1} \cap \cdots \cap A_{v_n}$ is Prüfer in R. Let $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$ be compatible. Then there is some $x \in R$ such that $v_1(x - a_1) \geq \alpha_1, \ldots, v_n(x - a_n) \geq \alpha_n$.*

Proof. We can adapt the proof of Theorem 6.11. By Proposition 6.10 and [Vol. I, Corollary I.5.3] it is enough to show the case $n = 2$. So let $(\alpha_1, \alpha_2, a_1, a_2) \in \Gamma_{v_1} \times \Gamma_{v_2} \times R^2$ be compatible. We set $I_k := \{y \in R \mid v_k(y) \geq \alpha_i\}$ for $1 \leq k \leq 2$. By Lemma 6.7 we have to show that $a_1 - a_2 \in I_1 + I_2$.

Case 1: $v_k(a_1 - a_2) \geq \alpha_k$ for $k = 1$ or $k = 2$. Then $a_1 - a_2 \in I_k \subset I_1 + I_2$ and we are done.

Case 2: $v_k(a_1 - a_2) < \alpha_k$ for $1 \leq k \leq 2$. Then

$$(v_1 \vee v_2)(a_1 - a_2) = f_{v_1,v_2}(\alpha_1) = f_{v_2,v_1}(\alpha_2)$$

by Remark 2. Let $\gamma_k := \alpha_k - v_k(a_1 - a_2) > 0$. Then $f_{v_1,v_2}(\gamma_1) = f_{v_2,v_1}(\gamma_2) = 0$. So $(\gamma_1, 0)$ and $(0, \gamma_2)$ are compatible. Since $A := A_{v_1} \cap A_{v_2}$ is Prüfer we find by Corollary 5.12 $x_1, x_2 \in R$ such that $v_1(x_1) = \gamma_1, v_2(x_1) = 0$ and $v_1(x_2) = 0, v_2(x_2) = \gamma_2$. We have $v_1(x_1(a_1 - a_2)) = \alpha_1$. Now we can proceed as in the proof of Theorem 6.11. □

The main result is that the reinforced approximation theorem holds for pairwise non-isomorphic non-trivial Manis valuations $v_1, \ldots, v_n$ if $A_{v_1} \cap \ldots \cap A_{v_n}$ is Prüfer in R.

Theorem 7.10. *Let $v_1, \ldots, v_n$ be pairwise non-isomorphic non-trivial Manis valuations on R such that $A_{v_1} \cap \cdots \cap A_{v_n}$ is Prüfer in R. Then the reinforced approximation theorem holds for $v_1, \ldots, v_n$.*

Proof. Without restriction we write the valuations as $v_1, \ldots, v_k, w_1, \ldots, w_l$ such that the following properties hold.

i) v_i and v_j are incomparable for $i \neq j$.
ii) There is $\varphi : \{1, \ldots, l\} \to \{1, \ldots, k\}$ such that $v_{\varphi(j)} \leq w_j$ for all $1 \leq j \leq l$.

Let $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l, a_1, \ldots, a_k, b_1, \ldots, b_l) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_k} \times \Gamma_{w_1} \times \ldots \times \Gamma_{w_l} \times R^{k+l}$ be compatible. Since $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l)$ is compatible there is by Corollary 5.12 some $x' \in R$ such that $v_i(x') = \alpha_i$ for $1 \leq i \leq k$ and $w_j(x') = \beta_j$ for $1 \leq j \leq l$. As in the proof of Theorem 6.4 we find for $1 \leq i \leq k$ $\varepsilon_i > \alpha_i$ such that $(\varepsilon_1, \ldots, \varepsilon_k, \beta_1, \ldots, \beta_l, a_1, \ldots, a_k, b_1, \ldots, b_l)$ is compatible. By Proposition 9 there is some $x'' \in R$ such that $v_i(x'' - a_i) \geq \varepsilon_i$ for all $1 \leq i \leq k$ and $w_j(x'' - b_j) \geq \beta_j$ for all $1 \leq j \leq l$. Let $y := x' + x''$. Then $v_i(y - a_i) = \alpha_i$ for $1 \leq i \leq k$ and $w_j(y - b_j) \geq \beta_j$ for $1 \leq j \leq l$. For $1 \leq i \leq k$ we set

$$\Delta_i := \bigcap_{j \neq i} H_{v_i,v_j} \cap \bigcap_{1 \leq j \leq l} H_{v_i,w_j}.$$

Then Δ_i is a convex subgroup of Γ_{v_i} distinct from $\{0\}$. Hence we find some $\delta_i' \in \Delta_i$ such that $\delta_i' > 0$ and

$$\varepsilon_i' := \alpha_i + \delta_i' \notin \{v_i(y-b_1), \ldots, v_i(y-b_l)\}$$

for $1 \leq i \leq k$. Then $(\varepsilon_1', \ldots, \varepsilon_k', \beta_1, \ldots, \beta_l)$ is compatible. By Corollary 5.12 we find some $y' \in R$ such that $v_i(y') = \varepsilon_i'$ for $1 \leq i \leq k$ and $w_j(y') = \beta_j$ for $1 \leq j \leq l$. Let $x := y + y'$. Then $v_i(x - a_i) = \alpha_i$ for $1 \leq i \leq n$ and $w_j(x - b_j) \geq \beta_j$ for $1 \leq j \leq l$. We show that $w_j(x - b_j) = \beta_j$ for $1 \leq j \leq l$. To see this we fix $1 \leq j \leq l$. Since $\varepsilon_{\varphi(j)}' \neq v_{\varphi(j)}(y - b_j)$ we have

$$v_{\varphi(j)}(x - b_j) = v_{\varphi(j)}(y - b_j + y') = \min\{v_{\varphi(j)}(y - b_j), \varepsilon_{\varphi(j)}'\}.$$

We distinguish two cases.

