

Chapter 2

Haar Wavelets

2.1 Haar Wavelets and their Integrals

This section is based on paper [4]. Let us consider the interval $x \in [A, B]$, where A and B are given constants. We define the quantity $M = 2^J$, where J is the maximal level of resolution. The interval $[A, B]$ is divided into $2M$ subintervals of equal length; the length of each subinterval is $\Delta x = (B - A)/(2M)$. Next two parameters are introduced: $j = 0, 1, \dots, J$ and $k = 0, 1, \dots, m - 1$ (here the notation $m = 2^j$ is introduced). The wavelet number i is identified as $i = m + k + 1$.

The i -th Haar wavelet is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1(i), \xi_2(i)), \\ -1 & \text{for } x \in [\xi_2(i), \xi_3(i)), \\ 0 & \text{elsewhere,} \end{cases} \quad (2.1)$$

where

$$\begin{aligned} \xi_1(i) &= A + 2k\mu\Delta x, & \xi_2(i) &= A + (2k + 1)\mu\Delta x, \\ \xi_3(i) &= A + 2(k + 1)\mu\Delta x, & \mu &= M/m. \end{aligned} \quad (2.2)$$

These equations are valid if $i > 2$. The case $i = 1$ corresponds to the scaling function: $h_1(x) = 1$ for $x \in [A, B]$ and $h_1(x) = 0$ elsewhere. For $i = 2$ we have $\xi_1(2) = A$, $\xi_2(2) = 0.5(2A + B)$, $\xi_3(2) = B$. The parameters j and k have concrete meaning. The support (the width of the i -th wavelet) is

$$\xi_3(i) - \xi_1(i) = 2\mu\Delta x = (B - A)m^{-1} = (B - A)2^{-j} \quad (2.3)$$

It follows from here that if we increase j then the support decreases (the wavelet becomes more narrow). By this reason it is called the *dilatation parameter*. The other parameter k localises the position of the wavelet in the x -axis; if k changes from 0 to $m - 1$ the initial point of the i th wavelet $\xi_1(i)$ moves from $x = A$ to $x = [A + (m - 1)B]/m$. The integer k is called the *translation parameter*.

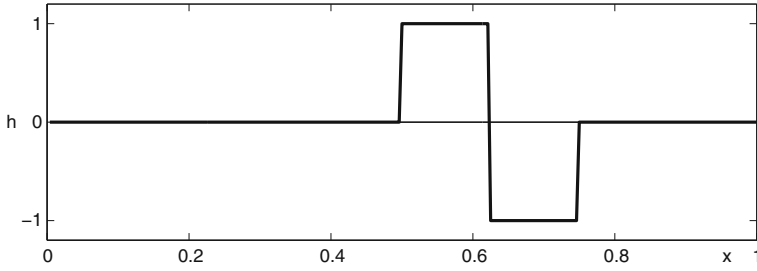


Fig. 2.1 Haar wavelet for $J = 2; i = 7$

Let us take an example. If $j = 2, J = 2, k = 2, A = 0, B = 1$ we have $m = M = 2^2 = 4, \mu = 1, \Delta x = 0.125$ and the wavelet number is $i = 7$. According to (2.2) $\xi_1(7) = 0.5, \xi_2(7) = 0.625, \xi_3(7) = 0.75$. This wavelet is plotted in Fig. 2.1.

Eight first wavelets h_1 - h_8 are shown in Fig. 2.2.

If the maximal level of resolution J is prescribed then then it follows from (2.1) that

$$\int_A^B h_i(x) h_l(x) dx = \begin{cases} (B - A)2^{-j} & \text{for } l = i, \\ 0 & \text{for } l \neq i. \end{cases} \quad (2.4)$$

So we see that the Haar wavelets are orthogonal to each other.

