

Chapter 2

Basic Mathematical Concepts

2.1 Introduction

Just like in other areas of science and engineering, mathematics also plays an important role in the area of human reliability, error, and human factors. The history of mathematics may be traced back to more than 2,200 years to the development of our day-to-day used number symbols. The very first evidence of the use of these symbols is found on stone columns erected around 250 BC by the Scythian emperor of India named Asoka [1].

However, the development of the probability field is relatively new, and its history may be traced back to the writings of Girolamo Cardano (1501–1576) in which he considered some interesting probability issues [1, 2]. Blaise Pascal (1623–1662) and Peirre de Fermat (1601–1665) solved the problem of dividing the winnings in a game of chance, independently and correctly [2]. The first formal treatise on probability based on the Pascal-Fermat correspondence was written by Christiaan Huygens (1629–1695) in 1657 [2]. Needless to say, additional information on historical developments in the area of mathematics is available in Refs. [1, 2].

This chapter presents basic mathematical concepts considered useful in performing human reliability and error analysis in the area of power generation.

2.2 Sets and Boolean Algebra Laws

A set may be defined as any well-defined collection or list of objects. Usually, the objects comprising the set are known as its elements. Normally, capital letters such as X , Y , and Z are used to denote sets and their elements by the lower-case letters such as a , b , and c .

Two basic set operations are referred to as the union of sets and the intersection of sets. Either of the following two symbols is used to denote the union of sets [3]:

- +
- \cup

For example, if $X + Y = Z$, it simply means that all the elements in set X or in set Y or in both sets (i.e. X and Y) are contained in set Z .

Similarly, either of the following two symbols is used to denote the intersection of sets:

- \cap
- dot (\cdot)

For example, if $A \cap B = C$, it simply means that set C contains all elements which belong to both sets A and B . However, when sets A and B have no common elements, then these two sets are referred to as mutually exclusive or disjoint sets or events.

Boolean algebra plays an important role in probability theory and reliability-related studies and is named after a mathematician named George Boole (1813–1864). Some of its laws are as follows [3–5]:

Cummutative Law

$$X + Y = Y + X \quad (2.1)$$

$$X.Y = Y.X \quad (2.2)$$

where

X is an arbitrary set or event.

Y is an arbitrary set or event.

$+$ denotes the union of sets.

\cdot denotes the intersection of sets. It is to be noted that sometimes, Eq. (2.2) is written without the dot, but it still conveys the same meaning.

Associative Law

$$(XY)Z = X(YZ) \quad (2.3)$$

$$(X + Y) + Z = X + (Y + Z) \quad (2.4)$$

where

Z is an arbitrary set or event.

Idempotent Law

$$X + X = X \quad (2.5)$$

$$XX = X \quad (2.6)$$

Absorption Law

$$X + (XY) = X \quad (2.7)$$

$$X(X + Y) = X \quad (2.8)$$

Distributive Law

$$X(Y + Z) = XY + XZ \quad (2.9)$$

$$X + YZ = (X + Y)(X + Z) \quad (2.10)$$

2.3 Probability Definition and Properties

Probability may be defined as follows [4, 6]:

$$P(X) = \lim_{n \rightarrow \infty} \left(\frac{N}{n} \right) \quad (2.11)$$

where

$P(X)$ is the probability of occurrence of event X .

N is the number of times event X occurs in the n repeated experiments.

Some of the basic properties of probability are presented below [4, 6].

- The probability of occurrence of event, say Y , is

$$0 \leq P(Y) \leq 1. \quad (2.12)$$

- The probability of occurrence and non-occurrence of an event, say Y , is always

$$P(Y) + P(\bar{Y}) = 1 \quad (2.13)$$

where

$P(Y)$ is the probability of occurrence of event Y .

$P(\bar{Y})$ is the probability of non-occurrence of event Y .

- Probability of the sample space S is

$$P(S) = 1. \quad (2.14)$$

- Probability of negation of the sample space S is

$$P(\bar{S}) = 0. \quad (2.15)$$

- The probability of the union of m independent events is

$$P(Y_1 + Y_2 + \cdots + Y_m) = 1 - \prod_{i=1}^m (1 - P(Y_i)) \quad (2.16)$$

where

$P(Y_i)$ is the probability of occurrence of event Y_i ; for $i = 1, 2, 3, \dots, m$.

- The probability of the union of m mutually exclusive events is given by

$$P(Y_1 + Y_2 + \cdots + Y_m) = \sum_{i=1}^m P(Y_i). \quad (2.17)$$

- The probability of an intersection of m independent events is given by

$$P(Y_1 Y_2 \dots Y_m) = P(Y_1)P(Y_2) \dots P(Y_m). \quad (2.18)$$

Example 2.1 Assume that a system used in a power generation plant is composed of three subsystems Y_1 , Y_2 , and Y_3 and which must be operated by three independent operators. For the successful operation of the system, all the three operators must perform their tasks correctly. The reliabilities of operators, operating subsystems Y_1 , Y_2 , and Y_3 are 0.95, 0.92, and 0.9, respectively.