Case 1: $v_{\varphi(j)}(x - b_j) = \varepsilon_{\varphi(j)}'$. Then

$$\begin{aligned} w_j(x - b_j) &= f_{v_{\varphi(j)}, w_j}(v_{\varphi(j)}(x - b_j)) = f_{v_{\varphi(j)}, w_j}(\varepsilon_{\varphi(j)}') \\ &= f_{v_{\varphi(j)}, w_j}(v_{\varphi(j)}(y')) = w_j(y') = \beta_j. \end{aligned}$$

Case 2: $v_{\varphi(j)}(x - b_j) < \varepsilon_{\varphi(j)}'$ (and $v_{\varphi(j)}(x - b_j) = v_{\varphi(j)}(y - b_j)$). Then

$$\begin{aligned} w_j(x - b_j) &= f_{v_{\varphi(j)}, w_j}(v_{\varphi(j)}(x - b_j)) \leq f_{v_{\varphi(j)}, w_j}(\varepsilon_{\varphi(j)}') \\ &= f_{v_{\varphi(j)}, w_j}(v_{\varphi(j)}(y')) = w_j(y') = \beta_j. \end{aligned}$$

But $w_j(x - b_j) \geq \beta_j$ by above. Hence equality holds. □

Corollary 7.11. *Let $v_1, \ldots, v_n$ be pairwise non-isomorphic Manis valuations on R. If $A_{v_1} \cap \ldots \cap A_{v_n}$ is Prüfer in R then the reinforced approximation theorem holds for $v_1, \ldots, v_n$.*

Proof. We can assume that $v_1, \ldots, v_k$ are trivial and $v_{k+1}, \ldots, v_n$ are non-trivial for some $0 \leq k \leq n$. We do induction on k.

$k = 0$: This is covered by Theorem 10.

$k - 1 \to k$: After switching the order we can assume that $\operatorname{supp} v_1$ is minimal in $\{\operatorname{supp} v_i \mid 1 \leq i \leq k\}$ with respect to inclusion. Let $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$ be compatible. Note that $\alpha_i = 0$ for $1 \leq i \leq k$. The tuple $(\alpha_2, \ldots, \alpha_n, a_2, \ldots, a_n) \in \Gamma_{v_2} \times \ldots \times \Gamma_{v_n} \times R^{n-1}$ is also compatible. By the inductive hypothesis there is some $x' \in R$ with $v_i(x' - a_i) = \alpha_i$ for $2 \leq i \leq n$. If $v_1(x' - a_1) = 0$ we are done. Assume that $v_1(x' - a_1) = \infty$. We have $\operatorname{supp} v_1 \neq \operatorname{supp} v_i$ for all $2 \leq i \leq k$ by assumption. By Theorem 4.18 and Corollary 4.9 $v_1, \ldots, v_n$ have the inverse property. Therefore Claim 2 in the proof of Theorem 4.18 holds. Hence there is some $x'' \in R$ with

$$v_1(x'') = 0, v_2(x'') = \infty, \ldots, v_k(x'') = \infty, v_{k+1}(x'') > \alpha_{k+1}, \ldots, v_n(x'') > \alpha_n.$$

Let $x := x' + x''$. Then $v_i(x - a_i) = \alpha_i$ for all $1 \leq i \leq n$. $\square$

Remark 7.12. It is in general necessary in Corollary 11 that the valuations are non-isomorphic.

Proof. Let R be the field $\mathbb{Z}/2\mathbb{Z}$. Let v be the trivial valuation on R (with $\operatorname{supp} v = \{0\}$). Then the reinforced approximation theorem does not hold for v, v for the following reason: The tuple $(0, 0, 1, 0) \in \Gamma_v \times \Gamma_v \times R^2$ is clearly compatible but there is no $x \in R$ such that $v(x-1) = 0$ and $v(x) = 0$ since there is no $x \in R$ with $x - 1 = 1$ and $x = 1$. $\square$

The converse of Theorem 10 holds if the valuations are PM. We need the notion of maximal dominance.

Definition 3. A Manis valuation v on R is said to be *maximally dominant* if for all non-trivial Manis valuations w on R with $v \leq w$ the ideal $\mathfrak{p}_w$ is maximal in A_w.

Remark 7.13.

i) A trivial valuation is maximally dominant.
ii) Let v be a non-trivial Manis valuation on R that is maximally dominant. Then $\mathfrak{p}_v$ is a maximal ideal of A_v.

Proposition 7.14. *Let v be a non-trivial Manis valuation on R. Then the following are equivalent:*

(1) A_v is Prüfer in R (i.e. v is a PM-valuation).
(2) v is maximally dominant.
(3) $\mathfrak{p}_v$ is a maximal ideal of A_v and for any maximal ideal $\mathfrak{m}$ of A_v with $\mathfrak{m} \neq \mathfrak{p}_v$ there is no prime ideal $\mathfrak{p}$ of A_v such that $\operatorname{supp} v \subsetneqq \mathfrak{p} \subset \mathfrak{p}_v$ and $\mathfrak{p} \subset \mathfrak{m}$.

Proof. Let $A := A_v$.

(1) $\Rightarrow$ (2): Let w be a non-trivial Manis valuation on R with $v \leq w$. Then $A_w \supset A_v$ by Proposition 1.8 and hence A_w is Prüfer by [Vol. I, Corollary I.5.3]. Therefore $\mathfrak{p}_w$ is a maximal ideal of A_w by [Vol. I, Corollary III.1.4] and [Vol. I, Proposition I.2.3].

(2) $\Rightarrow$ (3): By Remark 13(ii) $\mathfrak{p}_v$ is a maximal ideal of A. Let $\mathfrak{m}$ be a maximal ideal of A with $\mathfrak{m} \neq \mathfrak{p}_v$. Suppose that there is some prime ideal $\mathfrak{p}$ of A such that $\operatorname{supp} v \subsetneqq \mathfrak{p} \subset \mathfrak{p}_v$ and $\mathfrak{p} \subset \mathfrak{m}$. By Scholium 1.10 there is a coarsening w of v such that $A_w = A_{[\mathfrak{p}]}$ and $\mathfrak{p}_w = \mathfrak{p}$ (namely $v^{\mathfrak{p}}$, cf. Definition 2 in Sect. 1). By Proposition 1.12 w is non-trivial. By assumption $\mathfrak{p}$ is a maximal ideal of $A_w =: B$. By [Vol. I, Proposition I.1.17] $v|_B$ is Manis.

