Chapter 2
Linear Optimization

Abstract The purpose of this chapter is to cover the basic concept of linear optimization analysis and its applications in water resources engineering. The graphical and simplex solution methods for solving linear optimization problems are illustrated step by step. In addition, the applications of simplex method in solving water distribution network and one and two dimensional confined aquifer optimization problems using the Solver tool in Excel are presented.

2.1 Linear Programming

A linear optimization problem can be defined as solving an optimization problem in which the objective function and all associated constraints are linear. The linear optimization is also known as linear programming (LP) and it can be defined as the process of minimizing or maximizing a linear function to find the optimum value (minimum or maximum) for linear and non-negative constraints. The term *programming* here implies the way of planning and organizing (formulation) to find the optimal solutions, and it is different from its meaning in coding and computer programming. In general, this method is a relatively simple technique to find realistic solutions for a wide range of optimization problems and includes three essential elements listed below:

1. Identify decision variables: decision variables are the unknown variables of the problem statement that need to be determined to solve the problem. Defining decision variables precisely is a fundamental step in formulating a linear optimization model.
2. Obtain the objective function: in this step we need to define the objective of desired problem statement which shows the main goal of the decision-maker. Afterward, the relations between decision variables and the objective should be accurately determined. It should be noted that the function cannot include any nonlinear component such as exponential, products, or division of variables,

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and variables under a root sign. All variables only must be added or subtracted in a linear fashion.

3. Determine the constraints: constraints explain the requirements that desired problem shall meet, and it can be in the forms of either equalities (=) or inequalities ($\leq$, $\geq$).

The general form of linear programming model can be written as:

$$\min \mathrm{f}(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{2.1}$$

Subject to the following constraints:

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k \\ x_i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, n \end{gathered} \tag{2.2}$$

where, x_i are decision variables and a, b, c are known constants. The above equations also can be presented in the matrix form as:

$$\min \mathrm{f}(X) = c^T X \tag{2.3}$$

in which,

$$c = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$$

Subject to the following constraints:

$$aX = b \tag{2.4}$$

$$X \geq 0 \tag{2.5}$$

where,

$$a = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & & & . \\ . & . & & & . \\ a_{k1} & a_{k2} & . & . & a_{kn} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$$

It is important to note that maximization of any objective function is equivalent to negative minimization of that function. Hence, a maximization problem can be simply converted to a minimization problem in any linear programming.

$$\max \mathrm{f}(\mathrm{X}) = \min\ -\mathrm{f}(\mathrm{X}) \tag{2.6}$$

In addition, the constraints sometimes are presented in the form of inequalities, while, they can be simply presented in the form of equalities by adding or subtracting slack variables (s) as:

$$\begin{array}{ll} \text{inequality constraint} & a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \le b_k \\ \text{equality constraint} & a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n + s = b_k \end{array} \tag{2.7}$$

or

$$\begin{array}{ll} \text{inequality constraint} & a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \ge b_k \\ \text{equality constraint} & a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n - s = b_k \end{array} \tag{2.8}$$

The solution to an optimization problem can be categorized into four types as: (1) a unique optimal solution, (2) an infeasible solution, (3) an unbounded solution, and (4) multiple solutions. The unique optimal solution is obtained when all of the constraints are satisfied and the minimum or maximum values of the objective function are precisely determined. In this case, the set of all feasible solutions is called the feasible region. On the other hand if we cannot find any solution that satisfies the desired constraints, the problem is called infeasible. Therefore, the feasible region is empty and there is no optimal solution since there is no solution in this condition. The unbounded solution in LP problems happens when the objective value is feasible while its value increases or decreases indefinitely and approaches to negative or positive infinity. And finally, there are multiple solutions, if more than one solution can be found for the desired optimization problem. In this case, all values of objective function are equaled and can be considered as optimum value.

In the following sections, the graphical and simplex methods for linear programming are presented with a few examples for each technique.

2.2 Graphical Method

Graphical methods can be applied to solve linear optimization problems involving two decision variables and a limited numbers of constraints. As the graphical methods are visual approach, they can increase our understanding from the basics of linear programming and the steps to find the optimal value in an optimization problem. To be more familiar with the concept of these methods, a simple example is presented in the following section.

