

Chapter 2

Deformation

The velocity and acceleration vectors derived in the preceding chapter are important kinematic fields, but, in and of themselves, they are not capable of describing how a body \mathcal{B} deforms; they only describe how any particle \mathcal{P} within \mathcal{B} moves through ambient space \mathbb{R}^3 . In order to study *deformation*, one needs to quantify the change in shape of a body \mathcal{B} as it is transformed from some initial configuration Ω_0 into its final configuration Ω over some interval $[t_0, t]$ in time, which is the topic of this chapter.

Motion is described by a position vector, but not deformation. However, the difference between two position vectors associated with a pair of neighboring particles is such a measure. Through the relative motions of such differences, deformations can be quantified. Consider two particles \mathcal{P} and \mathcal{P}' that are neighbors to one another in body \mathcal{B} . Let the incremental *displacement vector* connecting particle \mathcal{P} to particle \mathcal{P}' in configuration Ω_0 be denoted by $d\mathbf{X} = \mathbf{X}' - \mathbf{X}$ and let the displacement vector that connects these same two particles in the current configuration Ω be denoted by $d\mathbf{x} = \mathbf{x}' - \mathbf{x}$, as shown in Fig. 2.1. Vectors $d\mathbf{X}$ and $d\mathbf{x}$ point from material particle \mathcal{P} to material particle \mathcal{P}' in configurations Ω_0 and Ω , respectively.

Applying the chain rule to the *law of continuous media* given on p. 6 produces a linear transformation or mapping between $d\mathbf{X} = dX^I \mathbf{e}_I$ in Ω_0 and $d\mathbf{x} = dx^i \mathbf{e}_i$ in Ω that is expressed as

$$dx^i = \frac{\partial \xi^i}{\partial X^I} dX^I \quad \text{or} \quad d\mathbf{x} = F_I^i dX^I \quad \text{with} \quad F_I^i = \frac{\partial \xi^i}{\partial X^I} \quad (2.1)$$

which, in accordance with Appendix B, maps the Lagrangian vector $d\mathbf{X}$ into its Eulerian vector $d\mathbf{x}$; specifically, it obeys the field-transfer operator

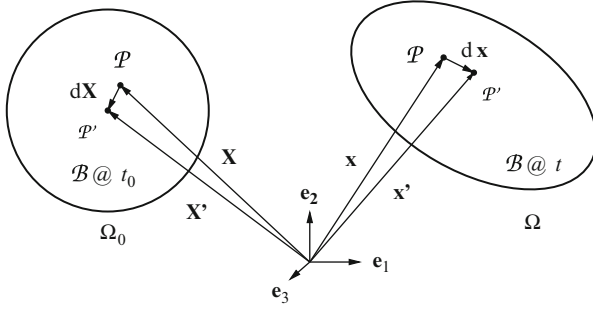


Fig. 2.1 Body \mathcal{B} is deformed from its original shape at time t_0 , i.e., from configuration Ω_0 , into a final deformed shape at time t , viz., into configuration Ω . Neighboring particles \mathcal{P} and \mathcal{P}' were at positions \mathbb{X} and \mathbb{X}' in Ω_0 and are now located at x and x' in Ω . The incremental vector $d\mathbb{X} = \mathbb{X}' - \mathbb{X}$ maps into vector $dx = x' - x$ according to Eq. (2.1), whose return mapping maps dx back into $d\mathbb{X}$ via Eq. (2.3). All fields are quantified against a common Cartesian basis (e_1, e_2, e_3) in \mathbb{R}^3

of a contravariant vector field described in Eq. (B.9) so that in matrix notation

$$\{dx(x, t)\} = [\mathbf{F}(\mathbb{X}, t)] \{d\mathbb{X}(\mathbb{X})\} \quad \text{with} \quad \mathbf{F}(\mathbb{X}, t) = \frac{\partial \xi(\mathbb{X}, t)}{\partial \mathbb{X}} \quad (2.2)$$

where tensor $\mathbf{F}(\mathbb{X}, t)$ is the *deformation gradient*. Tensor $\mathbf{F} = F_I^i e_i \otimes e^I$ is a two-state field with row index i in F_I^i associating with configuration Ω and column index I in F_I^i associating with configuration Ω_0 . Tensor \mathbf{F} is the fundamental field describing deformation. \mathbf{F} is normalized so that $\mathbf{F}(\mathbb{X}, t_0) = \mathbf{I} = \delta_I^i e_i \otimes e^I$ where \mathbf{I} is the *identity tensor* with components δ_I^i that are the Kronecker delta, which equal 1 whenever $i = I$ and are 0 otherwise. Here e_i and e_I are the contravariant base vectors in Ω and Ω_0 , while e^i and e^I are the covariant base vectors in Ω and Ω_0 . In Cartesian tensor analysis, $e_i \equiv e_I \equiv e^i \equiv e^I$. This is not the case in general tensor analysis [cf. Sokolnikoff (1964)]. The matrix representation $[\mathbf{F}]$ of tensor \mathbf{F} contains nine nonsymmetric components.

In like manner, a deformation gradient exists for the inverse motion. Applying the chain rule to the *law of continuous media* also allows one to write

$$d\mathbb{X}^I = \frac{\partial \Xi^I}{\partial x^i} dx^i \quad \text{or} \quad d\mathbb{X}^I = f_I^i dx^i \quad \text{with} \quad f_I^i = \frac{\partial \Xi^I}{\partial x^i} \quad (2.3)$$

which, in accordance with Appendix B, maps the Eulerian vector dx into its Lagrangian vector $d\mathbb{X}$; specifically, it obeys the field-transfer operator

of a contravariant vector field described in Eq. (B.10) so that in matrix notation

$$\{d\mathbb{X}(\mathbb{X})\} = [\mathbf{f}(x, t)] \{dx(x, t)\} \quad \text{with} \quad \mathbf{f}(x, t) = \frac{\partial \underline{\mathbb{X}}(x, t)}{\partial x}. \quad (2.4)$$

Like $\mathbf{F} = F_I^i e_i \otimes e^I$, $\mathbf{f} = f_i^I e_I \otimes e^i$ is a two-state tensor field.

Tensor \mathbf{F} maps the tangent vector $d\mathbb{X}$ from Ω_0 into its associated tangent vector dx in Ω . Tensor \mathbf{f} maps the tangent vector dx from Ω into its associated tangent vector $d\mathbb{X}$ in Ω_0 , which is the reverse of mapping \mathbf{F} .

2.1 Homogeneous Deformation

From the definition of a homogeneous motion given in Eq. (1.2), it follows that its spatial gradient is

$$\mathbf{F} = \frac{\partial x}{\partial \mathbb{X}} = \frac{\partial \underline{\mathbb{X}}(\mathbb{X}, t)}{\partial \mathbb{X}} = \Phi(t) \frac{\partial \mathbb{X}}{\partial \mathbb{X}} = \Phi(t) \mathbf{I} = \Phi(t) \quad (2.5)$$

and, therefore, any distortion that contributes to a homogeneous motion, i.e., Φ , equates with a deformation gradient \mathbf{F} that depends only upon time t .

A deformation is said to be *inhomogeneous* whenever its deformation gradient \mathbf{F} depends upon position \mathbb{X} as well as upon time t , e.g., the 2D quad element whose deformation gradient is contained within Eq. (1.31).

2.2 Conservation of Mass

From the conservation of mass—a law of physics—comes the governing field equation (Holzapfel 2000, pp. 131–135)

$$\frac{D\varrho}{Dt} + \text{div}(v)\varrho = 0 \quad (2.6)$$

where ϱ is the mass density and $\text{div}(v) = \partial v^i / \partial x^i$ is the divergence of the velocity vector, which is scalar valued. The solution to this differential equation can be written in the form of

$$\det \mathbf{F} = \frac{dv}{dV} = \frac{\varrho_0}{\varrho} > 0 \quad (2.7)$$

where dv/dV ratios the volume of an infinitesimal material element (whose centroid is located at particle \mathcal{P}) between its current value dv in Ω to its

reference value dV in Ω_0 , while the reciprocal ratio ϱ_0/ϱ is a quotient of that volume element's respective mass densities.

A direct consequence of the conservation of mass is that the deformation gradient cannot be singular, i.e., $\det \mathbf{F} \neq 0$, consequently, its inverse must exist

$$\mathbf{F}^{-1} = \left(\frac{\partial \xi(\mathbb{X}, t)}{\partial \mathbb{X}} \right)^{-1} = \frac{\partial \mathbb{X}}{\partial \xi(\mathbb{X}, t)} = \frac{\partial \Xi(x, t)}{\partial x} = \mathbf{f}(x, t) \quad (2.8)$$

whose components are

$$[\mathbf{F}^{-1}]_i^I = \frac{\partial \Xi^I}{\partial x^i} = f_i^I. \quad (2.9)$$

So one observes that \mathbf{F} has components F_I^i , while its inverse \mathbf{F}^{-1} has components $[\mathbf{F}^{-1}]_i^I$. The notation $[\mathbf{F}^{-1}]_i^I$ does **not** mean $1/F_I^i$. The superscript (or row) index of \mathbf{F}^{-1} associates with the material frame, while the subscript (or column) index belongs to the spatial frame, which is the opposite index pairing present in \mathbf{F} .

