Chapter 2
Avalanches in Multiplex and Interdependent Networks

G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev and J. F. F. Mendes

Abstract Many real-world complex systems are represented not by single networks but rather by sets of interdependent networks. In these specific networks, vertices in each network mutually depend on vertices in other networks. In the simplest representative case, interdependent networks are equivalent to the so-called multiplex networks containing vertices of one sort but several kinds of edges. Connectivity properties of these networks and their robustness against damage differ sharply from ordinary networks. Connected components in ordinary networks are naturally generalized to viable clusters in multiplex networks whose vertices are connected by paths passing over each individual sort of their edges. We examine the robustness of the giant viable cluster to random damage. We show that random damage to these systems can lead to the avalanche collapse of the viable cluster, and that this collapse is a hybrid phase transition combining a discontinuity and the critical singularity. For this transition we identify latent critical clusters associated with the avalanches triggered by a removal of single vertices. Divergence of their mean size signals the approach to the hybrid phase transition from one side, while there are no critical precursors on the other side. We find that this discontinuous transition occurs in scale-free multiplex networks whenever the mean degree of at least one of the interdependent networks does not diverge.

2.1 Introduction

The network representation of complex systems is successfully exploited in various sciences [1]. Numerous real-world systems, however, cannot be represented by a single network. Instead, they consist of several interacting networks. In simple sit-

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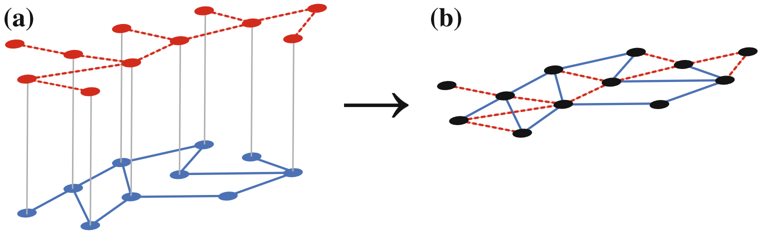

Fig. 2.1 **a** Two interdependent networks. A vertex in one network has a mutual dependence, represented by *grey vertical lines*, on zero or one vertex in the other network. **b** This can be reduced to a multiplex network by merging the mutually dependent vertices, and representing the edges of each network by different kinds or *colours* of edges

uations, these interactions can be represented by interlinks connecting vertices in different networks [2, 3]. When these interconnections and edges in all these networks are identical, then it is possible to describe the structural organization of this set of networks and the statistics of its connected components similarly to ordinary networks [4]. Here we consider significantly more interesting systems in which vertices in each network mutually depend on vertices in other networks in the sense that the removal (or, generally, change of the state) of a vertex in one network immediately leads to the removal (or change of the state) of its neighbour in another network. These interdependent networks describe numerous complex systems, both natural [5], and man-made [6, 7]. Importantly, the interdependencies can make a system more fragile: damage to one element can lead to avalanches of failures throughout the system [8, 9]. Recent theoretical investigations of interdependent networks consisting of two [10] or more [11] subnetworks have shown that small initial failures can cascade back and forth through the networks, leading, at some critical point, to the collapse of the whole system in a discontinuous phase transition.

In the original formulation of the problem [10] the researchers focused on the final result of the removal of a finite $1 - p$ fraction of vertices from one of the interdependent networks. This removal leads to a complicated infinite (for infinite networks) cascade in back-and-forth damage propagation. Below a critical point p_c, this cascade of failures eliminates the interdependent networks completely, while above the transition, the cascade sweeps out a finite fraction of the networks. Son et al. [12] showed the original approach of studying two interdependent networks can be simplified, if one uses the equivalence of a wide class of interdependent networks to a multiplex network problem. They proposed a simple mapping from the model used in [10] in which a vertex in one network has a mutual dependence on no more than one vertex in the other network, to a multiplex network with one kind of vertex but two kinds of edges. The mapping is achieved by simply merging the mutually dependent vertices from the two networks. Figure 2.1 explains this mapping. In graph theory, the multiplex networks are also called graphs with coloured edges.

As we will see, the phase transition in this system is discontinuous, and hybrid in nature, in contrast to ordinary percolation that occurs as a continuous phase transition. The difference between hybrid and continuous phase transitions is that the hybrid transition has a discontinuity like a first-order transition, but exhibits critical behavior near the transition, like a second-order transition. Moreover, the hybrid transition is asymmetric: critical correlations appear on only one side of the critical point, whereas they appear on both sides of a continuous phase transition. Another intriguing phenomenon appearing at the critical point of the hybrid transition is scale-invariant avalanches that are absent in a continuous phase transition. Each avalanche is triggered by removal of a single vertex and results in the elimination of multiple vertices. To highlight this principal difference from continuous phase transitions, let us compare with, for example, the continuous percolation phase transition. This is a second order phase transition in an equilibrium system. Percolation can be represented as the removal of uniformly randomly chosen vertices. Removal of a vertex can only split a cluster (connected component) into smaller clusters but, it cannot trigger an avalanche.