Example 2.1 Maximize the function f(x)

$$\mathrm{f(x)} = 15x_1 + 18x_2$$

Subject to the following constraints:

$$\begin{aligned} 2x_1 + 3x_2 &\leq 35 \\ 4x_1 + 2x_2 &\leq 50 \\ x_1 \geq 0 \quad \text{and} \quad x_2 &\geq 0 \end{aligned}$$

Solution: The first step is drawing the constraints to find the feasible region. In this case, we need to replace the inequality sign of each constrain with the equality sign as follow:

$$\begin{aligned} 2x_1 + 3x_2 &= 35 \\ 4x_1 + 2x_2 &= 50 \end{aligned}$$

Now, assume $x_1 = 0$ and solve for x_2 from first constraint equation and repeat the process again by assuming $x_2 = 0$ and solving for x_1. For the first constraint, we have:

$$\begin{aligned} &\text{If } x_1 = 0 \rightarrow x_2 = 11.66 \\ &\text{If } x_2 = 0 \rightarrow x_1 = 17.5 \end{aligned}$$

And, for the second constraint equation:

$$\begin{aligned} &\text{If } x_1 = 0 \rightarrow x_2 = 25 \\ &\text{If } x_2 = 0 \rightarrow x_1 = 12.5 \end{aligned}$$

Now, draw a line to connect the points (0, 11.66) to (17.5, 0) for the first constraint and (0, 25) to (12.5, 0) for the second constraint, as shown in the Fig. 2.1.

The shaded area in Fig. 2.1 shows the feasible region and all points that are in this domain satisfy both constraints of the model. The best points in the feasible region that maximize the function f(x) would be the optimal solution. For the sake of analysis, the feasible solution is redrawn as shown in Fig. 2.2. The intersection point between the two constraints can be calculated by solving the following two equations with two unknowns simultaneously as:

$$\begin{cases} 2x_1 + 3x_2 = 35 \\ \\ 4x_1 + 2x_2 = 50 \end{cases} \Rightarrow (x_1, x_2) = (10, 5)$$

Therefore, we need to draw a straight line from points (0, 11.66) to (10, 5), and (10, 5) to (12.5, 0).

To find the optimal solution in the feasible area, we assume a value for the objective function, for example f(x) = 270, and draw a line to see it is inside the

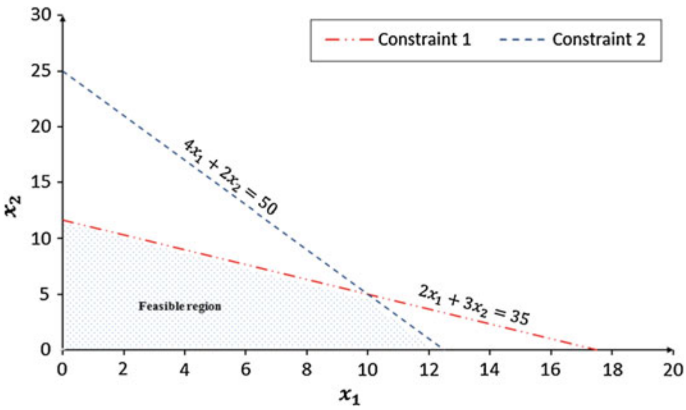


Fig. 2.1 The feasible solution domain for Example 2.1

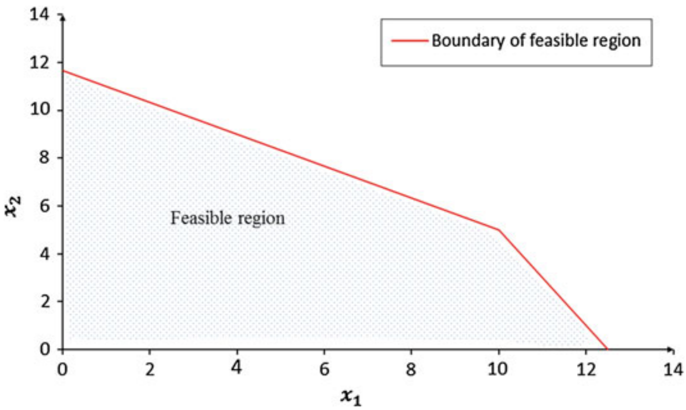


Fig. 2.2 The feasible region of the problem

feasible region for $15x_1 + 18x_2 = 270$ or not (Fig. 2.3). In this case, the line does not intersect the solution domain, therefore we know that the value for the objective function must be less than 270. It is important to note that all points with different values of x_1 and x_2 on this line have the same value of 270. We can consider other values for the objective function and draw new lines with lower values for f(x). The optimal point which is the maximum value will happen at the intersection of the last possible point in the feasible region and the associated objective line. In this problem, the maximum value occurs when $f(x) = 240$ since it is the last line that intersects the boundary of feasible solution domain (constraints). As noted above, the values of x_1 and x_2 can simply be calculated by solving the constraints equations simultaneously and the results will be $x_1 = 10$ and $x_2 = 5$.

