Application of φ-Sub-Gaussian Random Processes in Queueing Theory

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Abstract The chapter is devoted to investigation of the class $V(\varphi, \psi)$ of φ-sub-Gaussian random processes with application to queueing theory. This class of stochastic processes is more general than the Gaussian one; therefore, all results obtained in general case are valid for Gaussian processes by selection of certain Orlicz N-functions φ and ψ. We consider different queues filled by an aggregate of such independent sources and obtain estimates for the tail distribution of some extremal functionals of incoming random processes and their increments which describe behavior of the queue. We obtain the upper bound for the buffer overflow probability for the corresponding storage process and apply obtained result to the aggregate of sub-Gaussian generalized fractional Brownian motion processes.

1 Introduction

Consider a single server queue that is filled by the aggregate of $\mathbf{N}$ independent (uncorrelated) random sources $X_i = \{X_i(t), t \in T\}$, where T is some parametric set, e.g., an interval $[a, b]$. Our main interest is focused on studying the distribution of the following functionals that depend on the incoming aggregate and characterize the behavior of the queue:

$$\sup_{t \in T} \left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t) \right),$$

$$\sup_{s \le t,\ s,t \in T} \left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t) - \left(\sum_{i=1}^{\mathbf{N}} X_i(s) - f(s) \right) \right),$$

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and

$$\sup_{s,t\in T}\sup_{s\le t}\left(\sum_{i=1}^{\mathbf{N}} X_i(t)-f(t)-\left(\sum_{i=1}^{\mathbf{N}} X_i(s)-f(s)\right)\right),$$

where $f(t)$ is a continuous function which describes intensity of the queue serving.

Here we summarize recent studies [6–8, 13–18] for a general class of incoming processes X_i. We assume that incoming streams belong to the class $V(\varphi,\psi)$ and study the buffer overflow probability for the storage process $Q(t)$. Our results are illustrated by sub-Gaussian generalized fractional Brownian motion (GFBM). Recall that the normalized FBM process with Hurst parameter $H \in (0.5, 1)$ is the Gaussian centered process with stationary increments, continuous paths, and covariance function of the form

$$R_H(t,s)=\left(t^{2H}+s^{2H}-|s-t|^{2H}\right)/2. \tag{1}$$

Its long-range dependence and self-similarity properties make the FBM process a natural choice for modeling traffic through telecommunication networks (see more results on FBM storage models in [1, 10–12]).The paper is organized as follows. Section 2 is devoted to the general theory of φ-sub-Gaussian random processes and is based on the works [2–6, 8, 9, 13, 18]. In Sect. 3 we consider the storage process from the class $V(\varphi,\psi)$ with application to GFBM processes.

2 Random Variables from Spaces $\mathrm{Sub}_\varphi(\Omega)$, $\mathrm{SSub}_\varphi(\Omega)$, and Class $V(\varphi,\psi)$

This section contains some basic notions, definitions and properties of random variables, and processes from the spaces $\mathrm{Sub}_\varphi(\Omega)$, $\mathrm{SSub}_\varphi(\Omega)$, and the class $V(\varphi,\psi)$.

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a standard probability space and let (T,ρ) be a pseudometric (metric) compact space equipped by pseudometric (metric) ρ.

Definition 1 ([2]). Metric entropy with regard to pseudometric (metric) ρ or just metric entropy is a function

$$H_{(T,\rho)}(u)=H_T(u)=H(u)=\begin{cases}\log N_{(T,\rho)}(u), & \text{if } N_{(T,\rho)}(u)<+\infty\\ +\infty, & \text{if } N_{(T,\rho)}(u)=+\infty\end{cases},$$

where $N_{(T,\rho)}(u)=N_T(u)=N(u)$ denotes the least number of closed ρ-balls with radius u covering space (T,ρ).

Example 1. If T is an interval $[a, b]$ and ρ is the Euclidean distance, then

$$\ln\left(\max\left\{\frac{b-a}{2u}, 1\right\}\right) \le H(u) \le \ln\left(\frac{b-a}{2u}+1\right).$$

Definition 2 ([2]). A continuous even convex function φ is said to be an Orlicz N-function if it is strictly increasing for $x > 0$, $\varphi(0) = 0$ and

$$\frac{\varphi(x)}{x} \to 0 \text{ as } x \to 0 \quad \text{and} \quad \frac{\varphi(x)}{x} \to \infty \text{ as } x \to \infty.$$

Example 2. The following functions are N-functions.

- $\varphi(x) = \alpha|x|^{\beta}, \ \alpha > 0, \ \beta > 1;$
- $\varphi(x) = \exp\{|x|\} - |x| - 1;$
- $\varphi(x) = \exp\{\alpha|x|^{\beta}\} - 1, \ \alpha > 0, \ \beta > 1;$
- $\varphi(x) = \begin{cases} (e\alpha/2)^{2/\alpha} x^2, & |x| \le (2/\alpha)^{1/\alpha}; \\ \exp\{|x|^{\alpha}\}, & |x| > (2/\alpha)^{1/\alpha}, \ 0 < \alpha < 1. \end{cases}$

Condition Q ([3]). We say that an N-function φ satisfies Condition Q if

$$\liminf_{x \to 0} \frac{\varphi(x)}{x^2} = \alpha > 0. \tag{2}$$

It is permitted α to be infinite.

Example 3. Condition Q is fulfilled for N-function $\varphi(x) = c|x|^{\beta}, \ c > 0$, when $1 < \beta \le 2$ and is not fulfilled when $\beta > 2$.

Definition 3 ([2]). Let φ be an Orlicz N-function satisfying Condition Q. The random variable ξ belongs to the space $\mathrm{SUB}_{\varphi}(\Omega)$ (a space of φ-sub-Gaussian random variables), if it is centered, i.e., $\mathbf{E}\xi = 0$, the moment generating function $\mathbf{E}\exp\{\lambda\xi\}$ exists for all $\lambda \in R$ and there exists a positive constant a such that the following inequality

$$\mathbf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda)) \tag{3}$$

holds for all $\lambda \in R$.

