Chapter 2
Quasi-Neutrality and Magneto-Hydrodynamics

Abstract In this chapter, we justify firstly the massless-electron approximation from the general ion–electron electrodynamic model. Secondly, we present the quasi-neutrality approximation, which is the heart of most of the fluid models presented in this book; this approximation is rigorously proved by an asymptotic analysis where a small parameter related to the Debye length goes to zero. We then present the two-temperature Euler system which is the basic model for quasi-neutral plasmas; in this framework we deal also with thermal conduction and radiative coupling. Lastly, we introduce the well-known model called electron magneto-hydrodynamics (MHD) which is the fundamental model for all magnetized plasmas. We give some details about the related boundary conditions.

Some crucial mathematical properties related to the "ideal part" of the previous models are displayed at the end of this chapter.

Keywords Debye length • Massless-electron approximation • Quasi-neutrality approximation • Poisson equation • Two-temperature Euler system • Electron magneto-hydrodynamics • Ideal magneto-hydrodynamics • Boundary conditions for MHD

2.1 Massless-Electron Approximation

Before dealing with the massless-electron approximation, we first state the general electrodynamic model related to the *ion–electron Euler system* coupled with *Maxwell equations* (cf. the first subsection below). From this model, we may address either modelling based on high-frequency electron waves, called *Langmuir waves,* which develop at the electron time scale ω_p^{-1} (in this framework the ions are assumed to be either at rest or mobile; see Chap. 4 for this topic) or modelling related to the evolution of the ion population at a time scale much larger than ω_p^{-1}.

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In the second framework, which corresponds to the topic of the second subsection and the other sections of this chapter, we address an observation time scale T_{obs} that is in the order of $L_{\text{plasma}}/v_{\text{th,i}}$ or $L_{\text{plasma}}/v_{\text{Al}}$ (where L_{plasma} is a characteristic length of variations of the plasma density) and the picture is the following

$$T_{\text{obs}} \sim \frac{L_{\text{plasma}}}{v_{\text{th,i}}}, \frac{L_{\text{plasma}}}{v_{\text{Al}}} \gg \omega_p^{-1} = \frac{\lambda_D}{v_{\text{th,e}}}$$

Since one has always $\lambda_D \leq L_{\text{plasma}}$, one sees that $\frac{L_{\text{plasma}}}{v_{\text{th,i}}} \gg \frac{\lambda_D}{v_{\text{th,e}}}$, so the previous ordering is very general and corresponds to the case where electron inertia is neglected with respect to the ion inertia. Thus, we can make the *massless-electron approximation*. It consists of assuming that, in the ion–electron Euler–Maxwell system, the electron mass m_e is negligible compared to ion mass m_0; so the electron momentum balance equation reduces to the so-called *generalized Ohm's law* which links the electric field to the electron density (we stress that there are various ways to state such a Ohm's law depending on the physical effects to be accounted for). Moreover, if there is no external electromagnetic source, no phenomena travel at the speed of light and a simplified version of the Maxwell equations corresponding to an infinite speed of light may be used. So we are led to the so-called *ion Euler–Poisson system* [system $(\mathcal{M})$ below] where the electric field reduces to an electrostatic one; it is valid even if the Debye length λ_D is not very small with respect to the characteristic length.

Now, in the first subsection, we recall the general electrodynamic model. It is a classical one (cf. [38, 112]) although it is almost never used for numerical simulations because the order of magnitude of the characteristic times of the subsystems are very different. It is worth focusing on it because it gives the conservation balance for the mass, momentum, and energy of the two populations, and it enables us to derive numerous fluid models with formal asymptotics.

2.1.1 The Ion–Electron Electrodynamic Model

Let us state the system corresponding to the classical conservation laws for the two populations of ions and electrons. First of all, the continuity equations for both species are

$$\frac{\partial N_0}{\partial t} + \nabla.(N_0 \mathbf{U}) = 0, \tag{2.1}$$

$$\frac{\partial N_e}{\partial t} + \nabla.(N_e \mathbf{U}_e) = 0. \tag{2.2}$$

Denote by ν_{e0} the Coulomb collision coefficient, it is related to collision frequency between electrons and ions which is equal to $\nu_{e0}N_0/m_e$. Then, the ion momentum balance equation reads as

$$m_0\frac{\partial}{\partial t}(N_0\mathbf{U})+m_0\nabla.(N_0\mathbf{UU})+\nabla P_0=q_eZ(N_0\mathbf{E}+N_0\mathbf{U}\times\mathbf{B})-\nu_{e0}N_eN_0(\mathbf{U}-\mathbf{U}_e). \tag{2.3}$$

On the right-hand side, the first term corresponds to the Lorentz force and the second one to the Coulomb collisions between the two species; at this level it reduces to a simple friction force proportional to the relative velocity $(\mathbf{U}-\mathbf{U}_e)$ (but we stress that there are different expressions of this friction force).

Moreover, the ion internal energy equation reads classically

$$\frac{\partial}{\partial t}\mathcal{E}_0+\nabla.(\mathbf{U}\mathcal{E}_0)+P_0\nabla.\mathbf{U}=\Omega_{0e}, \tag{2.4}$$

where the term Ω_{0e} is related to the energy exchange between the ion and electron populations due to the Coulomb collisions. The relationship between this term and the corresponding term Ω_{e0} for the electrons will be given below [see relation (2.16)].

In the same way as above, the electron momentum balance equation reads as

$$m_e\frac{\partial}{\partial t}(N_e\mathbf{U}_e)+m_e\nabla.(N_e\mathbf{U}_e\mathbf{U}_e)+\nabla P_e=-q_e(N_e\mathbf{E}+N_e\mathbf{U}_e\times\mathbf{B})+\nu_{e0}N_eN_0(\mathbf{U}-\mathbf{U}_e), \tag{2.5}$$

and the electron internal energy equation

$$\frac{\partial}{\partial t}\mathcal{E}_e+\nabla.(\mathbf{U}_e\mathcal{E}_e)+P_e\nabla.\mathbf{U}_e+\nabla.\mathbf{q}_{\mathrm{th,e}}=\Omega_{e0}. \tag{2.6}$$

Here $\mathbf{q}_{\mathrm{th,e}}$ denotes the heat flux for the electron energy; its simplest expression is the so-called Spitzer flux which is proportional to $\nabla T_e^{7/2}$; the details about Spitzer flux are given below in Sect. 2.3.1, see also [111].

Of course, the previous energy equations may be stated in term of ion and electron total energy, i.e.,

$$\begin{aligned}\frac{\partial}{\partial t}(\mathcal{E}_0+m_0N_0\frac{1}{2}|\mathbf{U}|^2)+\nabla.(\mathbf{U}(\mathcal{E}_0+m_0N_0\frac{1}{2}|\mathbf{U}|^2))+\nabla.(\mathbf{U}P_0)\\=\Omega_{0e}+q_eN_0\mathbf{U}.\mathbf{E}-\nu_{e0}N_eN_0\mathbf{U}.(\mathbf{U}-\mathbf{U}_e),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\mathcal{E}_e+m_eN_e\frac{1}{2}|\mathbf{U}_e|^2)+\nabla.(\mathbf{U}_e(\mathcal{E}_e+m_eN_e\frac{1}{2}|\mathbf{U}_e|^2))+\nabla.(\mathbf{U}_eP_e)+\nabla\mathbf{q}_{\mathrm{th,e}}\\=\Omega_{e0}-q_eN_e\mathbf{U}_e.\mathbf{E}+\nu_{e0}N_eN_0\mathbf{U}_e.(\mathbf{U}-\mathbf{U}_e).\end{aligned} \tag{2.8}$$

We now deal with the Coupling with the Electromagnetic Fields.

First, using the electric current $\mathbf{J} = q_e Z N_0 \mathbf{U} - q_e N_e \mathbf{U}_e$, the electromagnetic fields $\mathbf{E}, \mathbf{B}$ satisfy the full Maxwell equations (respectively Maxwell–Ampère and Maxwell–Faraday relations)

$$\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} - \text{curl } \mathbf{B} + \mu^0 \mathbf{J} = 0, \tag{2.9}$$

$$\frac{\partial}{\partial t}\mathbf{B} + \text{curl } \mathbf{E} = 0. \tag{2.10}$$

It is also necessary to account for the electric Gauss relation

$$\varepsilon^0 \nabla.\mathbf{E} = q_e(ZN_0 - N_e). \tag{2.11}$$

and the magnetic Gauss relation which reads as $\nabla.\mathbf{B} = 0$.

According to the continuity equations (2.1) and (2.2), we see that the electric current satisfies the so-called *consistency relation* for electric charge

$$q_e \frac{\partial}{\partial t}(ZN_0 - N_e) + \nabla.\mathbf{J} = 0. \tag{2.12}$$

This consistency relation implies that relation (2.11) holds always if it holds at initial time (indeed, we have $\frac{\partial}{\partial t}\nabla.\mathbf{E} + \frac{1}{\varepsilon^0}\nabla.\mathbf{J} = 0$).

We now give the classical conversation relations related to this electrodynamic model.

(a) Momentum Balance Relation.

Adding (2.3) and (2.5), we get

$$\frac{\partial}{\partial t}(m_e N_e \mathbf{U}_e + m_0 N_0 \mathbf{U}) + \nabla.\,(m_e N_e \mathbf{U}_e \mathbf{U}_e + m_0 N_0 \mathbf{U}\mathbf{U}) + \nabla(P_0 + P_e) \tag{2.13}$$
$$= \varepsilon^0 \mathbf{E}(\nabla.\mathbf{E}) + \mathbf{J} \times \mathbf{B}$$

On the other hand, according to the Maxwell equations, one checks that the electromagnetic momentum $\mathbf{E} \times \mathbf{B}$ (equal to the Poynting vector, up a multiplicative constant) satisfies

$$\frac{1}{c^2\mu^0}\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) - \frac{1}{\mu^0}(\text{curl } \mathbf{B}) \times \mathbf{B} - \varepsilon^0(\text{curl } \mathbf{E}) \times \mathbf{E} + \mathbf{J} \times \mathbf{B} = 0.$$

Now, using the tensor $\mathbb{P}_B = \frac{1}{\mu^0}(\frac{1}{2}\mathbb{I}|\mathbf{B}|^2 - \mathbf{B}\mathbf{B})$ and identity (1.3), recall that

$$-\frac{1}{\mu^0}\text{curl } \mathbf{B} \times \mathbf{B} = \nabla.\mathbb{P}_B$$

so introducing the tensor $\mathbb{S} = \mathbb{P}_B + \varepsilon^0 \left(\frac{\mathbb{I}}{2}|\mathbf{E}|^2 - \mathbf{E}\mathbf{E}\right)$, we see that

$$\frac{1}{c^2\mu^0}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \nabla.\mathbb{S} + q_e\mathbf{E}(ZN_0 - N_e) + \mathbf{J}\times\mathbf{B} = 0,$$

therefore, we get a classical result of the conversation of global momentum

$$\frac{\partial}{\partial t}\left[\frac{1}{c^2\mu^0}(\mathbf{E}\times\mathbf{B}) + m_eN_e\mathbf{U}_e + m_0N_0\mathbf{U}_0\right] + \nabla.\left(\mathbb{S} + m_eN_e\mathbf{U}_e\mathbf{U}_e + m_0N_0\mathbf{U}\mathbf{U}\right) + \nabla(P_0 + P_e) = 0.$$

(b) Energy Balance Relation.

Using the classical vector identity $\nabla.(\mathbf{A}\times\mathbf{B}) = \mathbf{B}.\mathrm{curl}\,\mathbf{A} - \mathbf{A}.\mathrm{curl}\,\mathbf{B}$, we get for the electromagnetic energy,

$$\frac{\partial}{\partial t}\mathcal{E}_{\mathrm{electro}} + \nabla.(\frac{1}{\mu^0}\mathbf{E}\times\mathbf{B}) + \mathbf{J}.\mathbf{E} = 0, \quad \text{with } \mathcal{E}_{\mathrm{electro}} = \varepsilon^0\frac{|\mathbf{E}|^2}{2} + \frac{1}{\mu^0}\frac{|\mathbf{B}|^2}{2} \tag{2.14}$$

where $\frac{1}{\mu^0}\mathbf{E}\times\mathbf{B}$ is the Poynting vector. For the plasma energy balance, we first add (2.7) and (2.8), then on the right-hand side, we get the term

$$\mathbf{J}.\mathbf{E} + \Omega_{e0} + \Omega_{0e} - \nu_{e0}N_eN_0|\mathbf{U} - \mathbf{U}_e|^2$$

So we arrive at the following global energy balance relation

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)\left(\mathcal{E}_0 + m_0N_0\frac{1}{2}|\mathbf{U}|^2\right)\\&+\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}_e\cdot)\right)\times\left(\mathcal{E}_e + m_eN_e\frac{1}{2}|\mathbf{U}_e|^2\right) + \frac{\partial\mathcal{E}_{\mathrm{electro}}}{\partial t}\end{aligned} \tag{2.15}$$

$$+\nabla.\left(\frac{1}{\mu^0}\mathbf{E}\times\mathbf{B}\right) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}_e) + \nabla.\mathbf{q}_{\mathrm{th,e}} = \Omega_{e0} + \Omega_{0e} - \nu_{e0}N_eN_0|\mathbf{U} - \mathbf{U}_e|^2.$$

According to the energy conservation principle, the right-hand side must be zero; thus we must have

$$\Omega_{e0} = -\Omega_{0e} + \nu_{e0}N_eN_0|\mathbf{U} - \mathbf{U}_e|^2 \tag{2.16}$$

Moreover, the classical form of Ω_{0e} is the following

$$\Omega_{0e} = \omega_{0e}m_0N_0(T_e - T_0) \tag{2.17}$$

where ω_{0e} is a positive quantity with an inverse ${\omega_{0e}}^{-1}$ known as the characteristic time of temperature relaxation. Notice that expression (2.17) is obtained by taking the two first moments of the Vlasov–Fokker–Planck equations for the ion and electron population evolution (see the companion book on "Kinetic Models", [118]), where ω_{0e} depends on the electron temperature and the electron density, e.g., it may be assumed to be proportional to

$$N_e Z^2 (\log \Lambda)/T_e^{3/2}$$

2.1.2 *The Ion Euler System with Massless-Electron Approximation*

Consider the evolution of a plasma with an observation time $T_{\rm obs}$ large enough compared with the inverse of the plasma frequency (but with a Debye length λ_D not necessarily very small with respect to $L_{\rm plasma}$).

$$T_{\rm obs} \sim \frac{L_{\rm plasma}}{v_{\rm th,i}} \gg \omega_p^{-1}, T_{\rm obs} \gg \frac{L_{\rm plasma}}{c},$$
$$L_{\rm plasma} \sim \lambda_D \quad \text{or} \quad L_{\rm plasma} \gg \lambda_D.$$

This situation is very frequent. For instance in the ionosphere problems, recall some orders of magnitude: the Debye is in the order of 2.3×10^{-3} m and the inverse of the plasma frequency is in the order of 1.5×10^{-8} s, and for electrical discharge concerning small Earth-orbiting satellites the characteristic time is larger than 10^{-5} s and the characteristic length is larger than 10^{-2} m.

In the same way, for cavity plasmas in Inertial Confinement Fusion, the Debye length is about 2.3×10^{-9} m and the inverse of plasma frequency about 6×10^{-16} s, but the characteristic time is larger than 10^{-12} s and the characteristic length is larger than 10^{-5} m ; indeed the variation of laser intensity is in the order of 10^{-11} s and the size of the target is in the order of a few millimeters.

So, in this framework electron inertia may be neglected with respect to the ion inertia, and the characteristic speed of the phenomena is much smaller than the electron thermal speed (and, of course, the speed of light).

Let us first stress that we can assume that the speed of light c is infinite in the previous general electrodynamic model by assuming that the displacement current $\frac{1}{c^2\mu^0}\frac{\partial \mathbf{E}}{\partial t}$ in (2.9) is negligible with respect to the electric current $\mathbf{J}$. In this framework, the Maxwell equations and Gauss relations reduce to

$$\text{curl}\, \mathbf{B} = \mu^0 \mathbf{J}, \tag{2.18}$$

and

$$\begin{cases} \text{(i)} & \text{curl}\, \mathbf{E} = -\partial_t \mathbf{B}, \\ \text{(ii)} & \nabla.\mathbf{B} = 0, \\ \text{(iii)} & \varepsilon^0 \nabla.\mathbf{E} = q_e(ZN_0 - N_e). \end{cases} \tag{2.19}$$

Here there is no formal asymptotic analysis with a small parameter and we do not take care of the relation $\varepsilon^0\mu^0c^2 = 1$, in this approximation ε^0 and μ^0 are solid values (and we simply set $c^{-1} \simeq 0$). It is worth noticing that once the densities N_0, N_e are known one can find $\mathbf{B}$ and $\mathbf{E}$ satisfying (2.18) and (2.19) only if the electric current satisfies

$$\nabla.\mathbf{J} = 0.$$

Thus, the approximation "infinite speed of light" is not appropriated with any general model for the electrons: it is made when the electron velocity is evaluated thanks to the electric current $\mathbf{J}$, in particular in the framework of the massless-electron approximation.

With the approximation of "infinite speed of light", from (2.18), the electromagnetic momentum balance reduces to

$$\nabla.\mathbb{P}_B + \mathbf{J} \times \mathbf{B} = 0,$$

[using identity (1.3)].

Now, according to the above vector identity for $\nabla.(\mathbf{E} \times \mathbf{B})$, we see that the magnetic energy balance reads as

$$\frac{1}{2\mu^0}\frac{\partial}{\partial t}|\mathbf{B}|^2 + \frac{1}{\mu^0}\nabla.(\mathbf{E} \times \mathbf{B}) + \mathbf{J}.\mathbf{E} = 0. \tag{2.20}$$

Thus, for the global energy (the sum of ion energy, electron energy, and magnetic energy), we get

$$\left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_0+m_0N_0\frac{1}{2}|\mathbf{U}|^2)+\left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}_e\cdot)\right)(\mathcal{E}_e)+\nabla.(P_0\mathbf{U}+P_e\mathbf{U}_e)+\nabla.\mathbf{q}_{\text{th,e}}$$
$$= -\mathbf{J}.\mathbf{E} = -\frac{1}{2\mu^0}\frac{\partial}{\partial t}|\mathbf{B}|^2 - \nabla.\left(\frac{1}{\mu^0}\mathbf{E}\times\mathbf{B}\right).$$

This conservative balance is the same kind as the one displayed in (2.14) but $\mathcal{E}_{\text{electro}}$ is replaced by the magnetic energy $\frac{1}{2\mu^0}|\mathbf{B}|^2$.

In the framework of the massless-electron approximation, which is the framework of the rest of the chapter, the electron density and velocity are not characterized by the mass and momentum conservation laws. The picture is the following.

On the one hand, the electron density is evaluated thanks to the Poisson equation which is obtained by inserting an expression of the generalized Ohm's law (see (2.21) below) into the electric Gauss relation. On the other hand, we claim that the electric current $\mathbf{J}$ is given by (2.18) and the electron velocity defined by $N_e\mathbf{U}_e = ZN_0\mathbf{U}-q_e^{-1}\mathbf{J}$. Moreover, the magnetic field is the solution of the evolution equation obtained by inserting the expression of $\mathbf{E}$ given by (2.21) into the Maxwell–Faraday equation (2.19(i)).

More precisely, the massless-electron approximation corresponds to the case where the electron mass m_e is assumed to be very small with respect to the ion mass.

Then a formal asymptotics (corresponding to a small parameter $m_e/m_0 \to 0$) leads to the following model. Since the electron and ion density are in the same order of magnitude and the electron velocity $\mathbf{U}_e$ is in the same order of magnitude as the ion one $\mathbf{U}$, we check that the electron momentum is negligible with respect to the ion momentum and that the electron kinetic energy $m_e N_e \frac{1}{2}|\mathbf{U}_e|^2$ is negligible with respect to the ion kinetic energy. For a rigorous result, see the analysis in the one-dimensional framework in [63] or [71] and also [2].

Therefore, from the electron momentum balance equation (2.5), we have firstly the relation

$$\begin{aligned}\nabla P_e + q_e N_e \mathbf{E} + q_e Z N_0 \mathbf{U} \times \mathbf{B} - \mathbf{J} \times \mathbf{B} &= \nu_{e0} N_e N_0 (\mathbf{U} - \mathbf{U}_e) \\ &= \nu_{e0} N_0 (\mathbf{U}(N_e - Z N_0) + \mathbf{J} q_e^{-1}).\end{aligned}$$

But it is usual in the ion momentum equation (2.3) and in the previous equation to replace the term $\nu_{e0} N_0 (\mathbf{U} - \mathbf{U}_e)$ with a closure of the form $q_e \chi \mu^0 \mathbf{J}$ where χ is a positive function depending on the ion density and temperature. Then we get the relation

$$\nabla P_e + q_e N_e \mathbf{E} + q_e Z N_0 \mathbf{U} \times \mathbf{B} - \mathbf{J} \times \mathbf{B} - q_e N_e \chi \mu^0 \mathbf{J} = 0, \tag{2.21}$$

which is called the *generalized Ohm's law* (see e.g., [108] or [38]). Coefficient $\chi \mu^0$ is called the specific electric resistivity of the plasma.

Secondly, combining relation (2.21) with (2.3), we get a new relation for momentum balance

$$\begin{aligned}m_0 \frac{\partial}{\partial t}(N_0 \mathbf{U}) + m_0 \nabla.(N_0 \mathbf{U}\mathbf{U}) + \nabla(P_0 + P_e) &= \mathbf{J} \times \mathbf{B} + \varepsilon^0 \mathbf{E}(\nabla.\mathbf{E}) \\ &= -\nabla.\mathbb{P}_B + \varepsilon^0 \mathbf{E}(\nabla.\mathbf{E}). \end{aligned} \tag{2.22}$$

Thirdly, according to the previous remarks, the electron global energy $\left(\mathcal{E}_e + m_e N_e \frac{1}{2} |\mathbf{U}_e|^2\right)$ reduces to its internal energy $\mathcal{E}_e$. Thus, using (2.16), (2.8) reads now as follows

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_e) + \nabla.(P_e \mathbf{U}) - \nabla.\left(\frac{5}{2} T_e \frac{\mathbf{J}}{q_e}\right) \\ &\quad + \nabla.\mathbf{q}_{\mathrm{th,e}} = -\Omega_{0e} - q_e N_e \mathbf{U}_e.\mathbf{E} + q_e \mu^0 N_e \chi \mathbf{J}.\mathbf{U}.\end{aligned}$$

or equivalently

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_e) + \nabla.(P_e \mathbf{U}) - \nabla.\left(\frac{5}{2} T_e \frac{\mathbf{J}}{q_e}\right) + \nabla.\mathbf{q}_{\mathrm{th,e}} \\ &= -\Omega_{0e} + \mathbf{U}_e.\nabla P_e + q_e \mu^0 N_e \chi \mathbf{J}.(\mathbf{U} - \mathbf{U}_e)\end{aligned} \tag{2.23}$$

(indeed, according to (2.21), we have $q_e N_e \mathbf{U}_e.\mathbf{E} + \mathbf{U}_e.\nabla P_e - q_e \mu^0 N_e \chi \mathbf{J}.\mathbf{U}_e = 0$). Note that the term $\frac{5}{2}T_e$ in (2.23) stands for $(\mathcal{E}_e + P_e)/N_e$.

The ion internal energy $\mathcal{E}_0$ is given by (2.4).

The key point is to define the electron density N_e; this is done by gathering the electric Gauss relation and Ohm's law (2.21); see below.

Now, using the evaluation of $\mathbf{E}$ given by the generalized Ohm's law (2.21), the Maxwell–Faraday equation reads now as an evolution equation for the field

$$\frac{\partial}{\partial t}\mathbf{B} - \mathrm{curl}\,(\frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B}) = \mathrm{curl}\left(\frac{1}{q_e N_e}\nabla P_e - \frac{1}{\mu^0 q_e N_e}\mathrm{curl}\,\mathbf{B}\times\mathbf{B}\right) - \mathrm{curl}\,(\chi \mathrm{curl}\,\mathbf{B}) \tag{2.24}$$

It is called the diffusion magnetic equation. On the right-hand side, the last term is the usual resistive diffusion operator; moreover, the quadratic term, curl$(\frac{1}{N_e}$curl $\mathbf{B}\times\mathbf{B})$ is called the Hall's effect term (it is taken into account only if the electron density is small enough and if the magnetic field is strong enough).

Lastly, we have the Gauss relation

$$\nabla.\mathbf{B} = 0;$$

of course, if this relation holds at initial time, it holds at any time.