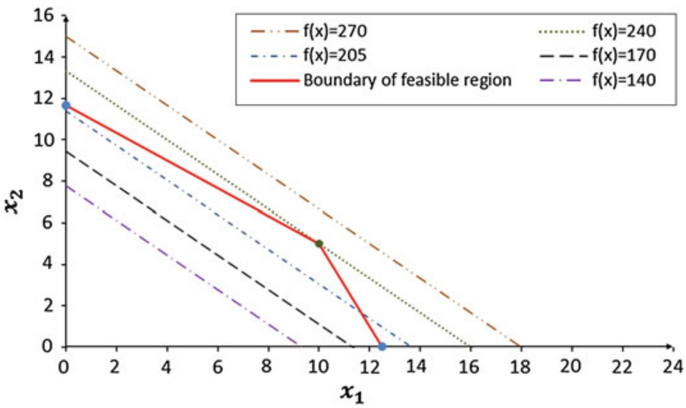


Fig. 2.3 The procedure of finding maximum of f(x)

This problem also can be solved as minimization problem by changing the sign of function f(x) into −f(x) and write it as following:

$$\min \ \ g(x) = -f(x) = -(15x_1 + 18x_2)$$

In this case we need to find the line that intersects the boundary of constraints to determine the minimum for the function g(x). Based on Fig. 2.4, the line g(x) = −240 is the last intersecting point, and so, it can be considered as the optimal value of the function g(x). As described above, the values of x_1 and x_2 can be simply calculated by solving the constraint equations simultaneously as $x_1 = 10$ and $x_2 = 5$. If the minimum value of g(x) or −f(x) is multiplied by a negative sign, it will result in the maximum for the function f(x).

2.3 Simplex Method

As noted above, the graphical method can only be used for LP problems with one or two decision variables, while many real LP problems involve more than two decision variables and so, we need to apply other optimization techniques to find the optimal solution. The simplex method is a well-known mathematical technique for solving LP models by constructing an acceptable solution domain and improving it step by step until the best solution is found and the optimum value is reached. The necessary steps in simplex method to find the optimal solution are presented in the following example.

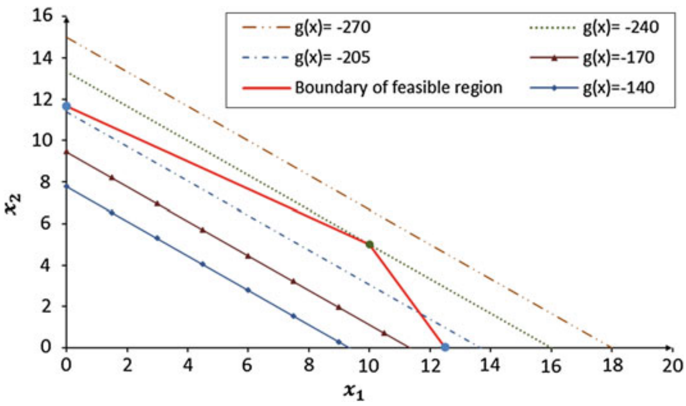


Fig. 2.4 The procedure of finding minimum of g(x)

Example 2.2 Maximize the objective function f(x) using the simplex method.

$$\max \ \mathrm{f}(x) = 2x_1 + 3x_2$$

Subject to

$$\begin{aligned} x_1 + x_2 &\leq 27 \\ 2x_1 + 5x_2 &\leq 90 \\ -x_1 + x_2 &\leq 11 \\ x_1 \geq 0 \quad &\text{and} \quad x_2 \geq 0 \end{aligned}$$

Solution: The following steps illustrate the whole process of solving a LP model using the simplex method.

Step 1: Convert desired LP problem into a standard form.

To convert a LP model to its standard form, all inequality constraints should be presented in the equality forms by considering the following conditions:

1. Adding a non-negative *slack* variable s_i for the constraints in the form of:

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,m}x_m \leq b_i$$

Hence, the constraint can be written as:

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots a + a_{i,m}x_m + s_i = b_i$$

2. Subtracting a non-negative *surplus* variable s_i for the constraints in the form of:

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,m}x_m \geq b_i$$

Therefore, the constraint can be written as:

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,m}x_m - s_i = b_i$$

Thus, transforming equalities into the standard form based on the above conditions can be written as:

$$\begin{cases} x_1 + x_2 + s_1 = 27 \\ 2x_1 + 5x_2 + s_2 = 90 \\ -x_1 + x_2 + s_3 = 11 \\ x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0 \end{cases}$$

It is important to note that non-negative constraints remain in inequality forms of $\geq$ or $\leq$. Figure 2.5 shows the feasible region and the corresponding constraints for example 2.2.