Theorem 1 ([2]). *The space $\mathrm{SUB}_{\varphi}(\Omega)$ is a Banach space with respect to the norm $\tau_{\varphi}(\xi) = \inf\{a \ge 0 \colon \mathbf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda)), \lambda \in R\}$ and the inequality*

$$\mathbf{E}\exp(\lambda\xi) \le \exp\big(\varphi\big(\lambda\tau_{\varphi}(\xi)\big)\big), \tag{4}$$

holds for all $\lambda \in R$. Moreover, for all $r > 0$ there exists constant $c_r > 0$ such that

$$(\mathbf{E}\xi^r)^{1/r} \le c_r \tau_{\varphi}(\xi). \tag{5}$$

When $\varphi(x) = x^2/2$, the space $\mathrm{SUB}_\varphi(\Omega)$ is called the space of *sub-Gaussian* random variables and is denoted by $\mathrm{SUB}(\Omega)$. The simplest examples of sub-Gaussian random variables are the following:

- Centered Gaussian random variables $\xi = N(0, \sigma^2)$ belong to the space $\mathrm{SUB}(\Omega)$ and $\tau(\xi) = (\mathbf{E}\xi^2)^{1/2}$.
- Let ξ be a centered bounded random variable, i.e., $\mathbf{E}\xi = 0$, and there exists number $c > 0$ that $|\xi| \le c$ almost surely. Then $\xi \in \mathrm{SUB}(\Omega)$ and $\tau(\xi) \le c$.

Theorem 2 ([8]). *Let $\xi \in \mathrm{SUB}_\varphi(\Omega)$. Then for all $\varepsilon > 0$ the following inequality holds:*

$$\mathbf{P}\{|\xi| > \varepsilon\} \le 2\exp\left\{-\varphi\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\}.$$

Definition 4 ([2]). Random process $X = \{X(t), t \in T\}$ is called a *φ-sub-Gaussian process* if for all $t \in T$ $X(t) \in \mathrm{SUB}_\varphi(\Omega)$.

Condition Σ. Suppose there exists such a continuous monotonically increasing function $\sigma = \{\sigma(h), h > 0\}$ that $\sigma(h) \to 0$, as $h \to 0$, and the following inequality for increments of the process is true:

$$\sup_{\rho(t,s) \le h} \tau_\varphi(Y(t) - Y(s)) \le \sigma(h). \tag{6}$$

If a process $X(t)$ is continuous in norm $\tau_\varphi(\cdot)$, then the function

$$\sigma(h) = \sup_{\rho(t,s) \le h} \tau_\varphi(Y(t) - Y(s))$$

satisfies Condition Σ.

Theorem 3 ([2]). *Let φ be an Orlicz N-function satisfying Condition Q and the function $\varphi(\sqrt{\cdot})$ be convex. Suppose that $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables from the space $\mathrm{SUB}_\varphi(\Omega)$. Then*

$$\tau_\varphi^2\left(\sum_{i=1}^n \xi_i\right) \le \sum_{i=1}^n \tau_\varphi^2(\xi_i). \tag{7}$$

Definition 5 ([4]). A family of random variables Δ from the space $\mathrm{SUB}_\varphi(\Omega)$ is called *strictly* $\mathrm{SUB}_\varphi(\Omega)$, if there exists a constant $C_\Delta > 0$ such that for arbitrary finite set $I : \xi_i \in \Delta, i \in I$ and for any $\lambda_i \in R$, the following inequality takes place:

$$\tau_\varphi\left(\sum_{i \in I} \lambda_i \xi_i\right) \le C_\Delta\left(\mathbf{E}\left(\sum_{i \in I} \lambda_i \xi_i\right)^2\right)^{1/2}. \tag{8}$$

If Δ is a family of strictly $\mathrm{SUB}_\varphi(\Omega)$ random variables, then linear closure $\overline{\Delta}$ of the family Δ in the space $L_2(\Omega)$ is also strictly $\mathrm{SUB}_\varphi(\Omega)$ family of random variables. Linearly closed families of strictly $\mathrm{SUB}_\varphi(\Omega)$ random variables form a space of *strictly φ-sub-Gaussian* random variables. This space is denoted by $\mathrm{SSUB}_\varphi(\Omega)$.

When $\varphi(x) = x^2/2$, the space $\mathrm{SSUB}_\varphi(\Omega)$ is called the space of *strictly sub-Gaussian* random variables and is denoted as $\mathrm{SSUB}(\Omega)$. The space of jointly Gaussian random variables belongs to the space $\mathrm{SSUB}(\Omega)$ and $\tau^2(\xi) = \mathbf{E}\xi^2$, i.e., $C_\Delta = 1$.

Definition 6 ([2]). A random process $X = \{X(t), t \in T\}$ is a *strictly φ-sub-Gaussian process* if the corresponding family of random variables belongs to the space $\mathrm{SSUB}_\varphi(\Omega)$.

Example 4 ([4]). Let φ be such an Orlicz N-function that the function $\varphi(\sqrt{\cdot})$ is convex and

$$X(t) = \sum_{k=1}^{\infty} \xi_k \phi_k(t),$$

where series $\sum_{k=1}^{\infty} \xi_k \phi_k(t)$ converges in mean square sense for all $t \in T$ and family $\{\xi_k, k \geq 1\}$ belongs to the space $\mathrm{SSUB}_\varphi(\Omega)$, for instance, $\{\xi_k, k \geq 1\}$ are independent random variables from $\mathrm{SSUB}_\varphi(\Omega)$. Then $X(t)$ is a strictly φ-sub-Gaussian random process.

Example 5 ([7]). We call the process $Z^H = (Z^H(t), t \in T)$ *strictly φ-sub-Gaussian GFBM* with Hurst index $H \in (0, 1)$, if Z^H is a strictly φ-sub-Gaussian process with stationary increments and covariance function as defined by (1).