(*a*) *The Ion Euler–Poisson Model Without Resistivity.*

For the sake of this presentation, we state first the Euler–Poisson model by neglecting the resistive term in Ohm's law, i.e., $\chi = 0$; afterwards we will reintroduce the resistivity. So, in this framework the Ohm's law reduces to

$$q_e(\mathbf{E} + \frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B}) = -\frac{1}{N_e}\nabla(N_e T_e) + \frac{1}{N_e}\mathbf{J}\times\mathbf{B}, \tag{2.25}$$

and, according to the electric Gauss relation, we get

$$-\frac{\varepsilon^0}{q_e}\nabla.(\frac{1}{N_e}\nabla(N_e T_e)) - \frac{\varepsilon^0}{q_e}\nabla.(q_e\frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B} - \frac{1}{N_e}\mathbf{J}\times\mathbf{B}) = q_e(ZN_0 - N_e).$$

Then, we are led to the following equation for defining the electron density N_e

$$-\frac{\lambda_D^2}{T_{\mathrm{ref}}}[\nabla.(T_e\nabla(\log N_e)) + \Delta T_e] - \frac{\lambda_D^2}{T_{\mathrm{ref}}}\nabla.\left(\frac{1}{N_e}(q_e Z N_0 \mathbf{U}\times\mathbf{B} - \mathbf{J}\times\mathbf{B})\right) = \frac{ZN_0}{N_{\mathrm{ref}}} - \frac{N_e}{N_{\mathrm{ref}}}. \tag{2.26}$$

It is a nonlinear Poisson equation and it is crucial in plasma modelling when the quasi-neutrality assumption is not valid. Of course, this nonlinear elliptic equation needs to be supplemented by boundary conditions, for instance, Neumann conditions if the electrostatic field may be set to zero on the boundaries. But in simulations where one has to account for probes or electrodes, one must use Dirichlet conditions or more implicit boundary conditions (e.g., related to the electric current); this kind of problem is related to the so-called plasma sheath theory (see, e.g., [3, 109]).

Denoting $N_e/N_{\text{ref}} = e^{\Phi}$, the nonlinear Poisson equation reads also as

$$-\frac{\lambda_D^2}{T_{\text{ref}}}\nabla.(T_e\nabla\Phi)+e^{\Phi}-\frac{\lambda_D^2}{T_{\text{ref}}}\nabla.((q_eZN_0\mathbf{U}\times\mathbf{B}-\mathbf{J}\times\mathbf{B})\frac{e^{-\Phi}}{N_{\text{ref}}}) = \frac{ZN_0}{N_{\text{ref}}}+\frac{\lambda_D^2}{T_{\text{ref}}}\Delta T_e \tag{2.27}$$

Note that relation $N_e = N_{\text{ref}}e^{\Phi}$, where Φ is an electric potential divided by a temperature, is called the Maxwell–Boltzmann relation; it is often used in physics literature when the magnetic effects are not important (then (2.25) reduces $q_e\mathbf{E} = -T_e\nabla\Phi - \nabla T_e$). Notice that if the reference density N_{ref} is modified, we must also modify the potential Φ by adding a constant; as a matter of fact, in physical applications, these constants are fixed by the boundary conditions (see, e.g., [65]).

Summary. Assuming that $\mathbf{B}$ solves (2.24), the model consists of the following Euler–Poisson system. Recall that $\mathcal{E}_e = \frac{3}{2}N_eT_e$, $P_e = T_eN_e$ and $\mathbf{J} = \frac{1}{\mu^0}\text{curl}\,\mathbf{B}$.

$$(\mathcal{M})\quad \boxed{\begin{aligned}
&\text{(i)}\quad \frac{\partial}{\partial t}N_0+\nabla.(N_0\mathbf{U})=0,\\
&\text{(ii)}\quad m_o\left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)(N_0\mathbf{U})+\nabla(P_0+P_e)+\nabla.\mathbb{P}_B=q_e\mathbf{E}(ZN_0-N_e),\\
&\text{(iii)}\quad \left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_0)+P_0\nabla.\mathbf{U}=\Omega_{0e},\\
&\text{(iv)}\quad \left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_e)+P_e\nabla.\mathbf{U}-\nabla.\left(\frac{5T_e}{2q_e}\mathbf{J}\right)+\nabla.\mathbf{q}_{\text{th,e}}\\
&\qquad =-\Omega_{0e}+\frac{\nabla P_e}{N_e}.\left(\mathbf{U}(ZN_0-N_e)-\frac{\mathbf{J}}{q_e}\right),\\
&\text{(v)}\quad -\frac{\lambda_D^2}{T_{\text{ref}}}\left[\nabla.(T_e\nabla(\log N_e))+\Delta T_e\right]-\frac{\lambda_D^2}{T_{\text{ref}}}\nabla.\left(\frac{1}{N_e}(q_eZN_0\mathbf{U}\times\mathbf{B}-\mathbf{J}\times\mathbf{B})\right)\\
&\qquad =\frac{ZN_0-N_e}{N_{\text{ref}}}.
\end{aligned}}$$

with

$$\mathbf{E}+\frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B} = -\frac{1}{q_e}\frac{\nabla P_e}{N_e}+\frac{1}{q_e}\frac{1}{N_e}\mathbf{J}\times\mathbf{B}. \tag{2.28}$$

Of course, this system needs to be supplemented with boundary conditions; for the nonlinear Poisson equation, the simplest one is the Neumann condition.

Equation (iv) reads also as (2.23). We can state once more an energy balance accounting only for ion and electron energy:

$$\left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)\left(\mathcal{E}_e+\mathcal{E}_0+\frac{m_0}{2}N_0\,|\mathbf{U}|^2\right)+\nabla.\left((P_e+P_0)\mathbf{U}+\frac{5T_e}{2q_e}\mathbf{J}\right)+\nabla.\mathbf{q}_{\text{th,e}}=\mathbf{J}.\mathbf{E}. \tag{2.29}$$

[indeed, if we multiply by $(ZN_0 - N_e)\mathbf{U}$ the relation defining the field $\mathbf{E}$, we get

$$-q_e(ZN_0 - N_e)\mathbf{U}.\mathbf{E} = -(ZN_0 - N_e)\mathbf{U}.\frac{\nabla P_e}{N_e} + (\frac{ZN_0}{N_e} - 1)\mathbf{U}.\mathbf{J}\times\mathbf{B} = \mathbf{J}.\mathbf{E}$$

moreover $ZN_0\dfrac{1}{N_e}\mathbf{U}.(\mathbf{J}\times\mathbf{B}) - \mathbf{J}.\nabla P_e\dfrac{1}{q_e N_e} = \mathbf{J}.\mathbf{E}$, so we have the previous balance by using equation (ii) multiplied by $\mathbf{U}$ and remembering that $\nabla.\mathbb{P}_B = -\mathbf{J}\times\mathbf{B}$. $\square$]

In the previous relation, term $\mathbf{J}.\mathbf{E}$ is the one which appears in the magnetic energy equation [see (2.20)].

Remark 1. To my knowledge, it is an open problem to make a rigorous asymptotic analysis leading to a massless-electron model in the multidimensional framework (i.e., to let the ratio of electron mass to ion mass m_e/m_0 converge to zero in the ion–electron Euler system); see [63] for a proof in a monodimensional case. $\square$

Notice that the nonlinear Poisson equation (v) may be replaced by a simpler one if necessary and in equation (iv), term $(ZN_0 - N_e)\mathbf{U}$ may be neglected with respect to $\mathbf{J}/q_e$.

This system looks quite complicated. As a matter of fact, it is an open problem to show its well-posedness from a mathematical point of view. Nevertheless, we have a result that is a clue in this direction: it shows that equation (v) above is well-posed and that its solution N_e is positive.

If we denote

$$\mathbf{Q} = \frac{1}{N_{\text{ref}}}(\mathbf{J}\times\mathbf{B} - q_e Z N_0 \mathbf{U}\times\mathbf{B}), \qquad g = \frac{1}{T_{\text{ref}}}\Delta T_e,$$

(2.26) reads as

$$-\lambda_D^2\nabla.(\frac{T_e}{T_{\text{ref}}}\nabla\log N_e) + \frac{N_e}{N_{\text{ref}}} + \lambda_D^2\frac{1}{T_{\text{ref}}}\nabla.(\mathbf{Q}\frac{N_{\text{ref}}}{N_e}) = \frac{ZN_0}{N_{\text{ref}}} + \lambda_D^2 g. \tag{2.30}$$

Let $\mathcal{O}$ be a bounded set with a smooth boundary and denote by $\mathbf{n}$ the outwards normal to the boundary $\partial\mathcal{O}$ of $\mathcal{O}$. Equation (2.30) needs to be supplemented with a boundary condition on $\partial\mathcal{O}$. For the sake of simplicity, we can take one of the following conditions on $\partial\mathcal{O}$

$$\text{(i)}\ \frac{\partial}{\partial\mathbf{n}}(N_e T_e) = 0, \qquad \text{or} \quad \text{(ii)}\ \frac{\partial}{\partial\mathbf{n}}N_e = 0. \tag{2.31}$$

The first condition corresponds to the case where the normal gradient of the electron pressure is zero, i.e., the electric field is tangential to the boundary (if there is no magnetic effect); the second condition is a simplification of the first one.

Assume that $N_0, g, \mathbf{Q}$ and $\nabla.\mathbf{Q}$ belong to $L^\infty(\mathcal{O})$, that N_0 is strictly positive, and that T_e is a strictly positive bounded function. So we have:

Proposition 1. *Assume that* $\inf_x(\frac{Z}{N_{ref}}N_0 + \lambda_D^2 g) > 0$. *For (2.30) supplemented with one of the conditions (2.31) on* $\partial\mathcal{O}$, *there is a unique solution* N_e *in the cone* *of functions of* $H^1(\mathcal{O})$ *which are strictly positive and bounded, if* $\lambda_D^2 \|\nabla.\mathbf{Q}\|_{L^\infty}$ *is small enough. Moreover, we have*

$$\frac{1}{N_{ref}} N_e(x) \geq \frac{1}{2} \inf_x \left(\frac{ZN_0}{N_{ref}} + \lambda_D^2 g \right)$$

Of course, we have also $T_e \nabla \log(N_e T_e) \in L^2(\mathcal{O})$.

We may also address this problem in the case where the spatial domain is the whole space $\mathbf{R}^3$ (or $\mathbf{R}^2$) and the result is the same, provided that N_0 is in $L^2(\mathcal{O})$.

It is worth noting that (2.30) is also sometimes called the Poincaré equation (cf. [96]), at least in its simple form $-\lambda_D^2 \Delta(\log N_e) + N_e = N_0$, and it arises in many physical areas.

(b) A Simplified Model without Magnetic Effects

In the case where the magnetic effects are neglected, we may address a simplified model based on the barotropic approximation for the ion pressure: it is given by a closure with respect to N_0, for instance, P_0 is defined by $\mathcal{P}_0(N) = \mu_p N^\gamma$ (where μ_p and $\gamma \geq 1$). In this model, there is only one energy equation (the coupling term Ω_{0e} disappears) and the four unknowns N_0, $\mathbf{U}$, T_e and N_e satisfy the system.

$$\frac{\partial}{\partial t} N_0 + \nabla.(N_0 \mathbf{U}) = 0, \tag{2.32}$$

$$m_0 \left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot) \right) (N_0 \mathbf{U}) + \nabla \mathcal{P}_0(N_0) = -ZN_0 \frac{1}{N_e} \nabla(N_e T_e), \tag{2.33}$$

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot) \right) \left(\frac{3}{2} N_e T_e \right) + N_e T_e \nabla.\mathbf{U} = (ZN_0 - N_e) \frac{1}{N_e} \mathbf{U}.\nabla(N_e T_e), \tag{2.34}$$

$$-\frac{\lambda_D^2}{T_{\mathrm{ref}}} \nabla.(\frac{1}{N_e} \nabla(N_e T_e)) = \frac{ZN_0 - N_e}{N_{\mathrm{ref}}}. \tag{2.35}$$

In the momentum equation, the right-hand side may also be stated in a conservative form, which reads as

$$-\nabla(N_e T_e) + \varepsilon^0 (\nabla.(\mathbf{EE}) - \frac{1}{2} \nabla.|\mathbf{E}|^2)$$

with $q_e \mathbf{E} = \dfrac{1}{N_e} \nabla(N_e T_e)$.

As usual, we need to supplement this system with boundary conditions on $\partial\mathcal{O}$. For the three first equations, it is sufficient to state two conditions (for this topic, see the last section of this chapter); so, for the sake of simplicity, we can state

$$\frac{\partial}{\partial \mathbf{n}} T_e = 0, \qquad \mathbf{n}.\mathbf{U} = 0.$$

For elliptic equation (2.35), we consider condition (2.31 (ii)) (which is now identical to (2.31 (i))

We also have to supplement this system with initial conditions, that is to say $N_0(0) = N_0^{\text{ini}}$, $\mathbf{U}(0) = \mathbf{U}^{\text{ini}}$, $T_e(0) = T_e^{\text{ini}}$, where these initial values satisfy the boundary conditions and N_0^{ini}, T_e^{ini} are strictly positive.

Then, we may define the ion internal energy by $\mathcal{E}_0(N) = P_0(N)/(\gamma-1)$ if $\gamma \neq 1$ and $\mathcal{E}_0(N) = \mu_p(N \log(N) + 1)$ otherwise, and we have the following result.

Proposition 2. *For system (2.32)–(2.35), supplemented with the previous boundary conditions, we have the energy balance relation*

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)\left(\frac{3}{2}N_eT_e + \mathcal{E}_0(N_0) + \frac{m_0}{2}N_0|\mathbf{U}|^2\right) + \nabla.(\mathbf{U}(P_e + \mathcal{P}_0(N_0))) = 0 \tag{2.36}$$

Moreover, as soon as the temperature T_e remains strictly positive and bounded, the electron density N_e is a positive bounded function such that $\nabla \log(N_eT_e) \in L^2(O)$.

Using the boundary conditions, property (2.36) implies that for all times t, we get

$$\int_{\mathcal{O}} \left(\frac{3}{2}N_eT_e + \mathcal{E}_0(N_0) + \frac{m_0}{2}N_0|\mathbf{U}|^2\right) dx = C_0$$

that is to say there is a good balance for the global energy which is $\frac{3}{2}N_eT_e+\mathcal{E}_0(N_0)+\frac{m_0}{2}N_0|\mathbf{U}|^2$.

(*c*) *The Ion Euler–Poisson Model with Resistivity.*

In order to account for the resistive term in the generalized Ohm's law (2.21), we need to modify the previous system. Due to (2.21), the following expression of $\mathbf{E}$

$$\mathbf{E} + \frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B} = -\frac{1}{q_e}\frac{\nabla P_e}{N_e} + \frac{1}{q_e}\frac{1}{N_e}\mathbf{J}\times\mathbf{B} + \chi\mu^0\mathbf{J}$$

may be plugged in Gauss relation, so we get the modified nonlinear Poisson

$$-\varepsilon^0\nabla.(\frac{1}{N_e}\nabla(N_eT_e)) - \varepsilon^0\nabla.(q_e\frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B} - \frac{1}{N_e}\mathbf{J}\times\mathbf{B}) + \varepsilon^0\nabla.(\mu^0 q_e\chi\mathbf{J}) = q_e^2(ZN_0 - N_e).$$

Or with the same notations as above

$$-\lambda_D^2\nabla.(\frac{T_e}{T_{\text{ref}}}\nabla\log N_e) + \frac{N_e}{N_{\text{ref}}} + \lambda_D^2\frac{1}{T_{\text{ref}}}\nabla.(\mathbf{Q}\frac{N_{\text{ref}}}{N_e}) = \frac{ZN_0}{N_{\text{ref}}} + \lambda_D^2(g - \mu^0 q_e\mathbf{J}.\nabla\chi).$$

From a mathematical point of view this equation is the same as in the case $\chi = 0$ (only the right-hand side is changed).

The model for the evolution of N_0, $\mathbf{U}$, $\mathcal{E}_0$ is the same as in system ($\mathcal{M}$); the only difference is that there is a resistive term in the evolution equation (v) for $\mathcal{E}_e$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_e) + P_e\nabla.\mathbf{U} - \nabla.\left(\frac{5T_e}{2q_e}\mathbf{J}\right) + \nabla.\mathbf{q}_{\text{th,e}} \\ = -\Omega_{0e} + \frac{\nabla P_e}{N_e}.\left(\mathbf{U}(ZN_0 - N_e) - \frac{\mathbf{J}}{q_e}\right) + q_e\mu^0 N_e\chi\mathbf{J}.(\mathbf{U} - \mathbf{U}_e).\end{aligned} \tag{2.37}$$

We can easily check that the global energy balance (2.29) is still true. It is worth noticing the supplementary term $N_e \chi \mathbf{J}.(\mathbf{U} - \mathbf{U}_e)$, which is called the Joule effect term. Indeed, we will see below that in the quasi-neutral case, $\mathbf{J}$ and $N_e(\mathbf{U} - \mathbf{U}_e)$ will be equal up to the constant q_e; then this term equal to $\chi|\mathbf{J}|^2$ will be positive. It corresponds to the heating of the plasma occurring due to the electric current and the energy exchange due to the friction between electrons and ions.

2.2 Quasi-Neutrality Approximation

As above, we make the massless-electron approximation, but we assume moreover that the Debye length λ_D is very small if compared to the characteristic length L_{plasma}. In this framework corresponding to the so-called *plasma approximation,* the picture is now

$$\lambda_D \ll L_{\text{plasma}}, \qquad T_{\text{obs}} \sim \frac{L_{\text{plasma}}}{v_{\text{th,i}}} \gg \omega_p^{-1}, \qquad T_{\text{obs}} \gg \frac{L_{\text{plasma}}}{c}$$

Then the electron density is close to the ion density $N_e \simeq ZN_0$ (this is the quasi-neutrality approximation), but as claimed previously the electric field cannot be set to zero.

Our aim is to explain how this quasi-neutrality may be justified from a mathematical point of view. This analysis will be performed from the ion Euler–Poisson system in two different physical frameworks

- The first case corresponds to the case where the magnetic phenomena are negligible and a simple Ohm's law suffices; see (2.38)
- The second case corresponds to the general case, where the magnetic field needs to be accounted for and Ohm's law reads as (2.25)

In both cases, starting from the Gauss relation

$$\lambda_D^2 \frac{q_e}{T_{\text{ref}}} \nabla.\mathbf{E} = \frac{1}{N_{\text{ref}}}(ZN_0 - N_e)$$

where the Debye length comes into sight, we derive a nonlinear Poisson equation of the form (2.26).

2.2.1 *Asymptotic Analysis in the Nonmagnetized Case*

We introduce a dimensionless spatial variable related to L_{Plasma} and we set $\lambda = \lambda_D / L_{\text{Plasma}}$ which is small with respect to 1.

Here, we assume that instead of (2.25) we simply have

$$q_e \mathbf{E} = -\frac{1}{N_e} \nabla(N_e T_e) \tag{2.38}$$

then the potential $\log(N_e/N_{\text{ref}})$ is solution to a nonlinear Poisson equation. Set

$$\Phi_\lambda = \log(N_e/N_{\text{ref}})$$

in order to make clear the dependency with respect to the parameter λ, so Poisson equation [which is a simplified form of (2.26)] reads as

$$-\lambda^2 \nabla . \left(\frac{T_e}{T_{\text{ref}}} \nabla \Phi_\lambda \right) = \frac{ZN_0}{N_{\text{ref}}} - e^{\Phi_\lambda} + \lambda^2 g \tag{2.39}$$

recalling that $g = \frac{1}{T_{\text{ref}}} \Delta T_e$. In the same way, in the sequel, denote the electric field by $\mathbf{E}_\lambda$ instead of $\mathbf{E}$.

The asymptotic analysis of this equation when the small parameter tends to 0 was addressed in [23] about 20 years ago in the case of a constant temperature. A wide range of literature is devoted to this subject; see, e.g., [110, 115] and the references therein. Notice also that it is interesting to address such an asymptotic analysis with a vanishing Debye length in order to design numerical schemes for Euler–Poisson models in the case where quasi-neutral and non–quasi-neutral regions coexist in the same simulation domain; see, e.g., [35, 36, 39].

Here for the sake of completeness we state a generalization of the initial result of [23] in the case of a nonconstant temperature. We first assume that the spatial domain $\mathcal{O}$ is a smooth bounded open set; of course, (2.39) needs to be supplemented with a boundary condition, and as previously we set

$$\frac{\partial \Phi_\lambda}{\partial \mathbf{n}} = 0; \tag{2.40}$$

This corresponds to the case of an insulating material. It also possible to address a boundary condition of the type Φ_λ equal to given data (see below). Now, to state a rigorous mathematical result, we need to make some technical assumptions. First, we assume

$$\inf_x T_e > 0, \ \inf_x N_0 > 0, \ T_e \in L^\infty(\mathcal{O}), \ N_0 \in H^1(\mathcal{O}) \cap L^\infty(\mathcal{O}), \ \Delta T_e \in L^\infty(\mathcal{O}). \tag{2.41}$$

Moreover, for the sake of simplicity, we impose

$$\frac{\partial}{\partial \mathbf{n}} N_0 = 0, \qquad \text{on } \partial\mathcal{O}.$$

Proposition 3. *Assume that assumptions (2.41) hold. When λ goes to 0, the unique solution Φ_λ of nonlinear equation (2.39) with the boundary condition (2.40) satisfies*[1]

$$N_{ref} \exp(\Phi_\lambda) \to ZN_0 \qquad \textit{in } L^2(\mathcal{O}),$$
$$\nabla \Phi_\lambda \to \nabla \log(ZN_0) \qquad \textit{in } L^2(\mathcal{O}) \textit{ weakly}.$$

[1] A sequence u_n converges weakly to u, if $\int u_n v \to \int uv$ for any v in L^2.

The proof is given below. Notice that there are also results with weaker assumptions on the regularity of N_0 (see the proof). This proposition shows first that the quasi-neutrality holds

$$N_e \simeq ZN_0, \qquad P_e \simeq ZN_0T_e.$$

Moreover, this implies that the electric field $\mathbf{E}_\lambda$ satisfies

$$q_e\mathbf{E}_\lambda = -T_e\nabla\Phi_\lambda - \nabla T_e \to -\frac{1}{N_0}\nabla(N_0T_e) \qquad \text{in } L^2(\mathcal{O}) \text{ weakly,}$$

that is to say, it may be approximated in the following way

$$q_e\mathbf{E}_\lambda \simeq -\frac{1}{N_0}\nabla(N_0T_e). \tag{2.42}$$

This last relation for the electric field is a very usual one. In the case where the electron temperature is almost constant, this relation reads $q_e\mathbf{E}_\lambda \simeq -T_e\nabla(\log N_0)$ as it appears in the beginning of the handbook [27].

Other boundary conditions of Dirichlet type may be addressed; for instance. for the modelling of a plasma surrounded by a conducting material. As a matter of fact, if a given electric potential may be applied on two different parts of the boundary that correspond to conducting material, we can make the following modelling: assume that Γ_1 and Γ_2 are two smooth parts of the boundary $\partial\mathcal{O}$ that are disconnected, such that the potential Φ is given by a constant: B_1 on Γ_1 and B_2 on Γ_2.

Then we have the following result.

Proposition 4. *With the same assumptions as above, there exists a unique solution Φ_λ in $H^1(\mathcal{O})$ to (2.39) with the boundary conditions*

$$\Phi_\lambda = B_1 \quad \text{on } \Gamma_1; \quad \Phi_\lambda = B_2 \quad \text{on } \Gamma_2; \quad \frac{\partial}{\partial \mathbf{n}}\Phi_\lambda = 0 \quad \text{on } \partial\mathcal{O}\setminus(\Gamma_1\cup\Gamma_2) \tag{2.43}$$

Moreover, when λ goes to 0, we have

$$N_{ref}e^{\Phi_\lambda} \to ZN_0 \qquad \text{in } L^2(\mathcal{O}), \tag{2.44}$$

$$\nabla\Phi_\lambda \to \nabla\log(ZN_0) \qquad \text{in } L^2(\mathcal{O}) \text{ weakly.} \tag{2.45}$$

Generally, there exists boundary layers on the parts Γ_1 and Γ_2, since $N_{\text{ref}}\exp(B_q)$ is not equal to ZN_0.

See also [3,64] for other boundary condition problems related to the electrostatic sheath phenomena. We stress that it may lead to technical difficulties related to boundary layers where the quasi-neutrality fails.