In the following we need the integrals of the Haar functions

$$p_{v,i}(x) = \underbrace{\int_A^x \int_A^x \dots \int_A^x}_{v\text{-times}} h_i(t) dt^v = \frac{1}{(v-1)!} \int_A^x (x-t)^{v-1} h_i(t) dt \quad (2.5)$$

$$v = 1, 2, \dots, n, \quad i = 1, 2, \dots, 2M.$$

Taking account of (2.1) these integrals can be calculated analytically; by doing it we obtain

$$p_{\alpha,i}(x) = \begin{cases} 0 & \text{for } x < \xi_1(i), \\ \frac{1}{\alpha!} [x - \xi_1(i)]^\alpha & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ \frac{1}{\alpha!} \{ [x - \xi_1(i)]^\alpha - 2[x - \xi_2(i)]^\alpha \} & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ \frac{1}{\alpha!} \{ [x - \xi_1(i)]^\alpha - 2[x - \xi_2(i)]^\alpha + [x - \xi_3(i)]^\alpha \} & \text{for } x > \xi_3(i). \end{cases} \quad (2.6)$$

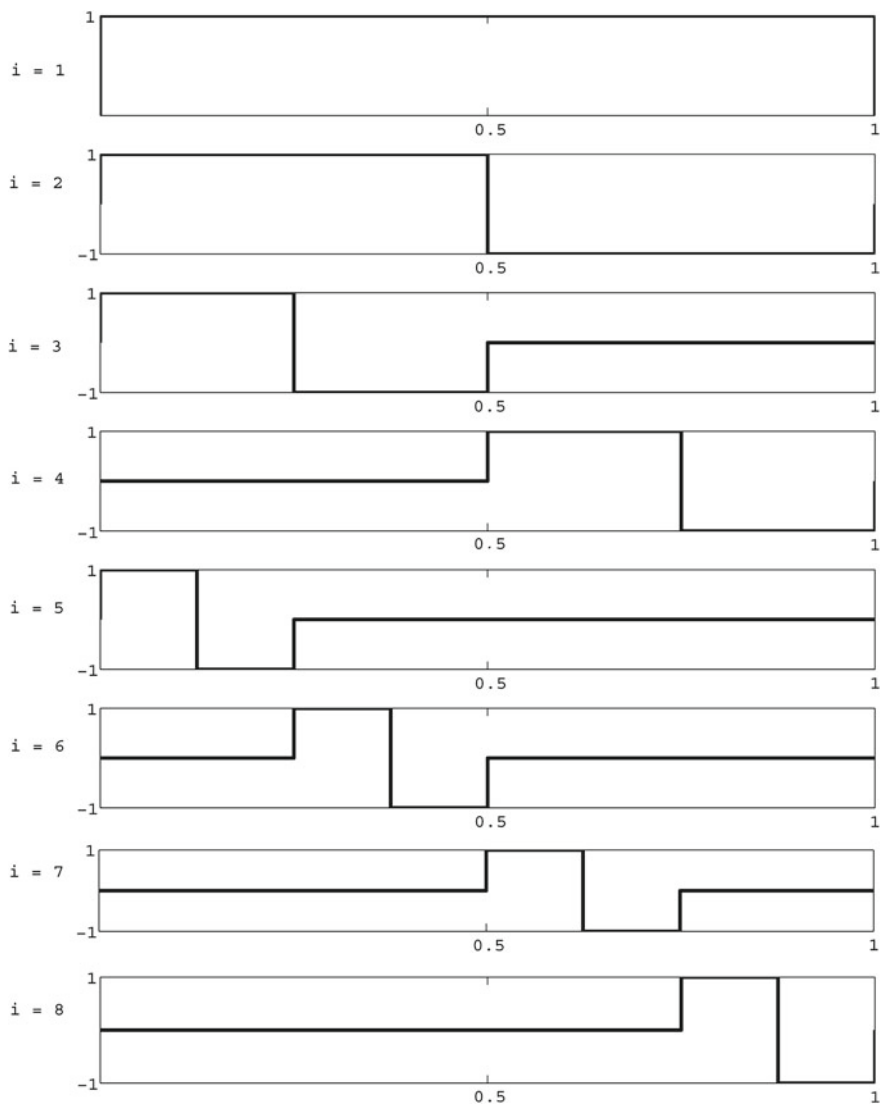


Fig. 2.2 Eight first Haar wavelets

These formulas hold for $i > 1$. In the case $i = 1$ we have $\xi_1 = A, \xi_2 = \xi_3 = B$ and

$$p_{\alpha,1}(x) = \frac{1}{\alpha!}(x - A)^\alpha. \quad (2.7)$$

2.2 Haar Matrices

If we want to use the Haar wavelets for the numerical solutions we must put them into a discrete form. There are different ways to do it; in this paper the collocation method is applied.