Step 2: Determine the basic and non-basic variables.

By considering f(x) as objective function and putting it along with the constraints, we will get the following system of linear equations:

$$Z - 2x_1 - 3x_2 = 0$$
$$\begin{cases} x_1 + x_2 + s_1 = 27 \\ 2x_1 + 5x_2 + s_2 = 90 \\ -x_1 + x_2 + s_3 = 11 \\ x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0 \end{cases}$$

The variables x_1 and x_2 are considered as *non-basic* variables and the other slack variables (s_1, s_2, and s_3) denoted as *basic* variables. In other words, the variables that only appear in one equation are basic and the other ones which are repeated in objective function and other equations are non-basic variables.

Step 3: Obtain entering and leaving variables.

The LP model in this problem includes five unknown variables ($n = 5$) x_1, x_2, s_1, s_2, and s_3 and three equations ($m = 3$) which are the constraints of the problem. As the numbers of unknown variables are more than equations ($n > m$), we will assume that the two non-basic variables are equal zero in order to find a basic solution for desired problem. It is important to note that the possible solution for LP model can be obtained if $n - m$ non-basic variables exist at the Zero level. By setting $x_1 = x_2 = 0$, we have: $f(x) = 0$, $s_1 = 27$, $s_2 = 90$, $s_3 = 11$.

Now the question is: can we still increase the objective function, or should this answer be considered as the optimal solution? By looking at the objective function equation, it can be seen that increasing x_1 or x_2 results in an increase in the values of f(x). Because both variables x_1 and x_2 have negative coefficients -2 and -3 respectively (or positive coefficients 2 and 3 in the original form as $f(x) = 2x_1 + 3x_2$), we still can increase the value of f(x) by setting higher values

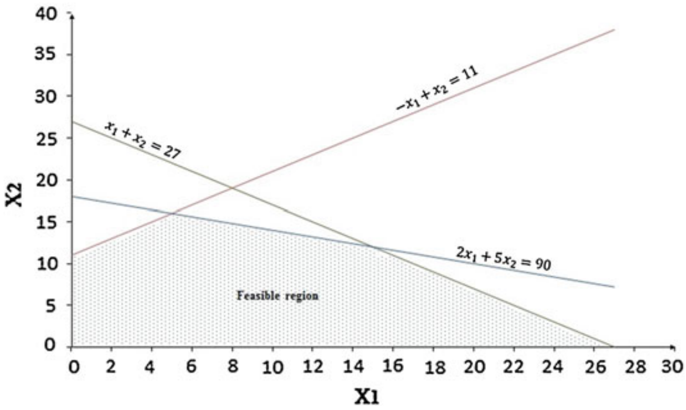


Fig. 2.5 The feasible region and corresponding constraints of f(x)

for non-basic variables. On the other hand, if all coefficients of the objective function are nonnegative, it can be concluded that the current basic solution, f(x) $= 0$ that is obtained by setting $x_1 = x_2 = 0$, is the optimum solution.

In this problem, the coefficients of both non-basic variable are negative, and hence, still it is possible to increase one of the variables x_1, or x_2 from zero to a higher value to increase the value of f(x). By increasing one of the non-basic variables from zero to a higher value, one of the basic variables should be pushed down to zero in order to maintain a feasible solution with $n - m$ non-basic variables. The selected non-basic variable which are going to be a basic variable, is called the *entering variable* and the basic variable that is changed to a non-basic variable is named *leaving variable*. The largest negative coefficient in the function f(x) or the largest positive coefficient in the original f(x) function can be selected as the entering variable in the maximization problems. The main reason for choosing a variable with the most negative coefficient is its potential to increase the objective function as much as possible. It is important to note that there is an inverse way to choose the entering variable for a minimization problem. In other words, the variable with largest positive coefficient of f(x) is selected as the entering variable. However, in the simplex method it is easier to convert minimization problems into a maximization problem and then find the optimum value of the desired objective function. The maximized function can be found by seeking the minimum of the negative of the same function by changing f(x) to −f(x).