Of course, we may address the case where the spatial domain $\mathcal{O}$ is the full space $\mathcal{O} = \mathbf{R}^3$ (or $\mathbf{R}^2$) and the asymptotic result may be proven in the same way.

Remark 2. Instead of (2.39), we may state the Poisson equation in the form

$$-\lambda^2 \nabla . \left(\frac{T_e}{T_{\text{ref}}} \nabla(\log P_e) \right) = \frac{Z N_0}{N_{\text{ref}}} - \frac{P_e}{N_{\text{ref}} T_e}, \qquad \frac{\partial P_e}{\partial \mathbf{n}} |_{\partial \mathcal{O}} = 0,$$

which is interesting if one prescribes a boundary condition on $\partial \mathcal{O}$ of the type $\frac{\partial P_e}{\partial \mathbf{n}} = 0$. Then, denoting $P_{e,\lambda}$ instead of P_e, Proposition 3 reads as follows: when λ goes to 0, the solution $P_{e,\lambda}$ of this equation satisfies

$$P_{e,\lambda} \to Z N_0 T_e \qquad \text{in } L^2(\mathcal{O}) \text{ strongly;}$$

$$\nabla(\log P_{e,\lambda}) \to \nabla(\log N_0 T_e) \qquad \text{in } L^2(\mathcal{O}) \text{ weakly.}$$

□

Remark 3. In the quasi-neutral approximation, determination of the electric field is a local problem in the interior of the domain, as it appears in (2.42); evaluation of $\mathbf{E}$ is performed with the local fluid variables only. But near the boundary $\partial \mathcal{O}$,this evaluation is more complex due to the boundary layers. □

2.2.2 Asymptotic Analysis in the Magnetized Case

We now address the more realistic case corresponding to a magnetized plasma (called also *current-carrying plasma*). We assume that the evolution of the field $\mathbf{B}$ is known (recall that the electron velocity is given by $\mathbf{U}_e = \frac{1}{N_e} N_0 \mathbf{U} - \frac{1}{N_e q_e} \mathbf{J}$).

We consider the generalized Ohm's law (2.25), i.e.,

$$q_e (\mathbf{E} + \frac{Z N_0}{N_e} \mathbf{U} \times \mathbf{B}) = -\frac{1}{N_e} \nabla(N_e T_e) + \frac{1}{N_e} \mathbf{J} \times \mathbf{B} \tag{2.46}$$

(if we want to account for the resistivity, it suffices to add to the right-hand side a term of the type $\chi \mu^0 \mathbf{J}$). The electron density $N_e = N_{\text{ref}} e^{\Phi_\lambda}$ is given by the solution of the nonlinear Poisson equation (2.27), which reads as

$$-\lambda^2 \nabla . (\frac{T_e}{T_{\text{ref}}} \nabla \Phi_\lambda) + e^{\Phi_\lambda} + \lambda^2 \frac{1}{T_{\text{ref}}} (\nabla . \mathbf{Q} e^{-\Phi_\lambda}) = \frac{Z N_0}{N_{\text{ref}}} + \lambda^2 g \tag{2.47}$$

Heuristically, if λ vanishes, we see that e^{Φ_λ} is very close to the function $\frac{Z N_0}{N_{\text{ref}}}$, i.e., the quasi-neutrality holds, and we may use the following approximations

$$N_e = N_{\text{ref}} e^{\Phi_\lambda} \simeq Z N_0, \qquad P_e \simeq Z N_0 T_e, \tag{2.48}$$

$$\frac{1}{N_e} \nabla N_e \simeq \frac{1}{N_0} \nabla N_0 \tag{2.49}$$

So, according to (2.46), we get

$$q_e(\mathbf{E}+\mathbf{U}\times\mathbf{B})\simeq -\frac{1}{N_0}\nabla(N_0T_e)+\frac{\mathbf{J}\times\mathbf{B}}{ZN_0}$$

Let us justify these approximations by an asymptotic analysis. So, we address equation (2.47) as above in a bounded open set $\mathcal{O}$, with boundary conditions (2.40)

Proposition 5. *Assume that (2.41) holds and that $g, \mathbf{Q}$ and $\nabla.\mathbf{Q}$ are bounded in L^∞. Then, when λ goes to 0, the unique solution Φ_λ to (2.47) with boundary condition (2.40) satisfies*

$$N_{ref}e^{\Phi_\lambda}\to ZN_0 \qquad \text{in } L^2(\mathcal{O}) \tag{2.50}$$

$$\nabla\Phi_\lambda\to\nabla\log N_0, \qquad \text{in } L^2(\mathcal{O}) \text{ weakly} \tag{2.51}$$

Thus, denoting the electric field by $\mathbf{E}_\lambda$ instead of $\mathbf{E}$, it satisfies

$$q_e\mathbf{E}_\lambda=-T_e\nabla\Phi_\lambda-\nabla T_e+\mathbf{Q}e^{-\Phi_\lambda}\to-\frac{1}{N_0}\nabla(N_0T_e)+\mathbf{Q}\frac{N_{\text{ref}}}{ZN_0} \quad \text{in } L^2(\mathcal{O}) \text{ weakly}$$

To justify the statement of quasi-neutral models, we must now address terms where the electric field appears in the general massless-electron model above. In particular, we need to deal with the term $\varepsilon^0\mathbf{E}_\lambda(\nabla.\mathbf{E}_\lambda)$ in the momentum equation (2.22). With the previous notations, we have $q_e\mathbf{E}_\lambda=-T_e\nabla\Phi_\lambda-\nabla T_e+\mathbf{Q}e^{-\Phi_\lambda}$ (recall that $\mathbf{Q}=0$ in the nonmagnetized case) and

$$\frac{\varepsilon^0}{N_{\text{ref}}}\mathbf{E}_\lambda(\nabla.\mathbf{E}_\lambda)=\left(T_e\nabla\Phi_\lambda+\nabla T_e-\mathbf{Q}e^{-\Phi_\lambda}\right)\frac{\lambda^2}{T_{\text{ref}}}\left(\nabla.(\frac{T_e}{T_{\text{ref}}}\nabla\Phi_\lambda)-\nabla.(\mathbf{Q}e^{-\Phi_\lambda})+g\right)$$

We now claim

Proposition 6. *Assume that the same assumptions as the one of Proposition 5 hold, then we have if $\lambda\to 0$,*

$$\frac{\varepsilon^0}{N_{\text{ref}}}\mathbf{E}_\lambda(\nabla.\mathbf{E}_\lambda)\to 0 \qquad \text{in } L^1(\mathcal{O}).$$

[**Indeed**, according to Proposition 5, we get

$$\lambda^2(\nabla.(\mathbf{Q}e^{-\Phi_\lambda})-\nabla.(\frac{T_e}{T_{\text{ref}}}\nabla\Phi_\lambda))=\frac{ZN_0}{N_{\text{ref}}}-e^{\Phi_\lambda}+\lambda^2 g\to 0 \qquad \text{in } L^2(\mathcal{O});$$

moreover, we know that $\nabla\Phi_\lambda$ is bounded in $L^2(\mathcal{O})$. Therefore, $\lambda^2\nabla\Phi_\lambda(\nabla.(\mathbf{Q}e^{-\Phi_\lambda})$ $-\nabla.(\frac{T_e}{T_{\text{ref}}}\nabla\Phi_\lambda))\to 0$ in L^1.$\square$]

Summary (About quasi-neutrality). The previous proposition means that either the plasma is resistive or not, and the term $\varepsilon^0\mathbf{E}(\nabla.\mathbf{E})$ may be neglected; then (2.22) becomes simply

$$m_0 \frac{\partial}{\partial t}(N_0 \mathbf{U}) + m_0 \nabla.(N_0 \mathbf{U}\mathbf{U}) + \nabla P_0 + \nabla P_e = \mathbf{J} \times \mathbf{B}. \tag{2.52}$$
$$= -\nabla.\mathbb{P}_B$$

Thus, the basic principles of quasi-neutral approximation may be summarized as follows.

- *One prescribes* $N_e = ZN_0$ and $P_e = ZN_0 T_e$.
- *For ion modelling, one has a classical fluid dynamic system, e.g., (2.52) for the momentum equation.*
- *The electrostatic field* $\mathbf{E}$ *is given by Ohm's law, which may read in the form:*

$$q_e Z N_0 (\mathbf{E} + \mathbf{U} \times \mathbf{B}) - \mathbf{J} \times \mathbf{B} + \nabla P_e = 0, \tag{2.53}$$

or, accounting for the resistive effect,

$$q_e Z N_0 (\mathbf{E} + \mathbf{U} \times \mathbf{B}) - \mathbf{J} \times \mathbf{B} + \nabla P_e = q_e Z N_0 \chi \mu^0 \mathbf{J}. \tag{2.54}$$

In these relations, the field $(\mathbf{E} + \mathbf{U} \times \mathbf{B})$ comes into sight; it is a natural quantity in all the MHD models. Moreover, it is worth noticing that the electron velocity never appears explicitly in these models (it is always evaluated thanks to $\mathbf{U}$ and $\mathbf{J}$) and that, generally, one has to account for the evolution of the electron temperature.

2.2.3 Proofs of the Propositions of Sects. 2.1 and 2.2

Proof of Proposition 1. Let us set $\lambda = \lambda_D$ which is fixed. For the sake of conciseness, we set $T_{\mathrm{ref}} = 1$ and $T_e = T$. Let us address equation (2.30) where N_e/N_{ref} is denoted by u ; moreover $f = \frac{ZN_0}{N_{\mathrm{ref}}} + \lambda^2 g$ and $r = -\nabla.\mathbf{Q}$ are assumed to be bounded. It reads

$$-\lambda^2 \nabla.\left(\frac{T}{u}\nabla u\right) + \lambda^2 \frac{1}{u^2}\mathbf{Q}.\nabla u + u = f + \lambda^2 \frac{r}{u}. \tag{2.55}$$

It is supplemented with Neumann boundary condition $\frac{\partial}{\partial \mathbf{n}} u = 0$; with the other boundary condition $\frac{\partial}{\partial \mathbf{n}}(Tu) = 0$, the proof is similar. There exist two constants α and f_∞ independent of λ such that

$$0 < 2\alpha = \inf_x f, \qquad f(x) \leq f_\infty$$

Denote also $T_{\mathrm{m}} = \inf T,\ q_\infty = \sup_{k=1,3} \|\mathbf{Q}_k\|_\infty$ and $r_\infty = \sup \|r\|_\infty$.

We will prove the existence and the uniqueness of a solution in the convex subset $K = \{v \in L^2(\mathcal{O})/\alpha \leq \inf v \leq \beta\}$ of the space $L^2(\mathcal{O})$, where β is defined below.

Existence of a Solution (Based on the Fixed-Point Schauder Theorem[2]). For all functions v in K, let us define the operator S from K into $L^2(\mathcal{O})$ defined by $S(v) = u$, where u is the unique solution in $H^1(\mathcal{O})$ to the linear equation

$$-\lambda^2\nabla.\left(\frac{T}{v}\nabla u\right) + \lambda^2\frac{1}{v^2}\mathbf{Q}.\nabla u + u = f + \lambda^2\frac{r}{v} \tag{2.56}$$

with boundary condition $\frac{\partial}{\partial \mathbf{n}}u = 0$. Since $|\frac{r}{v}| \leq \frac{r_\infty}{\alpha}$, according to the maximum principle,[3] we have

$$2\alpha - \lambda^2\frac{r_\infty}{\alpha} \leq u \leq f_\infty + \lambda^2\frac{r_\infty}{\alpha}.$$

We assume now that λ is smaller than $\lambda_0 = \alpha/\sqrt{r_\infty}$; therefore, we get

$$\alpha \leq u \leq \beta$$

where $\beta = f_\infty + \alpha$ and $u = S(v) \in K$.

To show that S is compact and continuous on $L^2(\mathcal{O})$, let us consider a sequence v_n in K that is bounded in $L^2(\mathcal{O})$ and we set $S(v_n) = u_n$. So we have

$$\lambda^2\left\langle\frac{T}{v_n}\nabla u_n, \nabla u_n\right\rangle + \lambda^2\left\langle\frac{1}{v_n^2}\mathbf{Q}.\nabla u_n, u_n\right\rangle + \|u_n\|^2 = \lambda^2\left\langle r\frac{1}{v_n}, u_n\right\rangle,$$

(in the sequel $\|\cdot\|$ means $\|\cdot\|_{L^2(\mathcal{O})}$). This implies

$$\lambda^2\frac{T_{\mathrm{m}}}{\beta}\|\nabla u_n\|^2 + \|u_n\|^2 \leq \lambda^2(C\|u_n\| + \frac{q_\infty}{\alpha^2}\|\nabla u_n\|\|u_n\|) \leq \lambda^2\left(\frac{T_{\mathrm{m}}}{2\beta}\|\nabla u_n\|^2 + \frac{\beta q_\infty^2}{2T_{\mathrm{m}}\alpha^4}\|u_n\|^2 + C\|u_n\|\right).$$

Therefore, the sequence ∇u_n is bounded in $L^2(\mathcal{O})$ and u_n is in a compact set of $L^2(\mathcal{O})$.

Assume now that the sequence converges to v_* in $L^2(\mathcal{O})$. We have seen that the sequence u_n is bounded in $H^1(\mathcal{O})$; thus a subsequence, still denoted u_n, converges to a function u_* in $L^2(\mathcal{O})$ and ∇u_n converges weakly to ∇u_*. Moreover, for each test function ψ we have

$$\lambda^2\left\langle\nabla u_n, \frac{T}{v_n}\nabla\psi\right\rangle + \lambda^2\left\langle\mathbf{Q}.\nabla u_n, \frac{1}{v_n^2}\psi\right\rangle + \langle u_n, \psi\rangle = \lambda^2\left\langle r\frac{1}{v_n}, \psi\right\rangle$$

[2]Let S be a mapping from a convex subset of a Banach space into itself, if S is continuous and compact with respect to the Banach topology, then S has a fixed point.

[3]See result 1 in the Appendix.

Now since the mappings $v \mapsto \frac{1}{v}$ and $v \mapsto \frac{1}{v^2}$ are Lipschitz on K, we know that $\frac{T}{v_n}\nabla\psi \to \frac{T}{v_*}\nabla\psi$ and $\frac{1}{v_n^2}\psi \to \frac{1}{v_*^2}\psi$ in L^2 strongly; thus we can pass to the limit in the previous relation and we get

$$\lambda^2 \left\langle \nabla u_*, \frac{T}{v_*}\nabla\psi \right\rangle + \lambda^2 \left\langle \mathbf{Q}.\nabla u_*, \frac{1}{v_*^2}\psi \right\rangle + \langle u_*, \psi\rangle = \lambda^2 \left\langle r\frac{1}{v_*}, \psi \right\rangle$$

Therefore, u_* is a solution to (2.56) with $v = v_*$. If another subsequence of u_n converges to a function u^* in $L^2(\mathcal{O})$, this function u^* is also a solution to (2.56) with $v = v_*$.

Since this solution is unique, we have $u^* = u_*$ and the entire sequence u_n converges to u_* in $L^2(\mathcal{O})$

$$S(v_n) \to u_* = S(v_*) \qquad \text{in } L^2(\mathcal{O})$$

The operator S is continuous from K into K endowed with the norm of $L^2(\mathcal{O})$. According to the Schauder theorem, there exists a fixed point $u = S(u)$; it belongs to $H^1(\mathcal{O})$ and is a solution of (2.55).

Uniqueness of the Solution. Assume that there exist two solutions u and U of (2.55) belonging to $H^1(\mathcal{O})$ which are strictly positive and bounded: for some α and β we have

$$\alpha \le u \le \beta, \qquad \alpha \le U \le \beta,$$

and they satisfy

$$-\lambda^2\nabla.\left(T\nabla(\log u - \log U)\right) + \lambda^2\frac{1}{u^2}\mathbf{Q}.\nabla u - \lambda^2\frac{1}{U^2}\mathbf{Q}.\nabla U + u - U = \lambda^2\left(\frac{r}{u} - \frac{r}{U}\right).$$

Then multiplying by $\log(u/U)$, we get

$$\lambda^2\left\langle \nabla\log\frac{u}{U}, T\nabla\log\frac{u}{U}\right\rangle - \lambda^2\left\langle \frac{1}{u} - \frac{1}{U}, \mathbf{Q}.\nabla(\log\frac{u}{U})\right\rangle + \left\langle u - U, \log\frac{u}{U}\right\rangle = \lambda^2\left\langle \frac{r}{u} - \frac{r}{U}, \log\frac{u}{U}\right\rangle. \tag{2.57}$$

If we had have $\mathbf{Q} = 0$, $r = 0$, the uniqueness would come from relation $(u - U)\log\frac{u}{U} \ge 0$. In the general case, denoting $z = (u - U)/U$, we see that $|z| \le \beta/\alpha$ and notice that

$$\left\langle \frac{1}{U(1+z)} - \frac{1}{U}, \mathbf{Q}.\nabla\log(1+z)\right\rangle \le C_0\|\nabla\log(1+z)\|\,\|z\|.$$

So, (2.57) implies

$$\lambda^2 T_{\mathrm{m}}\|\nabla\log(1+z)\|^2 + \langle Uz, \log(1+z)\rangle \le \lambda^2 C_0\|\nabla\log(1+z)\|\,\|z\| + \frac{r_\infty}{\alpha^2}\beta\lambda^2\langle z, \log(1+z)\rangle.$$

Then we have

$$\alpha \langle z, \log(1+z)\rangle \le \frac{\lambda^2}{4T_{\mathrm{m}}} C_0^2 \|z\|^2 + \frac{r_\infty}{\alpha^2}\beta\lambda^2 \langle z, \log(1+z)\rangle .$$

Thus, for $\lambda^2 \le \alpha^3/(2\beta r_\infty)$, we get

$$\langle z, \log(1+z)\rangle \le \frac{\lambda^2}{2T_{\mathrm{m}}} C_0^2 \|z\|^2.$$

and $z = 0$ if λ satisfies also $\lambda^2 C_0^2 \le 2T_{\mathrm{m}} \frac{\alpha}{\beta} \log(1+\frac{\beta}{\alpha})$. □

Proof of Proposition 2. According to (2.32) we have $N_0^{-1}(\partial_t + \mathbf{U}.\nabla)(N_0) = -\nabla.\mathbf{U}$, so using (2.33) we get in a classical way

$$N_0(\partial_t + \mathbf{U}.\nabla)(N_0^{-1}) = \nabla.\mathbf{U},$$

$$m_0 N_0(\partial_t + \mathbf{U}.\nabla)(\mathbf{U}) = -\nabla\mathcal{P}_0(N_0) - ZN_0\frac{1}{N_e}\nabla(N_eT_e).$$

Since $(N_0^{-1}\mathcal{E}_0(N_0))' = N_0^{-2}\mathcal{P}_0(N_0)$, we get

$$N_0(\partial_t + \mathbf{U}.\nabla)(m_0|\mathbf{U}|^2 + N_0^{-1}\mathcal{E}_0(N_0)) = N_0N_0^{-2}\mathcal{P}_0(N_0)(\partial_t + \mathbf{U}.\nabla)(N_0) - \mathbf{U}.\nabla\mathcal{P}_0(N_0) - ZN_0\frac{1}{N_e}\mathbf{U}.\nabla(N_eT_e),$$

$$= -\mathcal{P}_0(N_0)\nabla.\mathbf{U} - \mathbf{U}.\nabla\mathcal{P}_0(N_0) - ZN_0\frac{1}{N_e}\mathbf{U}.\nabla(N_eT_e)$$

Moreover, using the identity $N_0(\partial_t + \mathbf{U}.\nabla)w = \partial_t(N_0 w) + \nabla(N_0 w\mathbf{U})$, if we add this relation and (2.34), we get

$$(\partial_t + \nabla(\mathbf{U}.))(\mathcal{E}_e) + (\partial_t + \nabla(\mathbf{U}.))\left(m_0N_0\frac{1}{2}|\mathbf{U}|^2 + \mathcal{E}_0(N_0)\right) + N_eT_e\nabla.\mathbf{U} + \nabla(\mathcal{P}_0(N_0)\mathbf{U})$$

$$= -ZN_0\frac{1}{N_e}\mathbf{U}.\nabla(N_eT_e) + (ZN_0 - N_e)\frac{1}{N_e}\nabla(N_eT_e).\mathbf{U} = -\mathbf{U}.\nabla(N_eT_e).$$

Then, the result follows. □

Proof of Proposition 3. For λ small enough, the function $f_\lambda = ZN_0/N_{\mathrm{ref}} + \lambda^2 g$ satisfies $f_\lambda \ge 2\alpha$ for α strictly positive (independent of λ); it is bounded in L^2 uniformly with respect to λ. According to Proposition 1, there exists a unique solution Φ_λ in $H^1(\mathcal{O})$ to the equation

$$-\lambda^2\nabla.(T\nabla\Phi_\lambda) + e^{\Phi_\lambda} = f_\lambda. \tag{2.58}$$

and we have $\exp\Phi_\lambda \ge \alpha$.

We now multiply (2.58) by Φ_λ and integrate with respect to the space variable

$$\lambda^2\langle T\nabla\Phi_\lambda, \nabla\Phi_\lambda\rangle + \left\langle e^{\Phi_\lambda} - 1, \Phi_\lambda\right\rangle = \langle f_\lambda - 1, \Phi_\lambda\rangle .$$

Using the identity $(e^{\psi} - e^{\varphi})(\psi - \varphi) \geq (\psi - \varphi)^2 \min(e^{\psi}, e^{\varphi})$ for each ψ, φ (i.e., the convexity of the exponential function), we get

$$\lambda^2 T_{\mathrm{m}} \|\nabla \Phi_\lambda\|^2 + \alpha \|\Phi_\lambda\|^2 \leq \frac{\alpha}{2} \|\Phi_\lambda\|^2 + \frac{1}{2\alpha} \|f_\lambda - 1\|^2, \tag{2.59}$$

so there exists a constant C_0 (independent of λ) such that $\|\Phi_\lambda\| \leq C_0$. Using this bound, we get the estimate

$$\lambda^2 T_{\mathrm{m}} \|\nabla \Phi_\lambda\|^2 \leq C_0 \|f_\lambda - 1\|.$$

Thus, it exists C_1 such that

$$\lambda \|\nabla \Phi_\lambda\| \leq C_1. \tag{2.60}$$

We now denote $F = \log(ZN_0/N_{\mathrm{ref}})$, so multiplying (2.58) by $\Phi_\lambda - F$, we get

$$\lambda^2 \langle T \nabla \Phi_\lambda, \nabla (\Phi_\lambda - F) \rangle + \left\langle \Phi_\lambda - F, e^{\Phi_\lambda} - ZN_0/N_{\mathrm{ref}} \right\rangle = \lambda^2 \langle \Phi_\lambda - F, g \rangle .$$

Since $\langle T \nabla \Phi_\lambda, \nabla (\Phi_\lambda - F) \rangle \geq \langle T \nabla F, \nabla (\Phi_\lambda - F) \rangle$, according to the above mentioned property of the exponential function, we see that it exists α_0 independent of λ such that

$$\begin{aligned} \alpha_0 \|\Phi_\lambda - F\|^2 &\leq \left\langle \Phi_\lambda - F, e^{\Phi_\lambda} - ZN_0/N_{\mathrm{ref}} \right\rangle \\ &\leq \lambda^2 T_\infty \|\nabla \Phi_\lambda\| \|\nabla F\| + \lambda^2 \|\Phi_\lambda\| \|g\| - \lambda^2 \langle F, g \rangle \end{aligned}$$

Due to (2.60) and the hypothesis of the proposition, we have when λ goes to 0

$$\Phi_\lambda - F = \Phi_\lambda - \log(ZN_0/N_{\mathrm{ref}}) \to 0 \quad \text{in } L^2,$$

$$e^{\Phi_\lambda} - ZN_0/N_{\mathrm{ref}} \to 0 \quad \text{in } L^2. \tag{2.61}$$

Since $\nabla \Phi_\lambda$ is bounded in $L^2(\mathcal{O})$, according to result 4 in the Appendix, we get $\nabla \Phi_\lambda \rightharpoonup \nabla F$ in $L^2(\mathcal{O})$ weakly; so (2.45) holds. □

Remark 4. If the smoothness assumption on N_0 is false, i.e., ∇N_0 is not in $L^2(\mathcal{O})$, then using the bound of Φ_λ in $L^2(\mathcal{O})$, we may show that $\Phi_\lambda \to \log(ZN_0)$ in $L^2(\mathcal{O})$ weakly, but one can only prove that $\nabla \Phi_\lambda \to \nabla \log(ZN_0)$ in distribution meaning.