In order to give an example of such a process, let's consider a sequence of independent strictly φ-sub-Gaussian random variables $\{\eta_n, n = 1, 2, \ldots\}$ for which $\mathbf{E}\eta_n = 0$, $\mathbf{E}\eta_n^2 = 1$, and φ is such an N-function that function $\varphi(\sqrt{\cdot})$ is convex and $\tau_\varphi(\eta_n) \leq \tau < +\infty$. Then the process $Z^H(t) = \sum_{n=1}^{\infty} \lambda_n \eta_n \psi_n(t)$ is a centered strictly φ-sub-Gaussian random process with covariance function R_H from (1), if λ_n are eigenvalues and ψ_n are corresponding eigenfunctions of the following integral equation:

$$\psi(s) = \lambda^{-2} \int_0^T R_H(t, s) \psi(t)\, \mathrm{d}t.$$

Definition 7 ([6]). N-function φ is subordinated by an Orlicz N-function ψ ($\varphi \prec \psi$) if there are exist such numbers $x_0 > 0$ and $k > 0$ that $\varphi(x) < \psi(kx)$ for $x > x_0$.

Theorem 4 ([13]). *Let φ_1 and φ_2 be such N-functions that $\varphi_1 \prec \varphi_2$. If $\xi \in \mathrm{SUB}_{\varphi_1}(\Omega)$, then $\xi \in \mathrm{SUB}_{\varphi_2}(\Omega)$, and there exists such a constant c_{φ_1,φ_2} that $\tau_{\varphi_2}(\xi) \le c_{\varphi_1,\varphi_2}\tau_{\varphi_1}(\xi)$.*

Definition 8 ([6]). Let $\varphi \prec \psi$ are two Orlicz N-functions. Random process $X = \{X(t), t \in T\}$ belongs to class $V(\varphi,\psi)$ if for all $t \in T$ the random variable $X(t)$ is from $\mathrm{SUB}_{\psi}(\Omega)$ and, for all $s,t \in T$ increments $(X(t) - X(s))$ belong to the family $\mathrm{SUB}_{\varphi}(\Omega)$.

Example 6. Sub-Gaussian random processes belong to the class $V(\varphi,\varphi)$ with $\varphi(x) = x^2/2$.

Example 7. Let

$$X(t) = \xi_0 + \sum_{k=1}^{\infty} \xi_k f_k(t),$$

where φ is such an Orlicz N-function that $\varphi(\sqrt{\cdot})$ is a convex function. Let ξ_0 be a ψ-sub-Gaussian random variable and $\{\xi_k, k = 1,2,\dots\}$ be a sequence of φ-sub-Gaussian random variables such that $\sum_{k=1}^{\infty} \tau_\varphi(\xi_k)|f_k(t)| < \infty$. Then the process $X(t)$ belongs to the class $V(\varphi,\psi)$.

Condition *F*. A continuous function $f = \{f(t), t \in T\}$ satisfies Condition *F* if

$$|f(u) - f(v)| \le \delta(\rho(u,v)),$$

where $\delta = \{\delta(s), s > 0\}$ is some monotonically increasing nonnegative function.

Condition *R*. A continuous function $r = \{r(u), u \ge 1\}$ satisfies Condition *R* if $r(u) > 0$ when $u > 1$ and function $s(t) = r(\exp\{t\}), t \ge 0$, is convex.

Let $B \subset T$ be a compact set. In what follows we use the following notation:

- $\gamma(u) = \tau_\psi(X(u)) < \infty$;
- $\beta > 0$ is such a number that $\beta \le \sigma\left(\inf_{s\in B}\sup_{t\in B}\rho(t,s)\right)$;
- $B_t = \{u \in B : u \le t\}$;
- $\mathsf{L}(u) = \big(\mathsf{N}(u)^2 + \mathsf{N}(u)\big)/2$, where function $\mathsf{N}(u)$ is denoted in Definition 1.

Theorem 5 ([18]). *Let $X = \{X(t), t \in B\}$ be a separable random process from the class $V(\varphi,\psi)$ which satisfies Condition Σ. Let functions $f = \{f(t), t \in B\}$ and $r = \{r(u) : u \ge 1\}$ satisfy Conditions F and R, respectively. If*

$$\int_0^\beta r(\mathsf{N}(\sigma^{(-1)}(u)))\,\mathrm{d}u < \infty,$$

then for all $p \in (0,1)$ *and* $x > 0$ *the following inequality holds*

$$\mathbf{P}\left\{\sup_{t\in T}(X(t) - f(t)) > x\right\} \le \inf_{\lambda>0} Z_r(\lambda, p, \beta),$$

where

$$Z_r(\lambda, p, \beta)$$
$$= \exp\left\{\theta_\psi(\lambda, p) + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left(\sum_{k=2}^{\infty}\delta(\sigma^{(-1)}(\beta p^{k-1})) - x\right)\right\}$$
$$\times r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r(\mathsf{N}(\sigma^{(-1)}(u)))\,\mathrm{d}u\right), \tag{9}$$

$$\theta_\psi(\lambda, p) = \sup_{u\in T}\left((1-p)\psi\left(\frac{\lambda\gamma(u)}{1-p}\right) - \lambda f(u)\right). \tag{10}$$

Example 8 ([18]). Let $Z^H = \{Z^H(t), t \in [a,b]\}$, $0 \le a < b < \infty$ be a GFBM process from the class $V(\varphi, \psi)$ with Hurst index $H \in (0,1)$ and let $c > 0$ be a constant service rate. Then for all $p \in (0,1)$, $\beta \in \left(0, (b-a/2)^H\right]$, and $\lambda > 0$, the following inequality holds:

$$\mathsf{P}\left\{\sup_{a\le t\le b}\left(Z^H(t) - ct\right) > x\right\} \le (b-a)\left(\frac{e}{\beta p}\right)^{1/H}$$
$$\times \exp\left\{\frac{\lambda c(\beta p)^{1/H}}{C_\Delta(1-p^{1/H})} + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + (1-p)\theta_\psi(\lambda, p) - \frac{\lambda x}{C_\Delta}\right\}, \tag{11}$$

where $\theta_\psi(\lambda, p) = \sup_{a\le u\le b}\left(\psi\left(\frac{\lambda u^H}{1-p}\right) - \frac{\lambda c u}{C_\Delta(1-p)}\right)$.