Based on discussions above, the entering variable candidates in this problem are x_1 and x_2. As x_2 has the largest negative coefficient in $f(x) - 2x_1 - 3x_2 = 0$, it is selected as the entering variable. Once the *entering* variable is chosen, we need to determine one of the basic variables as *leaving* variable. The problem in the equality form can be written as:

$$\begin{cases} R_1 \rightarrow f(x) - 2x_1 - 3x_2 = 0 \\ R_2 \rightarrow x_1 + x_2 + s_1 = 27 \\ R_3 \rightarrow 2x_1 + 5x_2 + s_2 = 90 \\ R_4 \rightarrow -x_1 + x_2 + s_3 = 11 \end{cases}$$

where, R_i means the ith row of the equality system. To solve this LP model, the initial tableau corresponding to the equations can be represented as following:

The leaving variable can be selected by calculating the ratio of right side of equations (last column of table) to the non-negative coefficients of selected entering variable in the rows linked to the basic variables (here R_2, R_3, R_4). The entering variable and its coefficients are **bold** in Table 2.1. Finally, the leaving variable will be the current basic variable associated with the row with minimum ratio. In this problem the ratios are calculated as:

$$R_2 \rightarrow \frac{27}{1} = 27$$
$$R_3 \rightarrow \frac{90}{5} = 18$$
$$R_4 \rightarrow \frac{11}{1} = 11$$

The minimum value among the rations (27, 18, 11) is 11 and so, the current basic variable associated with this ratio which is s_3 is selected as the leaving variable and consequently become a non-basic variable.

Step 4: Determine pivot equation.

In this step, the row associated with the minimum ratio is selected as *pivot equation* and the coefficient of entering variable in the pivoting row is the *pivot element*. In this example, the pivot equation is R_4 and the pivot element is 1. To make a new simplex table, both side of pivot equation should be divided by pivot element to have a unit value for pivot element, and then, add or subtract multiples of the pivot equation to or from the other rows (here R_1, R_2 and R_3) in order to eliminate the selected entering variable (here x_2) from them. It is important to note that this method is called Gauss-Jordan elimination method. The equality equations after applying Gauss-Jordan elimination method will be changed as follows:

$$\begin{cases} R_1 + 3R_4 \rightarrow f(x) - 5x_1 + 3s_3 = 33 \\ R_2 - R_4 \rightarrow 2x_1 + s_1 - s_3 = 16 \\ R_3 - 5R_4 \rightarrow 7x_1 + s_2 - 5s_3 = 35 \\ R_4 \rightarrow -x_1 + x_2 + s_3 = 11 \end{cases}$$

The new coefficients of all basic and non-basic variables based on the simplex method are shown in the Table 2.2.

Table 2.1 Simplex tableau based on the coefficients of basic and non-basic variables

Equations	All variables →	f(x)	x_1	$\boldsymbol{x_2}$	s_1	s_2	s_3	Right side of equations
R_1	f(x)	1	−2	−3	0	0	0	0
	Basic variables							
R_2	s_1	0	1	**1**	1	0	0	27
R_3	s_2	0	2	**5**	0	1	0	90
R_4	s_3	0	−1	**1**	0	0	1	11

Table 2.2 Simplex tableau based on the new coefficients of basic and non-basic variables

Equations	All variables →	f(x)	$\boldsymbol{x_1}$	x_2	s_1	s_2	s_3	Right side of equations
$R_1 + 3R_4$	*f*(*x*)	1	−5	0	0	0	+3	33
	Basic variables							
R_2–R_4	s_1	0	**2**	0	1	0	−1	16
R_3–$5R_4$	s_2	0	**7**	0	0	1	−5	35
R_4	x_2	0	**−1**	1	0	0	1	11

Step 5: Find the optimal solution.

Once the new basic and non-basic variables are determined and the new simplex table is generated, we need to examine if the computed value for f(x) is an optimal solution or not. Based on Table 2.2, there is a non-basic variable (i.e., x_1) in the first row of the tableau with negative coefficient −5 that has potential to improve the value of f(x). In other words, for this maximization problem by setting higher values for non-basic variable x_1, the value of f(x) will be increased. Hence, the non-basic variable x_1 (it is **bold** in the Table 2.2) is considered as the entering variable and by using the same procedure described above the leaving variable should be selected. The ratios based on the new simplex table are:

$$R_2 - R_4 \rightarrow \frac{16}{2} = 8$$
$$R_3 - 5R_4 \rightarrow \frac{35}{7} = 5$$

As the leaving variable is chosen by dividing the ratio of right side of equations to the non-negative coefficients of entering variable in the constraint rows, the coefficient −1 should not be considered. The minimum of values (8, 5) is 5, and so, the current basic variable associated with this ratio, which is s_2, is chosen as the leaving variable. The pivot equation and pivot element here are $R_3 - 5R_4$ and 7, respectively. Afterward, both side of pivot equation is divided by pivot element to have a unit value for pivot element, and then, add or subtract multiples of the pivot equation to or from the other rows as $R_1 + 3R_4$, $R_2 - R_4$ and R_4 to eliminate the selected entering variable x_1 from them. The elimination procedure is:

$$\begin{cases} R_1' = (R_1 + 3R_4) + 5R_3' & \rightarrow \quad Z + \frac{5}{7}s_2 - \frac{4}{7}s_3 = 58 \\ R_2' = (R_2 - R_4) - 2R_3' & \rightarrow \quad s_1 - \frac{2}{7}s_2 + \frac{3}{7}s_3 = 6 \\ R_3' = \frac{1}{7}(R_3 - 5R_4) & \rightarrow \quad x_1 + \frac{1}{7}s_2 - \frac{5}{7}s_3 = 5 \\ R_4' = R_4 + R_3' & \rightarrow \quad x_2 + \frac{1}{7}s_2 + \frac{2}{7}s_3 = 16 \end{cases}$$