□

Proof of Proposition 4. We will use the following lemmas, the proof of which is given below.

Lemma 1. *Let Θ_λ be the solution of*

$$-\lambda^2 \Delta\Theta_\lambda + \Theta_\lambda = 0, \tag{2.62}$$

supplemented with the boundary condition: $\Theta_\lambda|_{\Gamma_1} = 1$, $\Theta_\lambda|_{\Gamma_2} = 0$, $\frac{\partial}{\partial \mathbf{n}}\Theta_\lambda|_{\partial\mathcal{O}\setminus\Gamma_1 \cup\Gamma_2} = 0$. *Then, we have*

$$\text{(i)} \quad \int_{\mathcal{O}} \Theta_\lambda \le C_* \lambda, \qquad \text{(ii)} \quad \|\nabla\Theta_\lambda\| \le C_0 \frac{1}{\lambda}, \qquad \text{(iii)} \quad \int_{\partial\mathcal{O}} \left|\frac{\partial\Theta_\lambda}{\partial \mathbf{n}}\right| \le C_1 \frac{1}{\lambda}.$$

Lemma 2. *With the assumptions of the proposition, the solution Φ_λ of (2.39) and (2.43) is such that there exists C_2 with*

$$\int_{\Gamma_1\cup\Gamma_2} \left|\frac{\partial\Phi_\lambda}{\partial \mathbf{n}}\right| \le C_2 \frac{1}{\lambda}.$$

As for the Neumann boundary condition, for all λ, there exists a unique solution Φ_λ to the Poisson equation

$$-\lambda^2\nabla.\,(T\nabla\Phi_\lambda) + e^{\Phi_\lambda} = f_\lambda$$

supplemented with (2.43). Moreover, for λ small enough, there exist two constants α, β (independent of λ) such that

$$\log\alpha \le \Phi_\lambda \le \log\beta.$$

Multiplying the Poisson equation by Φ_λ, we get as above

$$\lambda^2 \left\langle T, |\nabla\Phi_\lambda|^2 \right\rangle + \left\langle e^{\Phi_\lambda} - 1, \Phi_\lambda \right\rangle = \langle f_\lambda - 1, \Phi_\lambda\rangle + \lambda^2 \int_{\Gamma_1\cup\Gamma_2} \Phi_\lambda T \frac{\partial\Phi_\lambda}{\partial \mathbf{n}}.$$

Since $\left\langle e^{\Phi_\lambda} - 1, \Phi_\lambda \right\rangle \ge \alpha\Phi_\lambda^2$, according to Lemma 2, one sees that

$$\lambda^2 T_{\min}\|\nabla\Phi_\lambda\|^2 + \frac{\alpha}{2}\|\Phi_\lambda\|^2 \le C_3 + \lambda C_2 C_4,$$

therefore we get

$$\|\nabla\Phi_\lambda\| \le C_5/\lambda.$$

On the other hand, with $F = \log(ZN_0/N_{\text{ref}})$, multiplying the Poisson equation by $\Phi_\lambda - F$, we get

$$\lambda^2 \langle T\nabla(\Phi_\lambda - F), \nabla\Phi_\lambda\rangle + \left\langle \Phi_\lambda - F, e^{\Phi_\lambda} - \frac{ZN_0}{N_{\text{ref}}} \right\rangle =$$
$$\lambda^2 \sum_{q=1,2} \int_{\Gamma_q} (B_q - F) T \frac{\partial\Phi_\lambda}{\partial \mathbf{n}} + \lambda^2 \langle \Phi_\lambda - F, g\rangle$$

Thus, using the inequality $\langle T\nabla(\Phi_\lambda - F), \nabla\Phi_\lambda\rangle \geq \langle T\nabla(\Phi_\lambda - F), \nabla F\rangle$, we get according to Lemma 2:

$$\alpha\|\Phi_\lambda - F\|^2 \leq \lambda^2 T_\infty \|\nabla F\|\|\nabla\Phi_\lambda\| + \lambda C_2(\sup(B_1, B_2) + \|F\|_\infty)T_\infty + C_6\lambda^2.$$

So we see that $\|\Phi_\lambda - F\|^2 = O(\lambda)$ and the remaining part of the proposition follows. □

Proof of the Lemma 1. Denote by $\delta(x)$ the distance from x to the boundary Γ_1. Let now θ_λ be the solution of $-\lambda^2\Delta\theta_\lambda + \theta_\lambda = 0$, supplemented with the boundary condition $\theta_\lambda|_{\partial\mathcal{O}} = 1$. We have the usual bound (cf. Lemma 2 of [17])

$$0 \leq \theta_\lambda(x) \leq e^{-\delta(x)/4\lambda}, \qquad \text{for } \delta(x) \geq 4\lambda/3$$

Then, for each local map $\mathcal{O}_q$, one knows that there exists a local coordinate system x_1, x_2, x_3 such that the boundary is of the form $x_3 = f(x_1, x_2)$ and a constant r such that

$$\delta(x) \geq r(x_3 - f(x_1, x_2)).$$

so integrating over x, we get

$$\int_{\mathcal{O}\cap\mathcal{O}_q} \theta_\lambda(x)dx \leq \int_{\mathcal{O}\cap\mathcal{O}_q} e^{-\delta(x)/4\lambda}dx \leq \int\int_{x\in\mathcal{O}_q}\left(\int_0^{+\infty} e^{-r(x_3 - f(x_1,x_2))/4\lambda}dx_3\right)dx_1dx_2 \leq C\lambda.$$

Then $\int_{\mathcal{O}} \theta_\lambda(x)dx \leq C_*\lambda$. Now, since $0 \leq \Theta_\lambda \leq \theta_\lambda$, we get point (i).

Moreover, Θ_λ is the minimum function in $H^1(\mathcal{O})$ for the functional $\Theta \longmapsto \lambda^2\|\nabla\Theta\|^2 + \|\Theta\|^2$ with the constraints $\Theta_\lambda|_{\Gamma_1} = 1$, $\Theta_\lambda|_{\Gamma_2} = 0$. Thus, there exists C_0 such that $\lambda^2\|\nabla\Theta_\lambda\|^2 \leq C_0^2$ (it suffices to compare with a smooth function ζ such that $\zeta|_{\Gamma_1} = 1$, $\zeta|_{\Gamma_2} = 0$). Now consider a positive smooth test function ξ (independent of λ) such that $\xi|_{\Gamma_1} = 1$, $\xi|_{\Gamma_2} = 0$ and multiply (2.62) by ξ and integrate over the domain $\mathcal{O}$, we have

$$0 \leq \lambda^2\int_{\Gamma_1}\frac{\partial}{\partial\mathbf{n}}\Theta_\lambda = \lambda^2\int_{\mathcal{O}}\nabla\xi\nabla\Theta_\lambda dx + \int_{\mathcal{O}}\xi\Theta_\lambda dx \leq \lambda C_0\|\nabla\xi\| + \lambda C_*.$$

In the same way, multiplying (2.62) by η such that $\eta|_{\Gamma_1} = 0$, $\eta|_{\Gamma_2} = 1$ and integrating over the domain $\mathcal{O}$, we get

$$0 \leq -\lambda^2\int_{\Gamma_2}\frac{\partial}{\partial\mathbf{n}}\Theta_\lambda = -\lambda^2\int_{\mathcal{O}}\nabla\eta\nabla\Theta_\lambda dx - \int_{\mathcal{O}}\eta\Theta_\lambda dx \leq \lambda C_0\|\nabla\eta\| + \lambda C_*,$$

then the last point follows. □

Proof of Lemma 2. Since Φ_λ is bounded by $\log\beta$, we first notice that

$$1 + \Phi_\lambda \leq \exp(\Phi_\lambda) \leq \Phi_\lambda + \gamma.$$

with $\gamma = \max(\beta, 1 - \log\alpha)$

Let $a^1 = \inf_\lambda \inf_x (f_\lambda)$, and $a^2 = \sup_\lambda \sup_x (f_\lambda)$. Now, let ϕ^1 and ϕ^2 be the solutions of the linear problems

$$-\lambda^2 \nabla.\left(T\nabla\phi^1_\lambda\right) + \phi^1_\lambda = a^1 - \gamma$$
$$-\lambda^2 \nabla.\left(T\nabla\phi^2_\lambda\right) + \phi^2_\lambda = a^2 - 1$$

with the same boundary conditions $\phi^q_\lambda\big|_{\Gamma_1} = B_1$, $\phi^q_\lambda\big|_{\Gamma_2} = B_2$. So, according to the maximum and minimum principle, we check that

$$\phi^1_\lambda \le \Phi_\lambda \le \phi^2_\lambda.$$

Therefore, we get on each part Γ_1 and Γ_2 of the boundary

$$\left|\frac{\partial}{\partial \mathbf{n}}\Phi_\lambda\right| \le \max\left(\left|\frac{\partial}{\partial \mathbf{n}}\phi^1_\lambda\right|, \left|\frac{\partial}{\partial \mathbf{n}}\phi^2_\lambda\right|\right)$$

So if we prove that

$$\int_{\Gamma_1} \left|\frac{\partial}{\partial \mathbf{n}}\phi^q_\lambda\right| = O\left(\frac{1}{\lambda}\right), \qquad \text{for } q = 1, 2 \tag{2.63}$$

and the same bounds for Γ_2, then the lemma follows.

According to linearity properties, to show (2.63), it suffices to prove this relation for the following toy problem. So consider Ψ solution to

$$-\lambda^2 \nabla.\left(T\nabla\Psi\right) + \Psi = 0, \qquad \Psi|_{\Gamma_1} = 1, \qquad \Psi|_{\Gamma_2} = 0, \qquad \left.\frac{\partial}{\partial \mathbf{n}}\Psi\right|_{\partial\mathcal{O}\setminus\Gamma_2\cup\Gamma_1} = 0.$$

This solution is positive and bounded by 1. So, multiplying it by Θ_λ, we get

$$\int_{\Gamma_1} T\frac{\partial\Psi}{\partial \mathbf{n}} = \int_{\mathcal{O}} T\nabla\Psi.\nabla\Theta_\lambda + \frac{1}{\lambda^2}\left\langle \Psi, \Theta_\lambda\right\rangle.$$

and multiplying (2.62) by $T\Psi$,

$$\int_{\Gamma_1} T\Psi\frac{\partial\Theta_\lambda}{\partial \mathbf{n}} = \int_{\mathcal{O}} T\nabla\Psi.\nabla\Theta_\lambda + \frac{1}{\lambda^2}\left\langle T\Psi, \Theta_\lambda\right\rangle.$$

Therefore, we have

$$0 \le T_{\min}\int_{\Gamma_1} \frac{\partial\Psi}{\partial \mathbf{n}} \le T_\infty \int_{\Gamma_1}\left|\frac{\partial\Theta_\lambda}{\partial \mathbf{n}}\right| + \frac{1}{\lambda^2}\left\langle (1-T)\Psi, \Theta_\lambda\right\rangle + C.$$

Then, according to Lemma 1, we see that the right-hand side is a $O(\lambda^{-1})$ and that (2.63) holds for Γ_1. For Γ_2, it suffices to integrate the equation satisfied by Ψ over the domain $\mathcal{O}$ and to use Lemma 1. □

Proof of Proposition 5. Recall that $f_\lambda = ZN_0/N_{\mathrm{ref}} + \lambda^2 g$ is uniformly bounded, then the equation reads as

$$-\lambda^2\nabla.\,(T\nabla\Phi_\lambda) + \lambda^2\nabla.(e^{-\Phi_\lambda}\mathbf{Q}) + e^{\Phi_\lambda} = f_\lambda$$

According to Proposition 1, we know that $\frac{1}{2}(\inf_x f_\lambda(x)) = \alpha \le \exp\Phi_\lambda \le \beta$ with α, β independent from λ.Then, multiplying this equation by Φ_λ, we get

$$\lambda^2\left\langle T\nabla\Phi_\lambda, \nabla\Phi_\lambda\right\rangle + \left\langle\Phi_\lambda, e^{\Phi_\lambda} - 1\right\rangle = \left\langle\Phi_\lambda, f_\lambda - 1\right\rangle + \lambda^2\left\langle e^{-\Phi_\lambda}\mathbf{Q}, \nabla\Phi_\lambda\right\rangle$$
$$\lambda^2 T_{\mathrm{m}}\|\nabla\Phi_\lambda\|^2 \le C(1+\lambda^2) + \lambda^2 C\,\|\nabla\Phi_\lambda\|\|e^{-\Phi_\lambda}\|.$$

Then by the same argument as above, there exists a constant C_* independent from λ such that

$$\lambda\|\nabla\Phi_\lambda\| \le C_*$$

Moreover, multiplying by $\Phi_\lambda - F$, where $F = \log(f_\lambda)$, we get

$$\lambda^2\left\langle T\nabla\Phi_\lambda, \nabla\left(\Phi_\lambda - F\right)\right\rangle + \left\langle\Phi_\lambda - F, e^{\Phi_\lambda} - f_\lambda\right\rangle = \lambda^2\left\langle\Phi_\lambda - F, \nabla.(e^{-\Phi_\lambda}\mathbf{Q})\right\rangle.$$

Since $\left\langle T\nabla\Phi_\lambda, \nabla\left(\Phi_\lambda - F\right)\right\rangle \ge \left\langle T\nabla F, \nabla\left(\Phi_\lambda - F\right)\right\rangle$, we obtain

$$\alpha\|\Phi_\lambda - F\|^2 \le \left\langle\Phi_\lambda - g, e^{\Phi_\lambda} - f\right\rangle \le \lambda^2 T_\infty\|\nabla\Phi_\lambda\|\|\nabla F\| + \lambda^2(\|\Phi_\lambda\| + \|F\|)(C + CC_*).$$

thus $\|\Phi_\lambda - F\| \to 0$. Then, the result follows. $\square$

2.3 Two-Temperature Euler Models and Magneto-Hydrodynamics

We now perform the quasi-neutrality approximation that is justified above. With this approximation, different quasi-neutral models may be stated; these models depend on the physics to be accounted for. We address in the first subsection plasmas without magnetic effect, which leads to the usual two-temperature Euler system that we address in different physical situations.

In the second subsection, we describe the popular magneto-hydrodynamic (MHD) system, which is relevant if we must account for magnetic effects; we address this in the two-temperature framework, which enables us to emphasize the energy exchange among ion population, electron population, and magnetic energy. To get the one-temperature MHD system, it suffices to mingle the ion and electron temperature, as explained below.

It is worth noticing that the two-temperature Euler system with radiative coupling and thermal conduction is the basis of most of the computational codes for solar astrophysics and for the simulation of Inertial Confinement Fusion experiments, while the electron MHD system is the basis of all models of reduced MHD (see below) which are used in practice for the fluid simulations of the evolution of tokamak plasmas.

2.3.1 The Two-Temperature Euler System

Here, we do not account for the time evolution of the magnetic field. Firstly, we address the case where no electric current is accounted for and we state a simple system including the evolution of the ion temperature and of the electron temperature without radiative phenomenon. Secondly, we give some enlightenment on a simple thermal conduction model and on a radiative diffusion model. Lastly, we deal with the case of current-carrying plasmas.

As was explained above, since the quasi-neutrality approximation is valid, the general prescription is that $N_e = ZN_0$, $P_e = ZN_0T_e$ and the Ohm's law reduces to

$$q_e\mathbf{E} = -(ZN_0)^{-1}\nabla P_e = -{N_0}^{-1}\nabla(N_0T_e)$$

Now, the ion momentum equation reads

$$m_0\frac{\partial}{\partial t}(N_0\mathbf{U}) + m_0\nabla.(N_0\mathbf{U}\mathbf{U}) + \nabla P_0 + \nabla P_e = 0. \tag{2.64}$$

Recall that Ω_{0e} is proportional to $(T_e - T_0)$ (see formula (2.68) below) and the ion and the electron energy balance equations (2.4) and (2.37) simply read as

$$\frac{\partial}{\partial t}\mathcal{E}_0 + \nabla.(\mathcal{E}_0\mathbf{U}) + P_0\nabla.\mathbf{U} = \Omega_{0e}, \tag{2.65}$$

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathcal{E}_e\mathbf{U}) + P_e\nabla.\mathbf{U} + \nabla\mathbf{q}_{\text{th,e}} = -\Omega_{0e}. \tag{2.66}$$

Then, for the ion total energy balance, we get the relation

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)\left(\mathcal{E}_0 + m_0N_0\frac{1}{2}|\mathbf{U}|^2\right) + \nabla.(P_0\mathbf{U}) + \mathbf{U}.\nabla P_e = \Omega_{0e},$$

and for the global energy balance relation

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)\left(\mathcal{E}_0 + \mathcal{E}_e + m_0N_0\frac{1}{2}|\mathbf{U}|^2\right) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}} = 0. \tag{2.67}$$

It is classical to use the density $\rho = m_0 N_0$ and to introduce the specific internal energies

$$\varepsilon_0 = \frac{\mathcal{E}_0}{m_0 N_0}, \qquad \varepsilon_e = \frac{\mathcal{E}_e}{m_0 N_0} = \frac{3}{2}\frac{Z}{m_0} T_e.$$

Recall that the relation between pressure and internal energies is given by

$$P_e = \frac{2}{3}\mathcal{E}_e = \frac{2}{3}\rho\varepsilon_e = Z N_0 T_e, \qquad P_0 = (\gamma - 1)\mathcal{E}_0 = (\gamma - 1)\rho\varepsilon_0 = N_0 T_0.$$

(i.e., an equation of state of perfect gas law type). Then besides the continuity equation (2.1)

$$\frac{\partial}{\partial t} N_0 + \nabla.(N_0 \mathbf{U}) = 0,$$

the system that consists of (2.64)–(2.66) is closed. It is formally equivalent to the system (2.64), (2.66), and (2.67).

Now, introduce the Lagrangian derivative

$$\frac{D}{Dt}\bullet = \frac{\partial}{\partial t}\bullet + \mathbf{U}.\nabla(\bullet).$$

According to the continuity equation, we check that

$$\rho \frac{D}{Dt}\bullet = \frac{\partial}{\partial t}(\rho\bullet) + \nabla.(\mathbf{U}\rho\bullet).$$

Summary. Using the Lagrangian derivatives, the system may be stated as follows

$$(\mathcal{E}2\mathcal{T}) \boxed{\begin{array}{ll} \text{(i)} & \rho \dfrac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0, \\[2ex] \text{(ii)} & \rho \dfrac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) = 0, \\[2ex] \text{(iii)} & \rho \dfrac{D}{Dt}\varepsilon_0 + P_0 \nabla.\mathbf{U} = \Omega_{0e}, \\[2ex] \text{(iv)} & \rho \dfrac{D}{Dt}\varepsilon_e + P_e \nabla.\mathbf{U} + \nabla.\mathbf{q}_{\mathrm{th,e}} = -\Omega_{0e} \end{array}}$$

Of course, equation (iii) may be replaced by the following equation for the total energy

$$\rho \frac{D}{Dt}\left(\varepsilon_0 + \varepsilon_e + \frac{1}{2}|\mathbf{U}|^2\right) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}) + \nabla.\mathbf{q}_{\mathrm{th,e}} = 0$$

that is to say

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)\left(\mathcal{E}_0 + \mathcal{E}_e + \rho\frac{1}{2}|\mathbf{U}|^2\right) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}} = 0$$

It is worthwhile noticing that although the roles of ions and electrons seem to be similar in (2.65) and (2.66), it is not the case. Indeed, we know that when shocks occur, some information is missing if we state only the system (2.64)–(2.66). In its reduced form without the thermal conduction term $\mathbf{q}_{\text{th,e}}$, it has been analyzed recently from a mathematical point of view; it is related to the definition of the non-conservative products like $P_e\nabla.\mathbf{U}$ when $\mathbf{U}$ is discontinuous. It may be shown that one has to add some information on the entropy deposition, see [37] for example. Zel'dovich and Raizer noted a long time ago in [117] that this entropy deposition needs to be made on ion population only. This fact has been justified also in [32] with an asymptotic analysis (in the case where the Debye length goes to zero). In [31], it has been proved in one dimension that the solution of the ion Euler system coupled with the nonlinear Poisson equation converge to the solution of (2.1) and (2.64) in the case of constant temperatures.

Notice also that without accounting for the thermal conduction $\nabla.\mathbf{q}_{\text{th,e}}$, this system may enter into the framework of the well-posed Lagrangian system described in [44]. Its numerical treatment is a whole subject not covered in this book, but it is related to the hyperbolic property of this system; this property is analyzed in the last section of this chapter.

2.3.1.1 Accounting for the Thermal Conduction

More than 50 years ago, Spitzer [111] stated a usual formula for the electron thermal conduction in hot plasmas. In its simplest term, it reads as

$$\mathbf{q}_{\text{th,e}} = -\frac{\kappa}{m_0}\nabla T_e^{7/2}$$

where κ is a positive coefficient that is roughly speaking proportional to the inverse of the Coulomb logarithm $\log\Lambda$ (recall that $\log\Lambda$ is in the order of some units and may be considered as a very smooth function of the space variable).

Notice that the accurate modelling of the electron thermal conduction is a very active area; besides [111], see pioneering works such as [54, 79] and more recent works [94, 104] and the references therein (as a matter of fact, in hot plasmas the heat flux is not simply proportional to the gradient of the temperature, so it is called a nonlocal heat flux).

To achieve the statement of the evolution system ($\mathcal{E}2\mathcal{T}$), notice that the coupling term Ω_{0e} between the ion and electron temperatures reads generally as

$$\Omega_{0e} = \frac{\rho}{m_0}\zeta(T_e)(T_e - T_0) \tag{2.68}$$

where the inverse of the relaxation time ζ between the two temperature is given by

$$\zeta(T_e) = \rho\zeta_C / T_e^{3/2}$$

here the coefficient ζ_C is proportional to $Z^2(\log\Lambda)$, see [38, 111] for more details.

From $(\mathcal{E}2\mathcal{T})$, let us focus on the subsystem corresponding to the two-temperature evolution equations with thermal conduction and two-temperature coupling; it reads

$$\rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \frac{\rho}{m_0}\zeta(T_e)(T_e - T_0),$$
$$\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} - \nabla.(\frac{\kappa}{m_0}\nabla T_e^{7/2}) = \frac{\rho}{m_0}\zeta(T_e)(T_0 - T_e).$$

Here, we can consider a "hydrodynamics part" corresponding to the pressure terms and a "thermal part" corresponding to the conduction and the two-temperature coupling. This thermal part reads as follows

$$\rho\frac{3Z}{2}\frac{\partial}{\partial t}T_e - \nabla.(\kappa\nabla T_e^{7/2}) = \rho\zeta(T_e)(T_0 - T_e) \tag{2.69}$$

$$\rho\frac{3}{2}\frac{\partial}{\partial t}T_0 = \rho\zeta(T_e)(T_e - T_0), \tag{2.70}$$

where the density $\rho = \rho(x)$ is frozen and is strictly positive, κ is a strictly positive function depending on x, and ζ is a strictly positive function of x, T_e. We now consider this system on a bounded open set $\mathcal{O}$ with smooth boundary and state a result for its well-posedness. Of course, it needs to be supplemented with boundary conditions for T_e and initial conditions $T_e^{\text{ini}}, T_0^{\text{ini}}$. For the sake of simplicity, we consider the Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}T_e = 0$$

and we assume that two constants α and β exist and are such that

$$0 < \alpha \le T_e^{\text{ini}} \le \beta, \qquad \alpha \le T_0^{\text{ini}} \le \beta. \tag{2.71}$$

Now, we need to make technical assumptions. The function ζ satisfies the following properties: two constants K, κ_0 exist and are such that

$$\kappa(x) \ge \kappa_0 > 0;\ |\zeta(x, T_e) - \zeta(x, T_e')| \le K|T_e - T_e'| \text{ for all } x, T_e, T_e' \text{ s.t. } \alpha \le T_e, T_e' \le \beta \tag{2.72}$$

Then we have the following result (the proof of which is given at the end of the section).

Proposition 7. *Assume that (2.72) holds and the initial conditions satisfy (2.71) and belong to $H^1(\mathcal{O})$ and $L^2(\mathcal{O})$. Then for system (2.69), (2.70) and for any final time t_f, there is a unique solution $(T_e(t), T_0(t))$ belonging to the spaces $L^2(0,t_f;H^1(\mathcal{O})) \cap C(0,t_f,L^2(\mathcal{O}))$ and $C(0,t_f;L^2(\mathcal{O}))$ and such that*

$$\alpha \le T_e(t) \le \beta, \qquad \alpha \le T_0(t) \le \beta \tag{2.73}$$

Of course, according to (2.73), if the density ρ is bounded, the function $\zeta(x, T_e)$ which is proportional to $\rho(x)T_e^{-3/2}$ satisfies the Lipschitz condition (2.72).