Claim: $\operatorname{supp} v|_B = \mathfrak{p}$.

Proof of the Claim: We show that $\operatorname{supp} v|_B \subset \mathfrak{p}$. Let $x \in B \setminus \mathfrak{p}$. If $v(x) \leq 0$ then $v|_B(x) = v(x)$ and therefore $x \notin \operatorname{supp} v|_B$. So we assume that $v(x) > 0$. Then $x \in A \setminus \mathfrak{p}$. Since $\operatorname{supp} v \subset \mathfrak{p}$ and v is Manis there is some $y \in R$ with $v(y) = -v(x)$.

Hence $yx \in A$ and therefore $y \in A_{[\mathfrak{p}]} = B$. So $v|_B(x) = v(x) < \infty$. We show that $\mathfrak{p} \subset \operatorname{supp} v|_B$. Let $x \in \mathfrak{p}$. Then $v(x) > 0$. Assume that $x \notin \operatorname{supp} v|_B$. Then there is some $y \in B$ with $v(x) \leq -v(y)$. Since $B = A_{[\mathfrak{p}]}$ there is some $s \in A \setminus \mathfrak{p}$ with $ys \in A$. So $v(ys) \geq 0$ and therefore $v(s) \geq -v(y)$. We get $v(x) \leq v(s)$ and therefore $s \in \mathfrak{p}$ since $\mathfrak{p}$ is v-convex by [Vol. I, Proposition I.1.10], contradiction.

By the claim we obtain that $v|_B$ has maximal support (cf. [Vol. I, Definition 7 in I §1]). Clearly $A = A_{v|_B}$. By above $\mathfrak{p}_v$ and $\mathfrak{m}$ are prime ideals of A that contain $\operatorname{supp} v|_B$. By [Vol. I, Proposition I.1.11 v) $\Rightarrow$ vi)], the two ideals m and $\mathfrak{p}_v$ of A are $v|_B$ convex. Therefore they are comparable. This contradicts the fact that $\mathfrak{p}_v$ and $\mathfrak{m}$ are distinct maximal ideals of A.

(3) $\Rightarrow$ (1): Let $\mathfrak{m}$ be any maximal ideal of A. If $\mathfrak{m} = \mathfrak{p}_v$ then $A_{[\mathfrak{m}]} = A$ and $\mathfrak{m}_{[\mathfrak{m}]} = \mathfrak{m}$ by [Vol. I, Lemma III.1.0], so $(A_{[\mathfrak{m}]}, \mathfrak{m}_{[\mathfrak{m}]})$ is a Manis pair of R. Let now $\mathfrak{m} \neq \mathfrak{p}_v$. We show that $A_{[\mathfrak{m}]} = R$ and are done. Suppose that $A_{[\mathfrak{m}]} \neq R$ and let $x \in R \setminus A_{[\mathfrak{m}]}$. Then $(A : x) \subset \mathfrak{m}$. We have that $(A : x)$ is v-convex. Thus $\mathfrak{p} := \sqrt{(A : x)}$ is a v-convex prime ideal of A with $\mathfrak{p} \subset \mathfrak{m}$. We have $\mathfrak{p} \supsetneqq \operatorname{supp}(v)$ since v is Manis and $\mathfrak{p} \subset \mathfrak{p}_v$ since $(A : x) \subset \mathfrak{p}_v$ because of $x \notin A = A_{[\mathfrak{p}_v]}$. This contradicts (3). □

Note that (1) $\Rightarrow$ (3) in the previous proof would also follow from Theorem 4.16 with $A := A_v$ and Scholium 1.10 combined with Proposition 1.12.

Corollary 7.15. *Let v be a non-trivial discrete Manis valuation on R and $A := A_v$. Then A is Prüfer in R if and only if $\mathfrak{p}_v$ is a maximal ideal of A.*

Proof. Since v is discrete the set of all non-trivial Manis valuations coarser than v contains only v. We get the claim by Proposition 14. □

Theorem 7.16. *Let $v_1, \ldots, v_n$ be PM-valuations on R. If the reinforced approximation theorem holds for $v_1, \ldots, v_n$ then $A_{v_1} \cap \ldots \cap A_{v_n}$ is Prüfer in R.*

Proof. By Proposition 1.8 and Remark 5 we can assume that v_i, v_j are incomparable for $i \neq j$. By Remark 6 we can assume that v_i is non-trivial for all $1 \leq i \leq n$. Let $A := A_{v_1} \cap \ldots \cap A_{v_n}$. The approximation theorem in the neighbourhood of zero holds for $v_1, \ldots, v_n$ by Proposition 7. By Theorems 5.11 and 4.18 we get that v_i is A-essential for every $1 \leq i \leq n$. Therefore we show condition (ii) of Theorem 4.17 to get the claim. Let $\mathfrak{m}$ be a maximal ideal of A with $\mathfrak{m} \neq \operatorname{cent}_A(v_i)$ for $1 \leq i \leq n$. Let w be a non-trivial Manis valuation with $v_i \leq w$ for some $1 \leq i \leq n$. We have to show that $\operatorname{cent}_A(w) \not\subset \mathfrak{m}$. Without restriction we assume that $i = 1$ and set $v := v_i$. Assume that $\operatorname{cent}_A(w) \subset \mathfrak{m}$. Let $B_0 := A[\mathfrak{p}_w] = A + \mathfrak{p}_w$ and $\mathfrak{q}_0 := \mathfrak{m} + \mathfrak{p}_w$.

Claim 1: $\mathfrak{q}_0$ is a prime ideal of B_0 with $\mathfrak{q}_0 \cap A = \mathfrak{m}$.