The new coefficients of all new basic and non-basic variables are shown in the Table 2.3.

As it can be seen from this new tableaux, the value of the objective function is increased from 33 to 58. Now the question is "is this the optimal solution?" To find the appropriate answer for this question, we need to examine the coefficients of variables in the first row (R_1') and find the negative coefficient. If all coefficients are nonnegative, the optimization process is done and so, the final value of 58 will be the maximum for f(x). But, there still is a coefficient with negative value in the row of objective function of Table 2.3. Therefore, it can be concluded that s_3 is the entering variable and the leaving variable should be determined in this step. The ratios based on the previous simplex tableau are:

$$R_2' \rightarrow \frac{6}{3/7} = 14$$

$$R_4' \rightarrow \frac{16}{2/7} = 56$$

The minimum value of (14, 56) is 14 and so, the current basic variable associated with this ratio, which is s_1 is chosen as the leaving variable. The pivot equation and pivot element here are R_2' and 3/7 respectively. Now we will try to eliminate the selected entering variable s_1 from the equation system as follow:

$$\begin{cases} R_1'' = R_1' + \frac{4}{7}R_2'' & \rightarrow \quad Z + \frac{4}{3}s_1 + \frac{1}{3}s_2 = 66 \\ R_2'' = R_2' & \rightarrow \quad \frac{7}{3}s_1 - \frac{2}{3}s_2 + s_3 = 14 \\ R_3'' = R_3' + \frac{5}{7}R_2'' & \rightarrow \quad x_1 + \frac{5}{3}s_1 - \frac{1}{3}s_2 = 15 \\ R_4'' = R_4' - \frac{2}{7}R_2'' & \rightarrow \quad x_2 - \frac{2}{3}s_1 + \frac{1}{3}s_2 = 12 \end{cases}$$

The calculated coefficients for all new basic and non-basic variables are presented in the Table 2.4. As it can be seen from the objective coefficients in the row of R_1'', all coefficients are nonnegative and hence, no non-basic variable to increase the value of the objective function. Therefore, the current solution $x_1 = 15$ and $x_2 = 12$ are the optimum solution, and the maximum value of Z is 66.

The LP problems with many constraints and objective functions also can be solved quickly by applying powerful software like Excel. Excel contains a powerful tool, called Solver, to find the optimal solution of linear programming using the simplex method. This tool can be found on the Data tab of the Excel worksheet in which the opening window looks like Fig. 2.6. As seen from this figure, the first section of Solver Parameters window is *Set Objective*. In this part, we need to address the cell reference or name for the objective cell which contains a formula. Afterward, the value of the objective cell should be determined as: Max, Min, or a Value (the objective cell to be a certain value). The third part is *By Changing Variable Cells*

Table 2.3 New coefficients of basic and non-basic variables based on the simplex method

Equations	All variables →	Z	x_1	x_2	s_1	s_2	s_3	Right side of equations
R'_1	Z	1	0	0	0	$\frac{5}{7}$	$-\frac{4}{7}$	58
	Basic variables							
R'_2	s_1	0	0	0	1	$-\frac{2}{7}$	$\frac{3}{7}$	6
R'_3	x_1	0	1	0	0	$\frac{1}{7}$	$-\frac{5}{7}$	5
R'_4	x_2	0	0	1	0	$\frac{1}{7}$	$\frac{2}{7}$	16

Table 2.4 New coefficients of basic and non-basic variables based on the simplex method

Equations	All variables →	f(x)	x_1	x_2	s_1	s_2	s_3	Right side of equations
R''_1	f(x)	1	0	0	$\frac{4}{3}$	$\frac{1}{3}$	0	66
	Basic variables							
R''_2	s_3	0	0	0	$\frac{7}{3}$	$-\frac{2}{3}$	1	14
R''_3	x_1	0	1	0	$\frac{5}{3}$	$-\frac{1}{3}$	0	15
R''_4	x_2	0	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	0	12

that is used for choosing the decision variable cell ranges. It is important to note that the decision variables must be related to the objective cell. To enter all constraints of the problem, the *Subject to the Constraints* box should be applied. In this case, click the *Add* option and in the *Cell Reference* box, enter the cell reference of constraints. Excel provides the following three different techniques to solve an optimization problem: (1) Simplex LP, (2) GRG Nonlinear, and (3) Evolutionary approaches. The application of Excel in solving LP problems are illustrated in the following examples. The following example is a linear problem, and the Simplex LP method is selected to find the optimum value of profit (R). It should be noted that Solver is not a very robust tool for non-linear and complex optimization problems, but it is useful to find solution of simple problems.

