

Chapter 2

The Euler Equation

Abstract This chapter presents Euler equation without and with the presence of a magnetic field. The equation is written in both spherical and cylindrical coordinates, and applied to a stellar envelope. Some examples involving the solar wind are given, and a discussion is presented on the escape velocity from stars. The chapter ends with some further examples applied to hot and cool stars.

2.1 The Euler Equation

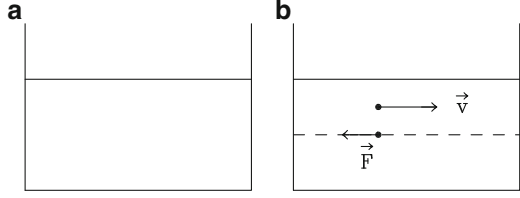
A fluid in equilibrium is not affected by any shearing stress, that is, there are no forces acting in the horizontal direction (Fig. 2.1a). If the fluid is in motion, these forces will be important, as a result of the *viscosity* of the fluid. Let us consider initially an *ideal fluid*, for which the viscosity is negligible. Therefore, the horizontal stress forces that arise between the moving fluid layers are negligible (Fig. 2.1b). In this case, apart from external forces, the only stress that has to be considered in the fluid is the *normal* stress, or pressure P . Pressure is a scalar quantity, and a function of the space coordinates and time, $P(x, y, z, t)$.

In the simplest cases, the fluid particles move in layers, or sheets, which constitute a *laminar* flow. If the particle trajectories are irregular, the flow is *turbulent*. In this book we will consider essentially turbulent free flows. In the general case, the displacements of a volume element in the fluid include its *translation*, *rotation*, and *deformation*. In this book, we will be interested principally in the translation motion, that is, we will consider non rotating fluids, in which the angular velocity of a fluid volume element is $\omega = 0$, and non-viscous fluids, in which the deformation (contraction or stretching) is negligible.

Let us consider again a volume V in the fluid. The total force acting on this volume due to the interactions with the remaining fluid particles is

$$-\oint P \mathbf{n} dS$$

Fig. 2.1 (a) Fluid in equilibrium. (b) Fluid moving under the action of viscous forces



(see Fig. 1.1), where the negative sign ($-$) is due to the fact that the force acts *on* the considered element. Applying the “gradient theorem” to a scalar quantity A , we have

$$\oint A \mathbf{n} dS = \int_V \nabla A dV, \quad (2.1)$$

so that

$$-\oint P \mathbf{n} dS = -\int_V \nabla P dV. \quad (2.2)$$

Therefore, the force by the remaining parts of the fluid on the volume element dV is $-\nabla P dV$, and the force per unit volume is simply $-\nabla P$. We can then express the momentum conservation by

$$\rho \times \text{acceleration} = -\nabla P \quad (2.3)$$

where ρ is again the fluid density. In the case of a particle of mass m under the action of a force $\mathbf{F}$ and moving with velocity $\mathbf{v}$, the acceleration is simply $\mathbf{a} = \mathbf{F}/m = d\mathbf{v}/dt$. For a “fluid particle”, the velocity variation has two components:

$$d\mathbf{v}_1 = \frac{\partial \mathbf{v}}{\partial t} dt, \quad (2.4)$$

which refers to the velocity variation at a *fixed point in space*, that is, having constant coordinates x, y, z in a time interval dt , and

$$d\mathbf{v}_2 = dx \frac{\partial \mathbf{v}}{\partial x} + dy \frac{\partial \mathbf{v}}{\partial y} + dz \frac{\partial \mathbf{v}}{\partial z}, \quad (2.5)$$

which refers to the velocity variation *at a given time* between two different points in space separated by

$$d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}, \quad (2.6)$$

which is the distance covered by the fluid particle in time dt . ($\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors of the directions x, y, z). Recalling that

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}, \quad (2.7)$$

we have

$$d\mathbf{r} \cdot \nabla = dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z}, \quad (2.8)$$

so that

$$d\mathbf{v}_2 = (d\mathbf{r} \cdot \nabla) \mathbf{v}. \quad (2.9)$$

The total variation $d\mathbf{v}$ is

$$d\mathbf{v} = d\mathbf{v}_1 + d\mathbf{v}_2 = \frac{\partial \mathbf{v}}{\partial t} dt + (d\mathbf{r} \cdot \nabla) \mathbf{v}. \quad (2.10)$$

The acceleration is then

$$\text{acceleration} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}. \quad (2.11)$$

The derivative in (2.11) is the *total derivative*, which is usually represented as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla). \quad (2.12)$$

The description of the fluid motion considering separately two terms, that is, $\partial/\partial t$ in a fixed position and $\mathbf{v} \cdot \nabla$ in a given time, corresponds to the *Eulerian description*. On the other hand, the description of the fluid motion in terms of the total derivative D/Dt corresponds to the *Lagrangian description*. In this case, we can imagine that we are following the motion of a fluid element. Naturally, the Lagrangian derivative D/Dt is related to the Eulerian derivatives by Eq. (2.12). Considering (2.11) and (2.3), the equation of motion is

$$\begin{cases} \rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla P \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P \\ \frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla P. \end{cases} \quad (2.13)$$

Equations (2.13) are different forms of the *Euler equation*, which was originally derived in 1755.

An important application of the equation of motion occurs when the fluid is in a gravitational field characterized by the acceleration (force per unit mass) $\mathbf{g}$. In this

case, each volume unit is under the action of the force $\rho \mathbf{g}$, which is the gravitational force acting per unit volume, so that the equation of motion (2.13) is written as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \mathbf{g} . \quad (2.14)$$

More generally, if the fluid is under the action of an external force field $\mathbf{F}$ (dyn/cm³), that is, $\mathbf{F}$ is the force acting on a unit volume, we have

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \frac{1}{\rho} \mathbf{F} . \quad (2.15)$$

2.1.1 Example: Static Star

Let us analyze a simple application of Eq. (2.14), considering a static star, that is, taking $\mathbf{v} = 0$. In this case, the pressure forces exactly balance the gravitational force in each fluid element constituting the star. We can then assume that the gravitational force can be derived from a gravitational potential such that $\mathbf{g} = -\nabla \phi$. The potential ϕ can be related to the gas density in the star by Poisson equation,

$$\nabla^2 \phi = 4 \pi G \rho . \quad (2.16)$$

From (2.14) with $\mathbf{v} = 0$, we have

$$\frac{1}{\rho} \nabla P = \mathbf{g} = -\nabla \phi .$$

Taking the divergence of this equation,

$$\nabla \cdot \left(\frac{1}{\rho} \nabla P \right) = -\nabla \cdot (\nabla \phi) = -\nabla^2 \phi ,$$

that is,

$$\nabla \cdot \left(\frac{1}{\rho} \nabla P \right) = -4 \pi G \rho . \quad (2.17)$$

In order to solve (2.17) we need an independent relation involving P and ρ , as for example the equation of state. In this book, however, we will be especially interested in expanding stellar envelopes, so that the more appropriate form of Euler equation is (2.14) or (2.15).

