Preface

This is the second volume of a book devoted to Prüfer extensions of rings (here always commutative and with 1) and to valuations on rings related to Prüfer extensions. Following the three Chaps. I–III of Volume I [KZ], the present volume contains three more Chaps. 1–3, and nevertheless by no means exhausts what is known nowadays about Prüfer extensions even by us.

While Chaps. I–III of Volume I are strongly intertwined, Chaps. 1–3 of Volume II are nearly independent of each other.

The reader should follow Sects. 1–3 of Chap. 1, but then may read the rest of Chaps. 1–3 in any order and also with long time intervals in between.

Also by no means the whole content of Volume I plays a role in Volume II. Needed are from Chap. I: §1–§5 and §6 up to Corollary 6.11; from Chap. II: §1, §2 up to Theorem 2.6, §3 up to Theorem 3.3, §5 up to Proposition 5.2, major parts of §6–§8, and §10 up to Theorem 10.2. In Chap. III only §1–§3 are relevant to Volume II, the one exception being the "PM-hull" $\mathrm{PM}(A, R, \mathfrak{p})$ of a pair $(A, \mathfrak{p})$ in a ring extension $A \subset R$, with $\mathfrak{p}$ a prime ideal, better a maximal ideal, of A (cf. [Vol. I, Theorem III.5.3][1]), which shows up in Chap. 1.

Nowadays there exists an enormously extended literature on Prüfer domains and a still large body of papers about the more general "Prüfer rings with zero divisors." In our framework Prüfer rings with zero divisors are the same objects as the Prüfer extensions $A \subset \mathrm{Quot}(A)$, or sometimes $A \subset Q(A)$, with A a commutative ring, $\mathrm{Quot}(A)$ its total ring of quotients and $Q(A)$ its complete ring of quotients (cf. [Vol. I, p. 39]). If we assume that A is an integral domain, we fall back on Prüfer domains. Then $\mathrm{Quot}(A) = Q(A)$.

On the other hand the literature on Prüfer extensions is meager. But Prüfer extensions seem to be mandatory in particular for large parts of real and p-adic algebra. All this is discussed in the Introduction to the book in Volume I. We urge the uninitiated reader to look first into the Introduction, also for understanding our goals and motivation in writing this book.

[1] This means Theorem 5.3 in Chap. III of Volume I.

For the expert we mention one item, which explicitly has nearly no presence in the literature on Prüfer extensions (but see [Huc, pp. 30–31] for Prüfer rings with zero divisors). This is the theory of *PM-valuations* and *PM-extensions* ("PM" is an acronym for "Prüfer–Manis.") A valuation v on a ring R is called PM, if v is Manis and the extension $A_v \subset R$ is Prüfer, and then $A_v \subset R$ is called a PM-extension. PM-extensions can be characterized in various ways without mentioning valuations. Perhaps the most striking one runs as follows: if A is a subring of a ring R, then $A \subset R$ is PM iff A is integrally closed in R and the set of subrings B of R with $A \subset B \subset R$ is totally ordered by inclusion ([Vol. I, Theorem III.3.1]) If $A \subset R$ is PM, then the Manis valuation v on R with $A_v = A$ is uniquely determined up to equivalence, cf. [Vol. I, Theorem III.3.12]. Manis valuations in general still show some pathologies, which are absent for PM-valuations. All these are explained in [Vol. I, Chap. III §1–§3].

All three chapters of Volume II deal with families of valuations instead of studying properties of just one valuation. PM-valuations play here center stage.

In Chap. 1 we aim at analyzing a Prüfer extension $A \subset R$ in terms of the set $S(R/A)$ of nontrivial PM-valuations v on R over A (i.e., with $A \subset A_v$), called the *restricted PM-spectrum* of R over A, in order to understand the lattice of overrings of A in R. We engage $S(R/A)$ as poset[2] ($v \leq w$ iff $A_v \subset A_w$), although it would be more comprehensive to view $S(R/A)$ as a topological space, namely as subspace of the valuation spectrum $\mathrm{Spv}(R)$, equipped with one of its well-established spectral topologies (cf. [HK]), whose specialization relation restricts to the partial ordering above. We exhibit several types of Prüfer extensions, where the poset viewpoint has seizable success.

Chapter 2 is devoted to approximation in a ring with respect to finite—and then also suitable infinite—systems of Manis valuations. Primordial approximation theorems already show up in Manis' seminal paper [M]. In Chap. 2 we embed the deep results of Gräter on approximation by a finite system of Manis valuations ([Gr], [Gr_1], [Gr_2], cf. also [Al_1]) in our framework. Gräter's theorems may be viewed as a grand generalization of the weak approximation theorems for finitely many valuations on a field, as presented, e.g., in the books [E] and [Rib]. They relate to arbitrary Prüfer extensions,[3] a novum in the literature on approximations in the beginning eighties of last century. It fits well with our thinking that all Manis valuations relevant in Gräter's papers turn out to be PM.