Notice that if assumption (2.72) is not true and if the initial conditions $(T_e^{\mathrm{ini}}, T_0^{\mathrm{ini}})$ are zero in a subdomain there are some technical difficulties to prove the uniqueness of the solution (up to my knowledge, it is an open problem).

2.3.1.2 Accounting for Radiative Coupling

In the case of hot plasmas, such as in stellar plasmas or in Inertial Confinement Fusion plasmas, radiative coupling has also to be taken into account. So, we describe here a simple model to address radiative phenomena: the frequency-dependent radiative diffusion. To account for these phenomena, we must introduce the radiative energy density $\mathcal{E}_\nu(t,x)$ corresponding to the energy of the photon population with frequency ν, at time t and position x; the frequency variable ν belongs to the half-line $\mathbf{R}^+$ (a photon of frequency ν has a energy equal to $h\nu$ where h is the Planck constant). Notice that $\mathcal{E}_\nu$ is evaluated in the fluid's reference frame.

Recall that each hot material emits spontaneously and continuously photonic radiation over a wide spectrum of frequency; if it is at local thermodynamic equilibrium, the emitted radiation is described by Planck's function which is a universal function depending only on the electron temperature T_e and frequency variable ν and which reads

$$B_\nu(T_e) = \frac{8\pi}{c^3} h\nu^3 (\exp(\frac{h\nu}{T_e}) - 1)^{-1}$$

up to a multiplicative factor $c/(4\pi)$. It is called black body emission. The integral of $B_\nu(T_e)$ over all the frequency is equal to $a_{\mathrm{r}} T_e^4$ (here a_{r} is the universal radiation constant).

Denote by $\sigma_\nu = \sigma_\nu(T_e)$ the frequency-dependent absorption coefficient (also called opacity); it depends on the temperature and on the material characteristic. For instance, for fully ionized plasma, we may use the so-called Kramer formula for the opacity

$$\sigma_\nu = C_0 \frac{1}{\nu^3} \frac{1}{{T_e}^{1/2}} (\exp(\frac{h\nu}{T_e}) - 1)$$

where the coefficient C_0 is a function of the atomic number of the material and its density. Moreover, denote by σ_{th} the so-called Thomson scattering coefficient given by $\sigma_{\text{Th}} = \sigma_{\text{cons}} N_e$ (where σ_{cons} is a universal constant).

Now, besides evolution equations for $\rho, \mathbf{U}, \varepsilon_0, \varepsilon_e$, we have to state the evolution equation for the radiative energy density $\mathcal{E}_\nu$; it reads

$$\frac{\partial}{\partial t}\mathcal{E}_\nu + \nabla.(\mathbf{U}\mathcal{E}_\nu) + \frac{1}{3}(\mathcal{E}_\nu - \frac{\partial(\nu\mathcal{E}_\nu)}{\partial \nu})\nabla.\mathbf{U} - \nabla.\left(\frac{c}{3(\sigma_\nu + \sigma_{\text{Th}})}\nabla\mathcal{E}_\nu\right) = c\sigma_\nu(B_\nu(T_e) - \mathcal{E}_\nu) + G_\nu(T_e, \mathcal{E}_\nu).$$

or in the Lagrangian framework

$$\rho\frac{D}{Dt}\left(\frac{\mathcal{E}_\nu}{\rho}\right) + \frac{1}{3}(\mathcal{E}_\nu - \frac{\partial(\nu\mathcal{E}_\nu)}{\partial \nu})\nabla.\mathbf{U} - \nabla.\left(\frac{c}{3(\sigma_\nu + \sigma_{\text{Th}})}\nabla\mathcal{E}_\nu\right) = c\sigma_\nu(B_\nu(T_e) - \mathcal{E}_\nu) + G_\nu(T_e, \mathcal{E}_\nu).$$

The term $\frac{\partial(\nu\mathcal{E}_\nu)}{\partial \nu}$ is related to the so-called Doppler effect, i.e., the frequency shift due to the expansion or compression of the matter. Moreover, the operator $G_\nu(T_e, \cdot)$ corresponds to the so-called Compton effect; it is defined by

$$G_\nu(T_e, \mathcal{E}_\nu) = \sigma_{\text{C}\nu}(4T_e\mathcal{E}_\nu - h\nu\mathcal{E}_\nu(1 + \frac{\mathcal{E}_\nu}{h\nu^3 c^{-3}})) + \frac{\partial}{\partial \nu}\left(\sigma_{\text{C}\nu} h\nu^2\mathcal{E}_\nu(1 + \frac{\mathcal{E}_\nu}{h\nu^3 c^{-3}}) + \sigma_{\text{C}\nu} T_e \nu^6 \frac{\partial}{\partial \nu}(\frac{\mathcal{E}_\nu}{\nu^4})\right)$$

($\sigma_{\text{C}\nu}$ is a coefficient close to the Thomson coefficient σ_{Th}); the Compton effect is negligible when the plasma is not very hot. One notices in these equations the radiative diffusion term $\nabla.\left(\frac{c}{3(\sigma_\nu + \sigma_{\text{th}})}\nabla\mathcal{E}_\nu\right)$ and the term $\sigma_\nu(B_\nu(T_e) - \mathcal{E}_\nu)$ related to the emission/absorption phenomena between radiation and matter.

Notice that the quantity

$$\mathcal{E}_r = \int_0^{+\infty} \mathcal{E}_\nu d\nu,$$

corresponds to the total radiative energy (evaluated in the fluid's reference frame) and we may define the radiative pressure by $\frac{1}{3}\mathcal{E}_r$.

Now, the radiative phenomena are coupled with the plasma model in the following way. In system ($\mathcal{E}2\mathcal{T}$), equation (iv) must account for the exchange term between radiation and matter; it reads now as

$$\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + c\int_0^{+\infty}[\sigma_\nu\mathcal{E}_\nu - \sigma_\nu B_\nu(T_e)]d\nu - \int_0^{+\infty} G_\nu(T_e, \mathcal{E}_\nu)d\nu$$

Moreover, equation (ii) is modified by the following way

$$\rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e + \frac{1}{3}\mathcal{E}_r) = 0.$$

Then, the energy balance equation reads

$$\rho\frac{D}{Dt}(\varepsilon_0 + \varepsilon_e + \frac{1}{2}|\mathbf{U}|^2 + \frac{\mathcal{E}_r}{\rho}) + \nabla.((P_0 + P_e + \frac{1}{3}\mathcal{E}_r)\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}} - \nabla.\int_0^{+\infty}\frac{c\nabla\mathcal{E}_\nu}{3(\sigma_\nu + \sigma_{\text{th}})}d\nu = 0. \tag{2.74}$$

For this modelling and the mathematical analysis, see, e.g., [87, 97] and more recently [78] and the references therein. Without accounting for the hydrodynamics part, it has been proved that with appropriated assumptions, the reduced system

$$\frac{\partial}{\partial t}\mathcal{E}_\nu - \nabla.\left(\frac{c}{3(\sigma_\nu + \sigma_{\rm th})}\nabla\mathcal{E}_\nu\right) = c\sigma_\nu(B_\nu(T_e) - \mathcal{E}_\nu),$$
$$\rho\frac{\partial}{\partial t}\varepsilon_e = c\int_0^{+\infty}\sigma_\nu\mathcal{E}_\nu d\nu - c\int_0^{+\infty}\sigma_\nu B_\nu(T_e)d\nu.$$

is well-posed; see [8]. But the full system is quite difficult to handle.

There is also a simpler radiative model called the gray-diffusion approximation which may be derived from the previous one by making some elementary closures. We emphasize it now (disregarding the Compton effect).

Instead of the frequency-dependent energy density $\mathcal{E}_\nu$, we need to consider only one evolution equation for the gray energy density $\mathcal{E}_r$. Then, we define the Planck averaged opacity $\sigma_P(T_e)$ as

$$\sigma_P(T_e) = \int\sigma_\nu B_\nu(T_e)d\nu.\left(\int B_\nu(T_e)d\nu\right)^{-1}$$

and the term $\int\sigma_\nu(B_\nu(T_e)-\mathcal{E}_\nu)d\nu$ for the emission/absorption phenomena is simply replaced by $\sigma_P(a_{\rm r}T_e^4 - \mathcal{E}_r)$. The evolution equation for $\mathcal{E}_r$ reads as

$$\frac{\partial}{\partial t}\mathcal{E}_r + \nabla.(\mathbf{U}\mathcal{E}_r) + \frac{1}{3}\mathcal{E}_r\nabla.\mathbf{U} - \nabla.\left(\frac{c}{3\sigma_R}\nabla\mathcal{E}_r\right) = c\sigma_P(a_{\rm r}T_e^4 - \mathcal{E}_r) \qquad (2.75)$$

or in the Lagrangian framework

$$\rho\frac{D}{Dt}\left(\frac{\mathcal{E}_r}{\rho}\right) + \frac{1}{3}\mathcal{E}_r\nabla.\mathbf{U} - \nabla.\left(\frac{c}{3\sigma_R}\nabla\mathcal{E}_r\right) = c\sigma_P(a_{\rm r}T_e^4 - \mathcal{E}_r),$$

where the Rosseland averaged opacity $\sigma_R(T_e)$ defined as

$$\frac{1}{\sigma_R(T_e)} = \int_0^{+\infty}\frac{1}{\sigma_\nu + \sigma_{\rm th}}\frac{\partial B_\nu}{\partial T_e}(T_e)d\nu.\left(\int_0^{+\infty}\frac{\partial B_\nu}{\partial T_e}(T_e)d\nu\right)^{-1}$$

See, e.g., [77, 97] or [21] for such a model.

If we disregard the thermal diffusive term, the previous system reads as

$$
(\mathcal{E}2\mathcal{TR}) \quad \boxed{\begin{aligned}
&\rho\frac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0,\\
&\rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e + \tfrac{1}{3}\mathcal{E}_r) = 0,\\
&\rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \tfrac{\rho}{m_0}\zeta(T_e)(T_e - T_0),\\
&\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} = \tfrac{\rho}{m_0}\zeta(T_e)(T_0 - T_e) + c\sigma_P(\mathcal{E}_r - a_{\mathrm{r}}T_e^4),\\
&\rho\frac{D}{Dt}\left(\frac{\mathcal{E}_r}{\rho}\right) + \tfrac{1}{3}\mathcal{E}_r\nabla.\mathbf{U} - \nabla.\left(\tfrac{c}{3\sigma_R}\nabla\mathcal{E}_r\right) = c\sigma_P(a_{\mathrm{r}}T_e^4 - \mathcal{E}_r).
\end{aligned}}
$$

Of course, there is a energy balance equation that reads as

$$
\rho\frac{D}{Dt}\left(\varepsilon_0 + \varepsilon_e + \frac{1}{2}|\mathbf{U}|^2 + \frac{\mathcal{E}_r}{\rho}\right) + \nabla.\left((P_0 + P_e + \frac{1}{3}\mathcal{E}_r)\mathbf{U}\right) - \nabla.\left(\frac{c}{3\sigma_R}\nabla\mathcal{E}_r\right) = 0.
$$

We may notice that in model $(\mathcal{E}2\mathcal{TR})$ as for system $(\mathcal{E}2\mathcal{T})$, there are difficulties when shocks occur which are related to the definition of the nonconservative products such as $P_e\nabla.\mathbf{U}$ or $\frac{1}{3}\mathcal{E}_r\nabla.\mathbf{U}$. As a matter of fact, we need to add some information on the entropy deposition: the two quantities $\mathcal{E}_r$ and $\mathcal{E}_e$ play a similar part, and from a physical point of view one needs to claim that the entropy deposition is made on the ion population only.

Remark 5. The usual Rosseland approximation corresponds to the fact that in system $(\mathcal{E}2\mathcal{TR})$ we set $\mathcal{E}_r = a_{\mathrm{r}}T_e^4$ and we gather together the two last equations and replace them by

$$
\rho\frac{D}{Dt}(\varepsilon_e + \frac{a_{\mathrm{r}}}{\rho}T_e^4) + (P_e + \frac{1}{3}a_{\mathrm{r}}T_e^4)\nabla.\mathbf{U} - \nabla.\left(\frac{ca_{\mathrm{r}}}{3\sigma_R}\nabla T_e^4\right) = \frac{\rho}{m_0}\zeta(T_e)(T_0 - T_e),
$$

For mathematical results related to the well-posedness of this system and to a justification of the Rosseland approximation, see, e.g., [8]. For some other models and numerical methods, see [15]. For details of the modelling of radiative hydrodynamics using with transport equations, see [25]. □

2.3.1.3 Accounting for Electric Current

We deal now with current-carrying plasmas: the magnetic field **B** is either an external field or is given by an evolution equation (see the following subsection where the electron magneto-hydrodynamics model is complete). For the sake of legibility, we disregard the radiative coupling phenomena.

So, we address a two-temperature quasi-neutral model with given fields **B** and **J** (with $\nabla.\mathbf{B} = 0$) and we take a generalized Ohm's law given by (2.54); that is to say

$$\mathbf{E} + \mathbf{U} \times \mathbf{B} - \frac{1}{q_e N_e}\mathbf{J} \times \mathbf{B} + \frac{1}{q_e N_e}\nabla P_e = \chi\mu^0 \mathbf{J}.$$

Recall that the ion momentum equation (2.22) reads

$$m_0 \frac{\partial}{\partial t}(N_0 \mathbf{U}) + m_0 \nabla.(N_0 \mathbf{U}\mathbf{U}) + \nabla P_0 + \nabla P_e = \mathbf{J} \times \mathbf{B} \tag{2.76}$$

or equivalently

$$m_0 \frac{\partial}{\partial t}(N_0 \mathbf{U}) + m_0 \nabla.(N_0 \mathbf{U}\mathbf{U}) + \nabla P_0 = q_e N_e (\mathbf{E} + \mathbf{U} \times \mathbf{B}) - q_e N_e \mu^0 \chi \mathbf{J}. \tag{2.77}$$

Let us go back now to the ion and electron energy equations (2.4) and (2.37). Thanks to the quasi-neutrality, since $\mathbf{U} - \mathbf{U}_e = \mathbf{J}(q_e N_e)^{-1}$, the ion energy equation reads

$$\frac{\partial}{\partial t}\mathcal{E}_0 + \nabla.(\mathcal{E}_0 \mathbf{U}) + P_0 \nabla.\mathbf{U} = \Omega_{0e}, \tag{2.78}$$

and the electron equation may read in one of the two equivalent forms

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathcal{E}_e \mathbf{U}) + P_e \nabla.\mathbf{U} + \nabla.\mathbf{q}_{\mathrm{th,e}} = -\Omega_{0e} + \nabla.\left(\frac{5}{2} T_e \frac{\mathbf{J}}{q_e}\right) - \frac{1}{N_e q_e}\mathbf{J}.\nabla P_e + \chi\mu^0 |\mathbf{J}|^2. \tag{2.79}$$

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathcal{E}_e \mathbf{U}) + P_e \nabla.\mathbf{U} + \nabla.\mathbf{q}_{\mathrm{th,e}} = -\Omega_{0e} + \nabla.\left(\frac{5}{2} T_e \frac{\mathbf{J}}{q_e}\right) + \mathbf{J}.(\mathbf{E} + \mathbf{U} \times \mathbf{B}).$$

One moves from one form to the other by using the following relation (obtained thanks to the Ohm's law)

$$\mathbf{J}.(\mathbf{E} + \mathbf{U} \times \mathbf{B}) = -\frac{1}{N_e q_e}\mathbf{J}.\nabla P_e + \chi\mu^0 |\mathbf{J}|^2.$$

Using the specific internal energies, the previous system reads as

$$
(\mathcal{E}2\mathcal{T}\mathbf{J})\quad
\begin{array}{ll}
\text{(i)} & \rho\dfrac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0,\\[2ex]
\text{(ii)} & \rho\dfrac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) = \mathbf{J}\times\mathbf{B},\\[2ex]
\text{(iii)} & \rho\dfrac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e},\\[2ex]
\text{(iv)} & \rho\dfrac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\mathrm{th,e}} = -\Omega_{0e} + \nabla.\left(\dfrac{5T_e}{2q_e}\mathbf{J}\right)\\
& \quad -\dfrac{1}{q_e N_e}\mathbf{J}.\nabla P_e + \chi\mu^0|\mathbf{J}|^2.
\end{array}
$$

The term $\chi\mu^0|\mathbf{J}|^2$ corresponds to the so-called Joule effect or Ohmic heating.

Of course, multiplying (2.76) by $\mathbf{U}$ and combining with (2.54), we get as above the global energy balance

$$
\frac{\partial}{\partial t}(\mathcal{E}_e + \mathcal{E}_0 + m_0N_0\frac{1}{2}|\mathbf{U}|^2) + \nabla.(\mathcal{E}_e\mathbf{U}_e + E_0\mathbf{U}) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}_e) + \nabla\mathbf{q}_{\mathrm{th,e}} = \mathbf{J}.\mathbf{E}
$$

(the term $\mathbf{J}.\mathbf{E}$ being related to the work of electromagnetic forces).

Remark 6. The two terms $\nabla.\left(\frac{5}{2q_e}T_e\mathbf{J}\right)$ and $\frac{1}{ZN_0q_e}\mathbf{J}.\nabla P_e$ are not always taken into account in the literature, but they appear, e.g., in [21, 38].

The first term may be considered as a part of the thermal conduction flux. But the second one is the counterpart of the term $\mathbf{J}.\mathbf{E}$ and it is necessary to have the right energy balance (recall that $-\frac{1}{ZN_0}\nabla P_e$ is the main term in the expression that defines $\mathbf{E}$). □

2.3.2 Electron Magneto-Hydrodynamics

In the framework of the quasi-neutral approximation, we now account for the full electron magneto-hydrodynamics , i.e., the magnetic field equation is coupled with the previous system: the field $\mathbf{B}$ obeys an evolution equation coming from the Maxwell–Faraday equation and the generalized Ohm's law.

Beside the first paragraph where the conductivity σ is a scalar, we address in a second paragraph the case where the conductivity is a tensor (which is generally used in the cases where the magnetic field is very strong).

2.3.2.1 Case with Scalar Conductivity

Here the electric resistivity[4] χ is of Spitzer type: i.e., it is the inverse of the electric conductivity σ (up to the constant μ^0); we have with the notations introduced above.

$$(\chi\mu^0)^{-1} = \sigma = Zq_e^2/\nu_{e0}$$

so it is also proportional to the Coulomb collision frequency $\tau_e = m_e/(\nu_{e0}N_{\mathrm{ref}})$.

Notice that the electric conductivity is strongly related to the conductivity function (4.11) given below by $\underline{\sigma}(\omega) = (ZN_0q_e^2)(\nu_{e0}N_0 - im_e\omega)^{-1}$ since the Debye length is negligible; as a matter of fact, we get

$$\mathrm{Re}(\underline{\sigma}(\omega)^{-1}) = \sigma^{-1} = \chi\mu^0. \tag{2.80}$$

When we take the "curl" of relation (2.54), we get

$$\mathrm{curl}(\mathbf{E} + \mathbf{U} \times \mathbf{B}) = -\mathrm{curl}(\frac{1}{q_eN_e}\nabla P_e) + \mathrm{curl}(\chi\mu_0\mathbf{J}) + \mathrm{curl}(\frac{1}{q_eN_e}\mathbf{J} \times \mathbf{B}).$$

Therefore, the Maxwell–Faraday equation leads to

$$\frac{\partial}{\partial t}\mathbf{B} - \mathrm{curl}(\mathbf{U} \times \mathbf{B}) + \mathrm{curl}\,(\chi\mathrm{curl}\,\mathbf{B}) = \mathrm{curl}\left(\frac{\nabla P_e}{q_eN_e}\right) - \mathrm{curl}\,\left(\frac{1}{q_eN_e\mu^0}\mathrm{curl}\,\mathbf{B} \times \mathbf{B}\right),$$

with the constraint that $\mathbf{B}$ has to satisfy the magnetic Gauss relation

$$\nabla.\mathbf{B} = 0. \tag{2.81}$$

Of course, if $\mathbf{B}$ is initially divergence-free, it remains divergence-free always. According to this constraint and identity (A.1), we may write the left-hand side of this evolution equation in other forms

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B} - \mathrm{curl}(\mathbf{U} \times \mathbf{B}) &= \frac{D}{Dt}\mathbf{B} + \mathbf{B}(\nabla.\mathbf{U}) - (\mathbf{B}.\nabla)\mathbf{U} \\ &= \rho\frac{D}{Dt}(\frac{\mathbf{B}}{\rho}) - (\mathbf{B}.\nabla)\mathbf{U}.\end{aligned}$$

To get the last line, relation $\nabla.\mathbf{U} = N_0D_tN_0^{-1} = \rho D_t\rho^{-1}$ has been used.

On momentum equation [$\mathcal{E}2\mathcal{T}\mathbf{J}$(ii)], since the current is curl $\mathbf{B}/\mu^0$, we display the magnetic pressure tensor $\nabla.\mathbb{P}_B$. Then, we get the following system for the evolution of $\rho, \mathbf{U}, \varepsilon_0, \varepsilon_e, \mathbf{B}$

[4] As a matter of fact, the electric resistivity is defined as $\mu^0\chi$; it is also denoted by η in some physics textbooks.

$$(\mathcal{MHD})\quad \begin{array}{ll} \text{(i)} & \rho\dfrac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0, \\[2ex] \text{(ii)} & \rho\dfrac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) + \nabla.\mathbb{P}_B = 0, \\[2ex] \text{(iii)} & \rho\dfrac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e}, \\[2ex] \text{(iv)} & \rho\dfrac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \nabla.\left(\dfrac{5T_e}{2q_e}\dfrac{\text{curl}\mathbf{B}}{\mu^0}\right) \\ & \quad -\dfrac{\nabla P_e}{q_e N_e}.\dfrac{\text{curl}\mathbf{B}}{\mu^0} + \dfrac{\chi}{\mu^0}\,|\text{curl}\mathbf{B}|^2 \\[2ex] \text{(v)} & \rho\dfrac{D}{Dt}\left(\dfrac{\mathbf{B}}{\rho}\right) - (\mathbf{B}.\nabla)\mathbf{U} + \text{curl}\,(\chi\text{curl}\mathbf{B}) = \text{curl}\left(\dfrac{\nabla P_e}{q_e N_e}\right) \\[2ex] & \quad -\text{curl}\left(\dfrac{1}{q_e N_e \mu^0}\text{curl}\mathbf{B}\times\mathbf{B}\right), \end{array}$$

with the constraint (2.81).

Of course, we may state this system in an Euler framework, it suffices to replace equation (i) by

$$\frac{\partial}{\partial t}\rho + \nabla.(\rho\mathbf{U}) = 0$$

and in the other equations to replace $\rho\dfrac{D}{Dt}X$ by $\frac{\partial}{\partial t}(\rho X) + \nabla.(\rho X\mathbf{U})$. Then the last equation reads as

$$\frac{\partial}{\partial t}\mathbf{B}+\nabla.(\mathbf{UB})-(\mathbf{B}.\nabla)\mathbf{U}+\text{curl}\,(\chi\text{curl}\mathbf{B}) = \text{curl}\left(\frac{\nabla P_e}{q_e N_e}\right) - \text{curl}\left(\frac{1}{q_e N_e\mu^0}\text{curl}\mathbf{B}\times\mathbf{B}\right)$$

In equation (v), the quadratic term (curl $\mathbf{B}\times\mathbf{B}$), which is related to the Hall effect, may be neglected if the magnetic field is not very strong. Note that there is no contribution of the Hall effect term in the energy balance (indeed, we have always $\mathbf{B}.\text{curl}(N_e^{-1}\text{curl}\mathbf{B}\times\mathbf{B}) = 0$).

The term $\nabla.(\dfrac{5T_e}{2q_e}\dfrac{\text{curl}\mathbf{B}}{\mu^0})$ is called the thermoelectric one; it is zero if the electron temperature is constant.

It is worth noticing that in equation (v) the term with $(\nabla P_e)/N_e$ corresponds to the so-called *self-generated* magnetic field and reads also as $\text{curl}(N_e^{-1}\nabla P_e) = \nabla P_e \times \nabla(N_e^{-1})$; it is an external source that is not zero if the gradient of the plasma density and the gradient of the electron temperature are not parallel.

It has its counterpart at the right-hand side of equation (iv). Therefore, it is possible to neglect these terms with $(\nabla P_e)/N_e$ both in equation (iv) and (v).