Proof of Claim 1: It is clear that $\mathfrak{q}_0$ is an ideal of B_0 with $\mathfrak{q}_0 \cap A = \mathfrak{m}$. We show that it is prime. Let $x = a + p, y = b + q \in B_0$ where $a, b \in A$ and $p, q \in \mathfrak{p}_w$ such that $xy \in \mathfrak{q}_0$. Let $p' := aq + bp + pq \in \mathfrak{p}_w$. Then $xy = ab + p'$. Since $xy \in \mathfrak{q}_0$ there is some $m \in \mathfrak{m}$ and some $q' \in \mathfrak{p}_w$ such that $ab + p' = m + q'$. Hence $ab - m \in \operatorname{cent}_A(w) \subset \mathfrak{m}$ and therefore $ab \in \mathfrak{m}$. Since $\mathfrak{m}$ is prime in A we get $a \in \mathfrak{m}$ or $b \in \mathfrak{m}$. This gives that $x \in \mathfrak{q}_0$ or $y \in \mathfrak{q}_0$.

By [LM, Theorem 10.6] there is a Manis pair $(B, \mathfrak{q})$ of A_w with $B_0 \subset B$ and $\mathfrak{q} \cap B_0 = \mathfrak{q}_0$. Hence $\mathfrak{q} \supset \mathfrak{p}_w$ and $\mathfrak{q} \cap A = \mathfrak{m}$.

Claim 2: $(B, \mathfrak{q})$ is a Manis pair of R.

Proof of Claim 2: By [Vol. I, Theorem I.2.4] we have to find for $x \in R \setminus B$ some $y \in \mathfrak{q}$ such that $xy \in B \setminus \mathfrak{q}$. If $x \in A_w$ we are done since $(B, \mathfrak{q})$ is a Manis pair in A_w. So we assume that $x \in R \setminus A_w$. Then there is some $p \in \mathfrak{p}_w$ with $xp \in A_w \setminus \mathfrak{p}_w$. By assumption v is PM. Hence it is maximally dominant by Proposition 14, (1) $\Rightarrow$ (2). So $\mathfrak{p}_w$ is a maximal ideal of A_w. Hence $xpA_w + \mathfrak{p}_w = A_w$ and we find $a \in A_w$ and $p' \in \mathfrak{p}_w$ with $xpa + p' = 1$. Since $\mathfrak{p}_w \subset \mathfrak{q}$ we get that $xpa \notin \mathfrak{q}$. Let $q := pa \in \mathfrak{p}_w$. Then $xq \in A_w \setminus \mathfrak{q}$. If $xq \in B$ we take $y := q$. If $xq \in A_w \setminus B$ there is some $q' \in \mathfrak{q}$ such that $xqq' \in B \setminus \mathfrak{q}$. Then we take $y := qq'$.

Let u be the Manis valuation on R corresponding to $(B, \mathfrak{q})$. Since w and u are non-trivial we get $u \leq w$ by Proposition 1.8.

Claim 3: $v_i \not\leq u$ for $1 \leq i \leq n$.

Proof of Claim 3: Assume that there is some $1 \leq i \leq n$ such that $v_i \leq u$. Then $\mathfrak{p}_u \subset \mathfrak{p}_{v_i}$. We have $\mathfrak{p}_u = \mathfrak{q}$ and obtain $\mathrm{cent}_A(u) = \mathfrak{q} \cap A = \mathfrak{m}$. Hence $\mathfrak{m} \subset \mathrm{cent}_A(v_i)$. This contradicts the fact that $\mathfrak{m}$ is a maximal ideal of A distinct from $\mathrm{cent}_A(v_i)$.

We may assume that there is some $1 \leq l \leq n$ such that

$$\{v_1, \ldots, v_l\} = \{v_i \mid v_i \leq w\}.$$

In the proof of Claim 2 we have seen that $\mathfrak{p}_w$ is a maximal ideal of A_w. Hence $K := A_w/\mathfrak{p}_w$ is a field. For $1 \leq i \leq l$ we define

$$\overline{v_i} : K \to \Gamma_{v_i} \cup \{\infty\}, \overline{v_i}(x + \mathfrak{p}_w) = \begin{cases} v_i(x) & x \in A_w \setminus \mathfrak{p}_w, \\ & \text{if} \\ \infty & x \in \mathfrak{p}_w. \end{cases}$$

and we set

$$\overline{u} : K \to \Gamma_u \cup \{\infty\}, \overline{u}(x + \mathfrak{p}_w) = \begin{cases} u(x) & x \in A_w \setminus \mathfrak{p}_w, \\ & \text{if} \\ \infty & x \in \mathfrak{p}_w. \end{cases}$$

Since $\mathfrak{p}_w$ is v_i-convex for every $1 \leq i \leq l$ and u-convex these are well defined valuations on k. We have $\mathfrak{p}_{\overline{v_i}} = \mathrm{cent}_{A_w}(v_i)/\mathfrak{p}_w = \mathfrak{p}_{v_i}/\mathfrak{p}_w$ for $1 \leq i \leq l$ and $\mathfrak{p}_{\overline{u}} = \mathrm{cent}_{A_w}(u)/\mathfrak{p}_w = \mathfrak{p}_u/\mathfrak{p}_w$. As in Claim 3 we see that $\overline{v_i} \not\leq \overline{u}$ for $1 \leq i \leq l$.

Claim 4: There is some $y \in K$ with $\overline{u}(y) < 0$ and $\overline{v_i}(y) = 0$ for all $1 \leq i \leq l$.

Proof of Claim 4: Since $\overline{v_i} \not\leq \overline{u}$ for all $1 \leq i \leq l$ we have that $\bigcap_{1 \leq i \leq l} H_{\overline{u}, \overline{v_i}} \neq \{0\}$. Let $\alpha \in \bigcap_{1 \leq i \leq l} H_{\overline{u}, \overline{v_i}}$ with $\alpha < 0$. Then $(\alpha, 0, \ldots, 0) \in \Gamma_{\overline{u}} \times \Gamma_{\overline{v_1}} \times \ldots \times \Gamma_{\overline{v_l}}$ is compatible. By Remark 3.1(d) valuations on a field have the inverse property.

Hence we find by Theorem 5.11 some $y \in K$ with $\overline{u}(y) = \alpha < 0$ and $\overline{v_i}(y) = 0$ for $1 \le i \le l$.