In this chapter we describe these discontinuous phase transitions. Our aim is to expand and deepen the understanding of the nature of the phase transition and the avalanche collapse in interdependent and multiplex networks. This understanding has been lacking until recently. We investigate the damage caused by the removal of a single node chosen at random from an infinite network. The removal of a single vertex causes an avalanche of damage (so named to distinguish it from the cascades of failures mentioned above, which are caused by the sudden removal of a finite fraction of the vertices in the network). Our method allows the identification of individual avalanches and the study of their structure.

Why is the problem of the avalanches triggered by the removal of a single vertex principally important and attractive for researchers? The reason is that the statistics of these individual avalanches reveals the critical divergence at the phase transition point. To understand a phase transition, it is not sufficient to obtain an equation showing the emergence of a non-zero order parameter. For continuous and hybrid transitions, one should also find the divergence of susceptibility associated with this transition, and also describe critical correlations. It is avalanches that are responsible for critical correlations. The mean size of the individual avalanches triggered by a randomly removed vertex plays a role of susceptibility and diverges at the critical point manifesting the hybrid transition. The second reason, with a practical perspective, is that knowledge of the organization of individual avalanches enables one to control them and increase robustness of the system.

In the remainder of this chapter, then, we will generally consider multiplex networks, but it should be noted that the results are identical to those for two interdependent networks as defined above, and may be qualitatively extended to interdependent networks in general. The results presented in this chapter are based on results obtained in our paper [13].

This chapter is organised as follows. In Sect. 2.2 we define the multiplex network model, and give an algorithm for identifying the viable clusters. In Sect. 2.3 we derive basic equations for the size of the giant viable cluster, and show how the location and scaling of the transition may be obtained. In Sect. 2.4 we analyse the structure and statistics of the avalanches associated with the transition. These results are extended

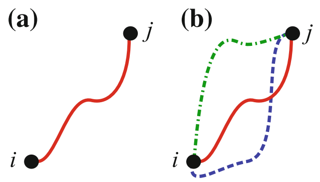

Fig. 2.2 **a** In an ordinary network, two vertices i and j belong to the same cluster if there is a path connecting them. **b** In a multiplex network, vertices i and j belong to the same viable cluster if there is a path connecting them for every kind of edge, following only edges of that kind. In the example shown, there are $m = 3$ kinds of edges. Vertices i and j are said to be 3-connected

to the special case of scale-free networks in Sect. 2.5. Results are summarised in Sect. 2.6.

2.2 Viable Clusters and Algorithm

In ordinary networks, two vertices are connected if there is a path between the vertices. Based on this notion, one introduces clusters of connected vertices and studies emergence of the giant connected component of a graph. In multiplex network, this notion of connection between vertices must be modified. We consider a set of vertices connected by m different types of edges. The connections are essential to the function of each site, so that a vertex is only *viable* if it maintains connections of every type to other viable vertices. A *viable cluster* is defined as follows: For every kind of edge, and for any two vertices i and j within a viable cluster, there must be a path from i to j following only edges of that kind. In other words, in multiplex network with m types of edges, two vertices are m-connected if for every type of edges there is a path between these vertices. Based on this definition, a viable cluster is then a cluster of m-connected vertices. Figure 2.2 explains the viable clusters. In a large system, we wish to find when there is a giant cluster of viable vertices. From this definition of viable clusters, it follows that any giant viable cluster is a subgraph of the giant connected component of each of the m networks formed by considering only a single type of edge in the multiplex network. The absence of, at least, any one giant connected component means the absence of the giant viable cluster. Note that viable clusters are simple generalization of clusters of connected vertices in ordinary networks with a single type of edges. The important difference is that in a multiplex networks we demand that vertices in a viable cluster must be connected by every type of edges (m-connected). It is this additional condition that leads to discontinuous emergence of the giant viable cluster as a result of a hybrid phase transition in contrast to a continuous phase transition in ordinary percolation.

The viable clusters of any size may be identified by an iterative pruning algorithm, based on the principles of percolation. Here we give such an algorithm for identifying viable clusters that may be implemented, for example, in a computer program for investigations of the resilience of real-world multiplex networks.