Regarding deformation gradients, the superscript index associates with the argument in the numerator (the dependent variable of the motion map), while the subscript index associates with the argument in the denominator (the independent variable of the motion map). Whenever the independent variable is \mathbb{X} (i.e., Lagrangian), then F_I^i are the components of \mathbf{F} ; likewise, whenever the independent variable is x (viz., Eulerian), then $[\mathbf{F}^{-1}]_i^I$ are the components of \mathbf{F}^{-1} .

In what follows, \mathbf{F}^{-1} will be used in place of \mathbf{f} , as they are two expressions of the same mapping. When describing fluids, \mathbf{f} is preferred over \mathbf{F} because one places oneself in Ω . When characterizing solids, which is the focus of this text, \mathbf{F} is preferred over \mathbf{f} because one places oneself in Ω_0 .

2.2.1 Isochoric Deformation

A deformation is said to be *isochoric* if it preserves its volume, which is also referred to in the thermodynamics literature as an isometric or isovolumetric process. Consequently, imposing a constraint of $\det \mathbf{F} = 1$, which implies that $dv = dV$ from Eq. (2.7), ensures that the corresponding deformation is isochoric.

2.3 Deformation Gradient as a Mapping Function

Tensor \mathbf{F} is a two-state field, i.e., it has one index in the material configuration Ω_0 , while the other index resides in the spatial configuration Ω . This is because the deformation gradient \mathbf{F} is a *transformation mapping*. Matrix $[\mathbf{F}]$ maps a tangent vector $\{d\mathbb{X}\}$ from the reference configuration Ω_0 into $\{d\mathbf{x}\}$, which is this tangent vector's representation in the current configuration Ω , according to Eq. (2.2). Physically, these are the same tangent vectors, but, mathematically, they are distinct.

Similarly, matrix $[\mathbf{F}^{-1}]^T$ maps the normal vector $\{N\} = \{d\mathcal{S}(\mathbb{X})/d\mathbb{X}\}$ to some material surface $\mathcal{S}(\mathbb{X})$ from a reference configuration Ω_0 into a normal vector $\{n\} = \{d\mathcal{S}(\underline{\mathbb{X}}(x, t))/d\mathbf{x}\}$ that resides within the current configuration Ω . Applying the chain rule to $\mathcal{S}(\underline{\mathbb{X}}(x, t))$, incorporating the *law of continuous media*, allows one to write

$$\frac{d\mathcal{S}}{d\mathbf{x}^i} = \frac{d\mathcal{S}}{d\mathbb{X}^I} \frac{\partial \mathbb{X}^I}{\partial x^i} \quad \text{or} \quad n_i = [\mathbf{F}^{-1}]_i^I N_I. \quad (2.10)$$

which, in accordance with Appendix B, maps the Lagrangian vector $d\mathcal{S}/d\mathbb{X}$ into its Eulerian vector $d\mathcal{S}/d\mathbf{x}$; specifically, it obeys the field-transfer operator of a covariant vector field described in Eq. (B.13) in that

$$\left\{ \frac{d\mathcal{S}(\underline{\mathbb{X}}(x, t))}{d\mathbf{x}} \right\} = \left\{ \frac{d\mathcal{S}(\mathbb{X})}{d\mathbb{X}} \right\} \left[\frac{d\underline{\mathbb{X}}(x, t)}{d\mathbf{x}} \right] = \left\{ \frac{d\mathcal{S}(\mathbb{X})}{d\mathbb{X}} \right\} [\mathbf{F}^{-1}] \quad (2.11)$$

or, equivalently,

$$\{n\} = \{N\} [\mathbf{F}^{-1}] \quad \text{so} \quad \{n(x, t)\} = [\mathbf{F}^{-1}(\mathbb{X}, t)]^T \{N(\mathbb{X})\}. \quad (2.12)$$

The dual mapping matrices of $[\mathbf{F}]$ and $[\mathbf{F}^{-1}]^T$ are the kinematic cornerstones of continuum mechanics; cf. Appendix B. The notation for the deformation gradient \mathbf{F} is made special, viz., it is typeset in an upright font instead of a slanted font like other tensor fields, precisely because the field-transfer operators $[\mathbf{F}]$ and $[\mathbf{F}^{-1}]^T$ map vector and tensor fields between the two configurations of a deformation marked by the end points of a motion over some interval in time, say $[t_0, t]$.

Vector fields that are pushed forward from Ω_0 into Ω via $[\mathbf{F}]$, in accordance with Eq. (2.2), can be pulled back from Ω into Ω_0 with the reverse mapping $[\mathbf{F}^{-1}]$. Such vectors are called *contravariant vector* fields. Likewise, vector fields that are pushed forward from Ω_0 into Ω via $[\mathbf{F}^{-1}]^T$, in accordance with Eq. (2.12), can be pulled back from Ω into Ω_0 with the reverse mapping $[\mathbf{F}]^T$. Such vectors are called *covariant vector* fields (Lodge

1974; Marsden and Hughes 1983; Sokolnikoff 1964). These mappings are discussed in more detail in Appendix B, wherein Figs. B.2 and B.3 are useful illustrations showing how these mappings apply to the transfer of field from one configuration into another.

Coming to an understanding of the concepts that are outlined in Appendix B, viz., configuration physics, is essential before you can come to a physical understanding of the mechanics that materials incur during finite deformations. Please study Appendix B before advancing.

2.4 Stretch and Rotation

At any particle \mathcal{P} in body \mathcal{B} , the deformation gradient \mathbf{F} that connects a reference configuration Ω_0 with its current configuration Ω will have a unique *polar decomposition*, as illustrated in Fig. 2.2, that is given by (Noll 1958)

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R} \quad (2.13)$$

where tensors $\mathbf{U}(\mathbb{X}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ are called the *right* and *left stretch* tensors, so named because they reside at the right and the left of the *rotation* tensor \mathbf{R} . Mathematically, tensors \mathbf{U} and \mathbf{v} are positive definite with matrix representations that are always symmetric, while tensor \mathbf{R} is orthogonal, all of which are consequences of the polar decomposition theorem from linear algebra. Consequently, $[\mathbf{U}] = [\mathbf{U}]^T$, $[\mathbf{v}] = [\mathbf{v}]^T$, and $\mathbf{R}^T\mathbf{R} \equiv \mathbf{R}\mathbf{R}^T = \mathbf{I}$ implying that $[\mathbf{R}]^T = [\mathbf{R}^{-1}]$ that, because $\det \mathbf{F} > 0$, requires $\det \mathbf{R} = 1$ since stretches \mathbf{U} and \mathbf{v} are positive definite, i.e., rotation \mathbf{R} is a proper orthogonal tensor. In component form, this decomposition is written as

$$F_I^i = R_I^j U_I^j = v_j^i R_I^j \quad (2.14)$$

where $\mathbf{R} = R_I^j \mathbf{e}_i \otimes \mathbf{e}^I$, $\mathbf{U} = U_I^j \mathbf{e}_I \otimes \mathbf{e}^J$, and $\mathbf{v} = v_j^i \mathbf{e}_i \otimes \mathbf{e}^j$ where in each the superscript is the row indexer and the subscript is the column indexer. Furthermore, $\mathbf{R}^{-1} = [R^{-1}]_i^I \mathbf{e}_I \otimes \mathbf{e}^i$ and, in accordance with Eq. (A.26),

$$\mathbf{R}^T = \delta_{ij} R_J^j \delta^{II} \mathbf{e}_I \otimes \mathbf{e}^i \quad (2.15)$$

so that $[R^{-1}]_i^I = \delta_{ij} R_J^j \delta^{II}$. The transpose of a mixed tensor field, like \mathbf{R}^T , is a bit tricky to work with in component notation, requiring extra care in its handling [cf. Appendix A and Marsden and Hughes (1983, pp. 48–49, 137)].

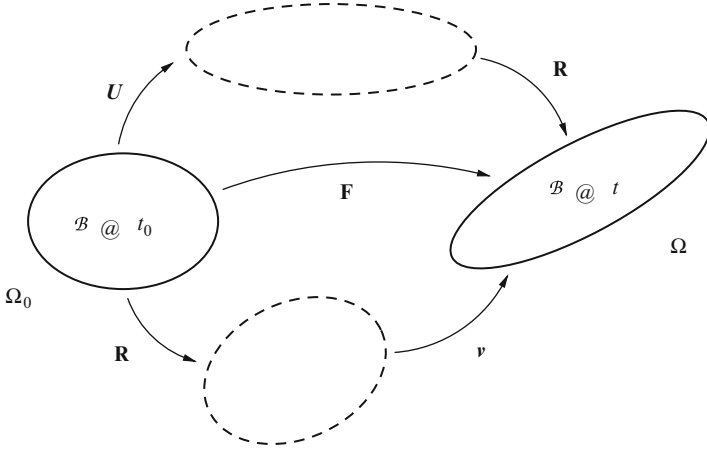


Fig. 2.2 Body \mathcal{B} is deformed from its original shape at time t_0 , i.e., from configuration Ω_0 , into a final deformed shape at time t , viz., into configuration Ω . Considering the field transfer of a contravariant vector illustrated in Fig. B.2, the deformation gradient \mathbf{F} can be decomposed in one of two ways: $\mathbf{F} = \mathbf{R}\mathbf{U}$ applies a Lagrangian stretch of \mathbf{U} from Ω_0 that is followed by a rotation of \mathbf{R} into Ω , whereas $\mathbf{F} = \mathbf{v}\mathbf{R}$ rotates out of Ω_0 via \mathbf{R} after which an Eulerian stretch of \mathbf{v} places \mathcal{B} into Ω . The intermediate configurations (drawn as *dashed ellipses*) are not physically realizable unless the deformation is uniform