From Claim 4 we obtain some $x \in A_w$ with $u(x) < 0$ and $v_i(x) = 0$ for $1 \le i \le l$. We fix this x.

Claim 5: Let $J := \{(i,j) \in \{1,\ldots,l\} \times \{l+1,\ldots,n\} \mid v_i, v_j \text{ dependent}\}$. Then there is some $z \in \mathfrak{p}_w$ with $z \in A_{v_i \vee v_j} \setminus \mathfrak{p}_{v_i \vee v_j}$ for all $(i,j) \in J$.

Proof of Claim 5: Assume that the assertion does not hold. Then

$$\mathrm{cent}_A(w) \subset \bigcup_{(i,j)\in J} \mathrm{cent}_A(v_i \vee v_j).$$

So there is a pair $(i,j) \in J$ with $\mathrm{cent}_A(w) \subset \mathrm{cent}_A(v_i \vee v_j)$. Then $A_{[\mathrm{cent}_A(w)]} \supset A_{[\mathrm{cent}_A(v_i \vee v_j)]}$. Since v_i is A-essential we get that w is also A-essential by Proposition 4.5. By the same argument we see that $v_i \vee v_j$ is A-essential. So $A_w \supset A_{v_i \vee v_j}$. We have seen at the beginning of the proof that the approximation theorem in the neighbourhood of zero holds for $v_1,\ldots,v_n$. Hence by Theorem 5.11 and Proposition 3.21 w, v_i, v_j have the inverse property. By Proposition 3.22 we obtain that $v_i \vee v_j \le w$. Therefore $v_j \le w$, contradiction.

By Claim 5 we find some $z \in \mathfrak{p}_w$ with $z \in A_{v_i \vee v_j} \setminus \mathfrak{p}_{v_i \vee v_j}$ for all $(i,j) \in J$. Let $\alpha_i := v_i(z)$ for $1 \le i \le l$. Then $\alpha_i > 0$ since $z \in \mathfrak{p}_w \subset \mathfrak{p}_{v_i}$ for $1 \le i \le l$. Since $z \in A_{v_i \vee v_j} \setminus \mathfrak{p}_{v_i \vee v_j}$ for all $(i,j) \in J$ the tuple

$$(\alpha_1,\ldots,\alpha_l,0,\ldots,0,x,\ldots,x,0,\ldots,0) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$$

is compatible. Since the reinforced approximation theorem holds for $v_1,\ldots,v_n$ there is some $a \in R$ with $v_i(a-x) = \alpha_i$ for $1 \le i \le l$ and $v_j(a) = 0$ for $l+1 \le j \le n$. For $1 \le i \le l$ we get $v_i(a) = v_i(a-x+x) = 0$ since $v_i(x) = 0$ and $\alpha_i > 0$. So $a \in A$. We show that $u(a-x) > 0$. We have $v \le w$. Therefore

$$w(a-x) = f_{v,w}(v(a-x)) = f_{v,w}(\alpha_1) = f_{v,w}(v(z)) = w(z) > 0$$

since $z \in \mathfrak{p}_w$. Since $u \le w$ we get $u(a-x) > 0$. So $u(a) = u(x+a-x) < 0$. Hence we have found some $a \in A$ with $u(a) < 0$. But $A \subset A_u$ by construction, contradiction. □

Corollary 7.17. *Let $v_1,\ldots,v_n$ be pairwise non-isomorphic non-trivial PM-valuations on R satisfying the reinforced approximation theorem. Let $w_1,\ldots,w_m$ be pairwise non-isomorphic trivial Manis valuations on R. Then the reinforced approximation theorem holds for $v_1,\ldots,v_n,w_1,\ldots,w_m$.*

Proof. By Theorem 16 $\bigcap_{1\le i\le n} A_{v_i}$ is Prüfer in R. Since $\bigcap_{1\le i\le n} A_{v_i} \cap \bigcap_{1\le j\le m} A_{w_j} = \bigcap_{1\le i\le n} A_{v_i}$ we get the claim by Corollary 11. □

Corollary 7.18. *Let $v_1, \ldots, v_n$ be pairwise independent PM-valuations on R. If the reinforced approximation theorem holds for $v_1, \ldots, v_n$ then also the general approximation theorem.*

Proof. By Theorem 16 $A_{v_1} \cap \ldots \cap A_{v_n}$ is Prüfer in R. By Theorem 6.11 the general approximation theorem holds for $v_1, \ldots, v_n$. ☐

Definition 4. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. We say that the *reinforced approximation theorem* holds for the family if for each $i_1, \ldots, i_n \in I$ and each compatible and $\{i_1, \ldots, i_n\}$-complete tuple $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times R^n$ there is some $x \in R$ with $v_{i_k}(x - a_k) = \alpha_k$ for $1 \leq k \leq n$ and $v_j(x) \geq 0$ for all $j \in I \setminus \{i_1, \ldots, i_n\}$.

Remark 7.19. Let $(v_i \mid i \in I)$ be a family of Manis valuations on R and let $i_1, \ldots, i_n \in I$. Let $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times R^n$ be compatible and $\{i_1, \ldots, i_n\}$-complete. Let $j_1, \ldots, j_m \in I \setminus \{i_1, \ldots, i_n\}$. Then

$$(\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, a_1, \ldots, a_n, 0, \ldots, 0) \in$$

$$\Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times \Gamma_{v_{j_1}} \times \ldots \times \Gamma_{v_{j_m}} \times R^{n+m}$$

is compatible and $\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$-complete.

Proof. Let $1 \leq k \leq n$ and $1 \leq l \leq m$. If $\alpha_k = 0$ then clearly $f_{v_{i_k}, v_{j_l}}(\alpha_k) = 0$. If $\alpha_k \neq 0$ then also $f_{v_{i_k}, v_{j_l}}(\alpha_l) = 0$ by Proposition 5.17 since $j_q \notin I(\alpha_i)$. By Proposition 5.17 we get also $(v_{i_k} \vee v_{j_l})(a_k) = 0$ in $\Gamma_{v_{i_k}, v_{j_l}}$ since $j_q \notin I(v_{i_k}(a_k))$. So the above tuple is compatible. It is clear by Definition 4 in Sect. 5 that the tuple is $\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$-complete. ☐

Remark 7.20. If I is finite then Definition 4 coincides with Definition 2.