Calculate the probability of successful operation of the system.

By substituting the given data values in Eq. (2.18), we get

$$\begin{aligned} P(Y_1 Y_2 Y_3) &= P(Y_1)P(Y_2)P(Y_3) \\ &= (0.95)(0.92)(0.9) \\ &= 0.7866. \end{aligned}$$

Thus, the probability of successful operation of the system is 0.7866.

2.4 Useful Mathematical Definitions

This section presents five mathematical definitions considered useful to perform human reliability-related studies in the area of power generation.

2.4.1 Definition I: Probability Density Function

For a continuous random variable, the probability density function is defined by [6, 7]

$$f(t) = \frac{dF(t)}{dt} \quad (2.19)$$

where

t is time (i.e. a continuous random variable).

$f(t)$ is the probability density function. In the area of human reliability, it is often referred to as error density function.

$F(t)$ is the cumulative distribution function.

Example 2.2 Assume that the error probability at time t (i.e. cumulative distribution function) of a power generating system operator is expressed by

$$F(t) = 1 - e^{-\theta t} \quad (2.20)$$

where

θ is the constant error rate of the power generating system operator.
 $F(t)$ is the cumulative distribution function or the operator error probability at time t .

Obtain an expression for the probability density function (i.e. in this case, the operator error density function) by using Eq. (2.19).

By inserting Eq. (2.20) into Eq. (2.19), we obtain

$$\begin{aligned} f(t) &= \frac{d(1 - e^{-\theta t})}{dt} \\ &= \theta e^{-\theta t} \end{aligned} \quad (2.21)$$

Thus, Eq. (2.21) is the expression for the operator error density function.

2.4.2 Definition II: Cumulative Distribution Function

Cumulative distribution function for a continuous random variable is expressed by [6, 7]

$$F(t) = \int_{-\infty}^t f(y) dy \quad (2.22)$$

where

y is a continuous random variable.
 $f(y)$ is the probability density function.

For $t = \infty$ in Eq. (2.22), we get

$$\begin{aligned} F(\infty) &= \int_{-\infty}^{\infty} f(y) dy \\ &= 1. \end{aligned} \quad (2.23)$$

It simply means that the total area under the probability density curve is always equal to unity.

Example 2.3 Prove Eq. (2.20) with the aid of Eq. (2.21).

Thus, for $t \geq 0$, by inserting Eq. (2.21) into Eq. (2.22), we obtain

$$\begin{aligned} F(t) &= \int_0^t \theta e^{-\theta t} dt \\ &= 1 - e^{-\theta t}. \end{aligned} \quad (2.24)$$

Both Eqs. (2.20) and (2.24) are identical.

2.4.3 Definition III: Expected Value

The expected value of a continuous random variable is expressed by [6, 7]

$$E(t) = \int_{-\infty}^{\infty} tf(t)dt \quad (2.25)$$

where

$E(t)$ is the expected value or mean value of the continuous random variable t . It is to be noted that in the area of human reliability and error, the expected value is known as the mean time to human error.

Example 2.4 Assume that a power generating system operator's error density function, for time $t \geq 0$, is expressed by Eq. (2.21) and the operator's error rate is 0.0005 errors/h. Calculate mean time to human error of the power generating system operator.

For $t \geq 0$, by inserting Eq. (2.21) into Eq. (2.25), we get

$$\begin{aligned} E(t) &= \int_0^{\infty} t \theta e^{-\theta t} dt \\ &= \frac{1}{\theta}. \end{aligned} \quad (2.26)$$

By substituting the given data value for θ in Eq. (2.26), we obtain

$$\begin{aligned} E(t) &= \frac{1}{0.0005} \\ &= 2,000 \text{ h.} \end{aligned}$$

Thus, the mean time to human error of the power generating system operator is 2,000 h.

2.4.4 Definition IV: Laplace Transform

The Laplace transform of the function $f(t)$ is defined by [8]

$$f(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (2.27)$$

where

s is the Laplace transform variable.

$f(s)$ is the Laplace transform of $f(t)$.

t is the time variable.

Laplace transforms of some frequently used functions to perform human reliability-related mathematical analysis in the area of power generation are presented in Table 2.1 [8, 9].

2.4.5 Definition V: Laplace Transform: Final-Value Theorem

If the following limits exist, then the final-value theorem may be expressed as follows:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} [sf(s)] \quad (2.28)$$

Example 2.5 Prove with the aid of the equation presented below that the right side of Eq. (2.28) is equal to its left side.

$$f(t) = \frac{c}{(c+d)} - \frac{c}{(c+d)} e^{-(c+d)t} \quad (2.29)$$

where c and d are constants.