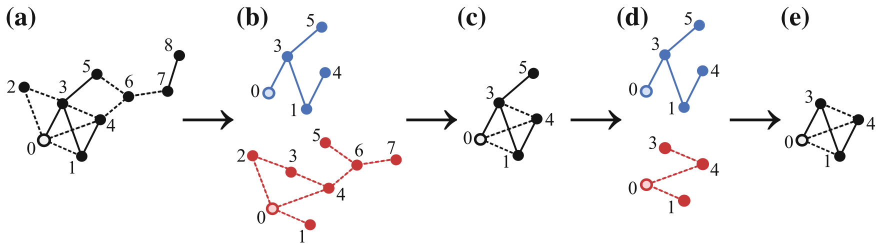

Fig. 2.3 An example demonstrating the algorithm for identifying a viable cluster in a small network with two kinds of edges. **a** In the original network, in step (i) we select vertex 0 as the test vertex. **b** In step (ii) we identify the clusters of vertices connected to 0 by each kind of edge. **c** Step (iii): the intersection of these two clusters forms becomes the new candidate set for the viable cluster to which 0 belongs. **d** We repeat steps (ii) using only vertices from the candidate set shown in **c**. Repeating step (iii), we find the overlap between the two clusters from **d**, shown in **e**. Further repetition of steps (ii) and (iii) does not change this cluster, meaning that the cluster consisting of vertices 0, 1, 3 and 4 is a viable cluster

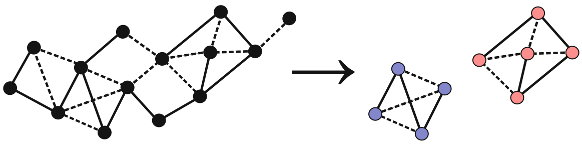

Fig. 2.4 A small network with two kinds of edges (*left*). Applying the algorithm described in the text, non-viable vertices are removed, leaving two viable clusters (*right*)

Consider a multiplex network, with vertices $i = 1, 2, \ldots, N$ connected by m kinds of edges labeled $s = a, b, \ldots$. Viable clusters in any multiplex network may be identified by the following algorithm.

(i) Choose a test vertex i at random from the network.
(ii) For each kind of edge s, compile a list of vertices that can be reached from i by following only edges of type s.
(iii) The intersection of these m lists forms a new candidate set for the viable cluster containing i.
(iv) Repeat steps (ii) and (iii) but traversing only the current candidate set. When the candidate set no longer changes, it is either a viable cluster, or contains only vertex i.
(v) To find further viable clusters, remove the viable cluster of i from the network (cutting any edges) and repeat steps (i)–(iv) on the remaining network beginning from a new test vertex.

Repeated application of this procedure will identify every viable cluster in the network. A simple example of the use of the algorithm to identify a small viable cluster is given in Fig. 2.3. The results of applying the algorithm to a graph containing two finite viable clusters is illustrated in Fig. 2.4.

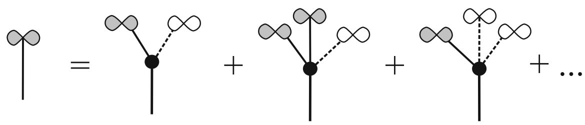

Fig. 2.5 Diagrammatic representation of Eq. (2.1) in a system of two interdependent networks a and b. The probability X_a, represented by a *shaded infinity symbol* can be written recursively as a sum of second-neighbor probabilities. *Open infinity symbols* represent the equivalent probability X_b for network b, which obeys a similar recursive equation. The filled *circle* represents the probability p that the vertex remains in the network

2.3 Hybrid Transition in Multiplex Networks

In this section we will study collapse of giant viable cluster in multiplex networks damaged by random removal of vertices. We will use the fraction p of vertices remaining undamaged as a control variable, however other control variables such as mean degree could also be used. As we will show below, in uncorrelated random networks the giant viable cluster collapses at a critical undamaged fraction p_c in a discontinuous hybrid transition, similar to that seen in the k-core or bootstrap percolation [14, 15].

Hybrid transitions, like those which occurs in the collapse of multiplex and interdependent networks, and associated avalanches, also occur in a wide variety of other systems. For example, a jump in activity in neural networks [16], population collapse in biological systems [17, 18], jamming and rigidity transitions and glassy dynamics [19, 20], and magnetic systems [21].

Let us construct the basic equations which allow us to analyse the hybrid transition. Consider the case of sparse uncorrelated networks, which are locally tree-like in the infinite size limit $N \to \infty$. In order to find the giant viable cluster, we take advantage of the locally tree-like property of the network, and define X_s, with the index $s \in \{a, b, \ldots\}$, to be the probability that, on following an arbitrarily chosen edge of type s, we encounter the root of an infinite sub-tree formed solely from type s edges, whose vertices are also each connected to at least one infinite subtree of every other type. We call this a type s infinite subtree. This is illustrated in Fig. 2.5, which shows the probability X_a as the sum of second-level probabilities in terms of X_a and X_b. The vector $\{X_a, X_b, \ldots\}$ plays the role of the order parameter. Writing this graphical representation in equation form, using the joint degree distribution $P(q_a, q_b, \ldots)$, we arrive at the self consistency equations

$$X_s = p \sum_{q_a, q_b, \ldots} \frac{q_s}{\langle q_s \rangle} P(q_a, q_b, \ldots) \big[1 - (1 - X_s)^{q_s - 1}\big] \prod_{l \neq s} \big[1 - (1 - X_l)^{q_l}\big]$$
$$\equiv \Psi_s(X_a, X_b, \ldots)\,. \tag{2.1}$$