Let us denote the grid points by

$$\tilde{x}_l = A + l\Delta x, \quad l = 0, 1, \dots, 2M \quad (2.8)$$

For the collocation points we take

$$x_l = 0.5(\tilde{x}_{l-1} + \tilde{x}_l), \quad l = 1, \dots, 2M \quad (2.9)$$

and replace $x \rightarrow x_l$ in Eqs. (2.1), (2.6) and (2.7). It is convenient to put these results into the matrix form. For this we introduce the Haar matrices H, P_1, P_2, \dots, P_v which are $2M \times 2M$ matrices. The elements of these matrices are $H(i, l) = h_i(x_l), P_v(i, l) = p_{vi}(x_l), v = 1, 2, \dots$

For illustration consider the case $A = 0, B = 1, J = 1$. Now $2M = 4$ and the grid points are $\tilde{x}_0 = 0, \tilde{x}_1 = 0.25, \tilde{x}_2 = 0.5, \tilde{x}_3 = 0.75, \tilde{x}_4 = 1$. By calculating the coordinates of the collocation points from (2.9) we find $x_1 = 0.125, x_2 = 0.375, x_3 = 0.625, x_4 = 0.875$. The Haar matrices H, P_1, P_2 are

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad P_1 = \frac{1}{8} \begin{pmatrix} 1 & 3 & 5 & 5 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad P_2 = \frac{1}{128} \begin{pmatrix} 1 & 9 & 25 & 49 \\ 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 \\ 0 & 0 & 1 & 7 \end{pmatrix} \quad (2.10)$$

2.3 Expanding Functions into the Haar Wavelet Series

Consider a square integrable function $f = f(x)$ for $x \in [A, B]$. This function can be expanded into the Haar wavelet series

$$f(x) = \sum_{i=1}^{2M} a_i h_i(x). \quad (2.11)$$

The symbol a_i denotes the Haar wavelet coefficients. The discrete form of (2.11) is (x_l are the collocation points):

$$\hat{f}(x_l) = \sum_{i=1}^{2M} a_i h_i(x_l). \quad (2.12)$$

The matrix form of (2.12) is

$$f = aH. \quad (2.13)$$

Here H is the Haar matrix; a and f are defined as $a = (a_i)$, $f = (f_i)$; both are $2M$ dimensional row vectors. Solving the matrix equation (2.13) with regard to the coefficient vector a we find (H^{-1} denotes the inverse of H)

$$a = fH^{-1}. \quad (2.14)$$

Replacing a into (2.11) we obtain the wavelet approximation of the function $f(x)$ for the level of resolution J . The question arises as to what the degree of exactness of the approximation is (2.11). There are different possibilities to estimate the error function Δ of the wavelet approximations. Here we define the error function as

$$\Delta = \int_A^B [f(x) - \hat{f}(x)]^2 dx, \quad (2.15)$$

where $\hat{f}(x)$ denotes the approximation of $f(x)$. The discrete form of (2.15) is

$$\Delta_J = \Delta x \sum_{l=1}^{2M} [f(x_l) - \hat{f}(x_l)]^2. \quad (2.16)$$

The Haar wavelets belong to the group of piecewise constant functions. It is known that if the function is sufficiently smooth, then the convergence rate for the piecewise constant function is $O(M^{-2})$; this result can be transferred also to the Haar wavelet approach. So it could be expected that by doubling the number of collocation points the error roughly decreases four times. Consider two examples.

Example 1: Let $f(x) = \sqrt{x}$ and $x \in (0, 1)$. The Haar matrix is put together as shown in Sect. 2.2. The wavelet coefficients were calculated according to (2.14) and for $J = 3$ they are plotted in Fig. 2.3. Wavelet approximation for some values of J are presented in Fig. 2.4.

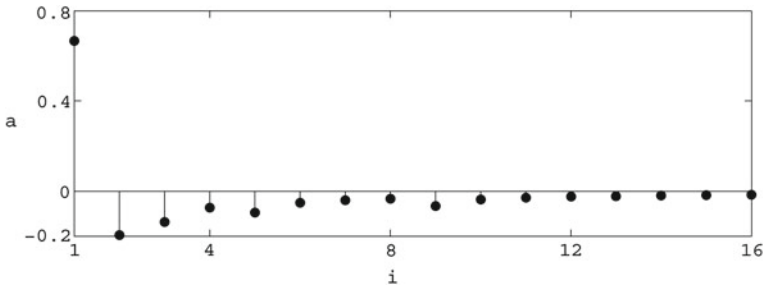


Fig. 2.3 Wavelet coefficients for the equation $f = \sqrt{x}$, $x \in (0, 1)$, $J = 3$