Contracting Eq. (2.13) with \mathbf{R}^{-1} from the right gives

$$\mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{R}^{-1} \quad \text{or} \quad v_j^i = R_I^i U_J^I [R^{-1}]_j^J, \quad (2.16)$$

while contracting Eq. (2.13) with \mathbf{R}^{-1} from the left produces

$$\mathbf{U} = \mathbf{R}^{-1}\mathbf{v}\mathbf{R} \quad \text{or} \quad U_J^I = [R^{-1}]_i^I v_j^i R_j^J. \quad (2.17)$$

An absence of rotation implies that $\mathbf{R}^{-1} \equiv \mathbf{R} = \mathbf{I}$. An absence of stretch implies that $\mathbf{U} \equiv \mathbf{v} = \mathbf{I}$. Consequently, an absence of both rotation and stretch implies that $\mathbf{F}^{-1} \equiv \mathbf{F} = \mathbf{I}$.

What Eq. (2.13) implies is that any deformation \mathbf{F} from Ω_0 to Ω can be decomposed into two different mapping sequences whose intermediate configurations are, in general, nonphysical, viz., they are not realizable. In the case of $\mathbf{F} = \mathbf{v}\mathbf{R}$, a rotation takes place first, followed by a stretch of \mathbf{v} . In the other case of $\mathbf{F} = \mathbf{R}\mathbf{U}$, a stretch of \mathbf{U} is imposed first, with a rotation following. Both sequence maps begin in the reference configuration Ω_0 , and both eventually end up in the spatial configuration Ω ; they just traverse different paths to get there, as illustrated in Fig. 2.2. This is a consequence of the fact that matrix multiplication does not commute. The

polar decomposition of the deformation gradient plays an important role in many theories describing the mechanics of materials (Holzapfel 2000; Ogden 1984; Simo and Hughes 1998), but they are not directly employed in the constitutive theories put forward in this text.

Stretch is a unique descriptor of deformation. After one selects a frame of reference, viz., material or spatial, unique components can be assigned to stretch. Consequently, *great value exists in measuring and reporting stretches, not strains, when documenting experiments done on soft solids.* Given components for one strain measure, it is not always possible to arrive at components for another strain measure without also knowing the components of stretch.

Contraction $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ has *covariant* components $R_I^i \delta_{ij} R_J^j = \delta_{IJ}$. Contraction $\mathbf{R} \mathbf{R}^T = \mathbf{I}$ has *contravariant* components $R_I^i \delta^{IJ} R_J^j = \delta^{ij}$. Kronecker deltas δ_{ij} and δ^{IJ} are the covariant and contravariant metrics of Cartesian space in the Eulerian and Lagrangian frames, respectively. An application of Eq. (A.42) to $\mathbf{R}^T \mathbf{R}$ specifies components $\delta^{IK} R_K^i \delta_{ij} R_J^j$ with the first Kronecker delta being able to contract out, thereby proving the stated result $R_I^i \delta_{ij} R_J^j$. Likewise, an application of Eq. (A.44) to $\mathbf{R} \mathbf{R}^T$ specifies components $R_I^i \delta^{IJ} R_J^k \delta_{kj}$ with the second Kronecker delta being able to contract out, thereby proving the stated result $R_I^i \delta^{IJ} R_J^j$.

2.5 Rate Fields

Taking the material derivative of Eq. (2.2), viz., taking the time derivative of the transfer of field $\{\mathbf{dx}\} = [\mathbf{F}] \{\mathbf{dX}\}$ at a fixed particle \mathcal{P} and, therefore, at fixed material coordinates \mathbf{X} so that $\dot{\mathbf{dX}} = \mathbf{0}$, one determines from the product rule that

$$\{\dot{\mathbf{dX}}\} = [\dot{\mathbf{F}}] \{\mathbf{dX}\} = [\dot{\mathbf{F}}] [\mathbf{F}^{-1}] [\mathbf{dx}] = [\mathbf{l}] \{\mathbf{dx}\} \quad (2.18)$$

where $\{\mathbf{dX}\} = [\mathbf{F}^{-1}] \{\mathbf{dx}\}$ follows from Eqs. (2.4) and (2.8). This expression defines the *Eulerian velocity gradient tensor*¹

$$\mathbf{l}(\mathbf{x}, t) = \dot{\mathbf{F}} \mathbf{F}^{-1} = \frac{\partial \dot{\underline{\underline{\mathbf{X}}}}(\mathbf{X}, t)}{\partial \mathbf{X}} \frac{\partial \underline{\underline{\mathbf{X}}}(\mathbf{x}, t)}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \text{grad}(\mathbf{v}) \quad (2.19)$$

¹The Eulerian velocity gradient $\mathbf{l} = \dot{\mathbf{F}} \mathbf{F}^{-1}$ is typically denoted as \mathbf{L} in the literature. That notation is reserved here for the Lagrangian velocity gradient defined in Eq. (2.24), which is a field not found in the literature to the best of the author's knowledge.

where

$$\boldsymbol{\ell} = \dot{\mathbf{F}}\mathbf{F}^{-1} \quad \text{has spatial components} \quad \ell_j^i = \dot{F}_I^i [F^{-1}]_j^I, \quad (2.20)$$

which plays the analogous role in rate-based theories that \mathbf{F} plays in deformation-based theories. Like the deformation gradient \mathbf{F} , the velocity gradient $\boldsymbol{\ell}$ is not symmetric. Unlike \mathbf{F} , the inverse of $\boldsymbol{\ell}$ does not exist.

It is useful to decompose the velocity gradient, not as a product as in the polar decomposition of \mathbf{F} , but rather as a sum, viz.,

$$\boldsymbol{\ell} = \mathbf{d} + \mathbf{w} \quad (2.21)$$

wherein

$$\mathbf{d} = \frac{1}{2}(\boldsymbol{\ell} + \boldsymbol{\ell}^T) \quad \text{has components} \quad d_j^i = \frac{1}{2}(\ell_j^i + \delta_{jk} \ell_\ell^k \delta^{\ell i}), \quad (2.22)$$

$$\mathbf{w} = \frac{1}{2}(\boldsymbol{\ell} - \boldsymbol{\ell}^T) \quad \text{has components} \quad w_j^i = \frac{1}{2}(\ell_j^i - \delta_{jk} \ell_\ell^k \delta^{\ell i}) \quad (2.23)$$

where $\mathbf{d}(\mathbf{x}, t)$ is the symmetric *stretching* tensor (also referred to in the literature as the rate-of-deformation tensor or the strain-rate tensor) whose components come from Eq. (A.27), while $\mathbf{w}(\mathbf{x}, t)$ is the skew-symmetric *vorticity* tensor (also referred to in the literature as the rate-of-rotation tensor or the spin tensor) whose components come from Eq. (A.29).

2.5.1 Lagrangian Velocity Gradient

It is useful to define what is, essentially, the *Lagrangian velocity gradient* tensor $\mathbf{L}(\mathbb{X}, t)$ via the formula

$$\mathbf{L} = \mathbf{F}^{-1} \dot{\mathbf{F}} \quad \text{with material components} \quad L_J^I = [F^{-1}]_I^I \dot{F}_J^I, \quad (2.24)$$

which, by an inspection of its uppercase indices, is seen to be a material field defined over the reference configuration Ω_0 . Recall that $\boldsymbol{\ell} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. It is a trivial matter to show that \mathbf{L} pushes forward into $\boldsymbol{\ell}$, and $\boldsymbol{\ell}$ pulls back into \mathbf{L} between configurations Ω_0 and Ω according to the paired mappings

$$\begin{aligned} [\boldsymbol{\ell}] &= [\mathbf{F}][\mathbf{L}][\mathbf{F}^{-1}] \quad \text{or} \quad \ell_j^i = F_I^i L_J^I F_J^J = \dot{F}_J^i [F^{-1}]_j^J, \\ [\mathbf{L}] &= [\mathbf{F}^{-1}][\boldsymbol{\ell}][\mathbf{F}] \quad \text{or} \quad L_J^I = F_I^I \ell_j^j F_J^j = [F^{-1}]_I^I \dot{F}_J^I, \end{aligned} \quad (2.25)$$

which are the field-transfer operators for a mixed tensor field defined in Eqs. (B.23)–(B.26).