More details on the GFBM process can be found in papers [7, 13–15].

Condition ΣN. We say that independent separable random processes $X_i = \{X_i(t), t \in B\}$ from classes $V(\varphi_i, \psi_i)$ defined on a compact set $B \subset T$ satisfy Condition ΣN if there exist such continuous monotone increasing functions $\{\sigma_i(h), h \ge 0\}$ that $\sigma_i(h) \to 0$ when $h \to 0$ and

$$\sup_{\rho(t,s)\le h} \tau_\varphi(X_i(t) - X_i(s)) \le \sigma_i(h), \tag{12}$$

$$\gamma_i(u) = \tau_\psi(X_i(u)) < \infty, \tag{13}$$

$$\sigma(h) = \sup_{1\le i\le \mathbf{N}} \sigma_i(h) < \infty, \quad i = \overline{1, \mathbf{N}}. \tag{14}$$

Theorem 6. *Let random process* $X(t) = \sum_{i=1}^{\mathbf{N}} X_i(t)$ *satisfy the assumptions of Theorem 5, where functions* $\gamma_i(u)$ *and* $\sigma(h)$ *are given in (13) and (14). Then for all* $p \in (0,1)$ *and* $x > 0$*, following inequalities hold:*

$$\mathbf{P}\left\{\sup_{t\in B}\left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t)\right) > x\right\} \le Z_r(p,\beta,x),$$

$$\mathbf{P}\left\{\inf_{t\in B}\left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t)\right) < -x\right\} \le Z_r(p,\beta,x),$$

$$\mathbf{P}\left\{\sup_{t\in B}\left|\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t)\right| > x\right\} \le 2Z_r(p,\beta,x),$$

where

$$Z_r(p,\beta,x) = r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r\left(N_B(\sigma^{(-1)}(u))\right)\mathrm{d}u\right)$$
$$\times \inf_{\lambda>0}\exp\left\{\theta_\varphi(\lambda,p) + p\sum_{i=1}^{\mathbf{N}}\varphi_i\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left(\sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) - x\right)\right\},$$
$$\theta_\psi(\lambda,p) = \sup_{u\in B}\left((1-p)\sum_{i=1}^{\mathbf{N}}\psi_i\left(\frac{\lambda\gamma_i(u)}{1-p}\right) - \lambda f(u)\right).$$

Proof. Let V_{ε_k} denote a set of the centers of closed balls with radii $\varepsilon_k = \sigma^{(-1)}(\beta p^k)$, $p \in (0,1)$, $k = 0,1,2,\ldots$, which forms minimal covering of the space (B,ρ). Number of elements in the set V_{ε_k} is equal to $N_{(B,\rho)}(\varepsilon_k) = N_B(\varepsilon_k)$.

It follows from Theorem 2 and Condition Σ that for any $\varepsilon > 0$

$$\mathbf{P}\{|X_i(t) - X_i(s)| > \varepsilon\}$$
$$\le 2\exp\left\{-\varphi_i\left(\frac{\varepsilon}{\tau_{\varphi_i}(X_i(t)-X_i(s))}\right)\right\} \le 2\exp\left\{-\varphi_i\left(\frac{\varepsilon}{\sigma(\rho(t,s))}\right)\right\}.$$

Therefore the processes $X_i(t)$ are continuous in probability, and the process $X(t) = \sum_{i=1}^{\mathbf{N}} X_i(t) - f(t)$ is continuous in probability as well. Hence the set $V = \bigcup_{k=1}^{\infty} V_{\varepsilon_k}$ is a set of separability of the process X and with probability one

$$\sup_{t\in T} X(t) = \sup_{t\in V} X(t). \tag{15}$$

Consider a mapping $\alpha_n = \{\alpha_n(t), n = 0, 1 \ldots\}$ of the set $V = \bigcup V_{\varepsilon_k}$ into the set V_{ε_n}, where $\alpha_n(t)$ is such a point from the set V_{ε_n} that $\rho(t, \alpha_n(t)) < \varepsilon_n$. If $t \in V_{\varepsilon_n}$, then $\alpha_n(t) = t$. If there exist several such points from the set V_{ε_n} that $\rho(t, \alpha_n(t)) < \varepsilon_n$, then we choose one of them and denote it $\alpha_n(t)$.

It follows from Chebyshev's inequality, Theorem 1, and Condition Σ that

$$\begin{aligned}
&\mathbf{P}\left\{|X_i(t) - X_i(\alpha_n(t))| > p^{\frac{n}{2}}\right\} \\
&\quad \le \frac{\mathbf{E}(X_i(t) - X_i(\alpha_n(t)))^2}{p^n} \le \frac{c_2^2 \tau_\varphi^2(Y(t) - Y(\alpha_n(t)))}{p^n} \le \frac{c_2^2 \sigma^2(\varepsilon_n)}{p^n} = c_2^2 \beta^2 p^n,
\end{aligned}$$

where c_2 is the constant from (5). This inequality implies that

$$\sum_{n=1}^{\infty} \mathbf{P}\left\{|X_i(t) - X_i(\alpha_n(t))| > p^{\frac{n}{2}}\right\} < \infty.$$

It follows from the Borel-Kantelli lemma that $X_i(t) - X_i(\alpha_n(t)) \to 0$ as $n \to \infty$ with probability one. Since the function f is continuous, then $X(t) - X(\alpha_n(t)) \to 0$ as $n \to \infty$ with probability one as well. Since the set V is countable, then $X(t) - X(\alpha_n(t)) \to 0$ as $n \to \infty$ for all t simultaneously.