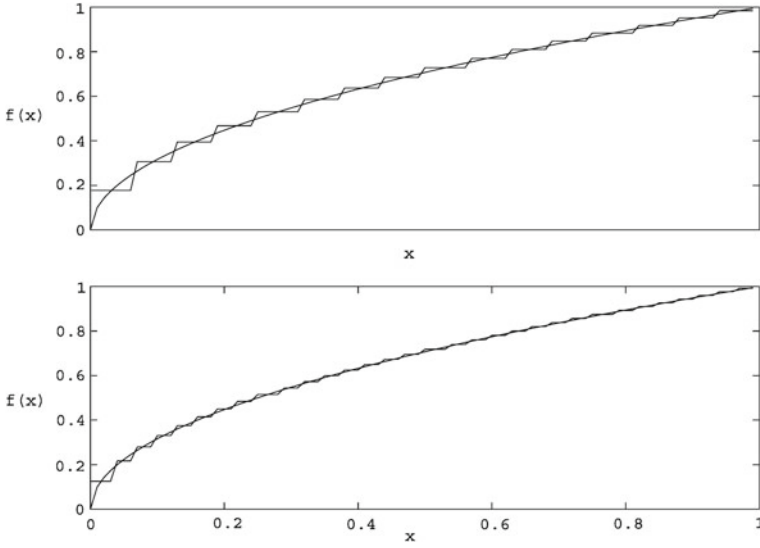


Fig. 2.4 Graph of the function $f = \sqrt{x}$ and its wavelet approximations for (a) $J = 3$, (b) $J = 4$

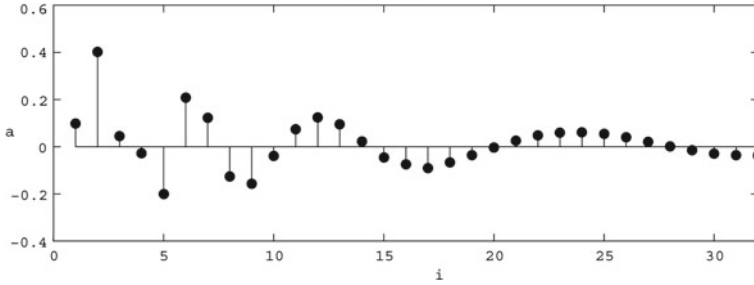


Fig. 2.5 Wavelet coefficients of the function $f = e^{-x} \sin 2\pi x$, $x \in [0, 1]$ for $J = 4$

Example 2: Similar calculations were carried out for $f(x) = e^{-x} \sin 2\pi x$, $x \in (0, 1)$. The results are plotted in Figs. 2.5 and 2.6.

Error estimates for these two problems are presented in Table 2.1.

It follows from Table 2.1 that the coefficient Δ_{J-1}/Δ_J is near to the predicted theoretical value 4.

From the analysis of these examples, a good feature of the Haar wavelets can be noticed. It follows from Figs. 2.3 and 2.5 that by increasing the wavelet number, the wavelet coefficients rapidly decrease and higher coefficients are practically zero. This obstacle enables to confine to a small number of terms in the wavelet series. The fact that the matrices H and H^{-1} contain many zeros makes the Haar wavelet transform faster when compared with other transforms.