As in the Eulerian frame, the velocity gradient of the Lagrangian frame can be additively decomposed into symmetric and skew-symmetric parts

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \quad (2.26)$$

wherein

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \quad \text{with} \quad D_J^I = \frac{1}{2}(\mathbf{L}_J^I + \delta_{JK} \mathbf{L}_L^K \delta^{LI}), \quad (2.27)$$

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) \quad \text{with} \quad W_J^I = \frac{1}{2}(\mathbf{L}_J^I - \delta_{JK} \mathbf{L}_L^K \delta^{LI}) \quad (2.28)$$

where \mathbf{D} and \mathbf{W} are the respective Lagrangian stretching and vorticity tensors. It follows that $[\boldsymbol{\ell}^T] = [\mathbf{F}][\mathbf{L}^T][\mathbf{F}^{-1}]$ and $[\mathbf{L}^T] = [\mathbf{F}^{-1}][\boldsymbol{\ell}^T][\mathbf{F}]$ and therefore

$$[\mathbf{d}] = [\mathbf{F}][\mathbf{D}][\mathbf{F}^{-1}] \quad \text{and} \quad [\mathbf{D}] = [\mathbf{F}^{-1}][\mathbf{d}][\mathbf{F}], \quad (2.29)$$

$$[\mathbf{w}] = [\mathbf{F}][\mathbf{W}][\mathbf{F}^{-1}] \quad \text{and} \quad [\mathbf{W}] = [\mathbf{F}^{-1}][\mathbf{w}][\mathbf{F}]. \quad (2.30)$$

2.5.2 Isochoric Deformations

Recall that an isochoric deformation requires that $\det \mathbf{F} = 1$. Applying Eq. (A.83) to \mathbf{F} leads to

$$\overline{\dot{\det \mathbf{F}}} = \det(\mathbf{F}) \operatorname{tr}(\dot{\mathbf{F}} \mathbf{F}^{-1}) = \det(\mathbf{F}) \operatorname{tr}(\boldsymbol{\ell}) \quad (2.31)$$

or, equivalently, from Eq. (A.84), one arrives at

$$\overline{\ln(\dot{\det \mathbf{F}})} = \operatorname{tr} \boldsymbol{\ell}. \quad (2.32)$$

If $\det \mathbf{F} = 1$ then $\overline{\dot{\det \mathbf{F}}} = 0$, thereby requiring that $\det(\mathbf{F}) \operatorname{tr}(\boldsymbol{\ell}) = 0$, but $\det \mathbf{F} = 1$ so $\operatorname{tr} \boldsymbol{\ell}$ must equal 0. Furthermore, from their definitions in Eqs. (2.19) and (2.24) and a property of the trace given in Eq. (A.70), it also follows that $\operatorname{tr} \boldsymbol{\ell} = \operatorname{tr}(\dot{\mathbf{F}} \mathbf{F}^{-1}) = \operatorname{tr}(\mathbf{F}^{-1} \dot{\mathbf{F}}) = \operatorname{tr} \mathbf{L}$.

Consequently, $\det \mathbf{F} = 1$ (or $\det \mathbf{F}^{-1} = 1$) and $\operatorname{tr} \mathbf{L} = 0$ (or $\operatorname{tr} \boldsymbol{\ell} = 0$) are equivalent constraints for imposing isochoric deformations. The former pair applies to formulations where the deformation gradients \mathbf{F} or \mathbf{F}^{-1} play the role of the independent variable for deformation, while the latter pair applies to formulations where the velocity gradients \mathbf{L} or $\boldsymbol{\ell}$ play the role of the independent variable for deformation. The first occurrence in each pairing applies for Lagrangian analysis, while the second occurrence in each pairing applies for Eulerian analysis.

2.6 Numerical Implementation

The polar decomposition theorem tells us that the rotation \mathbf{R} and stretch tensors \mathbf{U} and \mathbf{v} exist, but it does not tell us how to compute them. That is the topic of this section. Here we consider interfacing with, e.g., an updated

Lagrangian finite element code where the velocity gradient $\boldsymbol{\ell}$ is the independent kinematic variable [cf. Belytschko et al. (2000)]. The algorithms that follow employ a two-step Adams–Bashforth predictor followed by a trapezoidal corrector for numerical integration,² which is in keeping with the target application of an algorithm that resides within a finite element code.

2.6.1 Angular Velocity

The skew-symmetric *angular velocity* $\boldsymbol{\Omega} = \Omega_j^i \mathbf{e}_i \otimes \mathbf{e}^j$ is defined by

$$\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^{-1} \quad \text{with components} \quad \Omega_j^i = \dot{R}_I^i [R^{-1}]_j^I, \quad (2.33)$$

which has the same mathematical structure as $\boldsymbol{\ell} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. From the definitions for a polar decomposition of the deformation gradient and for the velocity gradient given in Eqs. (2.13) and (2.19) one arrives at a differential equation governing stretch, in particular

$$\dot{\mathbf{v}} = \boldsymbol{\ell} \mathbf{v} - \mathbf{v} \boldsymbol{\Omega} \quad (2.34)$$

whose skew-symmetric contribution describes the expression

$$\mathbf{v} \boldsymbol{\Omega} + \boldsymbol{\Omega} \mathbf{v} = \mathbf{v} \mathbf{w} + \mathbf{w} \mathbf{v} + \mathbf{z} \quad \text{with} \quad \mathbf{z} = d\mathbf{v} - \mathbf{v} d \quad (2.35)$$

where $\mathbf{w} = w_j^i \mathbf{e}_i \otimes \mathbf{e}^j$, $\mathbf{z} = z_j^i \mathbf{e}_i \otimes \mathbf{e}^j$, and $\boldsymbol{\Omega} = \Omega_j^i \mathbf{e}_i \otimes \mathbf{e}^j$ are each skew symmetric. Because these three tensors are skew, they can be expressed as vectors $\mathbf{w} = w^i \mathbf{e}_i$, $\mathbf{z} = z^i \mathbf{e}_i$, and $\boldsymbol{\omega} = \omega^i \mathbf{e}_i$ defined so as to obey maps

$$w^i = \epsilon^{ik}_j \omega_k^j, \quad z^i = \epsilon^{ik}_j z_k^j, \quad \text{and} \quad \omega^i = \epsilon^{ik}_j \Omega_k^j, \quad (2.36)$$

wherein ϵ^{ij}_k is the cyclic permutation operator with values: 1 if indices ijk are unique and cyclically ordered, i.e., 123, 231, or 312; −1 if they are unique with reverse cyclic ordering, viz., 132, 213, or 321; or 0 whenever two or three indices have the same indicial value. Note: $\epsilon^{ik}_j = -\epsilon^{ij}_k$. From here, Dienes (1979) derived the formula

$$\boldsymbol{\omega} = \mathbf{w} + (\text{tr}(\mathbf{v})\mathbf{I} - \mathbf{v})^{-1} \mathbf{z} \quad \text{wherein} \quad z^i = \epsilon^{ik}_j (d_\ell^j v_k^\ell - v_\ell^j d_k^\ell), \quad (2.37)$$

where $\boldsymbol{\omega}$ is the angular velocity vector.

These formulæ comprise Algorithm 2.1, which returns the angular velocity vector $\boldsymbol{\omega}$ given the velocity gradient $\boldsymbol{\ell}$ and stretch \mathbf{v} tensors in their

²One should not use forward Euler as one's numerical method for integrating ODEs, even though it was likely the method taught to you in your class on differential equations. This method has serious stability issues and has been the cause of numerous engineering failures, even disasters.

Algorithm 2.1 Vector of angular velocity ω

function ANGULARVELOCITY (ℓ, v)

$$d_j^i \leftarrow \frac{1}{2}(\ell_j^i + \delta_{jk} \ell_\ell^k \delta^{\ell i})$$

$$w_j^i \leftarrow \frac{1}{2}(\ell_j^i - \delta_{jk} \ell_\ell^k \delta^{\ell i})$$

$$w^i \leftarrow \epsilon_j^{ik} w_k^j$$

$$z^i \leftarrow \epsilon_j^{ik} (d_\ell^j v_k^\ell - v_\ell^j d_k^\ell)$$

$$\omega \leftarrow w + (\text{tr}(v)I - v)^{-1} z$$

return ω
end function

Eulerian form. The velocity gradient ℓ is the known, independent, kinematic variable in such formulations, while a predicted estimate for the stretch tensor v comes from Algorithm 2.3.

2.6.2 Rotation

The issue with applying a conventional numerical technique to integrate $\dot{\mathbf{R}} = \boldsymbol{\Omega} \mathbf{R}$ [from Eq. (2.33)] for updating the rotation tensor \mathbf{R} , without addressing the intrinsic character of \mathbf{R} , is an unavoidable degradation in the orthogonality of \mathbf{R} with increasing numbers of integration steps n taken, no matter how fine a step size Δt is used (Atluri and Cazzani 1995; Flanagan and Taylor 1987). The reason for this degradation is that the sum of two orthogonal tensors is not orthogonal. Algorithm 2.2 is free from this defect. It multiplies rotations, thereby maintaining orthogonality. The product of two orthogonal tensors is an orthogonal tensor, within numerical roundoff error.