Let t be an arbitrary point from the set V. Denote by $t_m = \alpha_m(t)$, $t_{m-1} = \alpha_{m-1}(t_m), \ldots, t_1 = \alpha_1(t_2)$ for any $m \ge 1$. Since for all $m \ge 2$

$$\begin{aligned}
X(t) &= X(t_1) + \sum_{k=2}^{m} (X(t_k) - X(t_{k-1})) + X(t) - X(\alpha_m(t)) \\
&\le \max_{u \in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)) + X(t) - X(\alpha_m(t))
\end{aligned}$$

we have

$$\begin{aligned}
X(t) &\le \lim_{m\to\infty} \inf \left(\max_{u \in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)) + X(t) - X(\alpha_m(t))) \right) \\
&= \lim_{m\to\infty} \inf \left(\max_{u \in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right). \qquad (16)
\end{aligned}$$

It follows from (15) and (16) that with probability one

$$\sup_{t \in T} X(t) \le \lim_{m\to\infty} \inf \left(\max_{u \in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right). \qquad (17)$$

Let $\{q_k, k = 1, 2, \dots\}$ be such a sequence that $q_k > 1$ and $\sum\limits_{k=1}^{\infty} q_k^{-1} \le 1$. It follows from the Hölder's inequality, the Fatou's lemma, and (17) that for all $\lambda > 0$

$$\begin{aligned}
&\mathbf{E}\exp\left\{\lambda \sup_{t\in T} X(t)\right\} \\
&\le \mathbf{E} \lim_{m\to\infty} \inf \exp\left\{\lambda\left(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right)\right\} \\
&\le \lim_{m\to\infty} \inf \mathbf{E}\exp\left\{\lambda\left(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right)\right\} \\
&\le \lim_{m\to\infty} \inf \left(\left(\mathbf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{1/q_1}\right. \\
&\quad \left.\times \prod_{k=2}^{m}\left(\mathbf{E}\exp\left\{q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right\}\right)^{1/q_k}\right) \\
&\le \left(\mathbf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{1/q_1} \\
&\quad \times \prod_{k=2}^{\infty}\left(\mathbf{E}\exp\left\{q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right\}\right)^{1/q_k}.
\end{aligned} \tag{18}$$

Consider each of the factors in the right-hand side of (18) separately. It follows from (4) that for all $1 \le i \le \mathbf{N}$

$$\mathbf{E}\exp\{q_1\lambda X_i(u)\} \le \exp\{\psi_i(q_1\lambda\gamma_i(u))\}$$

and

$$\mathbf{E}\exp\{q_k\lambda(X_i(u) - X_i(\alpha_{k-1}(u)))\} \le \exp\{\varphi_i(q_k\lambda\sigma_i(\varepsilon_{k-1}))\}.$$

Therefore,

$$\begin{aligned}
&\left(\mathbf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{1/q_1} \\
&\le \left(\sum_{u\in V_{\varepsilon_1}} \mathbf{E}\exp\left\{q_1\lambda \sum_{i=1}^{\mathbf{N}} X_i(u)\right\} \exp\left\{-q_1\lambda f(u)\right\}\right)^{1/q_1}
\end{aligned}$$

$$\leq \left(\sum_{u\in V_{\varepsilon_1}} \prod_{i=1}^{\mathbf{N}} \mathbf{E} \exp\left\{ q_1\lambda X_i(u) \right\} \exp\left\{ -q_1\lambda f(u) \right\} \right)^{1/q_1}$$

$$\leq \left(N_B(\varepsilon_1) \right)^{1/q_1} \exp\left\{ \frac{1}{q_1} \sup_{u\in B} \left(\sum_{i=1}^{\mathbf{N}} \psi_i(q_1\lambda\gamma_i(u)) - q_1\lambda f(u) \right) \right\}.$$

Using the assumption $|f(u) - f(v)| \leq \delta(\rho(u, v))$, we obtain that

$$\left(\mathbf{E} \exp\left\{ q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right\} \right)^{1/q_k}$$

$$\leq \left(N_B(\varepsilon_k) \max_{u\in V_{\varepsilon_k}} \mathbf{E} \exp\left\{ q_k\lambda \sum_{i=1}^{\mathbf{N}} \left(X_i(u) - X_i(\alpha_{k-1}(u)) \right) \right\} \right.$$

$$\left. \times \exp\left\{ -q_k\lambda\left(f(u) - f(\alpha_{k-1}(u)) \right) \right\} \right)^{1/q_k} \leq \left(N_B(\varepsilon_k) \right)^{1/q_k}$$

$$\times \left(\max_{u\in V_{\varepsilon_k}} \exp\left\{ \sum_{i=1}^{\mathbf{N}} \varphi_i\left(q_k\lambda\sigma(\varepsilon_{k-1}) \right) + q_k\lambda\delta(\rho(u, \alpha_{k-1}(u))) \right\} \right)^{1/q_k}$$

$$\leq \left(N_B(\varepsilon_k) \right)^{1/q_k} \exp\left\{ q_k^{-1} \sum_{i=1}^{\mathbf{N}} \varphi_i\left(q_k\lambda\beta p^{k-1} \right) + \lambda\delta\left(\sigma^{(-1)}(\beta p^{k-1}) \right) \right\}.$$

From inequality (18) after substitution of $q_k = p^{1-k}/(1-p)$, $k \geq 1$, we have

$$\mathbf{E} \exp\left\{ \lambda \sup_{t\in B} X(t) \right\}$$

$$\leq \exp\left\{ \sum_{k=2}^{\infty} (1-p)p^{k-1} \sum_{i=1}^{\mathbf{N}} \varphi_i\left(\frac{\lambda\beta}{1-p} \right) + \lambda \sum_{k=2}^{\infty} \delta\left(\sigma^{(-1)}(\beta p^{k-1}) \right) \right\}$$

$$\times \exp\left\{ \theta_\psi(\lambda, p) + \sum_{k=1}^{\infty} (1-p)p^{k-1} H_B\left(\sigma^{(-1)}(\beta p^k) \right) \right\}. \tag{19}$$

As in Theorem 5 Condition $\boldsymbol{R}$ implies the next inequality for the function $r(t)$

$$\exp\left\{ \sum_{k=1}^{\infty} (1-p)p^{k-1} H_B\left(\sigma^{(-1)}(\beta p^k) \right) \right\} \leq r^{(-1)}\left(\frac{1}{\beta p} \int_0^{\beta p} r\left(N_B\left(\sigma^{(-1)}(u) \right) \right) du \right). \tag{20}$$

So, we obtain the assertion of Theorem 6 from (19), (20), and Chebyshev's inequality.