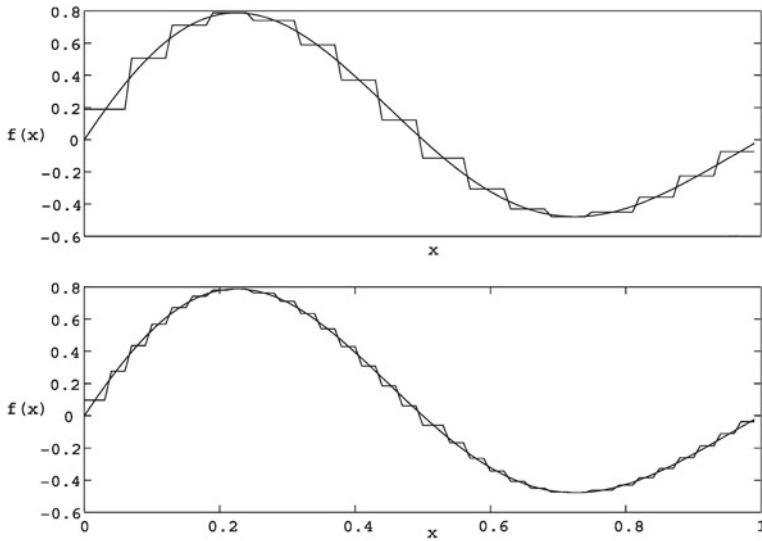


Fig. 2.6 Graph of the function $f = e^{-x} \sin 2\pi x$, $x \in [0, 1]$ and its wavelet approximations for (a) $J = 4$, (b) $J = 5$

Table 2.1 Error estimates Δ_J for the wavelet approximations

J	$2M$	$f = \sqrt{x}$		$f = \exp(-x) \sin x$	
		Δ_J	Δ_{J-1}/Δ_J	Δ_J	Δ_{J-1}/Δ_J
2	8	0.134	—	1.025	3.98
3	16	0.032	3.73	0.259	3.98
4	32	0.010	3.73	0.065	3.98
5	64	0.002	4.25	0.016	4.03

2.4 Non-uniform Haar Wavelets

The present section refers to the paper [5]. Usually it is assumed in the wavelet analysis that the interval $x \in [A, B]$ is distributed into subintervals of equal length. If the function to be expanded into the Haar wavelet series has singularities or the interval (A, B) is infinite, this approach may turn out not to fit (in the class of such problems belong e.g. vibrations under local excitation, boundary value problems, weakly singular integral equations, discontinuities and abrupt changes of the system). In these cases, it is suitable to increase the density of the collocation points in the region of rapid changes. This idea was realized by Dubeau et al. [3] who initiated the theory of non-uniform Haar wavelets. The following analysis is based on the papers of Lepik [4, 5].

We distribute the interval $x \in [A, B]$ optionally into $2M$ subintervals so that $\tilde{x}(0) = A$, $\tilde{x}(2M) = B$ and $\tilde{x}(l+1) > \tilde{x}(l)$ for $l = 0, 1, \dots, 2M-1$.

We define the i th wavelet as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ -c_i & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ 0 & \text{elsewhere.} \end{cases} \quad (2.17)$$

Here the following notations are introduced:

$$\begin{aligned} \xi_1(i) &= x(2k\mu), \quad \xi_2(i) = x[(2k+1)\mu], \\ \xi_3(i) &= x[2(k+1)\mu], \quad \mu = M/m. \end{aligned} \quad (2.18)$$

The coefficient c_i is calculated from the requirement

$$\int_A^B h_i(x) dx = 0 \quad (2.19)$$

which gives

$$c_i = \frac{\xi_2(i) - \xi_1(i)}{\xi_3(i) - \xi_2(i)}. \quad (2.20)$$

These equations hold if $i > 2$. For the cases $i = 1$ and $i = 2$ we have $\xi_1(1) = A$, $\xi_2(1) = \xi_3(1) = B$, $\xi_1(2) = A$, $\xi_2(2) = \tilde{x}(M)$, $\xi_3(2) = B$. By integrating (2.17) α times we obtain

$$p_{\alpha,i}(x) = \begin{cases} 0 & \text{for } x < \xi_1(i), \\ \frac{1}{\alpha!} [x - \xi_1(i)]^\alpha & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ \frac{1}{\alpha!} \{ [x - \xi_1(i)]^\alpha - (1 + c_i)[x - \xi_2(i)]^\alpha \} & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ \frac{1}{\alpha!} \{ [x - \xi_1(i)]^\alpha - (1 + c_i)[x - \xi_2(i)]^\alpha + c_i[x - \xi_3(i)]^\alpha \} & \text{for } x > \xi_3(i). \end{cases} \quad (2.21)$$

The collocation points are defined by (2.9). The Haar function $h(x)$ and the integrals $p_1(x), p_2(x)$ for $J = 2, A = 0, B = 1, x = [0.1 \ 0.2 \ 0.3 \ 0.4 \ 0.55 \ 0.7 \ 0.85]$ are plotted in Fig. 2.7.