Because the rotation tensor \mathbf{R} arising from a polar decomposition of the deformation \mathbf{F} is orthogonal, it can be described in an alternative form of

$$\mathbf{R} = e^{\theta \mathbf{Q}} \quad \text{so} \quad \mathbf{R} \mathbf{r} = \mathbf{r} \quad \text{with} \quad [\mathbf{Q}]_j^i = \epsilon_{kj}^i r^k, \quad \|\mathbf{r}\| = 1, \quad (2.38)$$

where \mathbf{Q} is a skew-symmetric tensor representation for the vector that is the axis \mathbf{r} about which rotation \mathbf{R} occurs, with \mathbf{r} being the only real-valued eigenvector of \mathbf{R} , and with θ corresponding to the angle of this rotation. From the eigenvector property of $\mathbf{R} \mathbf{r} = \mathbf{r}$, one determines that

$$\mathbf{R} = \mathbf{I} + \sin(\theta) \mathbf{Q} + (1 - \cos \theta) \mathbf{Q}^2, \quad (2.39)$$

while from the evolution of \mathbf{R} described by $\dot{\mathbf{R}} = \boldsymbol{\Omega} \mathbf{R}$ one arrives at

$$\omega = \dot{\theta} \mathbf{r} + \sin(\theta) \dot{\mathbf{r}} + (1 - \cos \theta) \mathbf{r} \times \dot{\mathbf{r}}, \quad (2.40)$$

which are Eqs. (2.16) and (8.25) of Atluri and Cazzani (1995), neither being simple to derive. Equation (2.39) is the matrix representation for a set of formulæ originally derived by Euler in 1775 (Cheng and Gupta 1989).

Following a suggestion by Dienes (2003), Eq. (2.40) can be rewritten as the matrix equation

$$\mathbf{A} \dot{\mathbf{r}} = \boldsymbol{\omega} - \dot{\theta} \mathbf{r} \quad \text{so that} \quad \dot{\mathbf{r}} = \mathbf{A}^{-1}(\boldsymbol{\omega} - \dot{\theta} \mathbf{r}) \quad (2.41)$$

with tensor \mathbf{A} having components

$$[\mathbf{A}] = \begin{bmatrix} \sin \theta & -(1 - \cos \theta) r_3 & (1 - \cos \theta) r_2 \\ (1 - \cos \theta) r_3 & \sin \theta & -(1 - \cos \theta) r_1 \\ -(1 - \cos \theta) r_2 & (1 - \cos \theta) r_1 & \sin \theta \end{bmatrix}. \quad (2.42)$$

By definition $\mathbf{r} \cdot \mathbf{r} = 1$ and, therefore, $2 \mathbf{r} \cdot \dot{\mathbf{r}} = 0$ that, from Eq. (2.41), requires $\mathbf{r} \cdot \mathbf{A}^{-1}(\boldsymbol{\omega} - \dot{\theta} \mathbf{r}) = 0$. But it can be shown that $\mathbf{r} \cdot \mathbf{A}^{-1} = \csc(\theta) \mathbf{r} \neq \emptyset$, which reflects the singularity of \mathbf{A} present at $\theta = 0$, while $\mathbf{r} \cdot \mathbf{A} = \sin(\theta) \mathbf{r}$. Consequently, it is sufficient to require $(\boldsymbol{\omega} - \dot{\theta} \mathbf{r}) = \emptyset$ to ensure $\mathbf{r} \cdot \mathbf{A}^{-1}(\boldsymbol{\omega} - \dot{\theta} \mathbf{r}) = 0$, thereby producing the anticipated result

$$\boldsymbol{\omega} = \dot{\theta} \mathbf{r} \quad \therefore \quad \dot{\theta} = \mathbf{r} \cdot \boldsymbol{\omega} = \|\boldsymbol{\omega}\| \quad \text{with} \quad \mathbf{r} = \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \quad (2.43)$$

which enables the rotation tensor \mathbf{R} to be quantified via Eq. (2.39). To the best of the author's knowledge, $\boldsymbol{\omega} = \dot{\theta} \mathbf{r}$ has been suggested in the literature, but not rigorously proven.

The axis of rotation \mathbf{r} cannot be oriented in an absence of rotation, viz., whenever $\dot{\theta} = \|\boldsymbol{\omega}\| = 0$. This condition associates with the point singularity of tensor \mathbf{A}^{-1} , with \mathbf{A} being defined in Eq. (2.42). This special case is readily handled, because $\mathbf{R} = \mathbf{I}$ in the absence of rotation. Parameter `tol` in Algorithm 2.2 handles this condition, where `tol` is set to a small positive number, e.g., machine precision ϵ_m .

The formulæ of this section present a strategy whereby the rotation tensor \mathbf{R} of Eq. (2.39), which is the rotation tensor within $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}$, can be acquired, given the angular velocity vector $\boldsymbol{\omega}$ is known a priori, which follows from Algorithm 2.1. This strategy is implemented in Algorithm 2.2, where a two-step Adams–Bashforth predictor and a trapezoidal corrector are used to numerically integrate the angle of rotation θ and to assign the axis of rotation \mathbf{r} . A forward Euler step is used to start the predictor.

Algorithm 2.2 Updating the rotation tensor **R**

```

var  $\omega_{n-1}$ ,  $\mathbf{R}_{n-1}$  ▷ stored variables

procedure INITROTATION ( $\Delta t$ ,  $\omega_0$ , var  $\mathbf{R}_0$ , var  $\mathbf{R}_1$ ) ▷  $n = 0$ 
   $\mathbf{R}_0 \leftarrow \mathbf{I}$ 
   $\dot{\theta} \leftarrow \|\omega_0\|$ 
  if  $|\dot{\theta}| > \text{tol}$  then ▷ forward Euler predictor
     $r^- \leftarrow \omega_0 / \dot{\theta}$ 
     $[\mathbf{Q}]_j^i \leftarrow \epsilon_{kj}^i r^k$ 
     $\mathbf{R}_1 \leftarrow \mathbf{I} + \sin(\dot{\theta} \Delta t) \mathbf{Q} + (1 - \cos(\dot{\theta} \Delta t)) \mathbf{Q}^2$ 
  else
     $\mathbf{R}_1 \leftarrow \mathbf{I}$ 
  end if
   $\omega_{n-1} \leftarrow \omega_0$  ▷ update stored variables
   $\mathbf{R}_{n-1} \leftarrow \mathbf{R}_0$ 
end procedure

procedure ROTATION ( $\Delta t$ ,  $\omega_n$ , var  $\mathbf{R}_{\Delta t}^-$ , var  $\mathbf{R}_{\Delta t}^+$ , var  $\mathbf{R}_n$ , var  $\mathbf{R}_{n+1}$ )
   $\dot{\theta}^- \leftarrow \|\frac{1}{2}(\omega_n + \omega_{n-1})\|$ 
  if  $|\dot{\theta}^-| > \text{tol}$  then ▷ run corrector over previous increment
     $r^- \leftarrow \frac{1}{2}(\omega_n + \omega_{n-1}) / \dot{\theta}^-$ 
     $[\mathbf{Q}^-]_j^i \leftarrow \epsilon_{kj}^i \{r^-\}^k$ 
     $\mathbf{R}_{\Delta t}^- \leftarrow \mathbf{I} + \sin(\dot{\theta}^- \Delta t) \mathbf{Q}^- + (1 - \cos(\dot{\theta}^- \Delta t)) \mathbf{Q}^- \cdot \mathbf{Q}^-$ 
  else
     $\mathbf{R}_{\Delta t}^- \leftarrow \mathbf{I}$ 
  end if
   $\mathbf{R}_n \leftarrow \mathbf{R}_{\Delta t}^- \mathbf{R}_{n-1}$ 
   $\dot{\theta}^+ \leftarrow \|\frac{1}{2}(3\omega_n - \omega_{n-1})\|$ 
  if  $|\dot{\theta}^+| > \text{tol}$  then ▷ run predictor over next increment
     $r^+ \leftarrow \frac{1}{2}(3\omega_n - \omega_{n-1}) / \dot{\theta}^+$ 
     $[\mathbf{Q}^+]_j^i \leftarrow \epsilon_{kj}^i \{r^+\}^k$ 
     $\mathbf{R}_{\Delta t}^+ \leftarrow \mathbf{I} + \sin(\dot{\theta}^+ \Delta t) \mathbf{Q}^+ + (1 - \cos(\dot{\theta}^+ \Delta t)) \mathbf{Q}^+ \cdot \mathbf{Q}^+$ 
  else
     $\mathbf{R}_{\Delta t}^+ \leftarrow \mathbf{I}$ 
  end if
   $\mathbf{R}_{n+1} \leftarrow \mathbf{R}_{\Delta t}^+ \mathbf{R}_n$ 
   $\omega_{n-1} \leftarrow \omega_n$  ▷ update stored variables
   $\mathbf{R}_{n-1} \leftarrow \mathbf{R}_n$ 
end procedure

```

Algorithm 2.2 requires, as input, the step size of integration Δt and the axis of angular velocity at the beginning of the step ω_n . The algorithm returns a corrected estimate for the rotation at the beginning of the step \mathbf{R}_n and a predicted estimate for the rotation at the end of the step \mathbf{R}_{n+1} . It also provides estimates for the incremental rotations $\mathbf{R}_{\Delta t}^-$ and $\mathbf{R}_{\Delta t}^+$ occurring over the prior $[t_n - \Delta t, t_n]$ and next $[t_n, t_n + \Delta t]$ integration steps, respectively. Rotation $\mathbf{R}_{\Delta t}^-$ is evaluated according to the integration rule of the corrector, while $\mathbf{R}_{\Delta t}^+$ is evaluated according to the integration rule of the predictor. These two incremental rotations find application when numerically integrating Eq. (2.34) for stretch.