With the aid of Table 2.1, we get the following Laplace transforms of Eq. (2.29)

$$f(s) = \frac{c}{s(c+d)} - \frac{c}{(c+d)} \cdot \frac{1}{(s+c+d)}. \quad (2.30)$$

By inserting Eq. (2.30) into the right side of Eq. (2.28), we get

$$\lim_{s \rightarrow 0} s \left[\frac{c}{s(c+d)} - \frac{c}{(c+d)(s+c+d)} \right] = \frac{c}{(c+d)}. \quad (2.31)$$

Table 2.1 Laplace transforms of some frequently used functions to perform human reliability-related mathematical analysis in the area of power generation

$f(t)$	$f(s)$
t^m , for $m = 0, 1, 2, 3, \dots$	$m!/s^{m+1}$
c , a constant	c/s
e^{-bt}	$\frac{1}{s+b}$
$\frac{df(t)}{dt}$	$s f(s) - f(0)$
$\frac{tf(t)}{dt}$	$-\frac{df(s)}{ds}$
$\alpha_1 f_1(t) + \alpha_2 f_2(t)$	$\alpha_1 f_1(s) + \alpha_2 f_2(s)$
te^{-bt}	$1/(s+b)^2$

By substituting Eq. (2.29) into the left side of Eq. (2.28), we obtain

$$\lim_{t \rightarrow \infty} \left[\frac{c}{(c+d)} - \frac{c}{(c+d)} e^{-(c+d)t} \right] = \frac{c}{(c+d)}. \quad (2.32)$$

As the right sides of Eqs. (2.31) and (2.32) are exactly the same, it proves that the right side of Eq. (2.28) is equal to its left side.

2.5 Probability Distributions

Over the years, a large number of probability distributions have been developed to perform various types of statistical analysis [10]. This section presents some of these probability distributions considered useful to perform human reliability-related probability analysis in the area of power generation.

2.5.1 Exponential Distribution

This is probably the most widely used probability distribution to perform various types of reliability-related studies [11]. Its probability density function is defined by

$$f(t) = \theta e^{-\theta t} \quad \text{for } t \geq 0, \theta > 0 \quad (2.33)$$

where

$f(t)$ is the probability density function.

t is the time variable.

θ is the distribution parameter.

By inserting Eq. (2.33) into Eq. (2.22), we obtain the following expression for cumulative distribution function:

$$\begin{aligned} F(t) &= \int_0^t \theta e^{-\theta t} dt \\ &= 1 - e^{-\theta t}. \end{aligned} \quad (2.34)$$

By substituting Eq. (2.33) in Eq. (2.25), we get the following expression for the distribution mean value:

$$\begin{aligned} m = E(t) &= \int_0^{\infty} t \theta e^{-\theta t} dt \\ &= \frac{1}{\theta} \end{aligned} \quad (2.35)$$

where

m is the distribution mean value.

Example 2.6 Assume that in a power generating station, the human error rate is 0.08 errors per week. Calculate the probability of an error occurrence during a 30-week period with the aid of Eq. (2.34).

Thus, we have

$$\begin{aligned} t &= 30 \text{ weeks and} \\ \theta &= 0.08 \text{ errors/week.} \end{aligned}$$

By substituting the above data values in Eq. (2.34), we get

$$\begin{aligned} F(30) &= 1 - e^{-(0.08)(30)} \\ &= 0.9093. \end{aligned}$$

This means that there is 90.93 % chance for the occurrence of human error during the 30-week period.

2.5.2 Rayleigh Distribution

This distribution is named after John Rayleigh (1842–1919), its founder, and is frequently used in reliability work and in the theory of sound [1]. The probability density function of the distribution is defined by

$$f(t) = \frac{2}{\alpha^2} t e^{-\left(\frac{t}{\alpha}\right)^2} \quad \text{for } t \geq 0, \alpha > 0 \quad (2.36)$$

where

t is time.

α is the distribution parameter.

By substituting Eq. (2.36) into Eq. (2.22), we obtain the following equation for the cumulative distribution function:

$$F(t) = 1 - e^{-\left(\frac{t}{\alpha}\right)^2}. \quad (2.37)$$

With the aid of Eqs. (2.25) and (2.36), we obtain the following expression for the expected value of t :

$$E(t) = \alpha \Gamma(3/2) \quad (2.38)$$

where

$$\Gamma(y) = \int_0^{\infty} t^{y-1} e^{-t} dt, \quad \text{for } y > 0. \quad (2.39)$$

2.5.3 Weibull Distribution

This distribution can be used to represent many different physical phenomena, and it is named after Weibull [12], a Swedish mechanical engineering professor, who developed it in the early 1950s. The probability density function of the distribution is defined by

$$f(t) = \frac{\beta t^{\beta-1}}{\theta^{\beta}} e^{-\left(\frac{t}{\theta}\right)^{\beta}} \quad t \geq 0, \beta > 0, \theta > 0 \quad (2.40)$$

where

t is time.