2.2 Euler Equation in the Presence of a Magnetic Field

We saw in Chap. 1 that the continuum hypothesis can be applied to several astrophysical situations, particularly if the particle mean free path, which is essentially established by collisions, is small relative to some typical dimension of the system. We can then say that collisions favour the behaviour of a system as a continuous fluid. Magnetic fields in a plasma are also able to keep charged particles locally confined for a sufficiently long time so that the system behaves as a fluid, even if the collisional processes are not efficient.

A detailed treatment of the astrophysical applications of magnetohydrodynamics (MHD) is beyond the limits of this book, and the interested reader may check some of the recent texts listed at the end of Chap. 1, such as [Choudhuri \(1998\)](#). In this section, we will limit ourselves to mentioning the main changes in Eq. (2.15) due to presence of a magnetic field. In this case, apart from the two thermodynamic quantities (usually the pressure P and density ρ) and the three velocity components, it is necessary to specify the vector magnetic field $\mathbf{B}$ so that the plasma can be described in a magnetohydrodynamic model. This type of model is frequently useful in the astrophysical applications, in particular in non-relativistic processes in which the plasma motions are essentially constant, or vary slowly with time, that is, when the time scale of the physical processes τ is much larger than the inverse of the plasma frequency, ω_p , or $\tau \gg 1/\omega_p$. These conditions are generally applied to the stellar winds, in the cases where the plasma motions are due to the action of mechanic and magnetic forces.

The continuity equation (1.6) remains the same, since the hypothesis of no sources or sinks is still valid. For the Euler equation (2.15) we may write

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \frac{1}{\rho} \mathbf{F} + \frac{1}{\rho c} \mathbf{J} \times \mathbf{B}, \quad (2.18)$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{J}$ is electric current density. As in the remaining applications in this book, we use the cgs system of units, in which the electric quantities are in e.s.u. and the magnetic quantities in e.m.u.

A detailed derivation of Eq. (2.18) from basic microscopic principles can be found in [Choudhuri \(1998, Chap. 13\)](#). In the MHD model we can write Maxwell equation without the displacement current term as

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (2.19)$$

that is, the current density $\mathbf{J}$ can be obtained from the magnetic field $\mathbf{B}$ and vice-versa. Substituting (2.19) in (2.18) we have

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \frac{1}{\rho} \mathbf{F} + \frac{1}{4\pi\rho} (\nabla \times \mathbf{B}) \times \mathbf{B} \quad (2.20)$$

Using the vector identity

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

and taking $\mathbf{A} = \mathbf{B}$, it is easy to show that

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla\left(\frac{B^2}{2}\right).$$

Substituting this equation in (2.20) we obtain the alternative form of Euler equation,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{\rho} \mathbf{F} - \frac{1}{\rho} \nabla \left(P + \frac{B^2}{8\pi} \right) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{4\pi\rho}. \quad (2.21)$$

From Eq. (2.21) we see that the term $B^2/8\pi$ corresponds to the magnetic pressure, introduced by the presence of the magnetic field. The last term in (2.21) is related to the tension along the magnetic field lines, that is, the magnetic field is associated to an isotropic pressure and also to a tension along the field lines.

2.2.1 Example: Solar Atmosphere

As an example, we may compare the magnetic pressure $B^2/8\pi$ with the gas pressure in a few points in the solar atmosphere. In the photosphere, the pressure varies by several orders of magnitude, but we can consider an average point where the total particle density is of the order of 10^{17} cm^{-3} and the temperature is about 6,000 K, so that the total pressure is 10^5 dyn/cm^2 . Similar pressures are obtained by a magnetic field of the order of 1,500 Gauss. These values are observed in regions of the solar atmosphere such the plages and solar faculae, and still higher values, $B \sim 3,000$ Gauss, are associated to the umbral regions of the sunspots. Here the temperature tends to be somewhat lower than in the photosphere, but the higher densities lead to pressures of the order of 10^5 dyn/cm^2 or higher. In the outer regions, which are associated to the solar wind, the field is probably lower by a few orders of magnitude, reaching values near the earth magnetic field, $B \sim 1$ Gauss. In this case, the magnetic pressure is of the order of 10^{-2} dyn/cm^2 , which can be compared with the gas pressure in a coronal region about 2 solar radii from the solar surface, where the density $n \sim 10^6 \text{ cm}^{-3}$, $T \sim 10^6 \text{ K}$ and the pressure is $P \sim 10^{-3} \text{ dyn/cm}^2$ (see Example 2.5.2).

Magnetic fields are also associated to the atmospheres and envelopes of red giant stars, although their magnitudes are uncertain. For example, in the envelope of a red giant with $T \sim 10^3 \text{ K}$ and $n \sim 10^8 \text{ cm}^{-3}$, the gas pressure is of the order of 10^{-5} dyn/cm^2 , which is equivalent to the magnetic pressure if the field is just $B \sim 10^{-2}$ Gauss, suggesting that magnetic fields effectively have some influence on the dynamics of the circumstellar envelopes.

2.3 Spherical Coordinates

Let us derive the equation of motion in the form (2.15) in a system of spherical coordinates (r, θ, ϕ) , defined in Fig. 2.2. The unit vectors $(\mathbf{n}, \boldsymbol{\ell}, \mathbf{m})$ are related to the unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ corresponding to the directions x, y, z by

$$\begin{cases} \mathbf{n} = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k} \\ \boldsymbol{\ell} = \cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} - \sin \theta \mathbf{k} \\ \mathbf{m} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j} . \end{cases} \quad (2.22)$$

The position vector $\mathbf{r}$ and velocity $\mathbf{v}$ are given by

$$\mathbf{r} = r \mathbf{n}(\theta, \phi) \quad (2.23)$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r} \mathbf{n} + r \dot{\mathbf{n}} . \quad (2.24)$$

On the other hand, from (2.22) we have the relations

$$\begin{cases} \frac{\partial \mathbf{n}}{\partial \theta} = \boldsymbol{\ell} \\ \frac{\partial \mathbf{n}}{\partial \phi} = \sin \theta \mathbf{m} , \end{cases} \quad (2.25)$$

so that

$$\dot{\mathbf{n}} = \frac{\partial \mathbf{n}}{\partial \theta} \dot{\theta} + \frac{\partial \mathbf{n}}{\partial \phi} \dot{\phi} = \dot{\theta} \boldsymbol{\ell} + \dot{\phi} \sin \theta \mathbf{m} , \quad (2.26)$$

and therefore

$$\mathbf{v} = \dot{r} \mathbf{n} + r(\dot{\theta} \boldsymbol{\ell} + \dot{\phi} \sin \theta \mathbf{m}) . \quad (2.27)$$