The multiplier p in Eq. (2.1) is the probability that the vertex remains in the network. The term $(q_s/\langle q_s \rangle) P(q_a, q_b, \ldots)$ gives the probability that on following an arbitrary

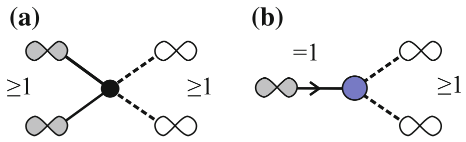

Fig. 2.6 Viable and critical viable vertices for two interdependent networks. **a** A vertex is in the giant viable cluster if it has connections of both kinds to giant viable subtrees, represented by *infinity symbols*, which occur with probabilities X_a (*shaded*) or X_b (*open*)—see text. **b** A critical viable vertex of type a has exactly one connection to a giant sub-tree of type a

edge of type s, we find a vertex with degrees $q_a, q_b, \ldots$, while $[1-(1-X_a)^{q_a}]$ is the probability that this vertex has at least one edge of type $a \neq s$ leading to the root of an infinite sub-tree of type a edges. This becomes $[1-(1-X_s)^{q_s-1}]$ when $a = s$. The argument leading to Eq. (2.1) is similar to that used in [12]. Later it will be useful to write the right-hand side of this equation as $\Psi_s(X_a, X_b, \ldots)$.

A vertex is then in the giant viable cluster if it has at least one edge of every type s leading to an infinite type s sub-tree (probability X_s), as shown in Fig. 2.6a.

$$S = p \sum_{q_a, q_b, \ldots} P(q_a, q_b, \ldots) \prod_{s=a,b,\ldots} \big[1-(1-X_s)^{q_s}\big], \tag{2.2}$$

which is equal to the relative size of the giant viable cluster of the damaged network.

A hybrid transition appears at the point where $\Psi_s(X_a, X_b, \ldots)$ first meets X_s at a non-zero value, for all s. This occurs when

$$\det[\mathbf{J} - \mathbf{I}] = 0 \tag{2.3}$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{J}$ is the Jacobian matrix $J_{ab} = \partial \Psi_b / \partial X_a$. The critical point p_c is found by solving Eqs. (2.1) and (2.3) together. To find the scaling near the critical point, we expand Eq. (2.1) about the critical value $X_s^{(c)}$. We find that

$$X_s - X_s^{(c)} \propto (p - p_c)^{1/2}. \tag{2.4}$$

This square-root scaling is the typical behaviour of the order parameter near a hybrid transition. In the next section we will show that this results from avalanches which diverge in size near the transition. The scaling of the size of the giant viable cluster, S, immediately follows

$$S - S_c \propto (p - p_c)^{1/2}. \tag{2.5}$$

A similar result is found for other control parameters, for example, mean degrees of the vertices.

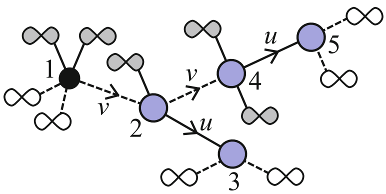

Fig. 2.7 A critical cluster. Removal of any of the shown viable vertices will result in the removal of all downstream critical viable vertices. Vertices 2–5 are critical vertices. Removal of the vertex labeled 1 will result in all of the shown vertices being removed (becoming non-viable). Removal of vertex 2 results in the removal of vertices 3, 4, and 5 as well, while removal of vertex 4 results only in vertex 5 also being removed. As before, *infinity symbols* represent connections to infinite viable subtrees. Other connections to non-viable vertices or finite viable clusters are not shown

2.4 Structure of Avalanches

Having established the behaviour of the order parameter, X_s, and the location of the hybrid transition, we now turn to examining avalanches, in order to understand the nature of the transition more completely. We focus on the case of two types of edges. Consider a viable vertex that has exactly one edge of type a leading to a type a infinite subtree, and at least one edge of type b leading to a type b infinite subtree. We call this a critical vertex of type a. It is illustrated in Fig. 2.6b. Critical vertices of type a will drop out of the viable cluster if they lose their single link to a type a infinite subtree. A vertex may have outgoing edges of this kind, so that removal of this vertex from the giant viable cluster also requires the removal of the critical vertices which depend on it. This is the way that damage propagates in the system. The removal of a single vertex can result in an avalanche of removals of critical vertices from the giant viable cluster. To represent this process visually, we draw a diagram of viable vertices and the edges between them. We mark the special critical edges, that critical viable vertices depend on, with an arrow leading to the critical vertex. An avalanche can only transmit in the direction of the arrows. For example, in Fig. 2.7, removal of the vertex labeled 1 removes the essential edge of the critical vertex 2 which thus becomes non-viable. Removal of vertex 2 causes the removal of further critical vertices 3 and 4, and the removal of 4 then requires the removal of 5. Thus critical vertices form critical clusters. At the head of each critical cluster is a 'keystone vertex' (e.g. vertex 1 in the figure) whose removal would result in the removal of the entire cluster. Graphically, upon removal of a vertex, we remove all vertices found by following the arrowed edges, which constitutes an avalanche. Note that an avalanche is a branching process. Removing a vertex may lead to avalanches along several edges emanating from the vertex (for example, in Fig. 2.7, removing vertex 2 leads to avalanches along two edges).As we approach the critical point from above, the avalanches increase in size. The mean size of avalanches triggered by a randomly removed vertex finally diverges in size at the critical point, which is the cause of the discontinuity in the size of the giant viable cluster, which collapses to zero. These avalanches are thus an inherent part of a hybrid transition.