θ and β are the scale and shape parameters, respectively.

By substituting Eq. (2.40) into Eq. (2.22), we obtain the following equation for the cumulative distribution function:

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}. \quad (2.41)$$

By inserting Eq. (2.40) into Eq. (2.25), we get the following equation for the expected value of t :

$$E(t) = \theta \Gamma\left(1 + \frac{1}{\beta}\right) \quad (2.42)$$

where

$\Gamma(\cdot)$ is the gamma function and is expressed by Eq. (2.39).

For $\beta = 2$ and 1, the Rayleigh and exponential distributions are the special cases of this distribution, respectively.

2.5.4 Bathtub Hazard Rate Curve Distribution

This probability distribution can be used to represent bathtub-shaped, decreasing and increasing, and increasing hazard/human error rates. It was developed in 1981 [13], and in the published literature, it is generally known as the Dhillon distribution/law/model [14–33].

The probability density function for the distribution is defined by [13].

$$f(t) = \beta \theta (\theta t)^{\beta-1} e^{-\{e^{(\theta t)^\beta} - (\theta t)^\beta - 1\}} \quad \text{for } \theta > 0, \beta > 0, t \geq 0 \quad (2.43)$$

where

t is time.

β and θ are the distribution shape and scale parameters, respectively.

By inserting Eq. (2.43) into Eq. (2.22), we get the following equation for the cumulative distribution function:

$$F(t) = 1 - e^{-\{e^{(\theta t)^\beta} - 1\}}. \quad (2.44)$$

At $\beta = 0.5$, this distribution gives the bathtub hazard rate curve, and at $\beta = 1$, it becomes the extreme value distribution. In other words, the extreme value distribution is the special case of this distribution at $\beta = 1$.

2.6 Solving First-Order Differential Equations with Laplace Transforms

Laplace transforms are an effective tool to find solution to linear first-order differential equations, and in human reliability-related studies, time-to-time linear first-order differential equations are solved. The application of Laplace transforms to find solution to a set of first-order differential equations describing the reliability of maintenance personnel in a power plant is demonstrated through the following example:

Example 2.7 Assume that the reliability of maintenance personnel in a power plant with respect to human error is described by the following two linear first-order differential equations:

$$\frac{dP_n(t)}{dt} + \theta_m P_n(t) = 0 \quad (2.45)$$

$$\frac{dP_e(t)}{dt} - \theta_m P_n(t) = 0 \quad (2.46)$$

where

$P_n(t)$ is the probability that the maintenance personnel are performing their tasks normally at time t .

$P_e(t)$ is the probability that the maintenance personnel have committed an error at time t .

θ_m is the constant error rate of the maintenance personnel.

At time $t = 0$, $P_n(0) = 1$, and $P_e(0) = 0$.

Find solutions to Eqs. (2.45) and (2.46) with the aid of Laplace transforms.

By taking the Laplace transforms of Eqs. (2.45) and (2.46) and then using the given initial conditions, we get

$$P_n(s) = \frac{1}{s + \theta_m} \quad (2.47)$$

$$P_e(s) = \frac{\theta_m}{s(s + \theta_m)} \quad (2.48)$$

where

s is the Laplace transform variable.

Taking the inverse Laplace transforms of Eqs. (2.47) and (2.48), we obtain

$$P_n(t) = e^{-\theta_m t} \quad (2.49)$$

$$P_e(t) = 1 - e^{-\theta_m t}. \quad (2.50)$$

Thus, Eqs. (2.49) and (2.50) are the solutions to Eqs. (2.45) and (2.46).

2.7 Problems

1. Write an essay on the history of mathematics.
2. Describe the following three laws:
 - Idempotent law.
 - Absorption law.
 - Distributive law.

3. What is the difference between mutually exclusive and independent events?
4. Discuss the basic properties of probability.
5. Define the following:
 - Cumulative distribution function.
 - Expected value.
6. Prove that the right-hand side of Eq. (2.10) is equal to its left-hand side.
7. Write down the probability density function for Rayleigh distribution.
8. What are the special case probability distributions of the Weibull distribution?
9. Using Eq. (2.27), obtain Laplace transform of the following function:

$$f(t) = 1 - e^{-\mu t} \quad (2.51)$$

where

t is a variable.

μ is a constant.

10. Prove Eq. (2.44) using Eqs. (2.22) and (2.43).

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