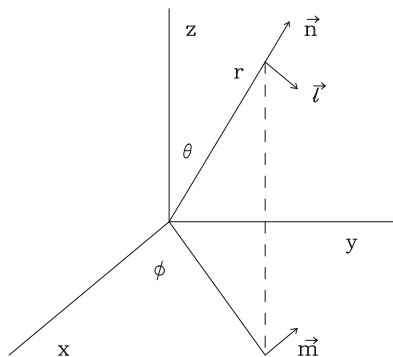


Fig. 2.2 Spherical coordinates and unit vectors in a cartesian axis system

We may write

$$\mathbf{v} = v_r \mathbf{n} + v_\theta \boldsymbol{\ell} + v_\phi \mathbf{m}, \quad (2.28)$$

where

$$\begin{cases} v_r = \dot{r} \\ v_\theta = r \dot{\theta} \\ v_\phi = r \dot{\phi} \sin \theta. \end{cases} \quad (2.29)$$

We have also

$$\begin{cases} \dot{v}_r = \ddot{r} \\ \dot{v}_\theta = \dot{r} \dot{\theta} + r \ddot{\theta} \\ \dot{v}_\phi = \dot{r} \dot{\phi} \sin \theta + r \ddot{\phi} \sin \theta + r \dot{\phi} \cos \theta \dot{\theta}. \end{cases} \quad (2.30)$$

Still from (2.22), we have

$$\begin{cases} \frac{\partial \boldsymbol{\ell}}{\partial \theta} = -\mathbf{n} \\ \frac{\partial \boldsymbol{\ell}}{\partial \phi} = \cos \theta \mathbf{m}, \end{cases} \quad (2.31)$$

so that

$$\dot{\boldsymbol{\ell}} = \frac{\partial \boldsymbol{\ell}}{\partial \theta} \dot{\theta} + \frac{\partial \boldsymbol{\ell}}{\partial \phi} \dot{\phi} = -\dot{\theta} \mathbf{n} + \dot{\phi} \cos \theta \mathbf{m}. \quad (2.32)$$

From (2.22), we may also write

$$\begin{cases} \frac{\partial \mathbf{m}}{\partial \theta} = 0 \\ \frac{\partial \mathbf{m}}{\partial \phi} = -\sin \theta \mathbf{n} - \cos \theta \boldsymbol{\ell}, \end{cases} \quad (2.33)$$

so that

$$\dot{\mathbf{m}} = \frac{\partial \mathbf{m}}{\partial \theta} \dot{\theta} + \frac{\partial \mathbf{m}}{\partial \phi} \dot{\phi} = -\dot{\phi} \sin \theta \mathbf{n} - \dot{\phi} \cos \theta \boldsymbol{\ell}. \quad (2.34)$$

Using (2.26), (2.28), (2.32) and (2.34), we have

$$\begin{aligned} \dot{\mathbf{v}} &= \ddot{\mathbf{r}} = \dot{v}_r \mathbf{n} + v_r \dot{\mathbf{n}} + \dot{v}_\theta \boldsymbol{\ell} + v_\theta \dot{\boldsymbol{\ell}} + \dot{v}_\phi \mathbf{m} + v_\phi \dot{\mathbf{m}} \\ &= \dot{v}_r \mathbf{n} + v_r (\dot{\theta} \boldsymbol{\ell} + \dot{\phi} \sin \theta \mathbf{m}) + \dot{v}_\theta \boldsymbol{\ell} + v_\theta (-\dot{\theta} \mathbf{n} + \dot{\phi} \cos \theta \mathbf{m}) \\ &\quad + \dot{v}_\phi \mathbf{m} + v_\phi (-\dot{\phi} \sin \theta \mathbf{n} - \dot{\phi} \cos \theta \boldsymbol{\ell}), \end{aligned}$$

or

$$\begin{aligned}\dot{\mathbf{v}} = & (\dot{v}_r - v_\theta \dot{\theta} - v_\phi \dot{\phi} \sin \theta) \mathbf{n} + (v_r \dot{\theta} + \dot{v}_\theta - v_\phi \dot{\phi} \cos \theta) \boldsymbol{\ell} \\ & + (v_r \dot{\phi} \sin \theta + v_\theta \dot{\phi} \cos \theta + \dot{v}_\phi) \mathbf{m} .\end{aligned}$$

Using Eq. (2.29)

$$\dot{\mathbf{v}} = \left(\dot{v}_r - \frac{v_\theta^2}{r} - \frac{v_\phi^2}{r} \right) \mathbf{n} + \left(\frac{v_r v_\theta}{r} + \dot{v}_\theta - \frac{v_\phi^2 \cos \theta}{r \sin \theta} \right) \boldsymbol{\ell} + \left(\frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cos \theta}{r \sin \theta} + \dot{v}_\phi \right) \mathbf{m} , \quad (2.35)$$

or still, using (2.29) and (2.30),

$$\begin{aligned}\dot{\mathbf{v}} = & (\ddot{r} - r \dot{\theta}^2 - r \dot{\phi}^2 \sin^2 \theta) \mathbf{n} + (r \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \dot{\phi}^2 \sin \theta \cos \theta) \boldsymbol{\ell} \\ & + (r \ddot{\phi} \sin \theta + 2 \dot{r} \dot{\phi} \sin \theta + 2 r \dot{\theta} \dot{\phi} \cos \theta) \mathbf{m} .\end{aligned} \quad (2.36)$$

Using (2.28) and the operator ∇ in spherical coordinates:

$$\nabla = \frac{\partial}{\partial r} \mathbf{n} + \frac{1}{r} \frac{\partial}{\partial \theta} \boldsymbol{\ell} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \mathbf{m} , \quad (2.37)$$

$$\mathbf{v} \cdot \nabla = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial}{\partial \phi} . \quad (2.38)$$

Applying this operator to vector $\mathbf{v}$,

$$\begin{aligned}(\mathbf{v} \cdot \nabla) \mathbf{v} = & \left(v_r \frac{\partial v_r}{\partial r} \mathbf{n} + v_r \frac{\partial v_\theta}{\partial r} \boldsymbol{\ell} + v_r \frac{\partial v_\phi}{\partial r} \mathbf{m} \right) + \left(\frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} \mathbf{n} + \right. \\ & \left. \frac{v_\theta}{r} \frac{\partial \mathbf{n}}{\partial \theta} v_r + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} \boldsymbol{\ell} + \frac{v_\theta^2}{r} \frac{\partial \boldsymbol{\ell}}{\partial \theta} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} \mathbf{m} + \frac{v_\theta v_\phi}{r} \frac{\partial \mathbf{m}}{\partial \theta} \right) + \\ & \left(\frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} \mathbf{n} + \frac{v_\phi}{r \sin \theta} \frac{\partial \mathbf{n}}{\partial \phi} v_r + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \boldsymbol{\ell} + \frac{v_\phi}{r \sin \theta} \frac{\partial \boldsymbol{\ell}}{\partial \phi} v_\theta + \right. \\ & \left. \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \mathbf{m} + \frac{v_\phi^2}{r \sin \theta} \frac{\partial \mathbf{m}}{\partial \phi} \right)\end{aligned}$$