2.6.3 Stretch

The left stretch tensor \mathbf{v} is governed by differential equation (2.34), whose solution can be acquired via, e.g., Algorithm 2.3. Stretch \mathbf{v} is required input for computing the axis vector for angular velocity ω using Algorithm 2.1. All numerical integrations taking place in Algorithm 2.3 are done in the updated Lagrangian configuration Ω_n affiliated with time t_n in accordance with Appendix B. Stretches belonging to the prior configuration Ω_{n-1} are pushed forward into Ω_n via the map $\mathbf{R}_{\Delta t}^-$, with the final result then being pushed forward from the updated Lagrangian frame Ω_n into the next Eulerian frame Ω_{n+1} using the map $\mathbf{R}_{\Delta t}^+$. It follows from Eq. (2.13) that $\mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{R}^{-1}$, recalling that $\mathbf{R}^{-1} = \mathbf{R}^T$, which makes our algorithm for integrating stretch rate fundamentally different from, say, an algorithm for integrating stress rate.

Algorithm 2.3 requires, as input, the step size of integration Δt , the angular velocity vector at the beginning of the step ω_n , the velocity gradient at the beginning of the step ℓ_n , the incremental rotations over the previous $\mathbf{R}_{\Delta t}^-$ and next $\mathbf{R}_{\Delta t}^+$ integration steps, and the left stretch tensor at the beginning of the step \mathbf{v}_n . The algorithm returns a corrected estimate for \mathbf{v}_n and a predicted estimate for \mathbf{v}_{n+1} for use in the next calling of Algorithm 2.1.

A different approach for quantifying the fundamental deformation fields of stretch and rotation, which is based upon the spectral decomposition theorem with the deformation gradient being the known kinematic variable, can be found in Simo and Hughes (1998, pp. 241–244).

Algorithm 2.3 Updating the left stretch tensor \mathbf{v}

```

var  $\mathbf{v}_{n-1}, \dot{\mathbf{v}}_{n-1}$  ▷ stored variables

procedure INITLEFTSTRETCH ( $\Delta t, \ell_0, \mathbf{var} \mathbf{v}_0, \mathbf{var} \mathbf{v}_1$ ) ▷  $n = 0$ 
     $\mathbf{v}_0 \leftarrow \mathbf{I}$ 
     $[\dot{\mathbf{v}}_0]_j^i \leftarrow \frac{1}{2} ([\ell_0]_j^i + \delta_{jk} [\ell_0]_\ell^k \delta^{\ell i})$ 
     $\mathbf{v}_1 \leftarrow \mathbf{I} + \Delta t \dot{\mathbf{v}}_0$  ▷ forward Euler predictor
     $\mathbf{v}_{n-1} \leftarrow \mathbf{v}_0$  ▷ update stored variables
     $\dot{\mathbf{v}}_{n-1} \leftarrow \dot{\mathbf{v}}_0$ 
end procedure

procedure LEFTSTRETCH ( $\Delta t, \omega_n, \ell_n, \mathbf{R}_{\Delta t}^-, \mathbf{R}_{\Delta t}^+, \mathbf{var} \mathbf{v}_n, \mathbf{var} \mathbf{v}_{n+1}$ )
     $[\Omega_n]_j^i \leftarrow \epsilon_{kj}^i \{\omega_n\}^k$ 
     $\dot{\mathbf{v}}_n \leftarrow \ell_n \mathbf{v}_n - \mathbf{v}_n \Omega_n$  ▷ evaluate
     $\mathbf{v}_n \leftarrow \mathbf{R}_{\Delta t}^- \mathbf{v}_{n-1} (\mathbf{R}_{\Delta t}^-)^T + \frac{\Delta t}{2} (\dot{\mathbf{v}}_n + \mathbf{R}_{\Delta t}^- \dot{\mathbf{v}}_{n-1} (\mathbf{R}_{\Delta t}^-)^T)$  ▷ correct
     $\dot{\mathbf{v}}_n \leftarrow \ell_n \mathbf{v}_n - \mathbf{v}_n \Omega_n$  ▷ reevaluate
     $\mathbf{U}_{n,n+1} \leftarrow \frac{\Delta t}{2} (3 \dot{\mathbf{v}}_n - \mathbf{R}_{\Delta t}^- \dot{\mathbf{v}}_{n-1} (\mathbf{R}_{\Delta t}^-)^T)$  ▷ predict
     $\mathbf{v}_{n+1} \leftarrow \mathbf{R}_{\Delta t}^+ (\mathbf{v}_n + \mathbf{U}_{n,n+1}) (\mathbf{R}_{\Delta t}^+)^T$  ▷ push from  $\Omega_n$  into  $\Omega_{n+1}$ 
     $\mathbf{v}_{n-1} \leftarrow \mathbf{v}_n$  ▷ update stored variables
     $\dot{\mathbf{v}}_{n-1} \leftarrow \dot{\mathbf{v}}_n$ 
end procedure

```

2.7 Examples

A fair number of fields can be used to describe deformation, each having its own purpose. The examples below will determine the deformation gradient \mathbf{F} and its inverse \mathbf{F}^{-1} ; its associated polar fields \mathbf{U} , \mathbf{v} , and \mathbf{R} ; and from $\dot{\mathbf{F}}$, the affiliated rate fields of \mathbf{L} , ℓ , \mathbf{d} , \mathbf{D} , \mathbf{w} , and \mathbf{W} as they apply to the BVPs studied in this text.

2.7.1 Uniaxial Extension

For the case of an isochoric uniaxial extension of an isotropic incompressible material, whose motion is given in Eqs. (1.10) and (1.11), the deformation gradient and its inverse are determined to have components

$$[\mathbf{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \quad \text{and} \quad [\mathbf{F}^{-1}] = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{1/2} & 0 \\ 0 & 0 & \lambda^{1/2} \end{bmatrix} \quad (2.44)$$

whose polar decomposition produces stretches and a rotation of³

$$[\mathbf{U}] \equiv [\mathbf{v}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \quad \text{while} \quad [\mathbf{R}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.45)$$

and whose rate fields, quantified via

$$[\dot{\mathbf{F}}] = \begin{bmatrix} \dot{\lambda} & 0 & 0 \\ 0 & -\dot{\lambda}/2\lambda^{3/2} & 0 \\ 0 & 0 & -\dot{\lambda}/2\lambda^{3/2} \end{bmatrix}, \quad (2.46)$$

become

$$[\mathbf{L}] \equiv [\boldsymbol{\ell}] \equiv [\mathbf{D}] \equiv [\mathbf{d}] = \begin{bmatrix} \dot{\lambda}/\lambda & 0 & 0 \\ 0 & -\dot{\lambda}/2\lambda & 0 \\ 0 & 0 & -\dot{\lambda}/2\lambda \end{bmatrix}. \quad (2.47)$$

No vorticity occurs because $\mathbf{w} = \boldsymbol{\ell} - \mathbf{d} = \mathbf{0}$; likewise, $\mathbf{W} = \mathbf{L} - \mathbf{D} = \mathbf{0}$. As a check, $\det \mathbf{F} = 1$ and $\text{tr } \mathbf{L} = 0$, so the prescribed deformation is isochoric.

2.7.2 Equi-biaxial Extension

For the case of an isochoric equi-biaxial extension of an isotropic incompressible material, whose motion is given in Eqs. (1.14) and (1.15), the deformation gradient and its inverse are determined to have components

$$[\mathbf{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix} \quad \text{and} \quad [\mathbf{F}^{-1}] = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} \quad (2.48)$$

whose polar decomposition produces stretches and a rotation of

$$[\mathbf{U}] \equiv [\mathbf{v}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix} \quad \text{while} \quad [\mathbf{R}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.49)$$

and whose rate fields, quantified via

$$[\dot{\mathbf{F}}] = \begin{bmatrix} \dot{\lambda} & 0 & 0 \\ 0 & \dot{\lambda} & 0 \\ 0 & 0 & -2\dot{\lambda}/\lambda^3 \end{bmatrix}, \quad (2.50)$$

³Shear-free motions experience no rotation, by definition, therefore, $\mathbf{R} = \mathbf{I}$ for all deformations belonging to this kinematic class of motions.

become

$$[\mathbf{L}] \equiv [\boldsymbol{\ell}] \equiv [\mathbf{D}] \equiv [\mathbf{d}] = \begin{bmatrix} \dot{\lambda}/\lambda & 0 & 0 \\ 0 & \dot{\lambda}/\lambda & 0 \\ 0 & 0 & -2\dot{\lambda}/\lambda \end{bmatrix}, \quad (2.51)$$

implying that no vorticity occurs, i.e., $\mathbf{w} \equiv \mathbf{W} = \mathbf{0}$. As a check, $\det \mathbf{F} = 1$ and $\text{tr } \mathbf{L} = 0$, so this deformation is isochoric, too.