Energy Balance

Let us recall that the magnetic energy is equal to $\frac{1}{2\mu^0}|\mathbf{B}|^2$. Recall that $\nabla\mathbf{U}$ denotes the tensor $(\nabla\mathbf{U})_{i,j} = \partial_{x_i}\mathbf{U}_j$. Let us multiply the last equation of $(\mathcal{MHD})$ by $\mathbf{B}/\mu^0$. For the first part of the magnetic energy balance relation, we get

$$\frac{1}{\mu^0}\mathbf{B}.\left(\frac{D}{Dt}\mathbf{B} + \mathbf{B}(\nabla.\mathbf{U}) - (\mathbf{B}.\nabla)\mathbf{U}\right) = \frac{D}{Dt}\left(\frac{|\mathbf{B}|^2}{2\mu^0}\right) + \frac{1}{\mu^0}[|\mathbf{B}|^2\,\nabla.\mathbf{U} - \mathbf{B}((\mathbf{B}.\nabla)\mathbf{U})]$$

$$= \rho\frac{D}{Dt}\left(\frac{|\mathbf{B}|^2}{2\rho\mu^0}\right) + \mathbb{P}_B : \nabla\mathbf{U},$$

indeed we have as usual $\rho\frac{D}{Dt}(\frac{|\mathbf{B}|^2}{2\rho}) = \frac{1}{2}\frac{D}{Dt}|\mathbf{B}|^2 - \frac{1}{2}|\mathbf{B}|^2(\nabla.\mathbf{U})$ and obviously $\mathbf{B}((\mathbf{B}.\nabla)\mathbf{U}) = \mathbf{BB} : \nabla\mathbf{U}$. For the second part, using identity (A.2), we get

$$\mathbf{B}.\text{curl}\left(\chi\mathbf{J} - \frac{1}{\mu^0}\frac{\nabla P_e}{q_e N_e} - \frac{1}{\mu^0 q_e N_e}\mathbf{J}\times\mathbf{B}\right) =$$
$$\nabla.\left(\left(\chi\mathbf{J} - \frac{1}{\mu^0}\frac{\nabla P_e}{q_e N_e} + \frac{1}{\mu^0 q_e N_e}\mathbf{J}\times\mathbf{B}\right)\times\mathbf{B}\right) + \mu^0\mathbf{J}.\left(\chi\mathbf{J} - \frac{1}{\mu^0}\frac{\nabla P_e}{q_e N_e}\right).$$

So we may state the magnetic energy balance equation

$$\rho\frac{D}{Dt}\left(\frac{|\mathbf{B}|^2}{2\rho\mu^0}\right) + \mathbb{P}_B : (\nabla\mathbf{U}) + \nabla.\left(\left(\chi\mathbf{J} - \frac{1}{\mu^0}\frac{\nabla P_e}{q_e N_e} + \frac{1}{\mu^0 q_e N_e}\mathbf{J}\times\mathbf{B}\right)\times\mathbf{B}\right) + \mu^0\chi|\mathbf{J}|^2 - \mathbf{J}.\frac{\nabla P_e}{q_e N_e} = 0.$$

Introduce the global energy

$$E_{\text{tot}} = \frac{1}{2\mu^0}|\mathbf{B}|^2 + \mathcal{E}_e + \mathcal{E}_0 + \frac{1}{2}m_0 N_0 |\mathbf{U}|^2.$$

Therefore, multiplying the momentum equation by $\mathbf{U}$ and using the tensor identity (A.5), we get finally

$$\rho\frac{D}{Dt}\left(\frac{E_{\text{tot}}}{\rho}\right) + \nabla.((P_0 + P_e)\mathbf{U}) + \nabla.(\mathbb{P}_B.\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}}$$
$$= \nabla.\left(\frac{\chi}{\mu^0}\mathbf{B}\times\text{curl}\mathbf{B} + \frac{1}{\mu^0}\mathbf{B}\times\left(-\frac{\nabla P_e}{q_e N_e} + \frac{1}{q_e N_e}\mathbf{J}\times\mathbf{B}\right)\right).$$

Since the right-hand side is a divergence of a vector, we see that this energy balance relation is a conservative one.

Remark 7. The usual simplified resistive electron-MHD system is obtained when one disregards the Hall effect terms and the self-generated magnetic field (this is justified in the cases where the magnetic field is not very strong) and the thermo-electric term (justified if the electron temperature is quite constant).

More precisely, it corresponds withdrawing the right hand side of equation [$\mathcal{MHD}$(v)] and keeping only the Joule effect term (and the two-temperature coupling) in the electron energy equation; so [$\mathcal{MHD}$(iv), (v)] are replaced by

$$\rho \frac{D}{Dt}\varepsilon_e + P_e \nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \frac{\chi}{\mu^0}\,|\text{curl}\mathbf{B}|^2 \tag{2.82}$$

$$\frac{\partial}{\partial t}\mathbf{B} - \text{curl}(\mathbf{U}\times\mathbf{B}) + \text{curl}\,(\chi\text{curl}\mathbf{B}) = 0. \tag{2.83}$$

Of course, the above global energy balance relation reads now as

$$\rho \frac{D}{Dt}\left(\frac{E_{\text{tot}}}{\rho}\right) + \nabla.((P_0+P_e)\mathbf{U}) + \nabla.(\mathbb{P}_B.\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}} = \nabla.\left(\frac{\chi}{\mu^0}\mathbf{B}\times\text{curl}\mathbf{B}\right). \quad \square$$

2.3.2.2 Case with a Tensor Conductivity

We now address the case of the strongly magnetized plasma and we want to determine a link between the electric current $\mathbf{J}$ and the fields $\mathbf{E}$ and $\mathbf{B}$. Therefore, we define the unit vector $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ and the electron Larmor frequency

$$\omega_{ce} = \frac{q_e}{m_e}|\mathbf{B}|$$

and we begin with the Ohm's law (2.54) which we write in the following form

$$\mathbf{J}\times\mathbf{b}\frac{m_e\omega_{ce}}{\nu_{e0}N_0} + \mathbf{J} - \frac{q_e^2 Z}{\nu_{e0}}(\mathbf{E}+\mathbf{U}\times\mathbf{B}) = \frac{q_e}{\nu_{e0}N_0}\nabla P_e. \tag{2.84}$$

Thanks to this expression, we now perform a closure by dropping the r.h.s. term related to the thermo-electric effect. By a classic way, $\mathbf{E}_\parallel = \mathbf{b}(\mathbf{E}.\mathbf{b})$ denotes the parallel component and $\mathbf{E}_\perp = \mathbf{b}\times(\mathbf{E}\times\mathbf{b})$ the perpendicular component of the electrostatic field. Since we have $\mathbf{b}\times(\mathbf{W}\times\mathbf{b})=\mathbf{W}$, for all vectors $\mathbf{W}$ orthogonal to $\mathbf{b}$, it may be checked that (2.84) leads to (χ_s is the scalar resistivity defined as above by $\nu_{e0}q_e^{-2}\frac{1}{Z\mu^0}$)

$$\mathbf{J} = \frac{1}{\mu^0\chi_s}\mathbf{E}_\parallel + \frac{1}{\mu^0\chi_s}\frac{1}{1+\frac{\omega_{ce}^2}{\nu^2}}(\mathbf{E}_\perp+\mathbf{U}\times\mathbf{B}) + \frac{1}{\mu^0\chi_s}\frac{1}{\frac{\nu}{\omega_{ce}}+\frac{\omega_{ce}}{\nu}}\mathbf{b}\times(\mathbf{E}_\perp+\mathbf{U}\times\mathbf{B})$$

This expresses the desired closure

$$\mathbf{J} = \underset{\leftrightarrow}{\sigma}(\mathbf{E}+\mathbf{U}\times\mathbf{B}) \tag{2.85}$$

where the tensor $\underset{\leftrightarrow}{\sigma}$ may read in the system of coordinates defined by $\mathbf{b}, \mathbf{b}\times(\hat{\mathbf{E}}\times\mathbf{b}), (\mathbf{b}\times\hat{\mathbf{E}})$ (where $\hat{\mathbf{E}} = \mathbf{E}\frac{1}{|\mathbf{E}|}$) as follows

$$\mu^0 \underset{\leftrightarrow}{\sigma} = \begin{bmatrix} \chi_s^{-1} & 0 & 0 \\ 0 & \chi_s^{-1}(1+\frac{\omega_{ce}^2}{\nu^2})^{-1} & 0 \\ 0 & 0 & \chi_s^{-1}(\frac{\nu}{\omega_{ce}}+\frac{\omega_{ce}}{\nu})^{-1} \end{bmatrix}.$$

Moreover, the resistivity tensor $\underset{\leftrightarrow}{\chi}$ is given by $\underset{\leftrightarrow}{\chi}\,\mu^0 = \underset{\leftrightarrow}{\sigma}^{-1}$ and reads as

$$\underset{\leftrightarrow}{\chi} = \begin{bmatrix} \chi_s & 0 & 0 \\ 0 & \chi_s(1+\frac{\omega_{ce}^2}{\nu^2}) & 0 \\ 0 & 0 & \chi_s(\frac{\nu}{\omega_{ce}}+\frac{\omega_{ce}}{\nu}) \end{bmatrix}.$$

We can now proceed as in the previous paragraph. Since

$$\text{curl}\,\mathbf{E} = -\text{curl}(\mathbf{U}\times\mathbf{B}) + \text{curl}\,(\mathbf{E}+\mathbf{U}\times\mathbf{B}) = -\text{curl}(\mathbf{U}\times\mathbf{B}) + \text{curl}\,(\underset{\leftrightarrow}{\sigma}^{-1}\mathbf{J}),$$

the Maxwell–Faraday equation leads to the evolution equation of the magnetic field.

$$\frac{\partial}{\partial t}\mathbf{B} - \text{curl}(\mathbf{U}\times\mathbf{B}) + \text{curl}\,(\underset{\leftrightarrow}{\chi}\,\text{curl}\mathbf{B}) = 0.$$

As above, this equation may read also as

$$\rho\frac{D}{Dt}(\frac{\mathbf{B}}{\rho}) - (\mathbf{B}.\nabla)\mathbf{U} + \text{curl}\left(\underset{\leftrightarrow}{\chi}\,\text{curl}\mathbf{B}\right) = 0 \tag{2.86}$$

We may state the MHD system in this framework

$$(\mathcal{MHD}_T)\quad \boxed{\begin{array}{ll} \text{(i)} & \rho\frac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0, \\ \text{(ii)} & \rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0+P_e) + \nabla.\mathbb{P}_B = 0, \\ \text{(iii)} & \rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e}, \\ \text{(iv)} & \rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \frac{1}{\mu^0}\text{curl}\mathbf{B}.\underset{\leftrightarrow}{\chi}.\text{curl}\mathbf{B}, \\ \text{(v)} & \rho\frac{D}{Dt}(\frac{\mathbf{B}}{\rho}) - (\mathbf{B}.\nabla)\mathbf{U} + \text{curl}\left(\underset{\leftrightarrow}{\chi}.\text{curl}\mathbf{B}\right) = 0. \end{array}}$$

Let us now focus on the Joule effect term

$$s_{\text{Joule}} = \frac{1}{\mu^0}\text{curl}\mathbf{B}.\underset{\leftrightarrow}{\chi}.\text{curl}\mathbf{B} = \mathbf{J}.(\mathbf{E} + \mathbf{U} \times \mathbf{B})$$

In the above-defined system of coordinates $\{\mathbf{b}, \mathbf{b}\times(\hat{\mathbf{E}}\times\mathbf{b}), (\mathbf{b} \times \hat{\mathbf{E}})\}$ we may compute this term. Since $\mathbf{J} = \underset{\leftrightarrow}{\sigma}.(\mathbf{E}+\mathbf{U}\times\mathbf{B})$, we get $(\mathbf{E}+\mathbf{U}\times\mathbf{B}) = \mu^0\chi_s\mathbf{J}_\parallel + \mu^0\chi_s(1+\frac{\omega_{ce}^2}{\nu^2})\mathbf{J}_\perp$ where $\mathbf{J}_\parallel = (\mathbf{J}.\mathbf{b}).\mathbf{b}.$ and $\mathbf{J}_\perp = \mathbf{J} - \mathbf{J}_\parallel$; then we see that

$$s_{\text{Joule}} = \mu^0\chi_s|\mathbf{J}_\parallel|^2 + \mu^0\chi_s(1 + \frac{\omega_{ce}^2}{\nu^2})|\mathbf{J}_\perp|^2.$$

We notice that the term with $(\frac{\nu}{\omega_{ce}} + \frac{\omega_{ce}}{\nu})$ in the resistivity tensor does not appear in this formula (this term corresponds to the component of $\underset{\leftrightarrow}{\chi}.\mathbf{J}$ which is orthogonal to $\mathbf{E}$).

Remark 8 (Energy Balance). We now state the magnetic energy balance. Since, according to (A.2), we have

$$\mathbf{B}.\text{curl}\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}\right) = \nabla.\left(\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}\right) \times \mathbf{B}\right) + \mu^0\mathbf{J}.\underset{\leftrightarrow}{\chi}\,\mathbf{J}$$

we get

$$\rho\frac{D}{Dt}\left(\frac{|\mathbf{B}|^2}{2\rho\mu^0}\right) + \mathbb{P}_B : \nabla\mathbf{U} + \nabla.\left(\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}\right) \times \mathbf{B}\right) + \mu^0\mathbf{J}.\underset{\leftrightarrow}{\chi}.\mathbf{J} = 0.$$

Thus, for the global energy balance relation, we get as above

$$\rho\frac{D}{Dt}\left(\frac{E_{\text{tot}}}{\rho}\right) + \nabla.((P_0 + P_e)\mathbf{U}) + \nabla.(\mathbb{P}_B.\mathbf{U}) = \nabla.\left(\mathbf{B} \times \left(\frac{1}{\mu^0}\underset{\leftrightarrow}{\chi}.\text{curl}\mathbf{B}\right)\right). \quad \square$$

Remark 9. As a matter of fact it is also possible to add in the r.h.s. of $[\mathcal{MHD}_T(\text{v})]$ the term $\text{curl}\left(\frac{\nabla P_e}{q_e N_e}\right)$, related to the self-generated magnetic field. □

Remark 10. As in the case of scalar conductivity, the conductivity tensor $\underset{\leftrightarrow}{\sigma}$ is closely related to the conductivity tensor function $\underset{\leftrightarrow}{\sigma}(\omega)$ proposed in Sect. 4.1.1 of Chap. 4 which links the envelope fields $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{E}}$ by a relation of the type $\tilde{\mathbf{J}} = \underset{\leftrightarrow}{\sigma}(\omega)\tilde{\mathbf{E}}$ (where the envelope fields are such that $\mathbf{E} = \tilde{\mathbf{E}}e^{i\mathbf{k}.\mathbf{x}-i\omega t}$, $\mathbf{J} = \tilde{\mathbf{J}}e^{i\mathbf{k}.\mathbf{x}-i\omega t}$). □

2.3.2.3 Boundary Conditions. Axi-Symmetric Geometry Case

(a) Boundary Conditions in the 3D Framework

We analyze here some different boundary conditions for the MHD models, but we disregard the two energy equations (indeed, for the sake of simplicity, one may assume that the ion and electron temperatures are somehow constant near the boundary). Roughly speaking, there are two kinds of material boundaries for such models: either the boundary corresponds to an insulation material or a conducting material. Of course, the wall material may also own magnetic properties; then we need to deal with impedance conditions on this boundary (but this is more tricky).

Since we assume that the temperatures are given, $c_s^2 \nabla \rho$ may replace $\nabla(P_e + P_0)$ with an appropriated sound velocity c_s, so we may address the following simplified problem near the boundary

$$\frac{\partial}{\partial t}\rho + \nabla.(\rho\mathbf{U}) = 0 \tag{2.87}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{U}) + \nabla.(\rho\mathbf{U}\mathbf{U}) + c_s^2\nabla\rho + \nabla.\mathbb{P}_B = 0 \tag{2.88}$$

$$\frac{\partial}{\partial t}\mathbf{B} - \mathrm{curl}(\mathbf{U}\times\mathbf{B}) + \mathrm{curl}\,(\chi \mathrm{curl}\mathbf{B}) = 0 \tag{2.89}$$

with $\mathbb{P}_B = \frac{1}{\mu^0}(\frac{\mathbb{I}}{2}|\mathbf{B}|^2 - \mathbf{B}\mathbf{B})$.

In the sequel, we denote as usual by $\mathbf{n}$ the outwards unit normal vector to the boundary Γ of domain $\mathcal{O}$.

It is worth noticing first that curl(χcurl $\mathbf{B}$) is a diffusion-like term; then for the boundary conditions, there are two different cases: firstly, χ is strictly positive near the boundary; secondly, χ is zero near the boundary. We focus here only on the first case corresponding to a resistive model, which is assumed to be valid up to the boundary. Therefore, we do not account for the existence of a "vacuum" near the boundary. ("Vacuum" means that the ion density is zero near the boundary. This case is quite difficult because the electron population is nonzero and a corresponding electric current needs to be addressed.)

Since χ is strictly positive, (2.89) is of parabolic type and its boundary condition is specific and may be imposed independently of the two other equations.

It is also worth recalling that system (2.87) and (2.88), which reads in the form

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{U}\end{pmatrix} + \nabla.\left(\mathbb{F}\begin{pmatrix}\rho\\ \rho\mathbf{U}\end{pmatrix}\right) = 0, \qquad \text{with } \mathbb{F}\begin{pmatrix}\rho\\ \rho\mathbf{U}\end{pmatrix} = \begin{pmatrix}\rho\mathbf{U}\\ \rho\mathbf{U}\mathbf{U} + c_s^2\rho\mathbb{I} + \mathbb{P}_B,\end{pmatrix}$$

is hyperbolic[5] and the eigenvalues of the Jacobian matrix are equal to $\mathbf{n}.\mathbf{U} \pm c_s$; indeed, $\mathbb{P}_B$ does not depend on ρ and $\mathbf{U}$, thus it does not come into the Jacobian matrix (notice that the computation of the eigenvalues for the full MHD system is made in Sect. 2.4 below). One knows (see, e.g., [60]) that if the flow near the boundary is supersonic with an outgoing velocity, i.e., $\mathbf{n}.\mathbf{U} > c_s$, then no boundary condition needs to be imposed; moreover, if it is subsonic, i.e., $|\mathbf{n}.\mathbf{U}| \leq c_s$, one must impose a scalar equation for the boundary condition.

Generally, the plasma flow is subsonic and we must handle one boundary condition related to the normal ion velocity $\mathbf{n}.\mathbf{U}$. This point is quite sensitive, since there are sheath effects according to the presence of a electric current at the boundary; moreover, the quasi-neutrality does not hold in the neighborhood of the conductor in a width of about a few tens of typical Debye lengths which is called the Langmuir sheath (and is related to the Child–Langmuir problem when an electric potential is imposed). There is a wide range of literature related to this problem; see, e.g., [28] for a review.

Nevertheless, in the case where there is no external electric circuit and the surface of the material is not insulated, there is a crude approximation known as the Bohm criterion, which claims that the ion velocity near the boundary must satisfy

$$\mathbf{n}.\mathbf{U} = c_s.$$

On the contrary, if the surface is perfectly insulated, we can assume that $\mathbf{n}.\mathbf{U} = 0$.

Let us address now the boundary condition problem for (2.89). But firstly, let us go back to the classical calculus for the magnetic energy. If we multiply (2.89) by $\mathbf{B}$ and integrate over the spatial domain, using the vector identity (A.5) and setting $S = -\int_{\mathcal{O}} \chi|\text{curl}\mathbf{B}|^2 dx + \int_{\mathcal{O}} (\mathbf{U} \times \mathbf{B}.\text{curl}\mathbf{B}) dx$ the usual inner term, we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{O}} |\mathbf{B}|^2 dx &= \int_{\mathcal{O}} \mathbf{B}.\text{curl}(\mathbf{U} \times \mathbf{B} - \chi\text{curl}\mathbf{B})\, dx = S + \int_{\Gamma} \mathbf{n}.(\mathbf{U} \times \mathbf{B} - \chi\text{curl}\mathbf{B}) \times \mathbf{B}\, d\Gamma(x) \\ &= S + \int_{\Gamma} ((\mathbf{U} \times \mathbf{B} - \chi\text{curl}\mathbf{B}) \times \mathbf{n})[(\mathbf{B} \times \mathbf{n}) \times \mathbf{n}]\, d\Gamma(x)\end{aligned}$$

[using the usual formula $\mathbf{n}.(\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \times \mathbf{n}).((\mathbf{B} \times \mathbf{n}) \times \mathbf{n})$]. This shows that the normal component of the Pointing vector $(\mathbf{U} \times \mathbf{B} - \chi\text{curl}\mathbf{B}) \times \mathbf{B}$ on the boundary is related to the tangential components of the two vectors $\mathbf{E} = -\mathbf{U} \times \mathbf{B} + \chi\text{curl}\mathbf{B}$ and $\mathbf{B}$.

We now address two kinds of boundary material.

1. *If it is a purely conductive material*, since the electric potential of the conductor is constant, we must have $\mathbf{E} \times \mathbf{n} = 0$, which implies the following condition.

[5] For a one-dimensional space variable, a system of the form $\partial_t \mathbf{Y} + \frac{\partial}{\partial x}(\mathbf{F}(\mathbf{Y})) = 0$ is called hyperbolic if all the eigenvalues of the Jacobian matrix $\partial \mathbf{F}/\partial \mathbf{Y}$ are real and there exists a complete set of eigenvectors. For a three-dimensional space variable, a system $\partial_t \mathbf{Y} + \Sigma_j \frac{\partial}{\partial x_j}(\mathbb{F}_j(\mathbf{Y})) = 0$ is called hyperbolic if one has the analogous property for the Jacobian matrix $\frac{\partial}{\partial \mathbf{Y}}(\omega_1 \mathbb{F}_1 + \omega_2 \mathbb{F}_2 + \omega_3 \mathbb{F}_3)$ for all coefficients $\omega_1, \omega_2, \omega_3$.

$$(\mathbf{U} \times \mathbf{B} - \chi \mathrm{curl}\mathbf{B}) \times \mathbf{n} = 0.$$

2. *If it is not a purely conductive material*, one needs to impose a boundary condition that links the tangential components of the electric field $\mathbf{E}$ and the magnetic one $\mathbf{B}$; so with a positive coefficient α depending on the material, we may impose the so-called impedance boundary condition

$$\alpha(\mathbf{U} \times \mathbf{B} - \chi \mathrm{curl}\mathbf{B}) \times \mathbf{n} + (\mathbf{B} \times \mathbf{n}) \times \mathbf{n} = 0.$$

This type of condition is usual in electromagnetism. Notice that the case $\alpha = 0$ corresponds to a pure insulated material.

Remark 11 (Transparent Boundary Conditions). Generally the simulation domain needs to be truncated and it is necessary to pay attention to the treatment of the artificial boundaries. For the subsystem (2.87) and (2.88), if the flow is hypersonic with an outgoing velocity, one must not impose any boundary condition; but if the flow is subsonic, one needs to impose a transparent boundary condition. One may address this problem by using a perfectly matched layer technique as in [11] (see, e.g., [89]); one may also use a technique related to the Riemann invariants of the system ($\rho \mathbf{n}.\mathbf{U} \pm \rho c_s$), see, e.g., [60].

In all cases, one needs to deal also with boundary conditions for the resistive part ($\mathrm{curl}(\chi \mathrm{curl}\mathbf{B})$) and the analysis made above needs to be adapted for dealing with this problem.