3 Storage Processes

This section is devoted to the study of a storage process with mixed input from class $V(\varphi, \psi)$.

Definition 9. We call a random process $Q(t) = \{Q(t), t \in T\}$ storage process of the queue with input process $X(t) = \{X(t), t \in T\}$ if

$$Q(t) = \sup_{s \le t} \big(X(t) - X(s) - (f(t) - f(s))\big), \quad s, t \in T, \tag{21}$$

where function $f(t)$ denotes intensity of queue serving.

If more work arrives than can be processed by the server, then the surplus is stored in a buffer of size $x \ge 0$. Obviously that part of the input received after the moment when buffer overflow is lost. Therefore estimation of the buffer overflow probability is an important task in the queueing theory. Such problem can also be reformulated in terms of the risk theory as estimation of the probability of bankruptcy for the corresponding risk process, e.g., [1, 15].

The following theorem gives an upper estimate for the tail distribution of a storage process with aggregate input

$$Q(t) = \sup_{s \le t} \left(\sum_{i=1}^{\mathbf{N}} (X_i(t) - X_i(s)) - (f(t) - f(s)) \right), \quad s, t \in B. \tag{22}$$

Theorem 7. *Let $X_i = \{X_i(t), t \in B\}$ be independent separable random processes from classes $V(\varphi_i, \psi_i)$ defined on a compact set $B \subset T$ and satisfying Condition $\Sigma\mathbf{N}$. Let $f = \{f(t), t \in B\}$ be a continuous function satisfying Condition F, and let $r = \{r(u), u \ge 1\}$ be a continuous function satisfying Condition R. If, in addition,*

$$R_\beta(t) = \int_0^\beta r(N_{B_t}(\sigma^{(-1)}(u)))\, \mathrm{d}u < \infty, \tag{23}$$

then for all $p \in (0, 1)$ and $x > 0$ the following inequality holds for the storage process $Q(t)$ defined in (22)

$$\mathbf{P}\{Q(t) > x\} \le Z_r(p, t, x), \tag{24}$$

where

$$Z_r(p,t,x) = r^{(-1)}\big(R_\beta(t)/(\beta p)\big)$$
$$\times \inf_{\lambda > 0} W(\lambda, p, t) \exp\left\{ p \sum_{i=1}^{\mathbf{N}} \varphi_i\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left(\sum_{k=2}^{\infty} \delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) - x\right)\right\},$$

$$W(\lambda, t, p) = \min\{U_1(\lambda, t, p);\;\; U_2(\lambda, t, p)\},$$

$$(U_1(\lambda, t, p))^{1/(1-p)}$$

$$= \int_1^{N_{B_t}\left(\sigma^{(-1)}(\beta p)\right)+1} \exp\left\{\sum_{i=1}^{\mathbf{N}} \varphi\left(\frac{\lambda\sigma\left(2x\sigma^{(-1)}(\beta p)\right)}{1-p}\right) + \frac{\lambda\delta(2x\sigma^{(-1)}(\beta p))}{1-p}\right\} \mathrm{d}x,$$

$$U_2(\lambda, t, p) = \left(N_{B_t}(\sigma^{(-1)}(\beta p))\right)^{1-p} \inf_{v>1} \exp\left\{(1-p)\sum_{i=1}^{\mathbf{N}} \psi_i\left(\frac{v\lambda\gamma(t)}{1-p}\right)/v\right.$$

$$\left. -\lambda f(t) + \max_{u\in B_t}\left((v-1)(1-p)\sum_{i=1}^{\mathbf{N}} \psi_i\left(\frac{v\lambda\gamma_i(t)}{(v-1)(1-p)}\right)/v + \lambda f(u)\right)\right.$$

The assertion of Theorem 7 follows from Theorem 6 and comes as a natural generalization of the results in [17] for a sum of random processes from classes $V(\varphi_i, \psi_i)$.

Theorem 8 also follows from Theorem 6 and can be easily obtained through the generalization of the results of paper [16] for an aggregate of random processes from classes $V(\varphi_i, \psi_i)$.

Theorem 8. *Let $X_i = \{X_i(t), t \in B\}$ be independent separable random processes from classes $V(\varphi_i, \psi_i)$ defined on a compact set $B \subset T$ and satisfying Condition $\Sigma\mathbf{N}$. Let $f = \{f(t), t \in B\}$ be a continuous function satisfying Condition F, and let $r = \{r(u), u \geq 1\}$ be a continuous function satisfying Condition R. If, in addition, the following condition holds*

$$\int_0^{\beta} r\left(L\left(\sigma^{(-1)}(u)\right)\right) \mathrm{d}u < \infty, \tag{25}$$

then for all $p \in (0, 1)$ and $x > 0$ following estimates hold for the storage process $Q(t)$ defined in (22)

$$\mathbf{P}\left\{\sup_{s\leq t;\, s,t\in B} Q(t) > x\right\} \leq Z_r(p, x), \tag{26}$$

$$\mathbf{P}\left\{\inf_{s\leq t;\, s,t\in B} Q(t) < -x\right\} \leq Z_r(p, x), \tag{27}$$

$$\mathbf{P}\left\{\sup_{s\leq t;\, s,t\in B} |Q(t)| > x\right\} \leq 2Z_r(p, x), \tag{28}$$

where

$$Z_r(p,t,x) = r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p^2} r\left(\mathsf{L}\left(\sigma^{(-1)}(u)\right)\right)\,\mathrm{d}u\right)$$

$$\times \inf_{\lambda>0} W(\lambda,p)\exp\left\{p\sum_{i=1}^{\mathbf{N}}\varphi_i\left(\frac{2\lambda\beta}{1-p}\right)+\lambda\left(2\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^k)\right)-x\right)\right\},$$

$$W(\lambda,p) = \left(\sum_{l=0}^{\mathsf{N}\left(\sigma^{(-1)}(\beta p)\right)-1}\left(\mathsf{N}\left(\sigma^{(-1)}(\beta p)\right)-l\right)\right.$$

$$\left.\times\exp\left\{\sum_{i=1}^{\mathbf{N}}\varphi_i\left(\frac{\lambda\sigma(2l\sigma^{(-1)}(\beta p))}{1-p}\right)+\frac{\lambda\delta(2l\sigma^{(-1)}(\beta p))}{1-p}\right\}\right)^{1-p}.$$

Example 9. Consider independent centered normalized GFBM processes $X_i(t) = \{X_i(t),\ t \in [a,b]\}$ from classes $V(\psi_i,\varphi_i)$ and N-functions $\varphi_i(x) = x^2/x_i^2$, where $x_i > 0$ are some constants, $i = \overline{1,\mathbf{N}}$. Let $f(t)$ be a continuous service function with the following property:

$$|f(t)-f(s)| \le c|t-s|^n, \quad t,s \in [a,b], \tag{29}$$

where $c > 0$ and $0 < n \le 1$ are some constants. It is easy to see that Condition F holds for the function f.