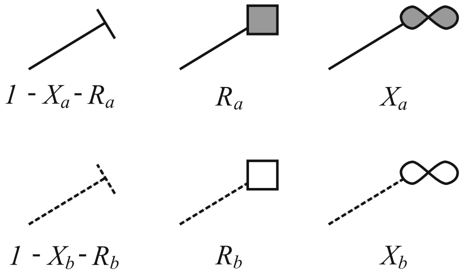

Fig. 2.8 Symbols used in the diagrams to represent key probabilities. *Solid lines* represent edges of type a, *dashed lines* represent edges of type b

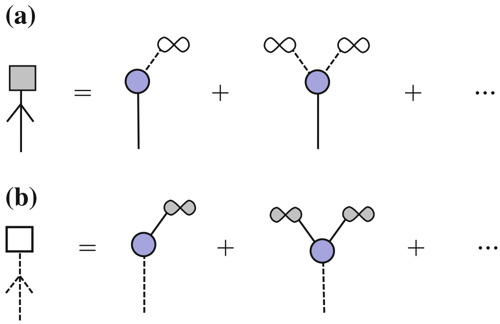

Fig. 2.9 **a** The probability R_a can be defined in terms of the second-level connections of the vertex found upon following an edge of type a. Note that possible connections to 'dead ends'—vertices not in the viable cluster (probability $1-X_a-R_a$ or $1-X_b-R_b$) are not shown. **b** The equivalent graphical equation for the probability R_b

We can use a generating function approach, similar to that developed by Newman [4] to calculate the sizes and structure of avalanches. There are three possibilities when following an arbitrarily chosen edge of a given type: (i) with probability X_s we encounter a type s infinite subtree (ii) with probability R_s we encounter a vertex which has a connection to an infinite subtree of the opposite type, but none of the same type. Such a vertex is part of the giant viable cluster if the parent vertex was; or (iii) with probability $1 - X_s - R_s$, we encounter a vertex which has no connections to infinite subtrees of either kind. These probabilities are represented graphically in Fig. 2.8. We will use these symbols in subsequent diagrams.

The probability R_a obeys

$$R_a = \sum_{q_a} \sum_{q_b} \frac{q_a}{\langle q_a \rangle} P(q_a, q_b)(1-X_a)^{q_a-1}\left[1-(1-X_b)^{q_b}\right] \tag{2.6}$$

and similarly for R_b. This equation is represented graphically in Fig. 2.9.

The generating function for the size of an avalanche triggered by removing an arbitrary type a edge which does not lead to an infinite type a subtree can be found by considering the terms represented in Fig. 2.10. The first term represents the probability, upon following an edge of type a (solid lines) of reaching a "dead end", that is, a vertex with no connection to a type b subtree (and hence is not a viable vertex). In other words, a critical cluster of size 0. The second term represents a critical cluster of size 1: the vertex encountered has a connection to the type b infinite subtree (infinity

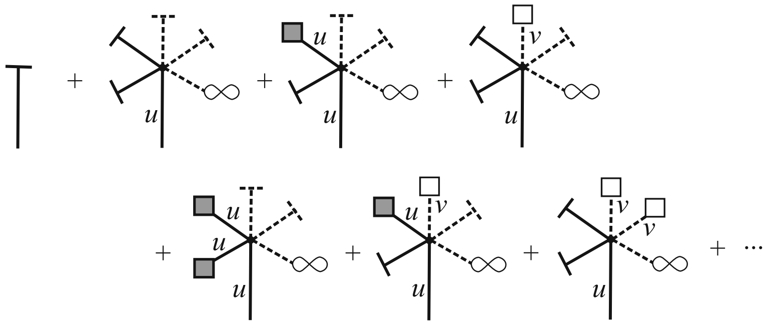

Fig. 2.10 Representation of the generating function $H_a(x, y)$ (*right-hand side* of Eq. 2.8) for the size of a critical cluster encountered upon following an edge of type a