2.5 Algorithms and Programs

All computations in this book were carried out with the aid of the MatLab (Matrix Laboratory) programs. The reason for this choice is the fact that many MatLab programs use the matrix representation; this essentially simplifies the programming and saves computing time. For several problems, as in the solution of linear equations, computing eigenvalues, matrix multiplication and inverse matrices etc. special MatLab programs are available. In addition, the graphics of MatLab allows us to prepare Figures of different form.

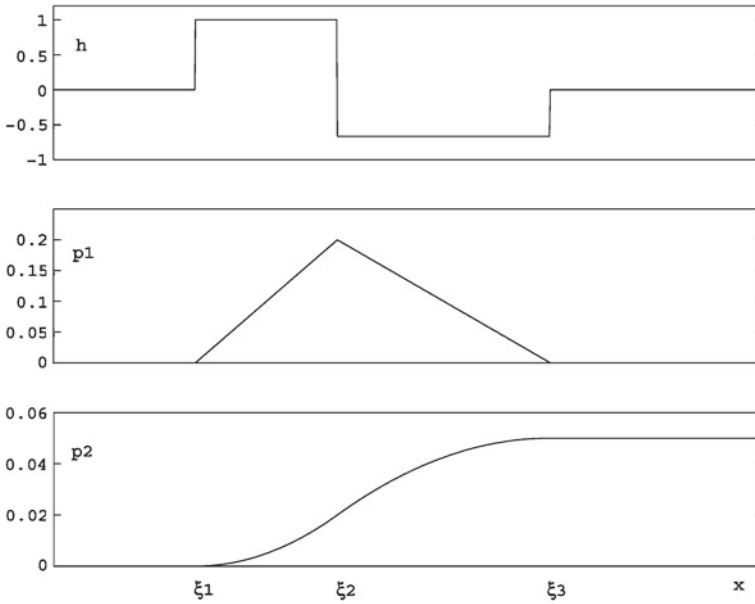


Fig. 2.7 Haar diagrams

There is a difference between the MatLab matrix symbolics and the conventional matrix representation. In MatLab, the vectors are treated as single row matrices, but conventionally as single column matrices. So the MatLab matrix equation $f = aH$, where a and f are row vectors, has in common use the form $\hat{f} = H^T \hat{a}$; here \hat{a} , \hat{f} are column vectors and H^T denotes the transverse of H .

In numerical problem solving by the Haar wavelet method, a lot of time is expended for the evaluation of the Haar matrices. Fortunately this process is the same for all problems solved in this book. Therefore it is reasonable to put together universal subprograms for it. Two of these programs are presented in the following.

Program 1: Uniform Haar method: calculation of the Haar matrices.

Code for Program 1.m

```
%Calculation of the integral matrices:uniform case
%Input:resolution level J
%Output: Haar matrices H, P1-P5
M=pow2(J);
M2=2*M;
dX=1/M2;
for l=1:M2
    X(1)=(1-0.5)*dX;
    H(1,l)=1;
    P1(1,l)=X(1);
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P2(1,1)=0.5*X(1)^2;
P3(1,1)=(1/6)*X(1)^3;
P4(1,1)=(1/24)*X(1)^4;
P5(1,1)=(1/120)*X(1)^5;
    if (X(1)<0.5)
        H(2,1)=1; P1(2,1)=X(1);
        P2(2,1)=0.5*X(1)^2;
        P3(2,1)=(1/6)*X(1)^3;
        P4(2,1)=(1/24)*X(1)^4;
        P5(2,1)=(1/120)*X(1)^5;
    elseif (X(1)>=0.5)
        H(2,1)=-1;
        P1(2,1)=1-X(1);
        P2(2,1)=0.25-0.5*(1-X(1))^2;
        P3(2,1)=0.25*(X(1)-0.5)+(1/6)*(1-X(1))^3;
        P4(2,1)=(X(1)-0.5)^2/8-(1-X(1))^4/24+1/192;
        P5(2,1)=(X(1)-0.5)^3/24+(1-X(1))^5/120+
            (X(1)-0.5)/192;
    end;end;
for j=1:J
    m=pow2(j);
    for k1=1:m
        k=k1-1;
        i=m+k1;
        ksi1=k/m;
        ksi2=(k+0.5)/m;
        ksi3=(k+1)/m;
        if ksi3==1
            h(i)=-1; else end;
        for l=1:M2
            if X(l)<ksi1
                H(i,l)=0; P1(i,l)=0; P2(i,l)=0;P3(i,l)=0;
                P4(i,l)=0; P5(i,l)=0;
            elseif X(l)<ksi2
                H(i,l)=1;
                P1(i,l)=X(l)-ksi1;
                P2(i,l)=0.5*(X(l)-ksi1)^2;
                P3(i,l)=(X(l)-ksi1)^3/6;
                P4(i,l)=(X(l)-ksi1)^4/24;
                P5(i,l)=(X(l)-ksi1)^5/120;
            elseif X(l)<ksi3
                H(i,l)=-1;
                P1(i,l)=ksi3-X(l);
                P2(i,l)=0.25/m^2-0.5*(ksi3-X(l))^2;

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P3(i,l)=(0.25/m^2)*(X(l)-ksi2)+(ksi3-X(l))^3/6;
P4(i,l)=(0.125/m^2)*(X(l)-ksi2)^2-
(ksi3-X(l))^4/24+1/(192*m^4);
P5(i,l)=(X(l)-ksi2)^3/(24*m^2)+(ksi3-X(l))^
5/120 +(X(l)-ksi2)/(192*m^4);
elseif X(l)>=ksi3
H(i,l)=0;
P1(i,l)=0;
P2(i,l)=0.5*((X(l)-ksi1)^2-2*(X(l)-ksi2)^2
+(X(l)-ksi3)^2);
P3(i,l)=(0.25/m^2)*(X(l)-ksi2);
P4(i,l)=(0.125/m^2)*(X(l)-ksi2)^2+1/(192*m^4);
P5(i,l)=(X(l)-ksi2)^3/(24*m^2)+(X(l)-ksi3)/
(192*m^4)+1/(384*m^5);
else;end;end;
end;end;