Proof. Let $I = \{1, \ldots, n\}$.

a) We show that the reinforced approximation theorem in the sense of Definition 4 implies the one in the sense of Definition 2. To see this let $(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$ be compatible. By Remark 5.19(a) it is $\{1, \ldots, n\}$-complete and we are done.
b) We show that the reinforced approximation theorem in the sense of Definition 2 implies the approximation theorem in the sense of Definition 4. For this let $i_1 \ldots, i_k \in \{1, \ldots, n\}$. Without restriction we can assume that $i_1 = 1, \ldots, i_k = k$. Let $(\alpha_1, \ldots, \alpha_k, a_1, \ldots, a_k) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_k} \times R^k$ be compatible and $\{1, \ldots, k\}$-complete. Then

$$(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0, a_1, \ldots, a_k, 0, \ldots, 0) \in \Gamma_{v_1} \times \ldots \times \Gamma_{v_n} \times R^n$$

is compatible by Remark 19. Hence there is some $x \in R$ such that $v_i(x - a_i) = \alpha_i$ for $1 \leq i \leq k$ and $v_i(x) = 0$ for $k + 1 \leq i \leq n$ and we are done. ☐

Proposition 7.21. *Let $(v_i \mid i \in I)$ be a family of Manis valuations on R. If the reinforced approximation holds then the approximation theorem in the neighbourhood of zero.*

Proof. Let $i_1,\dots,i_n \in I$. Let $(\alpha_1,\dots,\alpha_n) \in \Gamma_{v_{i_1}} \times \dots \times \Gamma_{v_{i_n}}$ be compatible and $\{i_1,\dots,i_n\}$-complete. Then

$$(\alpha_1,\dots,\alpha_n,0,\dots,0) \in \Gamma_{v_{i_1}} \times \dots \times \Gamma_{v_{i_n}} \times R^n$$

is compatible and $\{i_1,\dots,i_n\}$-complete. Since the reinforced approximation theorem holds there is some $x \in R$ such that $v_{i_k}(x) = \alpha_k$ for $1 \le k \le n$ and $v_j(x) \ge 0$ for $j \in I \setminus \{i_1,\dots,i_n\}$. $\square$

We extend the above relationship between the reinforced approximation theorem and Prüfer rings to families having finite avoidance.

Theorem 7.22. *Let $(v_i \mid i \in I)$ be a family of pairwise non-isomorphic Manis valuations on R having finite avoidance. If $\bigcap_{i\in I} A_{v_i}$ is Prüfer in R then the reinforced approximation theorem holds for the family.*

Proof. We set $A := \bigcap_{i\in I} A_{v_i}$. Let $i_1,\dots,i_n \in I$ and let $(\alpha_1,\dots,\alpha_n,a_1,\dots,a_n) \in \Gamma_{v_{i_1}} \times \dots \times \Gamma_{v_{i_n}} \times R^n$ be compatible and $\{i_1,\dots,i_n\}$-complete. Since A is Prüfer also $\bigcap_{k=1}^n A_{v_{i_k}}$ is Prüfer in R by [Vol. I, Corollary I.5.3]. By Corollary 5.12 the approximation theorem in the neighbourhood of zero holds for $v_{i_1},\dots,v_{i_n}$. Hence there are $y, y' \in R$ such that $v_{i_k}(y) = \alpha_i$ and $v_{i_k}(y') = -\alpha_i$ for $1 \le k \le n$. By Corollary 11 the reinforced approximation theorem holds for $v_{i_1},\dots,v_{i_n}$. Hence there is some $z \in R$ such that $v_{i_k}(z - a_i) = \alpha_i$ for $1 \le k \le n$. Since the family has finite avoidance there is a finite subset J of I containing $i_1,\dots,i_n$ such that $v_i(y) \ge 0, v_i(y') \ge 0, v_i(z) \ge 0$ and $v_i(a_k) \ge 0$ for all $i \in I \setminus J$ and all $1 \le k \le n$. Let $B := \bigcap_{i\in I\setminus J} A_{v_i}$. Then $y, y', z, a_1,\dots,a_n \in B$. We write $J \setminus \{i_1,\dots,i_n\}$ as $\{i_{n+1},\dots,i_m\}$ for some $m \ge n$. For $1 \le k \le m$ we set $w_k := v_{i_k}|_B$.

Claim 1: The valuations $w_1,\dots,w_k$ are pairwise non-isomorphic and Manis.

Proof of Claim 1: By Corollary 4.10 the valuations are Manis. Assume that there are $r \ne s$ such that $w_r \cong w_s$. Then $\mathfrak{p}_{w_r} = \mathfrak{p}_{w_s}$. Since $\mathfrak{p}_{w_r} = \mathfrak{p}_{v_{i_r}} \cap B$ and $A \subset B$ we obtain $\mathrm{cent}_A(v_{i_r}) = \mathrm{cent}_A(w_r)$. Similarly $\mathrm{cent}_A(v_{i_s}) = \mathrm{cent}_A(w_s)$. Hence $\mathrm{cent}_A(v_{i_r}) = \mathrm{cent}_A(v_{i_s})$. By Corollary 1.17 and Proposition 1.2 we get that v_{i_r} and v_{i_s} are isomorphic, contradiction.

Clearly $\bigcap_{1\le k\le m} A_{w_k} = A$. By [Vol. I, Corollary I.5.3] A is Prüfer in B. Hence the reinforced approximation theorem holds for $w_1,\dots,w_m$ by Corollary 11 and Claim 1. Since $y, y' \in B$ we have $\alpha_i \in \Gamma_{w_k}$ for $1 \le k \le n$. We set $\alpha_k := 0 \in \Gamma_{w_k}$ and $a_k := 0 \in B$ for $n+1 \le k \le m$.

Claim 2: The tuple $(\alpha_1,\dots,\alpha_m,a_1,\dots,a_m) \in \Gamma_{w_1} \times \dots \times \Gamma_{w_m} \times B^m$ is compatible.