symbol), but no further connections to viable vertices. Subsequent terms represent recursive probabilities that the vertex encountered has 1 (third and fourth terms), 2 (fifth, sixth, seventh terms) or more connections to further potential critical clusters. The variable u (for type a edges) or v (type b) are assigned to each such edge. The equation for this generating function can be written in terms of functions $F_a(x, y)$ and $F_b(x, y)$ which we define as follows:

$$F_a(x, y) \equiv \sum_{q_a}\sum_{q_b}\frac{q_a}{\langle q_a\rangle}P(q_a, q_b)x^{q_a-1}\sum_{r=1}^{q_b}\binom{q_b}{r}X_b^r y^{q_b-r} \tag{2.7}$$

and similarly for $F_b(x, y)$, by exchanging all subscripts a and b. While the function $F_a(x, y)$ does not necessarily represent a physical quantity or probability, we can see that it incorporates the probability of encountering a vertex with at least one child edge of type b leading to a giant viable subtree (probability X_b) upon following an edge of type a. All other outgoing edges then contribute a factor x (for type a edges) or y (type b).

In terms of these functions, we can write the generating function for the number of critical vertices encountered upon following an arbitrary edge of type a (that is, the size of the resulting avalanche if this edge is removed) as

$$H_a(u, v) = 1 - X_a - R_a + uF_a[H_a(u, v), H_b(u, v)] \tag{2.8}$$

and similarly for $H_b(u, v)$, the corresponding generating function for the size of the avalanche caused by removing a type b edge:

$$H_b(u, v) = 1 - X_b - R_b + vF_b[H_a(u, v), H_b(u, v)]. \tag{2.9}$$

These recursive equations can be understood by noting that $H_a(0, v) = 1 - X_a - R_a$ is the probability that an arbitrarily chosen edge leads to a vertex outside the viable cluster. Here u and v are auxiliary variables. Following through a critical cluster, a factor u appears for each arrowed edge of type a, and v for each arrowed edge of type b. For example, the critical cluster illustrated in Fig. 2.7 contributes a factor u^2v^2. The mean number of critical vertices reached upon following an edge of type a, i.e. the mean size of the resulting avalanche if this edge is removed, is given by $\partial_u H_a(1,1) + \partial_v H_a(1,1)$, where ∂_u signifies the partial derivative with respect to u.

Unbounded avalanches emerge at the point where $\partial_u H_a(1,1)$ [or $\partial_v H_b(1,1)$] diverges. Taking derivatives of Eq. (2.8),

$$\partial_u H_a(u,v) = F_a[H_a, H_b] + u\left\{\partial_u H_a \partial_x F_a[H_a, H_b] + \partial_u H_b \partial_y F_a[H_a, H_b]\right\} \tag{2.10}$$

$$\partial_v H_a(u,v) = u\left\{\partial_v H_a \partial_x F_a[H_a, H_b] + \partial_v H_b \partial_y F_a[H_a, H_b]\right\} \tag{2.11}$$

with similar equations for $\partial_u H_b(u,v)$ and $\partial_v H_b(u,v)$. Some rearranging gives

$$\partial_u H_a(1,1) = \frac{R_a + \partial_u H_b(1,1)\partial_y F_a(1-X_a, 1-X_b)}{1 - \partial_x F_a(1-X_a, 1-X_b)} \tag{2.12}$$

and

$$\partial_v H_a(1,1) = \frac{\partial_u H_a(1,1)\partial_x F_b(1-X_a, 1-X_b)}{1 - \partial_y F_b(1-X_a, 1-X_b)} \tag{2.13}$$

where we have used that $H_a(1,1) = 1 - X_a$ and $F_a(1-X_a, 1-X_b) = R_a$.

From Eqs. (2.1) and (2.7),

$$\partial_x F_a(1-X_a, 1-X_b) = \frac{\partial}{\partial X_a}\Psi_a(X_a, X_b) \tag{2.14}$$

$$\partial_y F_1(1-X_a, 1-X_b) = \frac{\langle q_a\rangle}{\langle q_b\rangle}\frac{\partial}{\partial X_a}\Psi_b(X_a, X_b), \tag{2.15}$$

and similarly for $\partial_x F_{1b}$ and $\partial_y F_{1b}$, which when substituted into (2.12) and (2.13) gives

$$\partial_u H_a(1,1) = \frac{R_a[1 - \frac{\partial}{\partial X_b}\Psi_b(X_a, X_b)]}{\det[\mathbf{J} - \mathbf{I}]}. \tag{2.16}$$

We see that the denominator exactly matches the left-hand side of Eq. (2.3), meaning that the mean size of avalanches triggered by random removal of vertices diverges exactly at the point of the hybrid transition.