```

Program 2: Non-uniform Haar method: calculation of the Haar matrices. Code for Program 2.m

```

%Calculation of the Haar matrices:non-uniform case
%Input: J, A, B; gridpoints x(1), x(2),...,x(M2)
%Output:Haar matrix H, P1 - P4
co=1/24;
M=pow2(J);
M2=2*M;
ksi1(1)=A; ksi2(1)=B; ksi3(1)=B;
ksi1(2)=A; ksi2(2)=x(M); ksi3(2)=B;
for j=1:J
m=pow2(j);
mu=round(M/m);
ksi1(m+1)=A;
ksi2(m+1)=x(mu);
ksi3(m+1)=x(2*mu);
for k=1:m-1
i=m+k+1;
ksi1(i)=x(2*k*mu);
ksi2(i)=x((2*k+1)*mu);
ksi3(i)=x(2*(k+1)*mu);
end;end;
xc(1)=0.5*(x(1)+A);
for l=2:M2;
xc(l)=0.5*(x(l)+x(l-1));
c(l)=(ksi2(l)-ksi1(l))/(ksi3(l)-ksi2(l));
end;

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for i=1:M2
    K(i)=0.5*(ksi2(i)-ksi1(i))*(ksi3(i)-ksi1(i));
for l=1:M2;
    X=xc(l);
if X<ksi1(i);
    H(i,l)=0;
    P1(i,l)=0; P2(i,l)=0;P3(i,l)=0;P4(i,l)=0;
elseif X<ksi2(i);

    H(i,l)=1;
    P1(i,l)=X-ksi1(i);
    P2(i,l)=0.5*(X-ksi1(i))^2;
    P3(i,l)=(X-ksi1(i))^3/6;
    P4(i,l)=co*(X-ksi1(i))^4;
elseif X<ksi3(i);
    H(i,l)=-c(i);
    P1(i,l)=c(i)*(ksi3(i)-X);
    P2(i,l)=K(i)-0.5*c(i)*(ksi3(i)-X)^2;
    P3(i,l)=K(i)*(X-ksi2(i))+(ksi3(i)-X)^3/6;
    P4(i,l)=co*((X-ksi1(i))^4-2*(X-ksi2(i))^4);
elseif X>=ksi3(i);
    H(i,l)=0;
    P1(i,l)=0; P2(i,l)=K(i);
    P3(i,l)=K(i)*(X-ksi2(i));
    P4(i,l)=co*((X-ksi1(i))^4-2*(X-ksi2(i))^4
    +(X-ksi3(i))^4);
else end;
end;
end;

```

2.6 Related Papers

Haar functions belong to a group of rectangular waves with magnitude ± 1 in some intervals and zeros elsewhere. From this group, besides Haar wavelets, the most known are the *block-pulse functions* and *Walsh functions*.