2.7.3 Simple Shear

From the planar motion describing a simple shear given in Eqs. (1.18) and (1.19), the deformation gradient and its inverse are determined to have components

$$[\mathbf{F}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{F}^{-1}] = \begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.52)$$

Derivation of the polar fields that associate with this deformation takes a bit of doing. Begin by considering a clockwise angle of rotation θ in the 1–2 plane described by

$$[\mathbf{R}] = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2.53)$$

which is easily shown to be proper orthogonal, i.e., $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ with $\det \mathbf{R} = 1$, and is, therefore, an admissible rotation. From the definition for stretch $\mathbf{U} = \mathbf{R}^{-1} \mathbf{F}$ obtained from the polar decomposition of the deformation gradient, i.e., Eq. (2.13), noting that $[\mathbf{R}^{-1}] = [\mathbf{R}]^T$, one gets

$$[\mathbf{U}] = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta + \gamma \cos \theta & 0 \\ \sin \theta & \cos \theta + \gamma \sin \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.54)$$

However, because $[\mathbf{U}] = [\mathbf{U}]^T$, it follows that $U_{12} = U_{21}$ and, as such, $\sin \theta = -\sin \theta + \gamma \cos \theta$, which can be solved for θ yielding $\theta = \tan^{-1}(\gamma/2)$, $-\pi/2 \leq \theta \leq \pi/2$. Recalling that $\tan \theta = \text{rise}/\text{run}$, it follows that the rise = γ and the run = 2, so the hypotenuse is $(4 + \gamma^2)^{1/2}$ and, therefore, $\cos \theta = 2/(4 + \gamma^2)^{1/2}$ and $\sin \theta = \gamma/(4 + \gamma^2)^{1/2}$. With the $\sin \theta$ and $\cos \theta$ now known

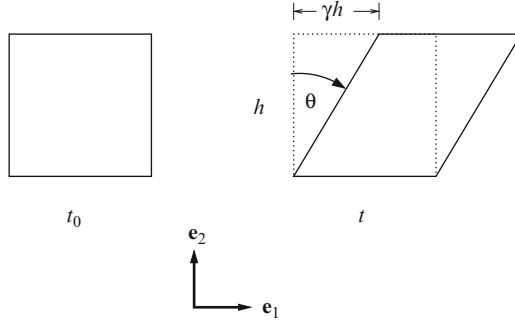


Fig. 2.3 Simple shear takes a square of dimension h (for height) and deforms it into a quadrilateral of the same area by a magnitude γ and angle θ of shearing

in terms of the deformation variable γ , whose geometric interpretation is drawn in Fig. 2.3, the rotation tensor is found to have components

$$[\mathbf{R}] = \frac{1}{(4 + \gamma^2)^{1/2}} \begin{bmatrix} 2 & \gamma & 0 \\ -\gamma & 2 & 0 \\ 0 & 0 & (4 + \gamma^2)^{1/2} \end{bmatrix} \quad (2.55)$$

from which the two stretch tensors are determined to have components

$$[\mathbf{U}] = \frac{1}{(4 + \gamma^2)^{1/2}} \begin{bmatrix} 2 & \gamma & 0 \\ \gamma & 2 + \gamma^2 & 0 \\ 0 & 0 & (4 + \gamma^2)^{1/2} \end{bmatrix} \quad (2.56)$$

and

$$[\mathbf{v}] = \frac{1}{(4 + \gamma^2)^{1/2}} \begin{bmatrix} 2 + \gamma^2 & \gamma & 0 \\ \gamma & 2 & 0 \\ 0 & 0 & (4 + \gamma^2)^{1/2} \end{bmatrix} \quad (2.57)$$

with their differences residing in the locations of the second-order term γ^2 found in the normal components.

Arriving at the velocity gradients is much more straightforward, viz.,

$$[\mathbf{L}] = [\mathbf{F}^{-1}\dot{\mathbf{F}}] = \begin{bmatrix} 1 - \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.58)$$

and

$$[\mathbf{l}] = [\dot{\mathbf{F}}\mathbf{F}^{-1}] = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 - \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.59)$$

where $[\mathbf{L}] \equiv [\boldsymbol{\ell}]$ is uncommon for deformations with off-diagonal terms, although that is the case here for simple shear. The symmetric and skew-symmetric parts of the velocity gradients are therefore given by

$$[\mathbf{D}] \equiv [\mathbf{d}] = \frac{1}{2} \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [\mathbf{W}] \equiv [\mathbf{w}] = \frac{1}{2} \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.60)$$

Consequently, from a kinematic perspective, simple shear finds distinction between its various deformation fields, important distinctions that are not present in uniaxial and biaxial deformations and which you can use to gain understanding when seeking insight into a material's behavior.

2.7.4 Homogeneous Planar Membranes

For the case of an isotropic planar membrane undergoing the isochoric homogeneous motion specified in Eqs. (1.22) and (1.26), the deformation gradient and its inverse are determined to have components

$$[\mathbf{F}] = \begin{bmatrix} \lambda_1 & \gamma_1 \lambda_2 & 0 \\ \gamma_2 \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \Lambda^{-1} \end{bmatrix} \quad \& \quad [\mathbf{F}^{-1}] = \frac{1}{\Lambda} \begin{bmatrix} \lambda_2 & -\gamma_1 \lambda_2 & 0 \\ -\gamma_2 \lambda_1 & \lambda_1 & 0 \\ 0 & 0 & \Lambda^2 \end{bmatrix} \quad (2.61)$$

whose time rate of change is

$$[\dot{\mathbf{F}}] = \begin{bmatrix} \dot{\lambda}_1 & \overline{\dot{\gamma}_1 \lambda_2} & 0 \\ \overline{\dot{\gamma}_2 \lambda_1} & \dot{\lambda}_2 & 0 \\ 0 & 0 & -\dot{\Lambda}/\Lambda^2 \end{bmatrix}, \quad (2.62)$$

which follow straightaway from Eqs. (1.24) and (1.25), while recalling that the areal stretch of Eq. (1.23) is given by $\Lambda = \lambda_1 \lambda_2 (1 - \gamma_1 \gamma_2)$ and, therefore,

$$\dot{\Lambda} = \overline{\dot{\lambda}_1 \lambda_2} (1 - \gamma_1 \gamma_2) - \lambda_1 \lambda_2 \overline{\dot{\gamma}_1 \gamma_2}. \quad (2.63)$$

The motion maps defined in Eqs. (1.22) and (1.26) are isochoric because $\det \mathbf{F} \equiv \det \mathbf{F}^{-1} = 1$ due to how F_{33} is defined.

Following the same line of reasoning that was used to derive the rotation and stretch tensors for simple shear, Freed et al. (2010) arrived at like values applicable for planar membranes, which, in the notation of this text, lead to polar descriptions where

$$[\mathbf{R}] = \frac{1}{\mathcal{E}} \begin{bmatrix} \lambda_1 + \lambda_2 & \gamma_1 \lambda_2 - \gamma_2 \lambda_1 & 0 \\ -(\gamma_1 \lambda_2 - \gamma_2 \lambda_1) & \lambda_1 + \lambda_2 & 0 \\ 0 & 0 & \mathcal{E} \end{bmatrix} \quad (2.64)$$

wherein

$$\mathcal{E} = \sqrt{(\lambda_1 + \lambda_2)^2 + (\gamma_1 \lambda_2 - \gamma_2 \lambda_1)^2} \quad (2.65)$$

normalizes the rigid-body rotation, i.e., it ensures that $\det \mathbf{R} = 1$. This expression for the rotation \mathbf{R} allows the right stretch tensor $\mathbf{U} = \mathbf{R}^T \mathbf{F}$ to be written as

$$[\mathbf{U}] = \frac{1}{\mathcal{E}} \begin{bmatrix} \lambda_1(\lambda_1 + \lambda_2) - \gamma_2 \lambda_1(\gamma_1 \lambda_2 - \gamma_2 \lambda_1) & & & \\ \lambda_1 \lambda_2(\gamma_1 + \gamma_2) & & & \\ 0 & & & \\ & \lambda_1 \lambda_2(\gamma_1 + \gamma_2) & 0 & \\ \lambda_2(\lambda_1 + \lambda_2) + \gamma_1 \lambda_2(\gamma_1 \lambda_2 - \gamma_2 \lambda_1) & 0 & & \\ 0 & \Lambda^{-1} \mathcal{E} & & \end{bmatrix}, \quad (2.66)$$

while the left stretch tensor $\mathbf{v} = \mathbf{R} \mathbf{F}^T$ has components

$$[\mathbf{v}] = \frac{1}{\mathcal{E}} \begin{bmatrix} \lambda_1(\lambda_1 + \lambda_2) + \gamma_1 \lambda_2(\gamma_1 \lambda_2 - \gamma_2 \lambda_1) & & & \\ \gamma_1 \lambda_2^2 + \gamma_2 \lambda_1^2 & & & \\ 0 & & & \\ & \gamma_1 \lambda_2^2 + \gamma_2 \lambda_1^2 & 0 & \\ \lambda_2(\lambda_1 + \lambda_2) - \gamma_2 \lambda_1(\gamma_1 \lambda_2 - \gamma_2 \lambda_1) & 0 & & \\ 0 & \Lambda^{-1} \mathcal{E} & & \end{bmatrix} \quad (2.67)$$

where $[\mathbf{U}]$ and $[\mathbf{v}]$ are similar, yet distinct.