The mean size of the avalanche triggered by the removal of a randomly chosen vertex can be related to the susceptibility of the giant viable cluster to random damage, similar to the susceptibility for ordinary percolation. In the latter case, the susceptibility is defined as the mean size of the cluster to which a randomly chosen vertex belongs [22]. Due to the similarity of Eq. (2.4) to the k-core version [23], we can expect that, at the critical point $p = p_c$, the size distribution of avalanches triggered by randomly removed vertices obeys a power law $p(s) \propto s^{-\sigma}$ with $\sigma = 3/2$.

2.5 Avalanches in Scale-Free Networks

In ordinary and k-core percolation, networks with degree distributions that are asymptotically power laws $P(q) \sim q^{-\gamma}$ may exhibit qualitatively different transitions from those described above, especially when $\gamma < 3$. To investigate such effects in the giant viable cluster, we consider two uncorrelated scale-free networks, so $P(q_a, q_b) = P_a(q_a)P_b(q_b)$, having powerlaw degree distributions with fixed minimum degree $q_0 = 1$ (then $\langle q\rangle \approx (\gamma - 1)q_0/(\gamma - 2)$), so that

$$P_s(q_s) = \zeta(\gamma_s)q^{-\gamma_s} \tag{2.17}$$

where s takes the values a or b, and $\zeta(\gamma)$ is the Riemann zeta function. As before, we apply random damage to the system as a whole as a control parameter, so that vertices survive with probability p.

First consider the case that at least one of the degree distribution exponents γ_a and γ_b is greater than three. The giant viable cluster is necessarily a subgraph of the overlap between the giant-components of each graph. We know from ordinary percolation that for $\gamma > 3$, the giant component appears at a finite value of p [24]. It follows that the giant viable cluster, also, cannot appear from $p = 0$; there must be a finite threshold p_c, (with a hybrid transition). This is true even if one of the networks has $\gamma_s < 3$.

The more interesting case is when $\gamma_a, \gamma_b < 3$, when the percolation threshold is zero for each network when considered separately. Let us write $\gamma_a = 2 + \delta_a$ and $\gamma_b = 2 + \delta_b$, and examine the behavior for small δ_a and δ_b. We proceed by assuming that in this situation, for p near p_c, Eq. (2.1) have a solution with small $X_a, X_b \ll 1$. Writing only leading orders of X_a and X_b, and δ_a and δ_b, we find that

$$\Psi_a(X_a, X_b) = p\frac{\pi^2}{6\,\delta_b}X_a^{\delta_a}\left(X_b - X_b^{1+\delta_b}\right) \tag{2.18}$$

and similarly for $\Psi_b(X_a, X_b)$. The location of the critical point is found from Eq. (2.3) which becomes

$$\delta_a + \delta_b = p\frac{\pi^2}{6}X_a^{\delta_a}X_b^{\delta_b}\left(\frac{X_a}{X_b} + \frac{X_b}{X_a}\right). \tag{2.19}$$

Substituting Eq. (2.18) into (2.1) and solving with Eq. (2.19), we find X_s and S at p_c. We find in general that the hybrid transition persists for $\delta_a, \delta_b \neq 0$, that is $p_c > 0$, but that the height of the discontinuity $X_s^{(c)}$ at the hybrid transition becomes extremely small for small δ small. In experiments or simulations, this could be misinterpreted as evidence of a continuous phase transition.

To illustrate the results in this case, we describe two representative examples. First, we fix δ_b at some small value, and examine the limit $\delta_a \to 0$, so that $\delta_a \ll \delta_b$. That is, $\gamma_a \to 2$ while $\gamma_b > 2$. We find that the location, p_c, of the transition tends to a finite value as $\delta_a \to 0$, proportional to the larger δ_b,

$$p_c = \sqrt{\frac{1-2f}{f(1-f)^2}\frac{\delta_b}{\zeta(2)}} \approx 1.19\delta_b, \tag{2.20}$$

where $f \approx 0.236$. The values of X_a and X_b become very small at the critical point, $X_b = f^{1/\delta_b}$ and $X_a \approx 1.5X_b$, meaning the size of the giant viable cluster at the critical point is exponentially small

$$S_c = \left(\frac{1-2f}{f}\right)^{3/2} f^{2/\delta_b} = Ae^{-B/\delta_b} \tag{2.21}$$

where $A \approx 3.36$ and $B \approx 2.89$. We see that a hybrid transition occurs, albeit with an extremely small discontinuity, at a non-zero threshold p_c as long as at least one of δ_a and δ_b is not equal to zero.

To examine the case that both δ_a and δ_b tend to zero, we consider the symmetric case $\delta_a = \delta_b \equiv \delta$. Then $X_a = X_b \equiv X$.