Using (2.25), (2.31) and (2.33), we have

$$\begin{aligned}(\mathbf{v} \cdot \nabla) \mathbf{v} = & \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{r} \right) \mathbf{n} + \\ & \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi^2 \cos \theta}{r \sin \theta} \right) \boldsymbol{\ell} +\end{aligned}$$

$$\left(v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cos \theta}{r \sin \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) \mathbf{m} \quad (2.39)$$

or

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = \left(\mathbf{v} \cdot \nabla v_r - \frac{v_\theta^2}{r} - \frac{v_\phi^2}{r} \right) \mathbf{n} + \left(\mathbf{v} \cdot \nabla v_\theta + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) \boldsymbol{\ell} + \left(\mathbf{v} \cdot \nabla v_\phi + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot \theta}{r} \right) \mathbf{m} . \quad (2.40)$$

Considering (2.28), the first term of the first member of Eq. (2.15) can be written as

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial v_r}{\partial t} \mathbf{n} + \frac{\partial v_\theta}{\partial t} \boldsymbol{\ell} + \frac{\partial v_\phi}{\partial t} \mathbf{m} . \quad (2.41)$$

From (2.37), the pressure term in (2.15) can be written as

$$\nabla P = \frac{\partial P}{\partial r} \mathbf{n} + \frac{1}{r} \frac{\partial P}{\partial \theta} \boldsymbol{\ell} + \frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} \mathbf{m} . \quad (2.42)$$

The force per unit volume can be written as

$$\mathbf{F} = F_r \mathbf{n} + F_\theta \boldsymbol{\ell} + F_\phi \mathbf{m} . \quad (2.43)$$

Rearranging expressions (2.39)–(2.43), we can write the equations separately, according to components $\mathbf{n}$, $\boldsymbol{\ell}$, $\mathbf{m}$:

Radial component, unit vector $\mathbf{n}$:

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{1}{\rho} F_r \quad (2.44)$$

$$\frac{\partial v_r}{\partial t} + \mathbf{v} \cdot \nabla v_r - \frac{v_\theta^2 + v_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{1}{\rho} F_r . \quad (2.45)$$

Polar component, unit vector $\boldsymbol{\ell}$:

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} = -\frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \frac{1}{\rho} F_\theta \quad (2.46)$$

$$\frac{\partial v_\theta}{\partial t} + \mathbf{v} \cdot \nabla v_\theta + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} = -\frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \frac{1}{\rho} F_\theta . \quad (2.47)$$

Azimuthal component, unit vector $\mathbf{m}$:

$$\begin{aligned} \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot \theta}{r} \\ = -\frac{1}{\rho r \sin \theta} \frac{\partial P}{\partial \phi} + \frac{1}{\rho} F_\phi \end{aligned} \quad (2.48)$$

$$\frac{\partial v_\phi}{\partial t} + \mathbf{v} \cdot \nabla v_\phi + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot \theta}{r} = -\frac{1}{\rho r \sin \theta} \frac{\partial P}{\partial \phi} + \frac{1}{\rho} F_\phi . \quad (2.49)$$

2.3.1 Example: Azimuthal Symmetry

Assuming that v_ϕ and its derivatives are zero, there are only two equations:

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{1}{\rho} F_r \quad (2.50)$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \frac{1}{\rho} F_\theta . \quad (2.51)$$

2.3.2 Example: Spherical Symmetry

Assuming that v_θ and its derivatives are also zero, Eq. (2.50) can be written as

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{1}{\rho} F \quad (2.52)$$

where we dropped subscript r .

2.3.3 Example: Spherical Symmetry and Steady State

In this case, $\partial v / \partial t = 0$ in (2.52), so that

$$v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} + \frac{1}{\rho} F . \quad (2.53)$$

This is frequently the form of the Euler equation that is used in the study of stellar winds.

2.3.4 Example: Hydrostatic Equilibrium

A particular case of (2.53) is that of *hydrostatic equilibrium*, where $v = 0$ in all points of the fluid. In this case,

$$\frac{dP}{dr} = F . \quad (2.54)$$

We will consider this situation again in Sect. 2.5.

2.4 Cylindrical Coordinates

Let us concisely repeat the procedure of Sect. 2.3 in the case of a system of cylindrical coordinates (R, ϕ, z) , with unit vectors $(\mathbf{h}, \mathbf{m}, \mathbf{k})$ as shown in Fig. 2.3. The relations between the cartesian coordinates (x, y, z) and (R, ϕ, z) are given by Eq. (1.22). The relations between the unit vectors $(\mathbf{h}, \mathbf{m}, \mathbf{k})$ and the corresponding vectors of the cartesian system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are:

$$\begin{cases} \mathbf{h} = \cos \phi \mathbf{i} - \sin \phi \mathbf{j} \\ \mathbf{m} = \sin \phi \mathbf{i} + \cos \phi \mathbf{j} \\ \mathbf{k} = \mathbf{k} \end{cases} \quad (2.55)$$

from which we obtain the derivatives

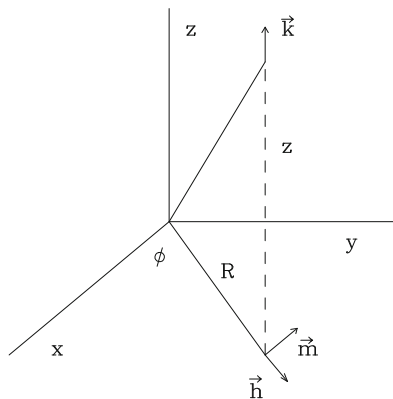


Fig. 2.3 Cylindrical coordinates and unit vectors in a cartesian axis system

$$\begin{cases} \frac{d\mathbf{h}}{d\phi} = \mathbf{m} \\ \frac{d\mathbf{m}}{d\phi} = -\mathbf{h} . \end{cases} \quad (2.56)$$

The position vector $\mathbf{r}$ and the velocity vector $\mathbf{v}$ are given by

$$\mathbf{r} = R \mathbf{h} + z \mathbf{k} \quad (2.57)$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{R} \mathbf{h} + R \dot{\phi} \mathbf{m} + \dot{z} \mathbf{k} . \quad (2.58)$$

Analogously to Eq. (2.28), the velocity can be written as

$$\mathbf{v} = v_R \mathbf{h} + v_\phi \mathbf{m} + v_z \mathbf{k} , \quad (2.59)$$

from which we see that $v_R = \dot{R}$, $v_\phi = R \dot{\phi}$ and $v_z = \dot{z}$.