Example 2.3 The proper monitoring of earth dams and safety evaluation of the large structure under operational conditions require using a number of instruments such as piezometers to monitor the earthen embankment pore water pressures for potential engineering improvement (filter systems, cut off walls, sheet piles, low permeability apron, etc.) to prevent failure. The Caspian Company has two production lines and produces two types of piezometers called "P_1" and "P_2". The P_1 production line has a capacity of 25 piezometers per day, whereas the daily capacity of the P_2 line is only 35 piezometers. The labors requirement to produce P_1 and P_2 are 2 man-hours and 3 man-hours, respectively. The maximum capacity of Caspian Company is 140 labor hours per day to produce two types of piezometers. Determine the daily production, if the profit for the P_1 piezometer is $20 and for the P_2 piezometer is $25.

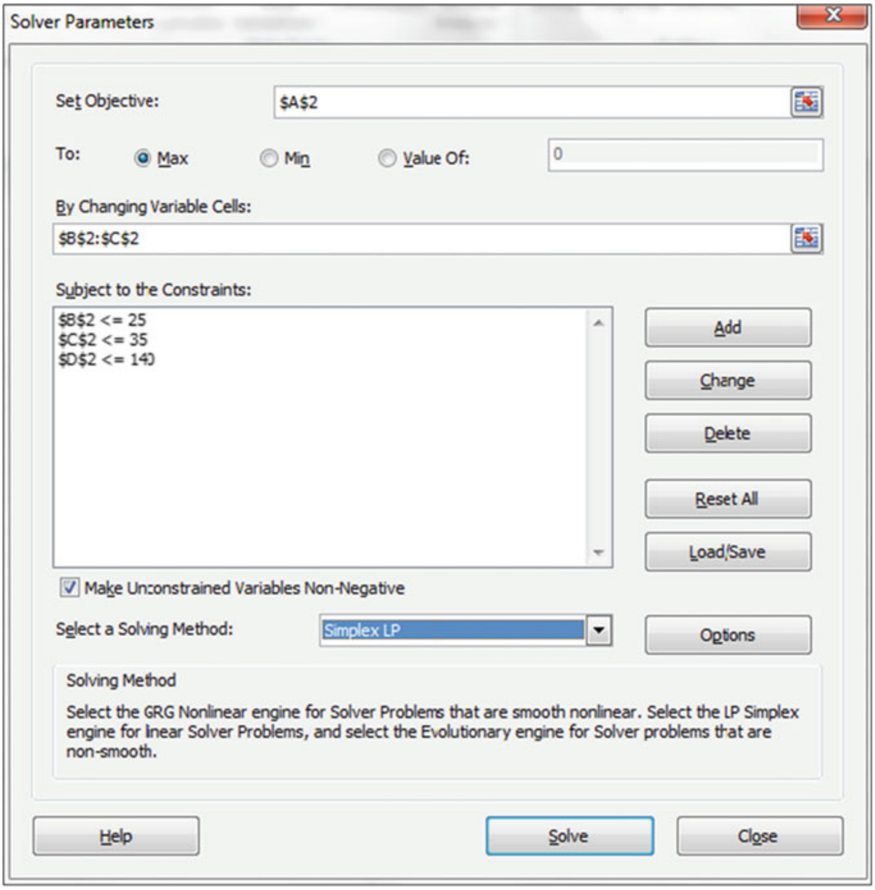


Fig. 2.6 The Solver parameters in Excel 2010

Solution:

1. The first step is to determine the objective and associated constraints of example problem. It is obvious that the objective function is maximizing the profit $R(\$)$ subject to the following constraints:

   (a) The number of P_1 piezometers (n_{p1}) produced each day are less than or equal to 25,
   (b) The number of P_2 piezometers (n_{p2}) produced each day are less than or equal to 35,
   (c) The total number of labor hours is a linear function of 2 and 3 man-hours for production of piezometers, that is: $2n_{p1} + 3n_{p2} \leq 140$.