Let Condition ΣN be fulfilled for processes X_i. Then the following estimates follow from Theorem 8.

Theorem 9. *Let $X_i(t) = \{X_i(t),\ t \in [a,b]\}$ be independent GFBM processes with Hurst indexes $H_i \in (0,1)$ from classes $V(\psi_i,\varphi_i)$ defined by Orlicz N-functions $\varphi(x) = x^2/x_i^2$, $i = \overline{1,\mathbf{N}}$, and let $f = \{f(t), t \in [a,b]\}$ be a continuous function which satisfies (29). Then for all*

$$p \in \left(0, \min\left\{(2/3)^{H_{\max}}; 1/\beta\right\}\right] \quad \text{and} \quad x \ge 0 \tag{30}$$

the following estimates hold

$$\mathbf{P}\left\{\sup_{s\le t;s,t\in B}\left(\sum_{i=1}^{\mathbf{N}}(X_i(t)-X_i(s))-(f(t)-f(s))\right) > x\right\} \le Z(p,x),$$

$$\mathbf{P}\left\{\inf_{s\le t;s,t\in B}\left(\sum_{i=1}^{\mathbf{N}}(X_i(t)-X_i(s))-(f(t)-f(s))\right) < -x\right\} \le Z(p,x),$$

$$\mathbf{P}\left\{\sup_{s\le t;s,t\in B}\left|\sum_{i=1}^{\mathbf{N}}(X_i(t)-X_i(s))-(f(t)-f(s))\right| > x\right\} \le 2Z(p,x),$$

where $H_{\min} = \min_{i=\overline{1,\mathbf{N}}} H_i$, $H_{\max} = \max_{i=\overline{1,\mathbf{N}}} H_i$,

$$Z(p,x) = \frac{(b-a)^2}{2}(\beta pe)^{2/H_{\max}} \left(\sum_{l=0}^{\frac{b-a}{2(\beta p)^{1/H_{\max}}}} \left(\frac{b-a}{2(\beta p)^{1/H_{\max}}} + 1 - l \right) \right.$$
$$\left. \times \exp\left\{ -\frac{\left(x - c(2l)^n (\beta p)^{n/H_{\max}} - 2c(\beta p^2)^{n/H_{\max}}/(1-p^{n/H_{\max}})\right)^2}{4p\beta^2 \sum_{i=1}^{\mathbf{N}} x_i^{-2}(4+p(2l)^{2H_i})} \right\} \right)^{1-p}.$$

Proof. From (8), (12), and (14), we obtain the following bounds for sub-Gaussian increments of the aggregate $\sum X_i(t)$:

$$\sigma(u) = \sup_{i=\overline{1,\mathbf{N}}} \{\sigma_i(u)\} = \sup_{i=\overline{1,\mathbf{N}}} u^{H_i} = \begin{cases} u^{H_{\min}}, & 0 \le u \le 1, \\ u^{H_{\max}}, & u > 1, \end{cases} \tag{31}$$

and

$$\sigma^{(-1)}(u) = \begin{cases} u^{1/H_{\max}}, & 0 \le u \le 1, \\ u^{1/H_{\min}}, & u > 1. \end{cases} \tag{32}$$

Let $r(u) = u^\alpha$, $0 < \alpha < \frac{H}{2}$. If $p \le \min\left\{(2/3)^{H_{\max}}; 1/\beta\right\}$, then $\frac{b-a}{2u^{1/H_{\max}}} > \frac{3}{2}$, since $u \le (2/3)^{H_{\max}}(b-a/2)^{H_{\max}} \le p\beta \le 1$. Therefore, we obtain

$$\begin{aligned} & r^{(-1)}\left(\frac{1}{\beta p} \int_0^{\beta p} r\left(L\left(\sigma^{(-1)}(u)\right)\right) \mathrm{d}u \right) \\ & \le \left(\frac{1}{\beta p} \int_0^{\beta p} \left(\left(\frac{b-a}{2u^{1/H_{\max}}} + 1 \right)^2 + \frac{b-a}{2u^{1/H_{\max}}} + 1 \right)^\alpha / 2^\alpha \, \mathrm{d}u \right)^{1/\alpha} \\ & \le \frac{1}{2} \left(\frac{1}{\beta p} \int_0^{\beta p} \left(\frac{b-a}{2u^{1/H_{\max}}} + \frac{3}{2} \right)^{2\alpha} \mathrm{d}u \right)^{1/\alpha} < \frac{1}{2} \left(\frac{1}{\beta p} \int_0^{\beta p} \left(\frac{b-a}{u^{1/H_{\max}}} \right)^{2\alpha} \mathrm{d}u \right)^{1/\alpha} \\ & = \frac{(b-a)^2}{2}(\beta p)^{2/H_{\max}} \left(1 - \frac{2\alpha}{H_{\max}} \right)^{-1/\alpha} \to \frac{(b-a)^2}{2}(\beta pe)^{2/H_{\max}}, \ \alpha \to 0. \end{aligned} \tag{33}$$

Also

$$\sum_{k=1}^\infty \delta\left(\sigma^{(-1)}(\beta p^k)\right) = \sum_{k=1}^\infty c(\beta p^k)^{n/H_{\max}} = \frac{c\beta^{n/H_{\max}} p^{n/H_{\max}}}{1 - p^{n/H_{\max}}}. \tag{34}$$

Applying (33), (34), and the following chain of transforms to Theorem 5, we obtain the assertion of Theorem 9 (Fig. 1).