The gradient in cylindrical coordinates is:

$$\nabla = \frac{\partial}{\partial R} \mathbf{h} + \frac{1}{R} \frac{\partial}{\partial \phi} \mathbf{m} + \frac{\partial}{\partial z} \mathbf{k} , \quad (2.60)$$

so that the operator $\mathbf{v} \cdot \nabla$ appearing in (2.15) is written as

$$\mathbf{v} \cdot \nabla = v_R \frac{\partial}{\partial R} + \frac{v_\phi}{R} \frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z} . \quad (2.61)$$

Applying this operator to vector $\mathbf{v}$, we obtain the equivalent expression to (2.39),

$$\begin{aligned} (\mathbf{v} \cdot \nabla) \mathbf{v} = & \left(v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} + v_z \frac{\partial v_R}{\partial z} \right) \mathbf{h} + \\ & \left(v_R \frac{\partial v_\phi}{\partial R} + \frac{v_R v_\phi}{R} + \frac{v_\phi}{R} \frac{\partial v_\phi}{\partial \phi} + v_z \frac{\partial v_\phi}{\partial z} \right) \mathbf{m} + \left(v_R \frac{\partial v_z}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_z}{\partial \phi} + v_z \frac{\partial v_z}{\partial z} \right) \mathbf{k} . \end{aligned} \quad (2.62)$$

The term $\partial \mathbf{v} / \partial t$ in (2.15) can be written in terms of the cylindrical coordinates, analogously to Eq. (2.41), and the same procedure can be applied to an external force $\mathbf{F}$, with components F_R, F_ϕ, F_z . Applying again the gradient operator (2.60) to the pressure term P , and using (2.62), we can write Euler equation for the three components (R, ϕ, z) as:

Radial component, unit vector $\mathbf{h}$:

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} + v_z \frac{\partial v_R}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial R} + \frac{1}{\rho} F_R \quad (2.63)$$

Azymuthal component, unit vector $\mathbf{m}$:

$$\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_R v_\phi}{R} + \frac{v_\phi}{R} \frac{\partial v_\phi}{\partial \phi} + v_z \frac{\partial v_\phi}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial \phi} + \frac{1}{\rho} F_\phi \quad (2.64)$$

Vertical component, unit vector $\mathbf{k}$:

$$\frac{\partial v_z}{\partial t} + v_R \frac{\partial v_z}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_z}{\partial \phi} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{1}{\rho} F_z. \quad (2.65)$$

2.4.1 Example: Azymuthal Symmetry

Assuming that v_ϕ and its derivatives are zero, there are only two equations:

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + v_z \frac{\partial v_R}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial R} + \frac{1}{\rho} F_R \quad (2.66)$$

$$\frac{\partial v_z}{\partial t} + v_R \frac{\partial v_z}{\partial R} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{1}{\rho} F_z. \quad (2.67)$$

2.4.2 Example: The Unidimensional Case

We can imagine that the galactic disk is an infinitely large cylinder, so that its physical properties vary only with the height relative to the reference plane, and there is only one equation that can be written as

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{1}{\rho} F, \quad (2.68)$$

where we dropped the subscript z .

2.5 Euler Equation in a Stellar Envelope

Let us consider a stellar envelope with spherical symmetry and in steady state, taking into account the stellar gravitational force. From (2.43), we can write for the gravitational force per unit volume

$$\begin{cases} \mathbf{F} = F_r \mathbf{n} \\ F_r = F = -\frac{G M_* \rho}{r^2}, \end{cases} \quad (2.69)$$

where M_* is the mass of the central object. Substituting in (2.53):

$$\begin{cases} v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{G M_*}{r^2} \\ \rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{G M_* \rho}{r^2} . \end{cases} \quad (2.70)$$

Calling g_* the stellar gravitational acceleration,

$$g_* = \frac{G M_*}{r^2} , \quad (2.71)$$

we obtain

$$\begin{cases} v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - g_* \\ \rho v \frac{dv}{dr} = -\frac{dP}{dr} - \rho g_* . \end{cases} \quad (2.72)$$

We will be interested in problems where, apart from the gravitational attraction g_* , there is another external force per unit mass in the same direction but in opposition to the gravitational force. Such force may be, for instance, due to momentum transfer from the stellar radiation field to the gas in an expanding envelope. Calling g_r the acceleration (force per unit mass) due to this process, we have in the spherically symmetric and stationary case:

$$\begin{aligned} v \frac{dv}{dr} &= -\frac{1}{\rho} \frac{dP}{dr} - g_* + g_r \\ &= -\frac{1}{\rho} \frac{dP}{dr} - (g_* - g_r) \\ &= -\frac{1}{\rho} \frac{dP}{dr} - g_{ef} , \end{aligned} \quad (2.73)$$

where we have introduced the effective gravity

$$g_{ef} = g_* - g_r . \quad (2.74)$$

Defining the Γ parameter by the ratio

$$\Gamma_r = \frac{g_r}{g_*} , \quad (2.75)$$

we may write

$$g_{ef} = g_* \left(1 - \frac{g_r}{g_*} \right) = g_* (1 - \Gamma_r) \quad (2.76)$$

and the equation of motion (2.73) is written as

$$v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - g_*(1 - \Gamma_r) . \quad (2.77)$$

2.5.1 Example: Envelope in Hydrostatic Equilibrium

The simplest case of Eq. (2.77) is the hydrostatic equilibrium under the action of gravity. In this case, $\Gamma_r = 0$ e $v = 0$ in (2.77), so that

$$\frac{1}{\rho} \frac{dP}{dr} = -g_* = -\frac{G M_*}{r^2}$$

or

$$\frac{dP}{dr} = -\rho g_* = -\frac{G M_* \rho}{r^2} , \quad (2.78)$$

which is the well-known hydrostatic equilibrium equation generally used in stellar structure theory. According to (2.78), the stellar fluid is in equilibrium if the pressure forces, characterized by the term dP/dr , exactly match the gravitational force. Since the product $\rho g_* > 0$, we must have $dP/dr < 0$, that is, in order to attain equilibrium, the pressure must decrease away from the center of the star.