A second source for our Chap. 2 is a paper by Griffin from the late sixties [G_3], where—in succession of work by Ribenboim—a "reinforced approximation theorem" has been proved and analyzed for certain infinite systems of valuations on the quotient fields of suitable Prüfer domains, there called "generalized Krull rings." This theorem generalizes the classical strong approximation theorem over a

[2] = partially ordered set.

[3] Gräter uses the equivalent term "R-Prüferring," which we also use alternatively to "Prüfer extension," cf. [Vol. I, Definition 1 in I §5].

Dedekind domain and then Krull domain (cf. [Bo, Chap. 7], [E]). In [G_2] Griffin has then established approximation theorems for a special class of Prüfer extensions.

In [Gr_2] Gräter shows the "reinforced approximation theorem" for arbitrary Prüfer extensions given by finitely many valuations and expands this also to Prüfer extensions "of finite type." Moreover, he elaborates deep connections between his various approximation theorems in [Gr]–[Gr_2]. His results have been incorporated only once in a book, namely the book [Al-M] by Alajbegović and Močkoř. There approximation theorems are investigated also in other systems, not only rings. We give a new presentation of Gräter's results, stressing thereby the case of possibly infinite families "with finite avoidance."[4]

We hope that Chap. 2 provides easier access to Gräter's approximation theorems and their enlargements than [Al-M], due to the fact that we can refer to a much more complete general theory of Prüfer extensions than present in [Al-M]. Nevertheless the way from classical weak and strong approximation to the "reinforced approximation theorem" at the end of Chap. 2 remains long and needs patience by the reader. Among all chapters in the book Chap. 2 is the most demanding.

Chapter 3 aims at applications of the Prüfer theory to arbitrary (commutative) ring extensions. For a given ring extension $A \subset R$ we produce a Prüfer extension $B \subset T$ in various ways, together with a homomorphism $j : R \to T$ mapping A into B, called a *Kronecker extension* of $A \subset R$. Kronecker extensions generalize the Kronecker function rings (e.g., [Gi, §32], [Ha-K]) in the classical literature.

The restricted PM-spectrum $S(T/B)$ of a Kronecker extension $B \subset T$ gives us a family $(w \circ j \mid w \in S(T/B))$ of valuations on R over A, which can serve for a valuation theoretic description of various A-modules in R, since $S(T/B)$ does this so well for B-modules in T.

For constructing Kronecker extensions we use—suitably defined—*star operations* $I \mapsto I^*$ on the semiring $J(A, R)$ of all A-submodules of R (cf. Chap. 3, Sect. 3) and then also "partial star operations" defined on certain subsets of $J(A, R)$, cf. Chap. 3, Sect. 7.

In the classical setting, where A is a domain and R its quotient field, such star operations have been defined on the set of fractional ideals of A, leading to a *star multiplicative ideal theory* with a very extended literature. In Chap. 3 we display, beside Kronecker extensions, the basics of such a theory in arbitrary ring extensions. Everything is more complex than in the classical theory, due to the fact that for a ring extension $A \subset R$ a principal ideal of A usually is not R-invertible. Nevertheless the pattern is natural and manageable. But we go in the book only so far, that a reader with experience in the classical star multiplicative ideal theory can get convinced, that establishing analogues and/or generalizations of many of the deeper results in the classical theory is possible and worth the labor.

[4] A new term by us; Griffin and most other authors speak instead of "Prüfer domains of finite character." Gräter uses the term "R-Prüfer rings of finite type." Our term aims at catching more general situations.

The interested reader should first have a look at the Summary of Chap. 3 below (pp. 123–125) to get a more detailed impression, what can be found in this long chapter and what cannot.

Acknowledgments

Among the many mathematicians of present time, whose work is related to this volume, we feel special indebtness to Professors Franz Halter-Koch and Joachim Gräter. Halter-Koch had the insight to define Kronecker function rings axiomatically without using star operations. This has been instrumental for Chap. 3. Gräter obtained already more than 30 years ago, starting with his dissertation TU Braunschweig 1980, a full-fledged deep and general approximation theory for Manis valuations in Prüfer extensions, the essential content of present Chap. 2 (see above). We are sorry to say that there are other important results by Gräter on Prüfer extensions [Gr3], which we could not incorporate in the volume, to keep it in reasonable size.

We are grateful to our friend Digen Zhang, coauthor of Volume I, who has accompanied early versions of Chaps. 1–3 until about 2005, but then has been called to other duties.

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