Let us divide the interval $x \in [0, L]$ into N parts of equal length and denote $\Delta x = L/N$. The block-pulse functions (BPF) are defined in the interval $x \in [0, L]$ by

$$b_i(x) = \begin{cases} 1 & \text{for } (i-1)\Delta x \leq x < i\Delta x, \\ 0 & \text{elsewhere, } i = 1, 2, \dots, N. \end{cases} \quad (2.22)$$

Since the block functions form a complete set of orthogonal functions then an arbitrary function $f(x)$ can be expanded by

$$f(x) = CB(x). \quad (2.23)$$

Here C denotes the coefficients vector $C = [c_1, c_2, \dots, c_N]^T$ and $B(x) = [b_1(x), b_2(x), \dots, b_N(x)]^T$.

Taking into account the orthogonality of the BPF we find

$$c_i = \frac{1}{L} \int_0^L f(x) b_i(x) dx. \quad (2.24)$$

The system of Walsh functions $w_n(x)$ is defined on $[0, 1]$ by

$$w_n(x) = (-1)^S, \quad (2.25)$$

where

$$S = \sum_{i=0}^k \sigma_{i+1} n_{-i}, \quad (2.26)$$

Here $\sigma_1, \sigma_2, \dots$ is the dyadic expansion of x with infinitely many zeros and

$$n = n_0 + 2n_{-1} + \dots + 2^k n_{-k} \quad (2.27)$$

is the dyadic expansion of a positive integer n .

Several authors have made use of the block-pulse and Walsh functions for solving differential and integral equations.

Xiang et al. [9] proposed a Haar-type orthogonal matrix (HTOM), which contains Haar, Walsh and other orthogonal matrices intervenient of the former two. The authors assert that HTOM is very suitable for engineering applications.

Stankovic and Falkowski [8] proposed different generalizations of the Haar transform as sign Haar transform, Haar functions, p -adic groups, Zhang-Moraga functions.

Next the epoch-making paper by Chen and Hsiao [2] is referenced (note that here the symbols may have different meaning as in the present book).

Let us denote the Haar functions by $h_i(t)$, $i = 0, 1, \dots, m-1$ and define the m -dimensional column vector

$$H_m(t) = [h_1(t), h_2(t), \dots, h_{m-1}(t)]^T. \quad (2.28)$$

The integral of this vector will be marked as

$$P_m H_m = \int_0^1 H_m(t) dt. \quad (2.29)$$

Here P_m is a $m \times m$ matrix, which is called the *operational matrix of integration*, it can be calculated as

$$P_m = \left[\int_0^1 H_m(t) dt \right] H_m^{-1}. \quad (2.30)$$

If $m = 1$ or $m = 2$ we find $P_1 = [0.5]$, $P_2 = 0.25[2 \ -1; \ 1 \ 0]$. Chen and Hsiao proposed a formula which allows successively evaluate higher order matrices. For this purpose the m -square matrix P_m is divided into four submatrices

$$P_m = \begin{pmatrix} P_{am} & P_{bm} \\ P_{cm} & P_{dm} \end{pmatrix}, \quad (2.31)$$

where

$$\begin{aligned} P_{am} &= P_{m/2}, & P_{bm} &= -\frac{1}{2m}H_{m/2}, \\ P_{cm} &= \frac{1}{2m}H_{m/2}^{-1}, & P_{dm} &= 0. \end{aligned} \quad (2.32)$$

If the matrices $P_{m/2}$, $H_{m/2}$ are known we can easily put together the higher order matrix P_m . This approach has been applied in several papers.

Maleknejad and Mirzaee [6] used the term *rationalized Haar wavelets* which should accent the fact that to be compared with the earlier conception about Haar wavelets, the rational numbers were deleted and the integral powers of two were introduced. The difference between the ordinary and rationalized Haar wavelet is mainly symbolic—the conventional notation for Haar wavelets H is replaced by RH .

Aziz et al. [1] and Siraj-ul-Islam et al. [7] investigated integrals of the type

$$\int_A^B f(x)h_i(x)dx \quad (2.33)$$

They worked out a procedure for which numerical integration of such integrals is avoided. This approach is valuable for more complicated functions $f(x)$.

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