Proof of Claim 2: Let $1 \le k < l \le m$. We show that $(\alpha_k,\alpha_l,a_k,a_l)$ is compatible. We distinguish three cases.

Case 1: $1 \le k, l \le n$. Then $w_k(y) = \alpha_k$ and $w_l(y) = \alpha_l$, so (α_k, α_l) is compatible. We have $w_k(z - a_k) = \alpha_k$ and $w_l(z - a_l) = \alpha_l$, so $(\alpha_k, \alpha_l, a_k, a_l)$ is compatible by Remark 3(ii).

Case 2: $1 \le k \le n$ and $n + 1 \le l \le m$. Since $i_l \notin I(\alpha_k)$ we get $f_{v_{i_k}, v_{i_l}}(\alpha_k) = 0$ in $\Gamma_{v_{i_k}, v_{i_l}}$ by Proposition 5.17. Hence $f_{w_k, w_l}(\alpha_i) = 0$ in Γ_{w_k, w_l} by Proposition 4.13. So (α_k, α_l) is compatible. Since $i_l \notin I(v_{i_k}(a_i))$ we get $f_{v_{i_k}, v_{i_l}}(v_{i_k}(a_k)) = 0$ in $\Gamma_{v_{i_k} \vee v_{i_l}}$ by Proposition 5.17. By Proposition 4.13 we obtain $f_{w_k, w_l}(w_k(a_k)) = 0$ in Γ_{w_k, w_l}. Since $f_{w_k, w_l}(\alpha_k) = 0$ in Γ_{w_k, w_l} as just seen we get the claim.

Case 3: $n + 1 \le k < l \le m$. This is obvious.

Since the reinforced approximation theorem holds for $w_1, \ldots, w_m$ there is some $x \in B$ such that $w_k(x - a_k) = \alpha_k$ for $1 \le k \le m$. Then $v_{i_k}(x - a_k) = \alpha_k$ for $1 \le k \le n$. It remains to show that $v_i(x) \ge 0$ for all $i \in I \setminus \{i_1, \ldots, i_n\}$. If $i = i_k$ for some $n + 1 \le k \le m$ then $v_i(x) = w_k(x) = 0$. If $i \notin J$ then $v_i(x) \ge 0$ since $x \in B$. $\square$

Theorem 7.23. *Let $(v_i \mid i \in I)$ be a family of PM-valuations on R with finite avoidance. If the reinforced approximation theorem holds for the family then $\bigcap_{i \in I} A_{v_i}$ is Prüfer in R.*

Proof. We can clearly assume that v_i is non-trivial for all $i \in I$ (cf. Remark 6). Let $A := \bigcap_{i \in I} A_{v_i}$. The approximation theorem in the neighbourhood of zero holds for $(v_i \mid i \in I)$ by Proposition 21. By Theorems 5.22 and 4.21 every v_i is A-essential. Therefore we show condition (ii) of Theorem 4.17 to get the claim. Let $\mathfrak{m}$ be a maximal ideal of A with $\mathfrak{m} \neq \mathrm{cent}_A(v_i)$ for all $i \in I$. Let w be a non-trivial Manis valuation with $v_{i_1} \le w$ for some $i_1 \in I$. We have to show that $\mathrm{cent}_A(w) \not\subset \mathfrak{m}$. We set $v := v_{i_1}$. By assumption v is PM. Hence it is maximally dominant by Proposition 14, (1) $\Rightarrow$ (2). So $\mathfrak{p}_w$ is a maximal ideal of A_w.

Assume that $\mathrm{cent}_A(w) \subset \mathfrak{m}$. Let $B_0 := A[\mathfrak{p}_w] = A + \mathfrak{p}_w$ and $\mathfrak{q}_0 := \mathfrak{m} + \mathfrak{p}_w$. As in the proof of Theorem 16 (Claim 1 and Claim 2) we find a Manis pair $(B, \mathfrak{q})$ of R such that $B_0 \subset B$ and $\mathfrak{q} \cap B_0 = \mathfrak{q}_0$. Then $\mathfrak{q} \supset \mathfrak{p}_w$ and $\mathfrak{q} \cap A = \mathfrak{m}$. Let u be the Manis valuation on R corresponding to $(B, \mathfrak{q})$. Since w and u are non-trivial we get $u \le w$ by Proposition 1.8. Again as in the proof of Theorem 16 (Claim 3) we have $v_i \not\le u$ for all $i \in I$. By Remark 4.14 the set $J := \{i \in I \mid v_i \le w\}$ is finite. Let $J = \{i_1, \ldots, i_n\}$. By above $\mathfrak{p}_w$ is a maximal ideal of A_w. Hence $K := A_w/\mathfrak{p}_w$ is a field. For $1 \le k \le n$ we define

$$\overline{v_{i_k}} : K \to \Gamma_{v_i} \cup \{\infty\}, \overline{v_{i_k}}(x + \mathfrak{p}_w) = \begin{cases} v_{i_k}(x) & \text{if } x \in A_w \setminus \mathfrak{p}_w, \\ \infty & \text{if } x \in \mathfrak{p}_w. \end{cases}$$

and we set

$$\overline{u} : K \to \Gamma_u \cup \{\infty\}, \overline{u}(x + \mathfrak{p}_w) = \begin{cases} u(x) & \text{if } x \in A_w \setminus \mathfrak{p}_w, \\ \infty & \text{if } x \in \mathfrak{p}_w. \end{cases}$$

Since $\mathfrak{p}_w$ is v_{i_k}-convex for every $1 \leq k \leq n$ and u-convex these are well defined valuations on K. We have $\mathfrak{p}_{\overline{v_{i_k}}} = \mathrm{cent}_{A_w}(v_{i_k})/\mathfrak{p}_w = \mathfrak{p}_{v_{i_k}}/\mathfrak{p}_w$ for $1 \leq k \leq n$ and $\mathfrak{p}_{\overline{u}} = \mathrm{cent}_{A_w}(u)/\mathfrak{p}_w = \mathfrak{p}_u/\mathfrak{p}_w$. As above we get that $\overline{v_{i_k}} \not\leq \overline{u}$ for $1 \leq k \leq n$. As in the proof of Theorem 16 (Claim 4) we find some $y \in K$ with $\overline{u}(y) < 0$ and $\overline{v_{i_k}}(y) = 0$ for all $1 \leq k \leq n$. Hence we find some $x \in A_w$ with $u(x) < 0$ and $v_{i_k}(x) = 0$ for $1 \leq k \leq n$. We fix this x.