2.5.2 Example: Hydrostatic Solar Corona

The solar atmospheric layers include a photosphere, a transition region and a hot corona, where the temperature may reach values $T \simeq 1.5 \times 10^6$ K with a density of protons and electrons $n \simeq 4 \times 10^8 \text{ cm}^{-3}$. This hot gas expands through the interplanetary space, which essentially constitutes the *solar wind*, and the temperature decreases at large distances from the Sun. We can assume that the expansion is initiated at a reference distance given by $r_0 \simeq R_\odot = 6.96 \times 10^{10}$ cm. Let us make the simplifying assumption that the corona is in hydrostatic equilibrium and that the plasma is isothermal, so that the Earth, located at $1 \text{ AU} = 1.5 \times 10^{13}$ cm from the Sun, is inside a high temperature coronal material. We can use the hydrostatic equilibrium hypothesis in order to estimate the coronal gas density at the Earth's position. From (2.78), the hydrostatic equilibrium equation can be written

$$\frac{dP}{dr} = -\frac{G M_\odot n m_H}{r^2} . \quad (2.79)$$

Assuming that the plasma contains the same number of protons and electrons, we can write the equation of state in the form

$$P = 2 n k T , \quad (2.80)$$

where $k = 1.38 \times 10^{-16}$ erg/K is Boltzmann constant (see Chap. 3). From (2.79) and (2.80), we have

$$\frac{dP}{dr} = 2 k T \frac{dn}{dr} = - \frac{G M_{\odot} n m_H}{r^2} , \quad (2.81)$$

so that

$$\frac{dn}{n} = - \frac{G M_{\odot} m_H}{2 k T} r^{-2} dr = - \frac{1}{h} \left(\frac{r_0}{r} \right)^2 dr , \quad (2.82)$$

where we have introduced the *scale height* h given by

$$\frac{1}{h} = \frac{G M_{\odot} m_H}{2 k T r_0^2} . \quad (2.83)$$

Integrating (2.82) between r_0 and r , where the particle density has values n_0 and n , respectively, we obtain

$$n(r) = n_0 \exp \left[- \frac{r_0^2}{h} \left(\frac{1}{r_0} - \frac{1}{r} \right) \right] . \quad (2.84)$$

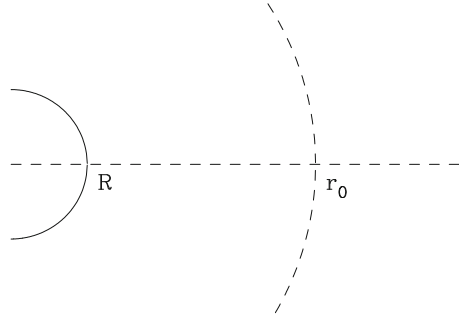
Taking $n_0 \simeq 4 \times 10^8 \text{ cm}^{-3}$ and $r_0 \simeq 6.96 \times 10^{10} \text{ cm}$, we get $h \simeq 9.0 \times 10^9 \text{ cm} = 9.0 \times 10^4 \text{ km}$. Therefore, at $r = 1 \text{ AU}$ the coronal gas density is $n(1 \text{ AU}) \simeq 2 \times 10^5 \text{ cm}^{-3}$. In fact, the constant temperature hypothesis in the whole corona is unrealistic, and a better approximation gives lower values for the density at $r = 1 \text{ AU}$.

2.5.3 Example: Free-Falling Stellar Envelope

In Example 2.5.1, we have seen that a negative pressure gradient is necessary ($dP/dr < 0$) in order to attain hydrostatic equilibrium. Let us consider the case where the gradient goes abruptly to zero, leaving the gas in the envelope under the action of the gravity alone. From (2.77) we have

$$v \frac{dv}{dr} \simeq -g_* , \quad (2.85)$$

Fig. 2.4 Stellar envelope in free fall from an initial position to the stellar radius R



so that the gas in the envelope falls onto the star in free fall. Considering an initial point at distance r_0 from the star, where $v_0 = 0$, let us obtain the velocity of the fluid element as it approaches the stellar surface, where $r = R$ (Fig. 2.4).

Integrating (2.85) and assuming $g_* \simeq G M_*/R^2 \simeq \text{constant}$, where M_* is the stellar mass, we find

$$\int_0^{v(R)} v \, dv = - \int_{r_0}^R g_* \, dr \simeq g_*(r_0 - R)$$

(see Exercise 2.1). The result is well-known,

$$v(R)^2 \simeq 2 g_*(r_0 - R) , \quad (2.86)$$

and the velocity is directed towards the stellar center. The time needed for the gas initially at r_0 to fall on the stellar surface can be obtained from Eq. (2.77), recalling that, in this case, $v \, dv/dr = v(dv/dt)/(dr/dt) = dv/dt$, and

$$\frac{dv}{dt} \simeq -g_* , \quad (2.87)$$

from which we have

$$t \simeq \frac{v(R)}{g_*} . \quad (2.88)$$

Considering the case of a red giant star with $M = 1 M_\odot$, $R = 100 R_\odot$ and $r_0 \simeq 2 R$, we have $v(R) \simeq 62 \text{ km/s}$ and $t \simeq 26 \text{ days}$, that is, the collapse of the envelope in these conditions would be relatively fast.

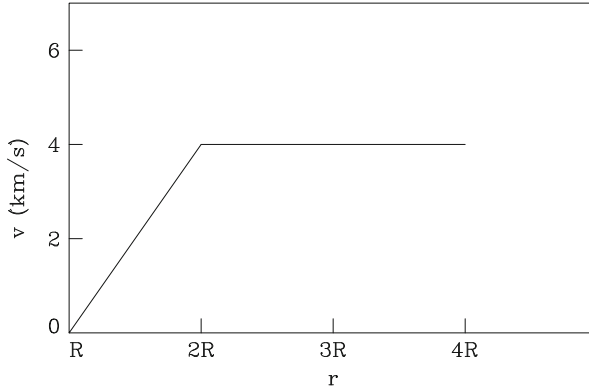


Fig. 2.5 Schematic velocity profile for an accelerated envelope

2.5.4 Example: Accelerated Envelope

Let us now consider the opposite situation of the previous example, in which the gravitational term is negligible compared to the pressure forces in a region of the envelope characterized by the width Δr . In this case, from (2.77),

$$v \frac{dv}{dr} \simeq -\frac{1}{\rho} \frac{dP}{dr} . \quad (2.89)$$

Considering that the average acceleration is simply given by $a \simeq |dP/dr|/\rho$, the gas is accelerated until it reaches the final velocity

$$v_f^2 \simeq 2 \left| \frac{dP}{dr} \right| \frac{\Delta r}{\rho} . \quad (2.90)$$

We can roughly estimate the final velocity in the circumstellar envelope of a red giant star using the equation of state (see Eq. 2.80), so that

$$v_f^2 \simeq 2 \frac{P}{\rho} \simeq 2 \frac{k T}{m_H} , \quad (2.91)$$

where $T \simeq 10^3$ K is the average temperature in the envelope, and $v_f \simeq 4$ km/s. This relation is simply a rough estimate, useful in order to obtain orders of magnitude. In contrast with Example 2.5.2, the gas is neutral, and the protons and electrons are locked in the hydrogen atoms or even in molecules. The behaviour of $v(r)$ is shown schematically by Fig. 2.5 for $\Delta r \simeq R$.