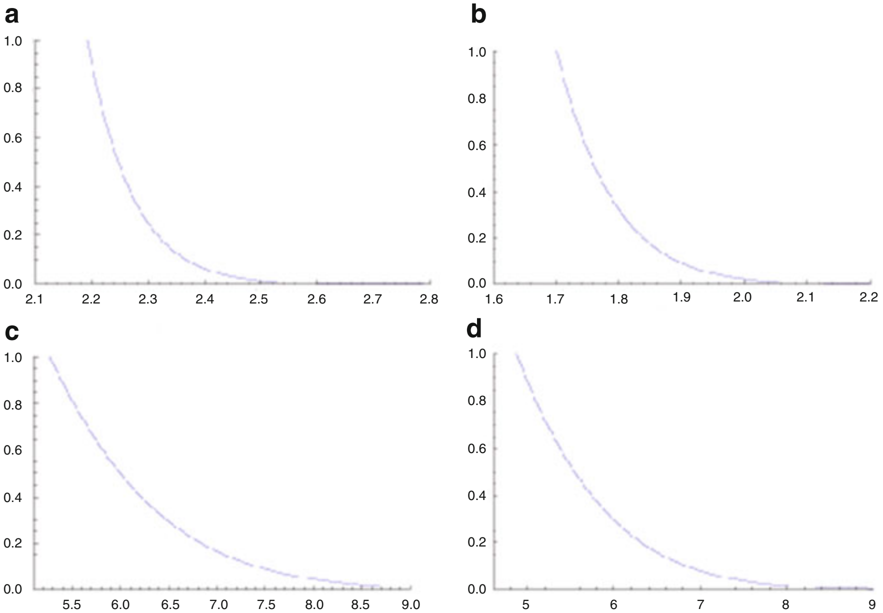

Fig. 1 Here is depicted upper estimating function $Z(p,x)$ from Theorem 9 for various sets of sub-Gaussian GFBM incoming processes considered on interval $[a,b]=[0,1]$ with the following values: $c=1$, $n=1$ (i.e., $f(t)=t$), $x_i=\sqrt{2}$, $\beta=\left(\frac{b-a}{2}\right)^{H_{\max}}=(0.5)^{H_{\max}}$, $p=0.25$ (**a**) $\mathbf{N}=1$, $H=0.5$ (**b**) $\mathbf{N}=1$, $H=0.75$ (**c**) $\mathbf{N}=5$, $H_i=0.9$ (**d**) $\mathbf{N}=5$, $H_i\in\{0.5,0.6,0.7,0.8,0.9\}$

$$
\begin{aligned}
&\inf_{\lambda>0}\exp\left\{p\sum_{i=1}^{\mathbf{N}}\varphi_i\left(\frac{2\lambda\beta}{1-p}\right)+\lambda\left(2\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^k)\right)-x\right)\right\}W(\lambda,p)\\
&=\inf_{\lambda>0}\exp\left\{\sum_{i=1}^{\mathbf{N}}\frac{4p\lambda^2\beta^2}{x_i^2(1-p)^2}+\lambda\left(\frac{2c\beta^{n/H_{\max}}p^{n/H_{\max}}}{1-p^{n/H_{\max}}}-x\right)\right\}\\
&\quad\times\left(\sum_{l=0}^{\frac{b-a}{2(\beta p)^{1/H_{\max}}}}\left(\frac{b-a}{2(\beta p)^{\frac{1}{H_{\max}}}}+1-l\right)\exp\left\{\sum_{i=1}^{\mathbf{N}}\frac{\lambda^2(2l)^{2H_i}(\beta p)^2}{x_i^2(1-p)^2}\right.\right.\\
&\quad\left.\left.+\frac{\lambda(2l)^n c(\beta p)^{\frac{n}{H_{\max}}}}{1-p}\right\}\right)^{1-p}\le\left(\sum_{l=0}^{\frac{b-a}{2(\beta p)^{1/H_{\max}}}}\left(\frac{b-a}{2(\beta p)^{\frac{1}{H\max}}}+1-l\right)\right.\\
&\quad\times\inf_{\lambda>0}\exp\left\{\lambda^2\sum_{i=1}^{\mathbf{N}}\left(\frac{4p\beta^2}{x_i^2(1-p)^2}+\frac{(2l)^{2H_i}(\beta p)^2}{x_i^2(1-p)^2}\right)\right.
\end{aligned}
$$

$$-\lambda\left(\frac{x}{1-p}-\frac{2c\beta^{n/H_{\max}}p^{n/H_{\max}}}{(1-p)(1-p^{n/H_{\max}})}-\frac{(2l)^n c(\beta p)^{n/H_{\max}}}{1-p}\right)\Bigg\}\Bigg)^{1-p}$$

$$\le \left(\sum_{l=0}^{\frac{b-a}{2(\beta p)^{1/H_{\max}}}}\left(\frac{b-a}{2(\beta p)^{1/H_{\max}}}+1-l\right)\right.$$

$$\left.\times \exp\left\{-\frac{\left(x-c(2l)^n(\beta p)^{n/H_{\max}}-2c(\beta p^2)^{n/H_{\max}}/(1-p^{n/H_{\max}})\right)^2}{4p\beta^2\sum_{i=1}^{\mathbf{N}} x_i^{-2}\left(4+p(2l)^{2H_i}\right)}\right\}\right)^{1-p}.$$

4 Conclusions

The paper summarizes some recent studies that have been made for a general class $V(\phi,\psi)$ of incoming processes X_i. As an example, we consider sub-Gaussian GFBM storage process with aggregated input formed by independent sources. We show that obtained estimate for the tail distribution of such storage process depends on the buffer size x as $o\left(\exp\left\{-\alpha x^2\right\}\right)$. Also we provide several illustrations of the buffer overflow probability for different values of Hurst parameter of the incoming processes.

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