Notice that in the case where $\chi = 0$ near the boundary, the full system (2.87)–(2.89) is hyperbolic, and the statement of the boundary conditions is different and depends on the flow characteristics—more precisely on the signs of the eigenvalues of this hyperbolic system (see Sect. 2.4). □

(b) The Two-Dimensional Axi-Symmetric Geometry

For many applications, e.g., for tokamak simulations or for Z-pinch simulations, one needs to deal with the previous system in this geometry. Let us give some notations: (r, z) denotes the coordinates (r is the distance to axis of axi-symmetric geometry), θ denotes the angular coordinate in the direction of rotation, and $\mathbf{e}_\theta$ denotes the unit vector at point (r, z, θ) in the direction of rotation. All the fields and functions are functions of (r, z) only. A vector field $\mathbf{A}$ in 3D may be decomposed into a toroidal component $A_\theta \mathbf{e}_\theta$ (where A_θ is a scalar function) and a two-dimensional poloidal component $\mathbf{A}_\Lambda = (A_r, A_z)$ which has no component according to $\mathbf{e}_\theta$; i.e., to say $\mathbf{A} = A_\theta \mathbf{e}_\theta + \mathbf{A}_\Lambda$. Recall that the divergence of the two-dimensional vector field is defined by

$$\nabla.\mathbf{A}_\Lambda = \frac{1}{r}\partial_r(rA_r) + \partial_z A_z$$

and for the curl we get

$$[\mathrm{curl}\mathbf{A}]_\theta = \nabla \times \mathbf{A}_\Lambda = \frac{\partial}{\partial z}A_r - \frac{\partial}{\partial r}A_z,$$

$$[\mathrm{curl}(A_\theta \mathbf{e}_\theta)]_r = -\frac{\partial}{\partial z}A_\theta, \qquad [\mathrm{curl}(A_\theta \mathbf{e}_\theta)]_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$$

Then the magnetic field $\mathbf{B}$ must be decomposed into its toroidal component B_θ and its poloidal component $\mathbf{B}_\Lambda = (B_r, B_z)$. The constraint $\nabla.\mathbf{B} = 0$ (in $\mathbf{R}^3$) reads now as

$$\nabla.\mathbf{B}_\Lambda = 0$$

Using this relation, the divergence of the magnetic tensor becomes

$$\begin{aligned}
\mu^0[\nabla.\mathbb{P}_B]_r &= \frac{B_\theta}{r}\frac{\partial}{\partial r}(rB_\theta) - B_z\frac{\partial}{\partial z}B_r + \frac{1}{2}\frac{\partial}{\partial r}B_z^2 \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left(|\mathbf{B}_\Lambda|^2 + B_\theta^2\right) + \frac{1}{r}B_\theta^2 - \frac{\partial}{\partial z}(B_r B_z) - \frac{1}{r}\frac{\partial}{\partial r}(rB_r^2) \\
\mu^0[\nabla.\mathbb{P}_B]_z &= \frac{1}{2}\frac{\partial}{\partial z}B_\theta^2 - B_r\frac{\partial}{\partial r}B_z + \frac{1}{2}\frac{\partial}{\partial z}B_r^2 \\
&= \frac{1}{2}\frac{\partial}{\partial z}\left(|\mathbf{B}_\Lambda|^2 + B_\theta^2\right) - \frac{1}{r}\frac{\partial}{\partial r}(rB_r B_z) - \frac{\partial}{\partial z}B_z^2.
\end{aligned}$$

The system $(\mathcal{MHD}_T)$ may now read as follows (with the notation $\tilde{\nabla}.\bullet = \frac{\partial}{\partial r}\bullet_r + \frac{\partial}{\partial z}\bullet_z$)

$$\rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) + \nabla.\mathbb{P}_B = 0,$$

$$\rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e},$$

$$\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\mathrm{th,e}} = -\Omega_{0e} + s_{\mathrm{Joule}},$$

$$\rho\frac{D}{Dt}\left(\frac{B_\theta}{\rho}\right) - B_\theta\frac{U_r}{r} - \tilde{\nabla}.\left(\underset{\leftrightarrow}{\chi}.\frac{1}{r}\nabla(rB_\theta)\right) = 0,$$

$$\rho\frac{D}{Dt}\left(\frac{\mathbf{B}_\Lambda}{\rho}\right) - (\mathbf{B}_\Lambda.\nabla)\mathbf{U} + \mathrm{curl}\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}_\theta\right) = 0, \qquad \mathbf{J}_\theta = \frac{\mathbf{e}_\theta}{\mu^0}(\mathrm{curl}\mathbf{B}_\Lambda)_\theta$$

where

$$s_{\mathrm{Joule}} = \frac{1}{\mu^0}\mathbf{J}_\theta.\underset{\leftrightarrow}{\chi}.\mathbf{J}_\theta + \frac{1}{\mu^0}\frac{1}{r^2}\nabla(rB_\theta).\underset{\leftrightarrow}{\chi}.\nabla(rB_\theta)$$

In a Euler framework, the two last equations read as

$$\frac{\partial B_\theta}{\partial t} + \nabla.(\mathbf{U}B_\theta) - B_\theta \frac{U_r}{r} - \tilde{\nabla}.\left(\underset{\leftrightarrow}{\chi}.\frac{1}{r}\nabla(rB_\theta)\right) = 0,$$

$$\frac{\partial \mathbf{B}_\Lambda}{\partial t} + \nabla.(\mathbf{U}\mathbf{B}_\Lambda) - (\mathbf{B}_\Lambda.\nabla)\mathbf{U} + \mathrm{curl}\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}_\theta\right) = 0.$$

It is worth noting that, in this model, the two components of the magnetic field are somehow no longer coupled, the Joule effect term being the sum of the contribution of the two components.

Notice that a lot of work has been made in order to state so-called reduced models of MHD in the axi-symmetric case, see, e.g., [45] and the references therein.

(c) Boundary Conditions in the Two-Dimensional Axi-Symmetric Framework

The boundary conditions for the magnetic field state for an insulated material is as follows

$$\mathbf{B}_\theta = 0;$$

moreover, there is no boundary condition for $\mathbf{B}_\Lambda$.

For the case of a conductive material, the conditions for the two components of $\mathbf{B}$ read as

$$\mathbf{n}\times(\mathbf{U}\times\mathbf{B}_\Lambda) - \chi\mathbf{n}\times\mathrm{curl}\mathbf{B}_\Lambda = 0, \qquad c_s\mathbf{B}_\theta + \chi[\mathbf{n}_z\frac{\partial}{\partial z}\mathbf{B}_\theta + \mathbf{n}_r\frac{1}{r}\frac{\partial}{\partial r}(r\mathbf{B}_\theta)] = 0.$$

indeed $(\mathbf{U}\times\mathbf{B}_\Lambda)$ and $\mathrm{curl}\mathbf{B}_\Lambda$ are parallel to $\mathbf{e}_\theta$. Therefore, we check that the boundary conditions for B_θ and for $\mathbf{B}_\Lambda$ are no longer coupled.

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Proof of Proposition 7. Let us set θ, T instead of T_e, T_0. Set also $m = 7/2$ and drop the coefficient $\frac{3}{2}$(it is possible with a change of time scaling). Then system (2.69) and (2.70) reads as

$$\rho Z\frac{\partial}{\partial t}\theta - \nabla.(\kappa m\theta^{m-1}\nabla\theta) = \rho\zeta(\theta)(T-\theta)$$

$$\rho\frac{\partial}{\partial t}T = \rho\zeta(\theta)(\theta - T)$$

Assume first that there exists a solution (θ, T) belonging to $L^2(0, t_f, H^1(\mathcal{O})) \cap C(0, t_f, L^2(\mathcal{O}))$, and $C(0, t_f, L^2(\mathcal{O}))$. The first key point is to prove the maximum principle (2.73).

For all functions Y, let us define the so-called sign $-$ function $s^-(Y)$ by

$$s^-(Y) = 0 \qquad \text{if } Y \geq 0, \qquad s^-(Y) = 1 \qquad \text{if } Y < 0.$$

then we have $s^-(Y)Y \leq 0$ for all Y. Notice that $s^-(Y)Y = 0$ is equivalent to $Y \geq 0$.

Let us denote

$$\tilde{\theta} = \theta - \alpha, \qquad \tilde{T} = T - \alpha$$

and set

$$I(t) \equiv Z \int_{\mathcal{O}} \rho s^-(\tilde{\theta}(t,x))\tilde{\theta}(t,x)dx + \int_{\mathcal{O}} \rho s^-(\tilde{T}(t,x))\tilde{T}(t,x)dx$$

We have, of course, $I(0) = 0$, so it suffices now to prove that

$$\partial_t I(t) \geq 0, \tag{2.90}$$

indeed, this implies $I(t) = 0$ and

$$\tilde{\theta} = \theta - \alpha \geq 0, \qquad \tilde{T} = T - \alpha \geq 0.$$

As a matter of fact, to be rigorous, we must introduce a regularized differentiable function $s^\varepsilon(Y)$ with compact support that is decreasing and that converges toward $s^-(Y)$. Then

$$I(t) \equiv \lim_{\varepsilon \to 0} \left(Z \int_{\mathcal{O}} \rho s^\varepsilon(\tilde{\theta}(t,x))\tilde{\theta}(t,x)dx + \int_{\mathcal{O}} \rho s^\varepsilon(\tilde{T}(t,x))\tilde{T}(t,x)dx \right)$$

Since $\frac{\partial s^\varepsilon}{\partial Y} \to \delta_{Y=0}$ (where δ is the Dirac distribution), we have $\frac{\partial}{\partial t}(Ys^\varepsilon(Y)) = s^\varepsilon(Y)\frac{\partial Y}{\partial t} + Y\frac{\partial s^\varepsilon}{\partial Y}\frac{\partial Y}{\partial t} \to s^-(Y)\frac{\partial Y}{\partial t}$, in the distribution meaning. Thus, we have

$$\lim_{\varepsilon \to 0} \rho Z \frac{\partial}{\partial t}(\tilde{\theta}s^\varepsilon(\tilde{\theta})) = \lim_{\varepsilon \to 0} s^\varepsilon(\tilde{\theta})\nabla.(\kappa m \theta^{m-1}\nabla\tilde{\theta}) + s^-(\tilde{\theta})\rho\zeta(\theta)(\tilde{T} - \tilde{\theta})$$

$$\lim_{\varepsilon \to 0} \rho \frac{\partial}{\partial t}(\tilde{T}s^\varepsilon(\tilde{T})) = s^-(\tilde{T})\rho\zeta(\theta)(\tilde{\theta} - \tilde{T})$$

and we get

$$\partial_t I(t) = \int_{\mathcal{O}} \rho\zeta(\theta)(\tilde{T} - \tilde{\theta})(s^-(\tilde{\theta}) - s^-(\tilde{T}))dx - \lim_{\varepsilon \to 0} \int_{\mathcal{O}} \kappa m \theta^{m-1}\nabla\tilde{\theta}.\nabla s^\varepsilon(\tilde{\theta})dx$$

For the first integral, to determine the sign of the integrand there are four cases according to the values of $s^-(\tilde{\theta})$ and $s^-(\tilde{T})$. If one has $s^-(\tilde{\theta}) = 1, s^-(\tilde{T}) = 0$, then we have $\tilde{\theta} < 0$ and $\tilde{T} \geq 0$ and $(\tilde{T} - \tilde{\theta})(s^-(\tilde{\theta}) - s^-(\tilde{T})) \geq 0$; in the case $s^-(\tilde{\theta}) = 0, s^-(\tilde{T}) = 1$, then we have $\tilde{T} < 0$ and $\tilde{\theta} \geq 0$ and $(\tilde{T} - \tilde{\theta})(s^-(\tilde{\theta}) - s^-(\tilde{T})) \geq 0$. Therefore,

$$\int \rho\zeta(\theta)(\tilde{T} - \tilde{\theta})(s^-(\tilde{\theta}) - s^-(\tilde{T}))dx \geq 0.$$

Let us focus on the second integral. We see that $\nabla\tilde{\theta}.\nabla s^{\varepsilon}(\tilde{\theta}) = (\partial_Y s^{\varepsilon})(\tilde{\theta})|\nabla\tilde{\theta}|^2 \leq 0$ for all function $\tilde{\theta}$ in $H^1(\mathcal{O})$. And passing to the limit, we get

$$\lim_{\varepsilon\to 0}\int \kappa m\theta^{m-1}\nabla\tilde{\theta}.\nabla s^{\varepsilon}(\tilde{\theta})dx \leq 0.$$

Therefore, (2.90) follows and $\theta(t) \geq \alpha$, $T(t) \geq \alpha$ for all t.

By the same technique, we can prove that for all t

$$\theta(t) \leq \beta, \qquad T(t) \leq \beta. \tag{2.91}$$

Now let us show an a priori estimate. Assume that $\theta(t), T(t)$ is a solution of (2.69) and (2.70) such that θ and T belong to $L^2(0, t_f, H^1(\mathcal{O}))$ and $L^2(0, t_f, L^2(\mathcal{O}))$. Multiplying (2.69) by θ and (2.70) by T and integrating over the domain $\mathcal{O}$,we get

$$\frac{Z}{2}\frac{\partial}{\partial t}\int \rho\theta^2 dx + \int \kappa m\theta^{m-1}|\nabla\theta|^2 dx = \int \rho\zeta(\theta)(T-\theta)\theta dx,$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int \rho T^2 dx = -\int \rho\zeta(\theta)(T-\theta)T dx.$$

Thus, since we have $\theta \geq \alpha$, we obtain

$$\frac{\partial}{\partial t}\left[\frac{Z}{2}\int \rho\theta^2 dx + \frac{1}{2}\int \rho T^2 dx\right] + \kappa_0 m\alpha^{m-1}\int |\nabla\theta|^2 dx \leq 0$$

then for all time intervals $[0, t_f]$ we have

$$\begin{aligned} &C_0\left\|\theta(t_f)\right\|^2_{L^2(\mathcal{O})} + C_0\left\|T(t_f)\right\|^2_{L^2(\mathcal{O})} + C\int_0^{t_f}\|\nabla\theta\|^2_{L^2(\mathcal{O})}dt \\ &\quad \leq \frac{Z}{2}\int \rho\theta^{\text{ini}2}dx + \frac{1}{2}\int \rho T^{\text{ini}2}dx \end{aligned} \tag{2.92}$$

Existence of a Solution. For system (2.69) and (2.70), there exists a weak solution (θ, T) belonging to $L^2(0, t_f; H^1(\mathcal{O})) \times L^2(0, t_f; L^2(\mathcal{O}))$, if (θ, T) satisfy

$$\frac{Z}{2}\frac{\partial}{\partial t}\int \rho\theta\phi dx + \int \kappa\nabla(\theta^m).\nabla\phi dx = \int \rho\zeta(\theta)(T-\theta)\phi dx$$

$$\frac{\partial}{\partial t}\int \rho T\psi dx = \int -\rho\zeta(\theta)(T-\theta)\psi dx$$

for all ψ, ϕ in $L^2(0, t_f; H^1(\mathcal{O})) \times L^2(0, t_f; L^2(\mathcal{O}))$.

We use a usual Galerkin method by approximating the original system by a system in a finite dimension.

More precisely, consider a family of finite-dimensional subspaces $(H^1_{(p)}(\mathcal{O}), L^2_{(p)}(\mathcal{O}))$ of $(H^1(\mathcal{O}), L^2(\mathcal{O}))$ such that $\cup_p(H^1_{(p)}(\mathcal{O}), L^2_{(p)}(\mathcal{O}))$ is dense in $(H^1(\mathcal{O}), L^2(\mathcal{O}))$. First, we can check that there exists a sequence of solutions

$(\theta_p(t), T_p(t))$ that are continuous from $[0, t_f]$ into $(H^1_{(p)}(\mathcal{O}) \times L^2_{(p)}(\mathcal{O}))$, which satisfy

$$\frac{Z}{2}\frac{\partial}{\partial t}\int \rho\theta_p\phi dx + \int m\kappa\theta_p^{m-1}\nabla\theta_p.\nabla\phi dx = \int \rho\zeta(\theta_p)(T_p - \theta_p)\phi dx \quad (2.93)$$

$$\frac{\partial}{\partial t}\int \rho T_p\psi dx = \int -\rho\zeta(\theta_p)(T_p - \theta_p)\psi dx \quad (2.94)$$

for all (ϕ, ψ) belonging to $(H^1_{(p)}(\mathcal{O}) \times L^2_{(p)}(\mathcal{O}))$. This is a system of ordinary differential equations.

Moreover, it is easy to check that these functions $(\theta_p(t), T_p(t))$ satisfy the bounds (2.91) and the analogous bounds of (2.92) since $\nabla((\theta_p)^m) = m\theta_p^{m-1}\nabla\theta_p$. So, we see that there exists a constant C independent from p such that

$$\int_0^{t_f} \|{\theta_p}^m\|^2_{H^1(\mathcal{O})} dt \leq C, \qquad \|\theta_p(t)\|_{L^\infty(\mathcal{O})} \leq \beta, \qquad \|T_p(t)\|_{L^\infty(\mathcal{O})} \leq \beta,$$

for all $t \in [0, t_f]$. Thus, there exists a subsequence still denoted by θ_p, T_p and two functions θ in $L^2(0, t_f; H^1(\mathcal{O})) \cap L^\infty(0, t_f; L^\infty(\mathcal{O})$ and T in $L^\infty(0, t_f; L^\infty(\mathcal{O})$ such that

$$\theta_p \to \theta \quad \text{and} \quad {\theta_p}^m \to \theta^m \text{ in } L^2(0, t_f; L^2(\mathcal{O})) \text{ strong}$$
$$\theta_p \to \theta \text{ in } L^\infty(0, t_f; L^\infty(\mathcal{O})) \text{ weak-*}$$
$$T_p \to T \text{ in } L^\infty(0, t_f; L^\infty(\mathcal{O})) \text{ weak-*}$$

and

$$\nabla({\theta_p}^m) \to \nabla(\theta^m) \text{ in } L^2(0, t_f; L^2(\mathcal{O})) \text{ weak.}$$

Since ζ is Lipschitz continuous, we have $\zeta(\theta_p) \to \zeta(\theta)$ in $L^2(0, t_f; L^2(\mathcal{O}))$ strongly and

$$\zeta(\theta_p)(T_p - \theta_p) \to \zeta(\theta)(T - \theta) \text{ in } L^2(0, t_f; L^2(\mathcal{O})) \text{ weak.}$$

(due to the usual result 4 stated in the appendix). Therefore, we can pass to the limit in (2.93) and (2.94); thus, (θ, T) is a weak solution of the systems (2.69) and (2.70).

Now, note that the equation satisfied by θ is of the type $\partial_t\theta + A(\theta) = \xi(\theta)$ where ξ is a Lipschitz continuous function and A is a continuous operator from H^1 into the space H^{-1}. Due to the classic Aubin–Lions lemma,[6] we know that θ belongs

[6]The Aubin–Lions lemma says that if $\theta \in L^2(0, t, H^1)$ and $\partial_t\theta \in L^2(0, t, H^{-1})$, then $\theta \in C(0, t, L^2)$.(H^{-1} is the dual space of H^1).

to $C(0,t_f;L^2(\mathcal{O}))$; thus, it is a classic solution in the space $L^2(\mathcal{O})$, i.e., we have $\theta(t)-\theta(0)=\int_0^t \xi(\theta(s))ds-\int_0^t A(\theta(s))ds$ for all t.

Moreover, according to result 1 in the appendix, we know that T belongs also to $C(0,t_f;L^2(\mathcal{O}))$.

Uniqueness. Assume that θ and T belong to $L^2(0,t_f;H^1(\mathcal{O}))\cap C(0,t_f;L^2(\mathcal{O}))$ and $C(0,t_f;L^2(\mathcal{O}))$ are solutions to the system (2.69), (2.70), and that $\hat{\theta}=\theta+Y$ and $\hat{T}=T+X$ belonging to the same spaces are also solutions to this system.Then we have

$$\rho Z\frac{\partial}{\partial t}Y-\nabla.(\kappa(\nabla\hat{\theta}^m-\nabla\theta^m))=\rho\zeta(\theta+Y)(T+X-\theta-Y)-\rho\zeta(\theta)(T-\theta)$$

$$\rho\frac{\partial}{\partial t}X=-\rho\zeta(\theta+Y)(T+X-\theta-Y)+\rho\zeta(\theta)(T-\theta)$$

Denote now $S(.)$ the sign function ($S(Y)=1$, if $Y\geq 0$ and $S(Y)=-1$, if $Y<0$) and $S^\varepsilon(.)$ is a regularized function of $S(.)$ that increases with compact support and such that $\lim_\varepsilon S^\varepsilon(Y)=S(Y)$. Then, multiplying by $S^\varepsilon(Y)$ and integrating over $\mathcal{O}$, we get, with $\xi_\theta(Y)=\zeta(\theta+Y)-\zeta(\theta)$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\partial}{\partial t}\int\rho ZYS^\varepsilon(Y)dx+\lim_\varepsilon\int\kappa\nabla((\theta+Y)^m-\theta^m).\nabla S^\varepsilon(Y)dx\\&=\int\rho\zeta(\hat{\theta})(X-Y)S(Y)dx+\int\rho\xi_\theta(Y)(T-\theta)S(Y)dx\end{aligned}$$

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial t}\int\rho XS^\varepsilon(X)dx=\int-\rho\zeta(\hat{\theta})(X-Y)S(X)dx-\int\rho\xi_\theta(Y)(T-\theta)S(X)dx$$

Therefore, using the same kind of argument as above, we get

$$\frac{\partial}{\partial t}\int\rho[Z|Y|+|X|]dx+\lim_\varepsilon B_\varepsilon=\int\rho\zeta(\hat{\theta})(X-Y)(S(Y)-S(X))dx+\int\rho\xi_\theta(Y)(T-\theta)(S(Y)-S(X))dx$$

where we have

$$\begin{aligned}B_\varepsilon&=\int m\kappa((\theta+Y)^{m-1}-\theta^{m-1})\nabla Y.\nabla S^\varepsilon(Y)dx\\&=\int m\kappa((\theta+Y)^{m-1}-\theta^{m-1})(\partial_Y S^\varepsilon)(Y)|\nabla Y|^2dx\to 0,\qquad\text{if }\varepsilon\to 0.\end{aligned}$$

We check that

$$\int\rho\zeta(\hat{\theta})(X-Y)(S(Y)-S(X))dx\leq 0.$$

Therefore, since ξ_θ is Lipschitz continuous and $|S(Y) - S(X)| \leq 2$, we have

$$\frac{\partial}{\partial t}\int \rho[Z|Y| + |X|]dx \leq C\int \rho|Y|dx \leq \frac{C}{Z}\int \rho[Z|Y| + |X|]dx$$

Now, according to Gronwal's lemma, we see that

$$\left(\int \rho[Z|\hat{\theta} - \theta| + |\hat{T} - T|]dx\right)(t) \leq e^{Ct/Z}\int \rho[Z|\hat{\theta}^{ini} - \theta^{ini}| + |\hat{T}^{ini} - T^{ini}|]dx.$$

and uniqueness follows. □

2.4 Analysis of the Hyperbolic Part of Systems ($\mathcal{E}2\mathcal{T}$) and ($\mathcal{MHD}$)

Here we are concerned only with the "ideal part" of systems ($\mathcal{E}2\mathcal{T}$) and ($\mathcal{MHD}$); the ideal part corresponds to keeping only the terms with first-order spatial derivatives and neglecting the right-hand-side terms (see below). Our aim is to check that this ideal part is hyperbolic: we need to check that the eigenvalues of the matrix of the first-order derivatives are real. To do this, we choose the Lagrangian framework. We introduce the specific volume

$$\tau = 1/\rho,$$

which is the good unknown function for the continuity equation; indeed, we have

$$\rho\frac{D}{Dt}\tau = \nabla.\mathbf{U}.$$

For the ideal part of two models, the electron energy equation reduces to

$$\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} = 0$$

and may be replaced by

$$\frac{D}{Dt}\varepsilon_e + P_e\frac{D}{Dt}\tau = 0.$$

But, since $\varepsilon_e = \frac{3}{2m_0}ZT_e$ and $P_e = ZN_0T_e = \frac{2}{3}\varepsilon_e/\tau$, it is equivalent to

$$\frac{D}{Dt}\log(\varepsilon_e\tau^{2/3}) = 0.$$

Then a natural quantity $\varepsilon_e \tau^{2/3} = \frac{3}{2} P_e \tau^{5/3}$ appears, called the electron entropy, and this entropy is preserved by the Lagrangian derivative (it is equal to the physical entropy up to the sign and a multiplicative constant).