Claim A: There is some $z \in \mathfrak{p}_w \setminus \operatorname{supp} w$ such that $J = I(v(z))$.

Proof of Claim A: Note that given $z \in \mathfrak{p}_w \setminus \operatorname{supp} w$ we have $J \subset I(v(z))$. To see this let $z \in \mathfrak{p}_w \setminus \operatorname{supp} w$. Then $0 < w(z) < \infty$. We have $v \vee v_{i_k} \leq w$ for all $1 \leq k \leq n$. Hence $0 < (v \vee v_{i_k})(z) < \infty$ for all $1 \leq k \leq n$. Therefore $f_{v,v_{i_k}}(v(z)) \neq 0$ for all $1 \leq k \leq n$. By Remark 5.17 we obtain $J \subset I(v(z))$.

By the above we have to find some $z \in \mathfrak{p}_w \setminus \operatorname{supp} w$ such that $I(v(z)) \subset J$. Let H be the convex subgroup of Γ_v such that $w = v/H$ (cf. Remarks 1.13(b)). Assume that there is no convex subgroup $\tilde{H}$ of Γ_v with $H \subsetneqq \tilde{H} \subsetneqq \Gamma_v$. Let then $z \in \mathfrak{p}_w \setminus \operatorname{supp} w$ be arbitrary. Since $0 < w(z) < \infty$ we have $v(z) \notin H$. By the assumption we get $H_{v(z)} = H$ (cf. Definition 4 in Sect. 5). Therefore $v/H_{v(z)} = v/H = w$. By Definition 4 in Sect. 5 we obtain

$$I(v(z)) = \{i \in I \mid v_i \leq v/H_{v(z)}\} = \{i \in I \mid v_i \leq w\} = J$$

and are done. So we assume that there is a convex subgroup $\tilde{H}$ of Γ_v with $H \subsetneqq \tilde{H} \subsetneqq \Gamma_v$. Let $\tilde{w} := v/\tilde{H}$. Then $\tilde{w}$ is non-trivial and $w \leq \tilde{w}$. By Remark 4.14 the set

$$\tilde{J} := \{i \in I \mid v_i \leq \tilde{w}\}$$

is finite and contains clearly J. Let $\tilde{J} \setminus J := \{i_{n+1}, \ldots, i_m\}$. Arguing similarly to above we obtain some $z \in A_{\tilde{w}}$ such that $w(z) > 0$ and $v_{i_{n+1}}(z) = \ldots = v_{i_m}(z) = 0$. Since $v_{i_k} \leq \tilde{w}$ for all $n+1 \leq k \leq m$ we obtain $z \notin \mathfrak{p}_{\tilde{w}}$. So $z \in \mathfrak{p}_w \setminus \mathfrak{p}_{\tilde{w}}$. Note that necessarily $\tilde{w}(z) = 0$ since $\tilde{w}(z) \leq 0$ and $w \leq \tilde{w}$. Since $\operatorname{supp} w = \operatorname{supp} \tilde{w}$ we have $z \in \mathfrak{p}_w \setminus \operatorname{supp} w$. Since $w(z) > 0$ and $\tilde{w}(z) = 0$ we have $H \subset H_{v(z)} \subsetneqq \tilde{H}$. So $w \leq v/H_{v(z)} < \tilde{w}$ and $v/H_{v(z)}(z) > 0$. Let $i \in I(v(z))$. Then $v_i \leq v/H_{v(z)}$. We get $v_i(z) > 0$. Moreover, $v_i \leq \tilde{w}$. So $i \in \tilde{J}$. But $v_{i_{n+1}}(z) = \ldots = v_{i_m}(z) = 0$. So $i \in J$ and Claim A is proven

We choose z as in Claim A. Let $\alpha_k := v_{i_k}(z)$ for $1 \leq k \leq n$. Since $z \in \mathfrak{p}_w \setminus \operatorname{supp} w$ we have $0 < w(z) < \infty$. Since $v_{i_k} \leq w$ for all $1 \leq k \leq n$ we get $0 < \alpha_k < \infty$ for all $1 \leq k \leq n$.

Claim B: The tuple $(\alpha_1, \ldots, \alpha_n, x, \ldots, x) \in \Gamma_{v_{i_1}} \times \ldots \times \Gamma_{v_{i_n}} \times R^n$ is compatible and J-complete.

Proof of Claim B. The tuple is clearly compatible. By Claim A $I(\alpha_1) = J$. Since $v_{i_k}(x) = 0$ for all $1 \leq k \leq n$ it remains to show that $I(\alpha_k) \subset J$ for all $2 \leq k \leq n$. Fix $2 \leq k \leq n$. Let $j \in I(\alpha_k)$. Then $v_j \leq v_{i_k}/H_{\alpha_{i_k}}$. By Claim A $i_k \in I(\alpha_1)$. Therefore $v_{i_k} \leq v_{i_1}/H_{\alpha_1}$. We conclude that $v_j \leq v_{i_1}/H_{\alpha_1}$ and so $j \in I(\alpha_1) = J$.

Applying the reinforced approximation theorem to the tuple of Claim B we can finish the proof as the proof of Theorem 16. □

Corollary 7.24. *Let* $(v_i \mid i \in I)$ *be a family of pairwise independent PM-valuations on R with finite avoidance. If the reinforced approximation theorem holds for* $(v_i \mid i \in I)$ *then also the general approximation theorem.*

Proof. By Theorem 23 $\bigcap_{i \in I} A_{v_i}$ is Prüfer in R. By Theorem 6.16 the general approximation theorem holds for $(v_i \mid i \in I)$. □

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