Equation (2.1) become a single equation,

$$\Psi(X) \approx p\frac{\zeta(2)}{\delta}\left(X^{1+\delta} - X^{1+2\delta}\right). \tag{2.22}$$

The discontinuity is found by requiring $\Psi'(X) = 1$ [from Eq. (2.3)] which condition becomes

$$\Psi'(X) \approx p\,\zeta(2)\left[(1+\delta)X^{\delta} - (1+2\delta)X^{2\delta}\right] = 1. \tag{2.23}$$

Solving these two equations, we find that $X_c = (1/2)^{1/\delta}$ and

$$p_c = \frac{24}{\pi^2}\delta \tag{2.24}$$

$$S_c = 4\left(\frac{1}{2}\right)^{2/\delta}. \tag{2.25}$$

The location of the hybrid transition tends to $p = 0$ as $\delta \to 0$, and the size of the 'jump' becomes very small even for nonzero δ, but vanishes completely as $\delta \to 0$. In Fig. 2.11 we plot the size of the giant viable cluster in this symmetric case for three values of γ. For values not close to two, the transition looks similar to that observed in, say, Erdős–Rényi graphs. As γ approaches 2, however, we see that the height

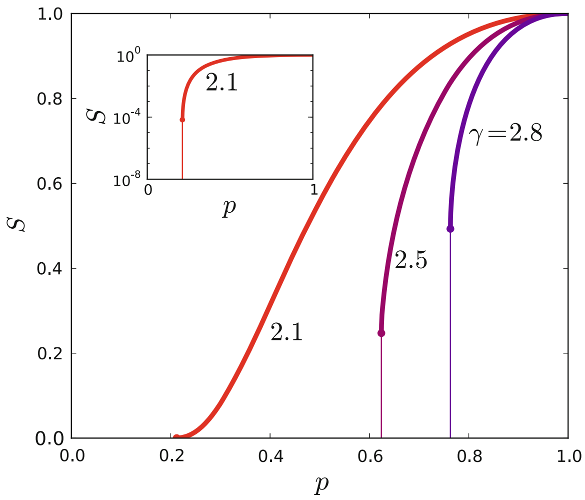

Fig. 2.11 Size of the giant viable cluster S as a function of the fraction p of vertices remaining undamaged for two symmetric powerlaw distributed networks with, from *right* to *left*, $\gamma = 2.8$, 2.5, and 2.1. The height of the jump becomes very small as γ approaches 2, but is not zero, as seen in the *inset*, which is S versus p on a logarithmic vertical scale for $\gamma = 2.1$

of the discontinuity becomes extremely small. Nevertheless, the square-root scaling and non-zero critical point are retained.

We can also examine the behaviour of X and S above the transition ($p > p_c$). Expanding $\Psi(X)$ about X_c we find that

$$\frac{X - X_c}{X_c} = \frac{12}{\pi^2 \, \delta \, p_c} \left(\frac{p - p_c}{p_c} \right)^{1/2} \tag{2.26}$$

which holds so long as $p - p_c \ll \delta^3$. That is, the scaling of the order parameter X, and hence the size of the giant viable cluster, S, is square-root in a narrow region of width $\mathscr{O}(\delta^3)$ above the hybrid transition. This region disappears as $\delta \to 0$.

2.6 Conclusions

In conclusion, we have studied the robustness of multiplex networks, which are networks with two or more different kinds of edges. There is a direct mapping between such multiplex networks and interdependent networks, in which vertices in one network depend on at most one vertex in another network. We found that the giant viable cluster of a multiplex network with two or more kinds of edges collapses with a discontinuous hybrid transition. The collapse occurs through avalanches which diverge in size when the transition is approached from above. We described critical clusters associated with these avalanches. The avalanches are responsible for both the critical scaling and the discontinuity observed in the size of the giant viable cluster. Remarkably, these specific clusters and avalanches in our problem turned out to be organized in a novel way, different from those in the k-core [15, 23] and bootstrap percolation [14] problems.

In contrast to ordinary networks, where two vertices are connected if there is a path between them, in multiplex network with m types of edges, two vertices are m-connected if for every kind of edge there is a path from one to another vertex. Based on this notion, we introduced *viable clusters* as clusters of m-connected vertices in multiplex network. This new notion of connectivity between vertices leads to the emergence in a multiplex network of a giant viable cluster in a hybrid phase transition in contrast to a continuous phase transition in ordinary percolation.

Surprisingly, when the degree distributions are asymptotically power-law $P(q) \propto q^{-\gamma}$ the critical point p_c (taking the undamaged fraction of vertices p as the control parameter) remains at a finite value even when the exponents γ of the degree distributions are below three, remaining finite until both exponents reach two, in agreement with an argument given in [10]. This is in stark contrast to ordinary percolation in complex networks, in which the threshold falls to zero as soon as γ reaches three [25, 26]. We show, further, that the nature of the transition does not change. Although the height of the discontinuity becomes extremely small near $\gamma = 2$, it remains finite near this limit (see Fig. 2.11). The critical clusters may have important practical applications, helping to identify vulnerabilities to targeted attack, as well as informing efforts to guard against such attack.

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