The observed winds in cool stars have profiles qualitatively similar to the figure. A variation of the velocity $v(r)$ as shown in Fig. 2.5 is known as a *velocity law*. The study of stellar winds frequently leads to relations of the type

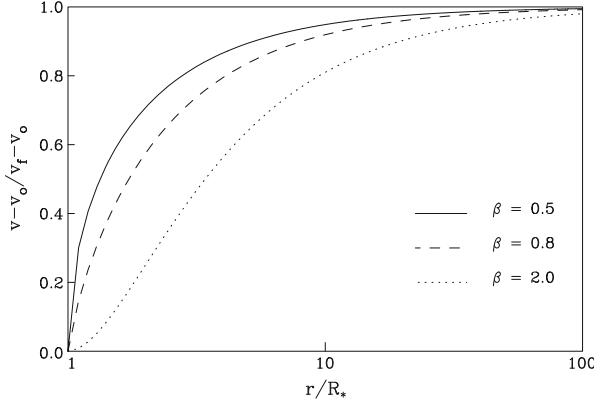


Fig. 2.6 Velocity profile for a beta law

$$v(r) = v_0 + (v_f - v_0) \left(1 - \frac{R}{r}\right)^\beta, \quad (2.92)$$

which is sometimes called a *β law*, where v_0 is the initial velocity of the wind, still in the photosphere, v_f is the final, or terminal velocity ($v_f \gg v_0$), R is a reference level, usually of the order of the stellar radius R_* , and β is a parameter that characterizes the acceleration of the envelope. We see that $v \rightarrow v_0$ for $r \rightarrow R$ and that $v \rightarrow v_f$ for $r \rightarrow \infty$. In the case of hot stars, $\beta \simeq 0.8$, while for cool stars we have typically $\beta \simeq 2$. Some examples of velocity laws for different values of the β parameter are shown in Fig. 2.6, which is considerably more realistic than the schematic variation shown in Fig. 2.5.

2.5.5 Example: The Escape Velocity

We have seen in Examples 1.7.2 and 1.7.3 that the winds in hot stars have terminal velocities of the order of $v_f \sim 2,000$ km/s, while the winds observed in red giant stars are considerably slower, with $v_f \sim 10$ km/s. We can compare these values with the *escape velocity* v_e from the atmospheres of these stars. Considering a particle of mass m and velocity v at a distance r from the center of a star with mass M_* and radius R_* , the escape velocity is defined by the condition $E_c = |E_p|$, where $E_c = (1/2)mv^2$ is the kinetic energy of the particle and $|E_p| = GM_*m/r$ is its potential energy at r , where we have $r \geq R_*$. The escape velocity at this position is then

$$v_e \simeq \sqrt{\frac{2GM_*}{r}}. \quad (2.93)$$

For a B-type supergiant, adopting $M_* \simeq 25 M_\odot \simeq 5 \times 10^{34}$ g and $r \simeq R_* \simeq 30 R_\odot \simeq 2 \times 10^{12}$ cm, we have $v_e \simeq 600$ km/s, that is, the terminal velocities obtained from the observations are higher than the escape velocity, $v_f \gg v_e$.

In the case of the red giants, the situation is somewhat more complex. Taking typical values for these stars, $M_* \simeq 1 M_\odot \simeq 2 \times 10^{33}$ g and $R_* \simeq 100 R_\odot \simeq 7 \times 10^{12}$ cm, we get $v_e \simeq 60$ km/s with $r \simeq R_*$, that is, in this case $v_f < v_e$. This question was solved from the observations of the α Her system by Armin Deutsch. In this system, a red giant star and a dwarf star rotate around their center of mass. The circumstellar spectral lines that characterize the red giant outer atmosphere were also observed in the companion star, that is, both stars were inside a giant envelope, originated by the giant star. In this way, the position r where we should apply the condition $E_c = |E_p|$ occurs at a much larger distance than the stellar radius, $r \gg R_*$, so that the escape velocity from the envelope is in fact lower than the value given above. For example, including in (2.93) the masses of both stars, assumed identical, the escape velocity $v_e = v_f = 10$ km/s is obtained at position $r \simeq 2GM_*/v_e^2 \simeq 5 \times 10^{14}$ cm, or about 70 stellar radii. The observed dimensions of the circumstellar envelopes are effectively of this order of magnitude, or even larger.

2.5.6 Example: The Γ Parameter in Hot Stars

We can obtain an estimate of the Γ parameter defined by (2.75) in the envelopes of hot stars, where the large electron density makes electron scattering an important source of opacity. We know that the cross section for electron scattering in the non relativistic case is simply the Thomson cross section, $\sigma_T = 6.65 \times 10^{-25}$ cm². In a layer of density ρ and electron density n_e , the electron scattering coefficient per mass is

$$\kappa_e \simeq \frac{n_e \sigma_T}{\rho}, \quad (2.94)$$

typically of the order of $\kappa_e \simeq 0.30$ cm²/g for a gas with normal chemical composition. Each absorbed (or scattered) photon with energy $h\nu$ has momentum $h\nu/c$, so that the ratio between the acceleration due to the scattering process and gravity is simply

$$\Gamma_e = \frac{g_e}{g_*} = \frac{L_* \kappa_e}{4\pi r^2 c} \frac{r^2}{G M_*} = \frac{L_* \kappa_e}{4\pi c G M_*}, \quad (2.95)$$

where L_* is the stellar luminosity (erg/s), $L_*/4\pi r^2$ is the stellar flux at distance r (erg cm⁻² s⁻¹), so that the term $L_* \kappa_e / 4\pi r^2 c$ is essentially the acceleration (force/mass) due to the electron scattering process. For solar-type stars, Γ_e is very small, $\Gamma_e \simeq 2 \times 10^{-5}$, but using appropriate values for hot stars of O, B spectral types, $L_* \sim 10^5$ to $10^6 L_\odot$ and $M_* \simeq 10$ to $50 M_\odot$, respectively, we obtain

$0.1 \leq \Gamma_e \leq 1$, that is, in these stars the acceleration due to electron scattering may be similar to the gravitational acceleration. Parameter Γ_e is called the Eddington parameter, and $\Gamma_e = 1$ defines the Eddington limit, in which case the star is gravitationally unbound. From (2.95) we have at the Eddington limit

$$\frac{L_* \kappa_e}{4 \pi c G M_*} = 1. \quad (2.96)$$

Recalling that for a spherical star $L_* = 4 \pi R_*^2 \sigma T_{eff}^4$ (see Eq. 1.31) and $g(R_*) = G M_*/R_*^2$, we have

$$g(R_*) = \frac{\sigma \kappa_e}{c} T_{eff}^4 \quad (2.97)$$

that is, plotting $\log g$ against $\log T_{eff}$ we get a straight line above which the star is gravitationally unbound,

$$\log g \simeq -15.25 + 4 \log T_{eff}. \quad (2.98)$$

For example, for $T_{eff} = 30,000$ K we have $\log g = 2.66$, or $g \simeq 460$ g/cm².