The components of the velocity gradients are easily acquired, too, being

$$[\mathbf{L}] = [\mathbf{F}^{-1} \dot{\mathbf{F}}] \\ = \frac{1}{\Lambda} \begin{bmatrix} \lambda_2 \dot{\lambda}_1 - \gamma_1 \lambda_2 \overline{\dot{\lambda}_1} & \lambda_2 \overline{\dot{\lambda}_2} - \gamma_1 \lambda_2 \dot{\lambda}_2 & 0 \\ \lambda_1 \overline{\dot{\lambda}_1} - \gamma_2 \lambda_1 \dot{\lambda}_1 & \lambda_1 \dot{\lambda}_2 - \gamma_2 \lambda_1 \overline{\dot{\lambda}_2} & 0 \\ 0 & 0 & -\dot{\Lambda} \end{bmatrix} \quad (2.68)$$

and

$$[\boldsymbol{\ell}] = [\dot{\mathbf{F}} \mathbf{F}^{-1}] \\ = \frac{1}{\Lambda} \begin{bmatrix} \lambda_2 \dot{\lambda}_1 - \gamma_2 \lambda_1 \overline{\dot{\lambda}_2} & \lambda_1 \overline{\dot{\lambda}_2} - \gamma_1 \lambda_2 \dot{\lambda}_1 & 0 \\ \lambda_2 \overline{\dot{\lambda}_1} - \gamma_2 \lambda_1 \dot{\lambda}_2 & \lambda_1 \dot{\lambda}_2 - \gamma_1 \lambda_2 \overline{\dot{\lambda}_2} & 0 \\ 0 & 0 & -\dot{\Lambda} \end{bmatrix} \quad (2.69)$$

whose components are obviously distinct in this case. A little effort, along with Eq. (2.63), leads to a verification of the isochoric constraints, viz., $\text{tr } \boldsymbol{\ell} = \text{tr } \mathbf{L} = 0$. The symmetric part of $\boldsymbol{\ell}$, i.e., the stretching, is given by

$$[\mathbf{d}] = \frac{1}{\Lambda} \begin{bmatrix} \lambda_2 \dot{\lambda}_1 - \gamma_2 \lambda_1 \overline{\dot{\lambda}_2} \\ \frac{1}{2}(\lambda_1 \overline{\dot{\lambda}_2} + \lambda_2 \overline{\dot{\lambda}_1} - \gamma_1 \lambda_2 \dot{\lambda}_1 - \gamma_2 \lambda_1 \dot{\lambda}_2) \\ 0 \\ \frac{1}{2}(\lambda_1 \overline{\dot{\lambda}_2} + \lambda_2 \overline{\dot{\lambda}_1} - \gamma_1 \lambda_2 \dot{\lambda}_1 - \gamma_2 \lambda_1 \dot{\lambda}_2) & 0 \\ \lambda_1 \dot{\lambda}_2 - \gamma_1 \lambda_2 \overline{\dot{\lambda}_1} & 0 \\ 0 & -\dot{\Lambda} \end{bmatrix}, \quad (2.70)$$

while its skew-symmetric part, viz., the vorticity, is described by

$$[\mathbf{w}] = \frac{1}{\Lambda} \begin{bmatrix} 0 \\ \frac{1}{2}(-\lambda_1 \overline{\dot{\lambda}_2} + \lambda_2 \overline{\dot{\lambda}_1} + \gamma_1 \lambda_2 \dot{\lambda}_1 - \gamma_2 \lambda_1 \dot{\lambda}_2) \\ 0 \\ \frac{1}{2}(\lambda_1 \overline{\dot{\lambda}_2} - \lambda_2 \overline{\dot{\lambda}_1} - \gamma_1 \lambda_2 \dot{\lambda}_1 + \gamma_2 \lambda_1 \dot{\lambda}_2) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.71)$$

Components for their Lagrangian counterparts \mathbf{D} and \mathbf{W} are left as an exercise.

2.8 Exercises

2.8.1 Pure Shear

Derive the components of the deformation gradient \mathbf{F} and its inverse \mathbf{F}^{-1} ; their associated polar fields, i.e., \mathbf{U} , \mathbf{v} , and \mathbf{R} ; and the rate fields that describe this deformation, viz., $\dot{\mathbf{F}}$, \mathbf{L} , $\boldsymbol{\ell}$, \mathbf{D} , \mathbf{d} , \mathbf{W} , and \mathbf{w} , for the motion of pure shear defined in Eqs. (1.32) and (1.33).

2.8.2 Biaxial Extension

Derive the components of the deformation gradient \mathbf{F} and its inverse \mathbf{F}^{-1} ; their associated polar fields, i.e., \mathbf{U} , \mathbf{v} , and \mathbf{R} ; and the rate fields that describe this deformation, viz., $\dot{\mathbf{F}}$, \mathbf{L} , $\boldsymbol{\ell}$, \mathbf{D} , \mathbf{d} , \mathbf{W} , and \mathbf{w} , for the motion of biaxial extension defined in Eqs. (1.34) and (1.35).

2.8.3 Extension Followed by Simple Shear

Derive the components of the deformation gradient \mathbf{F} and its inverse \mathbf{F}^{-1} ; their associated polar fields, i.e., \mathbf{U} , \mathbf{v} , and \mathbf{R} ; and the rate fields that describe this deformation, viz., $\dot{\mathbf{F}}$, \mathbf{L} , \mathbf{l} , \mathbf{D} , \mathbf{d} , \mathbf{W} , and \mathbf{w} for the second phase of the motion, viz., that of simple shearing, which follows the initial phase of an extension, as described in Eqs. (1.36) and (1.37). Here, in the second phase, the aspect ratio n and the stretch λ are held fixed, i.e., $\dot{n} = \dot{\lambda} = 0$.

2.8.4 Other Problems

1. In the two-dimensional case, show that for any two orthogonal tensors \mathbf{R}_1 and \mathbf{R}_2 described by angles of rotation θ_1 and θ_2 , their product $\mathbf{R}_1\mathbf{R}_2$ is another orthogonal tensor, but their sum $\mathbf{R}_1 + \mathbf{R}_2$ does not produce an orthogonal tensor.
2. Compute the components for the stretching \mathbf{D} and vorticity \mathbf{W} tensors of the Lagrangian frame for the homogeneous membrane. How do they compare with their Eulerian counterparts \mathbf{d} and \mathbf{w} found in Eqs. (2.70) and (2.71)?
3. Show that the components of the deformation gradient \mathbf{F} and its inverse \mathbf{F}^{-1} ; their associated polar fields, i.e., \mathbf{U} , \mathbf{v} , and \mathbf{R} ; and the rate fields that describe deformation, viz., $\dot{\mathbf{F}}$, \mathbf{L} , \mathbf{l} , \mathbf{d} , and \mathbf{w} , for (a) uniaxial extension, (b) equi-biaxial extension, (c) simple shear, (d) pure shear, (e) biaxial extension, and (f) extension followed by simple shear are each a special case of Eqs. (2.61)–(2.71), which describe their counterparts for planar membranes.
4. Show that the transpose of a covariant tensor is a covariant tensor and that the transpose of a contravariant tensor is a contravariant tensor. (Hint: Take the transpose of their respective field-transfer laws.) Also show that the inverse of a covariant tensor is a contravariant tensor and the inverse of a contravariant tensor is a covariant tensor, if they exist.
5. Show that the inverse of a mixed tensor is a mixed tensor. (Hint: Take the inverse of the field-transfer law for mixed tensors.) Show that the transpose of the field-transfer law for mixed tensors is not a mixed tensor in the sense of Eqs. (B.25) and (B.26). What does this imply? [It is for this reason that only one type of mixed tensor field is considered here, viz., right covariant, and it is why their transposes have tensor components defined according to Eq. (A.26)].
6. Show that, like the Lagrangian velocity gradient \mathbf{L} , the Lagrangian stretch \mathbf{U} and its inverse \mathbf{U}^{-1} obey the field-transfer operator of a mixed

tensor field with \mathbf{U} and \mathbf{U}^{-1} pushing from Ω_0 forward into Ω as \mathbf{v} and \mathbf{v}^{-1} according to Eq. (B.25). Conversely, show that the Eulerian stretches \mathbf{v} and \mathbf{v}^{-1} pull back from Ω into Ω_0 as mixed tensor fields producing \mathbf{U} and \mathbf{U}^{-1} according to Eq. (B.26). Do the stretches \mathbf{U} and \mathbf{v} and the velocity gradients \mathbf{L} and \mathbf{l} all obey the same field-transfer law? Are they the same or different from the transformation law between \mathbf{U}^{-1} and \mathbf{v}^{-1} ?

7. The components of the rotation tensor in Eq. (2.55) for simple shear were derived assuming a clockwise rotation in Eq. (2.53). Show that you would get the same result if, instead, Eq. (2.53) were to describe a counterclockwise rotation, i.e., $\theta \leftarrow -\theta$.
8. Consider the motion of a compressible isotropic solid that is uniaxially stretched according to the mapping

$$x_1 = \lambda_a X_1, \quad x_2 = \lambda_t X_2, \quad x_3 = \lambda_t X_3$$

where λ_a and λ_t are the axial and transverse stretches, respectively. What is this motion's deformation gradient \mathbf{F} ? Under what condition would this motion be isochoric?

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Freed, A.

2014, XXXVI, 364 p. 57 illus., 3 illus. in color., Hardcover

ISBN: 978-3-319-03550-5

A product of Birkhäuser Basel