Therefore, defining the specific total energy by

$$e = \varepsilon_0 + \varepsilon_e + \frac{1}{2}|\mathbf{U}|^2$$

the ideal part of system $(\mathcal{E}2\mathcal{T})$ has the following form

$$\begin{aligned} \rho \frac{D}{Dt}\tau - \nabla.\mathbf{U} &= 0, \\ \rho \frac{D}{Dt}\mathbf{U} + \nabla P_{\mathrm{p}} &= 0, \\ \rho \frac{D}{Dt} e + \nabla.(\mathbf{U} P_{\mathrm{p}}) &= 0, \\ \rho \frac{D}{Dt}\varepsilon_e \tau^{2/3} &= 0. \end{aligned}$$

Set $\mathbf{Y}$ as the vector (of dimension 6) of the physical state and $\mathbb{F}(\mathbf{Y})$ as the tensor, which are then characterized by

$$\mathbf{Y} = \begin{bmatrix} \tau \\ \mathbf{U} \\ e \\ \varepsilon_e \tau^{2/3} \end{bmatrix}, \qquad \text{and } \mathbb{F}(\mathbf{Y}) = \begin{bmatrix} -\mathbf{U} \\ P_{\mathrm{p}}\mathbb{I} \\ \mathbf{U} P_{\mathrm{p}} \\ 0 \end{bmatrix}.$$

(recall that $\mathbb{I}$ is the identity tensor), system $(\mathcal{E}2\mathcal{T})$ reads as

$$\rho \frac{D}{Dt}\left[\mathbf{Y}\right] + \nabla.\left[\mathbb{F}(\mathbf{Y})\right] = 0. \tag{2.95}$$

Now for $(\mathcal{MHD})$, the specific total energy is given by

$$e = \varepsilon_0 + \varepsilon_e + \frac{1}{2}\left|\mathbf{U}\right|^2 + \frac{1}{2\mu^0}\tau \left|\mathbf{B}\right|^2.$$

Using the identity $\nabla.(\mathbf{BU}) = \mathbf{U}(\nabla.\mathbf{B}) + (\mathbf{B}.\nabla)\mathbf{U}$ (and the fact that $\nabla.\mathbf{B} = 0$), the magnetic field equation is

$$\rho \frac{D}{Dt}(\tau \mathbf{B}) = \nabla.(\mathbf{BU}).$$

So, the ideal MHD system reads as (2.95) but the vector $\mathbf{Y}$ (of dimension 9) and $\mathbf{F}(\mathbf{Y})$ are now given by

$$\mathbf{Y} = \begin{bmatrix} \tau \\ \mathbf{U} \\ \tau\mathbf{B} \\ e \\ \varepsilon_e \tau^{2/3} \end{bmatrix}, \qquad \text{and } \mathbb{F} = \begin{bmatrix} -\mathbf{U} \\ P_\mathrm{p}\mathbb{I} + \mathbb{P}_B \\ -\mathbf{UB} \\ P_\mathrm{p}\mathbf{U} + \mathbb{P}_B.\mathbf{U} \\ 0 \end{bmatrix}.$$

2.4.1 On the Galilean Transformations

We now make a regular transformation of the unknown variables $\mathbf{Z} = \mathbf{Z}_\mathbf{u}(\mathbf{Y})$ corresponding to a fixed translation of vector $\mathbf{u}$, (i.e., a Galilean transformation); so the density ρ is not changed. Then system (2.95) becomes $\rho \dfrac{D}{Dt}[\mathbf{Z}] + \dfrac{\partial \mathbf{Z}}{\partial \mathbf{Y}} \nabla.\left[\mathbb{F}(\mathbf{Y}(\mathbf{Z}))\right] = 0$; since the Jacobian matrix $\frac{\partial \mathbf{Z}}{\partial \mathbf{Y}}$ is constant, the new system reads

$$\rho \frac{D}{Dt}[\mathbf{Z}] + \nabla.\left[\tilde{\mathbb{F}}(\mathbf{Z})\right] = 0$$

with $\tilde{\mathbb{F}}(\mathbf{Z}) = \dfrac{\partial \mathbf{Z}}{\partial \mathbf{Y}}.\mathbb{F}(\mathbf{Y}(\mathbf{Z}))$. The system is called "invariant through a Galilean transformation," if for any fixed vector $\mathbf{u}$, after the transformation, we have $\tilde{\mathbb{F}} = \mathbb{F}$.

For the ($\mathcal{MHD}$) model, consider now a Galilean transformation related to a translation $\mathbf{u}$; since we have $\tau' = \tau$, $\varepsilon'_e = \varepsilon_e$, $\varepsilon'_0 = \varepsilon_0$, $\mathbf{U}' = \mathbf{U} - \mathbf{u}$ and $\mathbf{B}' = \mathbf{B}$, the characteristic vector $\mathbf{Z} = \mathbf{Z}_\mathbf{u}(\mathbf{Y})$ reads

$$\mathbf{Z} = \begin{bmatrix} \tau' \\ \mathbf{U}' \\ \tau'\mathbf{B}' \\ e' \\ \varepsilon'_e(\tau')^{2/3} \end{bmatrix} = \begin{bmatrix} \tau \\ \mathbf{U} - \mathbf{u} \\ \tau\mathbf{B} \\ e + (\frac{1}{2}|\mathbf{u}|^2 - \mathbf{U}.\mathbf{u}) \\ \varepsilon_e(\tau)^{2/3} \end{bmatrix}$$

Of course, we have $P'_\mathrm{p} = P_\mathrm{p}, \mathbb{P}'_B = \mathbb{P}_B$. Since $\mathbf{u}$ is constant, we have $\partial_t \mathbf{u} = 0$ and $\nabla.\mathbf{u} = 0$; then we see that

$$\tilde{\mathbb{F}} = \begin{bmatrix} -\mathbf{U} \\ P_\mathrm{p}\mathbb{I} + \mathbb{P}_B \\ -\mathbf{UB} \\ P_\mathrm{p}\mathbf{U}' + \mathbb{P}_B.\mathbf{U}' \\ 0 \end{bmatrix} = \begin{bmatrix} -\mathbf{U}' \\ P_\mathrm{p}\mathbb{I} + \mathbb{P}_B \\ -\mathbf{U}'\mathbf{B}' \\ P_\mathrm{p}\mathbf{U}' + \mathbb{P}_B.\mathbf{U}' \\ 0 \end{bmatrix}.$$

Indeed, for the energy we have

$$\rho\frac{D}{Dt}(e-\mathbf{U}.\mathbf{u}) = \nabla.((P_\mathrm{p}\mathbb{I}+\mathbb{P}_B).\mathbf{U}) - \mathbf{u}.\nabla(P_\mathrm{p}\mathbb{I}+\mathbb{P}_B) = \nabla.((P_\mathrm{p}\mathbb{I}+\mathbb{P}_B).\mathbf{U}')$$

Thus, this model is invariant through this Galilean transformation.

For the model $(\mathcal{E}2\mathcal{T})$, it is the same calculus.

Moreover, for both models, it is crucial to notice that the electric field becomes $\mathbf{E}' = \mathbf{E} + \mathbf{u}\times\mathbf{B}$ and the generalized Ohm's law reads as

$$q_e Z N_0(\mathbf{E}' + \mathbf{U}'\times\mathbf{B}') + \nabla P_e = 0$$

One may also check that in these models, the flux tensor has the general form described in [43].

2.4.2 *Hyperbolic Properties of Both Models*

Our concern is analysis of the ideal part of both systems, so we focus on the Lagrangian framework in the one-dimensional case, i.e., all the physical quantities depend only on the x-variable. It is useful to introduce the new variable m defined by $dm = \rho dx$; the tensor $\mathbb{F}(\mathbf{Y})$ reduces to a vector $\mathbf{F}$ and the system for the ideal part reads now as

$$\frac{D}{Dt}\mathbf{Y} + \frac{\partial}{\partial m}\mathbf{F}(\mathbf{Y}) = 0. \tag{2.96}$$

(i) For Model $(\mathcal{E}2\mathcal{T})$.

Since τ, e depends only on the x-variable, the evolution of $\mathbf{U} = (U_x\, U_y, U_z)$ reduces to the one of U_x. So we have to deal with the system

$$\begin{aligned}
\frac{D}{Dt}\tau - \frac{\partial}{\partial m}U_x &= 0,\\
\frac{D}{Dt}U_x + \frac{\partial}{\partial m}(P_0 + P_e) &= 0,\\
\frac{D}{Dt}e + \frac{\partial}{\partial m}(U_x(P_0 + P_e)) &= 0,\\
\frac{D}{Dt}\log(\varepsilon_e\tau^{2/3}) &= 0.
\end{aligned}$$

It is a four-dimensional vector system of type (2.96) with

$$\mathbf{Y} = \begin{bmatrix}\tau\\ U_x\\ e\\ \varepsilon_e\tau^{2/3}\end{bmatrix}, \qquad \text{and } \mathbf{F}(\mathbf{Y}) = \begin{bmatrix}-U_x\\ P_\mathrm{p}\\ U_x P_\mathrm{p}\\ 0\end{bmatrix}$$

where we set (assuming that γ_0=5/3)

$$e = \varepsilon_p + \frac{1}{2}U_x^2, \qquad \varepsilon_p = \varepsilon_0 + \varepsilon_e, \qquad P_\mathrm{p} = P_0 + P_e = \frac{2}{3}\varepsilon_p \tau^{-1}.$$

Proposition 8. *The eigenvalues of the Jacobian matrix of the flux term* $\dfrac{\partial \mathbf{F}}{\partial \mathbf{Y}}$ *are*

$$-a_o, \qquad 0, \qquad 0, \qquad a_o, \qquad \textit{where } a_o^2 = \frac{5}{3}\rho P_p$$

It can be seen that for the double eigenvalue 0, the space of the eigenvector is of dimension two, so the system is hyperbolic. We recover that when the original system (2.95) is written in a Eulerian framework, the eigenvalues of the Jacobian matrix are $-a_o/\rho$, 0, 0, a_o/ρ (when the plasma velocity is zero). Note that

$$\frac{a_o}{\rho} = \sqrt{\frac{5}{3}\frac{P_\mathrm{p}}{\rho}}$$

is the usual expression for the sound speed (of course, the pressure taken into account is the total pressure).

(ii) For Model ($\mathcal{MHD}$).

Recall that $\mathbf{B} = (B_x, B_y, B_z)$ and $\mathbf{U} = (U_x, U_y, U_z)$ which depends only on the x-variable. According to the constraint on the magnetic field $\nabla.\mathbf{B} = 0$, we get $\frac{\partial}{\partial x}B_x = 0$; then we have also $\frac{\partial}{\partial t}B_x = 0$; so B_x is a given constant and the system reads as

$$\begin{aligned}
\frac{D}{Dt}\tau - \frac{\partial}{\partial m}U_x &= 0,\\
\frac{D}{Dt}U_x + \frac{\partial}{\partial m}(P_\mathrm{p} + P_\mathrm{mag}) &= 0,\\
\frac{D}{Dt}U_y - \frac{B_x}{\mu^0}\frac{\partial}{\partial m}B_y &= 0,\\
\frac{D}{Dt}U_z - \frac{B_x}{\mu^0}\frac{\partial}{\partial m}B_z &= 0,\\
\frac{D}{Dt}(B_y\tau) - B_x\frac{\partial}{\partial m}U_y &= 0,\\
\frac{D}{Dt}(B_z\tau) - B_x\frac{\partial}{\partial m}U_z &= 0,\\
\frac{D}{Dt}e + \frac{\partial}{\partial m}(U_x(P_\mathrm{p} + P_\mathrm{mag})) - \frac{\partial}{\partial m}\left((U_yB_y + U_zB_z)\frac{B_x}{\mu^0}\right) &= 0,\\
\frac{D}{Dt}(\varepsilon_e\tau^{2/3}) &= 0,
\end{aligned}$$

where $e = \varepsilon_p + \frac{1}{2}|\mathbf{U}|^2 + \frac{\tau}{2\mu^0}(|B_x|^2 + |B_y|^2 + |B_z|^2)$ and P_p, ε_p defined as above, $P_{\text{mag}} = \frac{1}{2\mu^0}(B_y^2 + B_z^2 - B_x^2)$.

Of course, if $B_x = 0$, we recover model $(\mathcal{E}2\mathcal{T})$. We are now concerned with the eigenvalues of the Jacobian matrix of this system. For the sake of simplicity, assume that U_z and B_z are zero. We simplify the notations by setting $U = U_x$, $V = U_y$, $M = B_y\tau$, $\beta = B_x$. So we get

$$\varepsilon_p = e - \frac{1}{2}U^2 - \frac{1}{2}V^2 - \frac{1}{2\mu^0\tau}M^2 - \frac{\tau}{2\mu^0}\beta^2;$$

$$P_{\text{p}} + P_{\text{mag}} = \frac{2}{3}\frac{1}{\tau}\varepsilon_p + \frac{1}{2\mu^0}\left(\frac{M^2}{\tau^2} - \beta^2\right) = \frac{2}{3}\frac{1}{\tau}\left(e - \frac{U^2}{2} - \frac{V^2}{2}\right) + \frac{1}{6\mu^0\tau^2}M^2 - \frac{5}{6\mu^0}\beta^2$$

and the six-dimensional vector system reads $\dfrac{D}{Dt}\mathbf{Y} + \dfrac{\partial}{\partial m}\mathbf{F}(\mathbf{Y}) = 0$, with

$$\mathbf{Y} = \begin{bmatrix} \tau \\ U \\ V \\ M \\ e \\ \varepsilon_e\tau^{2/3} \end{bmatrix}, \qquad \text{and } \mathbf{F}(\mathbf{Y}) = \begin{bmatrix} -U \\ P_{\text{p}} + P_{\text{mag}} \\ -\frac{\beta}{\mu^0}M\tau^{-1} \\ -\beta V \\ U(P_{\text{p}} + P_{\text{mag}}) - \frac{\beta}{\mu^0}VM\tau^{-1} \\ 0 \end{bmatrix}.$$

Proposition 9. *The six eigenvalues of the Jacobian matrix* $\left.\dfrac{\partial \mathbf{F}}{\partial \mathbf{Y}}\right|_{\mathbf{Y}}$ *are given by*

$$-\sqrt{X_{fast}/\rho}, \qquad \text{-}\sqrt{X_{slow}/\rho}, \qquad 0, \qquad 0, \qquad \sqrt{X_{slow}/\rho}, \qquad \sqrt{X_{fast}/\rho}$$

where

$$X_{slow} = \frac{1}{2}\left(\frac{a_0^2}{\rho} + \frac{|\mathbf{B}|^2}{\mu^0}\right) - \frac{1}{2}\left[\left(\frac{a_0^2}{\rho} + \frac{|\mathbf{B}|^2}{\mu^0}\right)^2 - 4\frac{a_0^2}{\rho}\frac{\beta^2}{\mu^0}\right]^{1/2}$$

$$X_{fast} = \frac{1}{2}\left(\frac{a_0^2}{\rho} + \frac{|\mathbf{B}|^2}{\mu^0}\right) + \frac{1}{2}\left[\left(\frac{a_0^2}{\rho} + \frac{|\mathbf{B}|^2}{\mu^0}\right)^2 - 4\frac{a_0^2}{\rho}\frac{\beta^2}{\mu^0}\right]^{1/2}.$$

Therefore, the six eigenvalues λ of the original problem are the following

$$-\sqrt{\rho X_{fast}}, \qquad -\sqrt{\rho X_{slow}}, \qquad 0, \qquad 0, \qquad \sqrt{\rho X_{slow}}, \qquad \sqrt{\rho X_{fast}}.$$

Remark 12. It may be checked that if $|\mathbf{B}|$ goes to zero, the two roots X_{slow} and X_{fast} converge to 0 and a_0^2/ρ, and the eigenvalues λ converge to

$$-a_0, \qquad 0, \qquad 0, \qquad 0, \qquad 0, \qquad a_0.$$

(recall that $\pm a_0$ are exactly the eigenvalues of the model ($\mathcal{E}2\mathcal{T}$)). One may prove (see [43]) that the dimension of the eigenspace corresponding to 0 is equal to 4, thus the system is strictly hyperbolic.

Assume on the contrary that the sound speed a_0 is very small compared to the magnetic energy; then we have $\left[\left(\frac{a_0^2}{\rho}+\frac{|\mathbf{B}|^2}{\mu^0}\right)^2-4\frac{a_0^2}{\rho}\frac{\beta^2}{\mu^0}\right]^{1/2}\simeq\frac{|\mathbf{B}|^2}{\mu^0}+\frac{a_0^2}{\rho}(1-2\beta^2/|\mathbf{B}|^2)$, then we get

$$\rho X_{slow}\simeq a_0^2\frac{\beta^2}{|\mathbf{B}|^2}$$

$$\rho X_{fast}\simeq\rho\frac{|\mathbf{B}|^2}{\mu^0}+a_0^2(1-\frac{\beta^2}{|\mathbf{B}|^2})$$

So, we see that the two eigenvalues λ related to X_{fast} are at first order equal to $\pm\sqrt{\rho/\mu^0}|\mathbf{B}|$ and the corresponding characteristic speeds are equal to $\pm|\mathbf{B}|/\sqrt{\rho\mu^0}$, that is to say the Alfven speed. □

2.4.3 Proofs of the Propositions of the Section

The following lemma will be useful.

Lemma 3. *If a system* $\frac{D}{Dt}[\mathbf{Y}]+\frac{\partial}{\partial m}[\mathbf{F}(\mathbf{Y})]=0$ *is invariant through a Galilean transformation, the eigenvalues of the Jacobian matrix* $\left.\frac{\partial\mathbf{F}}{\partial\mathbf{Y}}\right|_{\mathbf{Y}}$ *are the same as the ones of the Jacobian matrix* $\left.\frac{\partial\mathbf{F}}{\partial\mathbf{Y}}\right|_{\mathbf{Y}^0}$ *when* Y^0 *is the state obtained by setting* $U=0$.

Proof. It is clear that if we apply a transformation corresponding to a fixed translation of vector $\mathbf{u}$, the new system will have a Jacobian matrix

$$\left.\frac{\partial\tilde{\mathbf{F}}}{\partial\mathbf{Z}}\right|_{\mathbf{Z}}=\frac{\partial\mathbf{Z}}{\partial\mathbf{Y}}.\left.\frac{\partial\mathbf{F}}{\partial\mathbf{Y}}\right|_{\mathbf{Y}(\mathbf{Z})}.\frac{\partial\mathbf{Y}}{\partial\mathbf{Z}}$$

Since $\frac{\partial\mathbf{Z}}{\partial\mathbf{Y}}$ is the inverse of $\frac{\partial\mathbf{Y}}{\partial\mathbf{Z}}$, if λ is an eigenvalue for the matrix $\left.\frac{\partial\mathbf{F}}{\partial\mathbf{Y}}\right|_{\mathbf{Y}}$, it is also an eigenvalue for the matrix $\left.\frac{\partial\tilde{\mathbf{F}}}{\partial\mathbf{Z}}\right|_{\mathbf{Z}}$. For a given point $\mathbf{x}$, we can apply this remark by choosing $\mathbf{u}=-\mathbf{U}(\mathbf{x})$, then the state $\mathbf{Z}$ corresponds to the same state as the original one, but the velocity $\mathbf{U}'$ is equal to 0 at point $\mathbf{x}$. So, according to the fact that $\tilde{\mathbf{F}}=\mathbf{F}$, the lemma follows. □

Proof of Proposition 8. The Jacobian matrix of the flux term is the matrix $\dfrac{\partial \mathbf{F}}{\partial \mathbf{Y}}$ given by

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -\frac{2}{3}\varepsilon_t \tau^{-2} & -\frac{2}{3}U_x \tau^{-1} & \frac{2}{3}\tau^{-1} & 0 \\ -\frac{2}{3}U_x \varepsilon_t \tau^{-2} & \frac{2}{3}\varepsilon_t \tau^{-1} - \frac{2}{3}U_x^2 \tau^{-1} & \frac{2}{3}U_x \tau^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the characteristic polynomial is $\lambda \det(A - \lambda I) = 0$, where A is the same as above without the last row and column, i.e.,

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -\rho P_{\mathrm{p}} & -\frac{2}{3}U_x \rho & \frac{2}{3}\rho \\ -U_x \rho P_{\mathrm{p}} & P_{\mathrm{p}} - \frac{2}{3}U_x^2 \rho & \frac{2}{3}U_x \rho \end{bmatrix}$$

According to the previous lemma, one may evaluate the eigenvalues of A after setting U_x to zero and we get

$$\det(A - \lambda I) = \lambda^3 - \frac{5}{3}\rho P_{\mathrm{p}} \lambda. \qquad \square$$

Proof of Proposition 9. The Jacobian matrix of the flux term $\left[\frac{\partial \mathbf{F}}{\partial \mathbf{Y}}\right]$ reduces (after withdrawing the last column and the last row) to the matrix given by

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -\frac{2e-|\mathbf{U}|^2}{3\tau^2} - \frac{1}{3\mu^0\tau^3}M^2 & -\frac{2}{3\tau}U & -\frac{2}{3\tau}V & \frac{1}{3\mu^0}M\frac{1}{\tau^2} & \frac{2}{3}\frac{1}{\tau} \\ \frac{\beta}{\mu^0}M\frac{1}{\tau^2} & 0 & 0 & -\frac{\beta}{\mu^0}\frac{1}{\tau} & 0 \\ 0 & 0 & -\beta & 0 & 0 \\ -U\frac{2e-|\mathbf{U}|^2}{3\tau^2} + \frac{UM^2}{3\mu^0\tau^3} + \frac{\beta VM}{\mu^0\tau^2} & P_{\mathrm{p}} + P_{\mathrm{mag}} - \frac{2}{3\tau}U^2 & -\frac{2}{3\tau}UV - \frac{\beta}{\mu^0\tau}M & \frac{1}{3\mu^0\tau^2}UM - \frac{\beta}{\mu^0\tau}V & \frac{2}{3\tau}U \end{bmatrix}$$

According to the previous lemma, one may evaluate the eigenvalues of this matrix after setting $U = V = 0$. Then, going back to the physical variables, we have to compute the characteristic polynomial $\det(A_0 - \lambda I)$ where

$$A_0 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -\rho P_{\mathrm{p}} - \rho R & 0 & 0 & \frac{1}{3\mu^0}\rho B_y & \frac{2}{3}\rho \\ \frac{\beta}{\mu^0}\rho B_y & 0 & 0 & -\frac{\beta}{\mu^0}\rho & 0 \\ 0 & 0 & -\beta & 0 & 0 \\ 0 & P_{\mathrm{p}} + P_{\mathrm{mag}} & -\frac{\beta}{\mu^0}B_y & 0 & 0 \end{bmatrix}$$

[here we have set $R = \frac{1}{3\mu^0}(2B_y^2 + \beta^2)$] and this polynomial reads as

$$\begin{vmatrix} -\lambda & -1 & 0 & 0 & 0 \\ -\rho P_{\mathrm{p}} - \rho R & -\lambda & 0 & \frac{1}{3\mu^0}\rho B_y & \frac{2}{3}\rho \\ \frac{\beta}{\mu^0}\rho B_y & 0 & -\lambda & -\frac{\beta}{\mu^0}\rho & 0 \\ 0 & 0 & -\beta & -\lambda & 0 \\ 0 & P_{\mathrm{p}} + P_{\mathrm{mag}} & -\frac{\beta}{\mu^0}B_y & 0 & -\lambda \end{vmatrix} = 0$$

and we get after some tedious calculus

$$-\lambda^5 + \lambda^3\rho\left[P_{\mathrm{p}} + R + \frac{2}{3}P_{\mathrm{p}} + \frac{2}{3}P_{\mathrm{mag}} + \frac{\beta^2}{\mu^0}\right] + \lambda\rho^2\left[-\frac{2}{3}\frac{\beta^2}{\mu^0}(P_{\mathrm{p}} + P_{\mathrm{mag}}) + \frac{\beta^2}{\mu^{02}}B_y^2 - \frac{\beta^2}{\mu^0}(P_{\mathrm{p}} + R)\right] = 0.$$

We see that 0 is, of course, a root of this equation. The other roots are obtained by solving a second-degree equation with $\lambda^2 = \rho X$, where this equation in X reads as follows (using the notation $a_0^2 = \frac{5}{3}\rho P_p$ and noticing that $\frac{2}{3}P_{\mathrm{mag}} + R = \frac{B_y^2}{\mu^0}$)

$$X^2 - X\left[\frac{a_0^2}{\rho} + \frac{B_y^2 + \beta^2}{\mu^0}\right] + \left[\frac{a_0^2}{\rho}\frac{\beta^2}{\mu^0}\right] = 0.$$

Its discriminant is $\Delta = \left(\frac{a_0^2}{\rho} + \frac{B_y^2+\beta^2}{\mu^0}\right)^2 - 4\frac{a_0^2}{\rho}\frac{\beta^2}{\mu^0}$ [which is always greater than $(\frac{a_0^2}{\rho} - \frac{\beta^2}{\mu^0})^2$]. Since $B_z = 0$, notice that $\frac{B_y^2+\beta^2}{2\mu^0} = \frac{|\mathbf{B}|^2}{\mu^0}$ is the magnetic energy and the two roots are X_{slow} and X_{fast} given in the proposition. Then the eigenvalues of the initial Jacobian matrix are

$$-\sqrt{X_{fast}}, -\sqrt{X_{slow}}, 0, 0, \sqrt{X_{slow}}, \sqrt{X_{fast}}.$$

□

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