In terms of the Γ_e parameter, we can define an effective escape velocity given by

$$v_e = \left[\frac{2(1 - \Gamma_e) G M_*}{r} \right]^{1/2}. \quad (2.99)$$

Using data from Example 2.5.5, $M_* \simeq 25 M_\odot$ and $r \simeq 30 R_\odot$, taking $L_* \simeq 4 \times 10^5 L_\odot$ and $\kappa_e \simeq 0.30$ cm²/g, we get $\Gamma_e \simeq 0.37$; from (2.99), we find an effective escape velocity corresponding to 80 % of the value given by (2.93).

2.5.7 Example: The Γ Parameter in Cool Stars

We can use the same procedure of the previous example to estimate the Γ parameter in the case of cool giant stars. Considering that the contribution g_r to the effective gravity is due to the action of the stellar radiation on the dust grains embedded in the envelope (see Eqs. 2.74 and 2.75), Eq. (2.95) is still valid, replacing the scattering coefficient κ_e by the corresponding quantity for solid grains, κ_d , given by

$$\kappa_d \simeq \frac{\pi a^2 Q n_d}{\rho}, \quad (2.100)$$

where n_d the grain number density (cm⁻³), assuming spherical grains with radius a , and Q is an efficient factor for radiation pressure. Since $n_d = \rho_d / m_d$, where ρ_d is the grain density (g/cm³) and $m_d \simeq (4/3)\pi a^3 s_d$ is the mass of a grain with internal

density s_d , we can use average values adequate to silicate grains and estimate κ_d or the corresponding Γ parameter by

$$\Gamma_d = \frac{L_* \kappa_d}{4 \pi c G M_*} . \quad (2.101)$$

These values are: $a \simeq 1,000 \text{ \AA} = 10^{-5} \text{ cm}$, $s_d \simeq 3 \text{ g/cm}^3$ and $\rho/\rho_d \simeq 200$ (gas-grain ratio). We get $m_d \simeq 1.3 \times 10^{-14} \text{ g}$, $n_d/\rho \simeq 1/200 m_d \simeq 3.9 \times 10^{11} \text{ g}^{-1}$ and $\kappa_d \simeq 120 Q \text{ cm}^2/\text{g}$. Coefficient κ_d depends critically on the photon frequency, reflecting the dependency of the efficiency factor Q , which can reach values from about 0.001 up to unity, approximately. Taking values in the range $0.01 < Q < 1$ and considering a star with $L_* \simeq 10^3 L_\odot$ and $M_* \simeq 1 M_\odot$, from (2.101) we get $0.1 < \Gamma_d < 9$, that is, even in this case the added gravity term can be of the same order or higher than the stellar gravity.

2.5.8 Example: Expanding Planetary Nebula

An application of Example 2.5.4, where the gravitational attraction is not important, occurs in planetary nebulae, in which the envelope expands at an approximately constant velocity in the regions far away from the central star. For instance, a proton in the ionized region is typically located at a distance $r \simeq 0.2 \text{ pc}$ from the star. Considering an average mass $M_* \simeq 1 M_\odot$, the proton potential energy at this position is $|E_p| \simeq G M_* m_p / r \simeq 3.6 \times 10^{-16} \text{ erg}$. Taking typical expansion velocities for planetary nebulae, $v \simeq 20 \text{ km/s}$, the proton kinetic energy is $E_c \simeq (1/2) m_p v^2 \simeq 3.3 \times 10^{-12} \text{ erg}$, so that $E_c \gg |E_p|$, and the envelope escapes from the star. In other words, the expansion velocity is higher than the gas escape velocity, which is given by (2.93). In this case, $v_e \simeq 0.2 \text{ km/s}$, or $v \gg v_e$.

Exercises

2.1. Solve the case of Example 2.5.3 releasing the assumption that g_* is a constant. Show that in this case $v(R) \simeq 62/\sqrt{2} \simeq 44 \text{ km/s}$. What is the time needed for collapse?

2.2. Considering that the Sun is a spherical star with mass $M_\odot = 1.99 \times 10^{33} \text{ g}$ and radius $R_\odot = 6.96 \times 10^{10} \text{ cm}$, estimate the value of the pressure gradient in the Sun. Compare your result with the value obtained by the so-called “solar standard model” for the region in the solar interior where $r = R_\odot/2$, which is $dP/dr \simeq -1.3 \times 10^5 \text{ dyn/cm}^3$.

2.3. In a more realistic model for the solar corona, the temperature variation is given by $T(r) = T_0 (r_0/r)^{2/7}$, where $r_0 \simeq R_\odot = 6.96 \times 10^{10} \text{ cm}$ and $T_0 \simeq 1.5 \times 10^6 \text{ K}$ are the values of the position r and temperature T at a certain reference level near

the surface of the Sun. (a) What is the temperature of the coronal gas at the Earth's orbit? (b) Assume that the solar corona is in hydrostatic equilibrium. What is the variation of the coronal gas density with distance? What is the value of the density at $r = 1$ AU?

2.4. Consider a β velocity law for a stellar wind in which the reference level R is related to the stellar radius R_* by

$$R = R_* \left[1 - \left(\frac{v_0}{v_f} \right)^{\frac{1}{\beta}} \right]$$

(a) What is the value of R , in terms of the stellar radius, for a hot star of O spectral type, with effective temperature $T_{eff} = 40,000$ K, and a wind with terminal velocity 2,500 km/s and $\beta = 0.8$? Assume that the initial wind velocity is essentially given by the sound speed at the base of the circumstellar envelope, that is, $v_0^2 \simeq c_s^2 \simeq k T_{eff} / \mu m_H$, where $\mu \simeq 0.6$ is the mean molecular weight of the gas particles. (b) Plot the velocity $v(r)$ as a function of r/R_* . At what distance from the star, in terms of R_* , the wind reaches 60 % of the terminal velocity?

2.5. Hot stars with O, B spectral types and effective temperatures higher than 21,000 K present a ratio between the wind terminal velocity v_f and the effective escape velocity v_e given by $v_f/v_e \simeq 2.6$. The star ζ Pup has O4f spectral type, effective temperature $T_{eff} \simeq 42,000$ K, mass $M_* \simeq 60 M_\odot$, radius $R_* \simeq 20 R_\odot$ and a wind with terminal velocity $v_f \simeq 2,200$ km/s. (a) What is the average value of the Γ parameter for this star, defined as the ratio between the acceleration due to electron scattering and the stellar gravitational acceleration? (b) Considering that the total stellar luminosity is $L_* \simeq 8 \times 10^5 L_\odot$, what is the electron scattering coefficient (cm²/g) in the stellar envelope?

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Maciel, W.J.

2014, XVII, 178 p. 30 illus., 2 illus. in color., Softcover

ISBN: 978-3-319-04327-2