And the profit function in this example can be written as follow:

$$R(\$) = \left(n_{p_1} \times 20\$\right) + \left(n_{p_2} \times 25\$\right)$$

Then, the exact statement of this problem or the objective function is:

$$\max R(\$) = \left(n_{p_1} \times 20\$\right) + \left(n_{p_2} \times 25\$\right)$$

Subject to:

$$n_{p_1} \leq 25; \quad n_{p_2} \leq 35; \quad 2n_{p_1} + 3n_{p_2} \leq 140$$

Figure 2.7 shows the governing conditions in the Caspian Company to produce the piezometers. Based on the constraints, the feasible production combinations are the points in the shaded area of Fig. 2.7 and we need to find a point in this area that makes the highest profit. To solve this problem, different values for n_{p1} are considered, then associated n_{p2} are calculated, and finally the target point considering all constraints is found. Based on the presented results in Table 2.5, the acceptable ranges that meet all constraints of this problem are from 30 to 34, for n_{p2} and from 19 to 25 for n_{p1} which are shown in a gray color. According to this range, the highest profit for n_{p2} and n_{p1} are equal to 30 and 25, respectively.

The first column of Table 2.5 includes various numbers of P_2 piezometers (n_{p2}), while the second column (n_{p1}) is calculated based on the relationship between two variables P_1 and P_2. According to the results, if the Caspian Company produces 25 P_1 type and 30 P_2 type piezometers per day, the highest profit will occur.

In summary, the procedure for finding optimal solution using the Excel Solver can be explained as:

1. Set Objective: set the profit as target value that should be maximized,
2. By Changing Variables Cells: consider n_{p1} and n_{p2} as decision variables,
3. Subject to the Constraints: determine the constraints as: $n_{p1} \leq 25$, $n_{p2} \leq 35$, and $2n_{p1} + 3n_{p2} \leq 140$.

The achieved results using the Solver tool are the same as the computed results in the previous section. It is important to note that the existence of solution is only dependent on the defined constraints of the desired problem and it is not a function of objective function.

2. If the maximum capacity of Caspian Company is changed from 140 to 170 persons-hour of labor per day, there would not be feasible solution regarding this new production constraint, and there is no single point to satisfy all constraints. Figure 2.8 and Table 2.6 illustrate the feasible production combinations of piezometer productions by considering new values for the labor constraint.

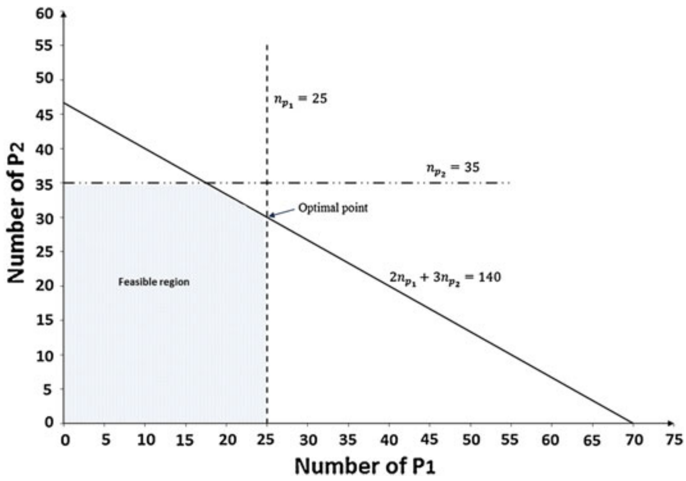


Fig. 2.7 The feasible production combinations for Caspian Company

Table 2.5 The possible combination of piezometer productions in Caspian Company

n_{p2}	$n_{p1} = (140 - 3n_{p2})/2$	R (\$)	n_{p2}	$n_{p1} = (140 - 3n_{p2})/2$	R (\$)
0	70	1,400	24	34	1,280
2	67	1,390	26	31	1,270
4	64	1,380	28	28	1,260
6	61	1,370	30	25	1,250
8	58	1,360	32	22	1,240
10	55	1,350	34	19	1,230
12	52	1,340	36	16	1,220
14	49	1,330	38	13	1,210
16	46	1,320	40	10	1,200
18	43	1,310	42	7	1,190
20	40	1,300	44	4	1,180
22	37	1,290	46	1	1,170

As can be seen from Table 2.6, there is no solution that satisfies all constraints, and hence, this problem is infeasible for the new value of labor constraint.

Using the Solver tool in Excel, the same results will be achieved and the program shows the following message as: *The Objective Cells values don't converge* (Fig. 2.9).

3. As mentioned previously, the unbounded solution happens when the feasible region (shaded area in Fig 2.7) is unbounded and so the value of objective

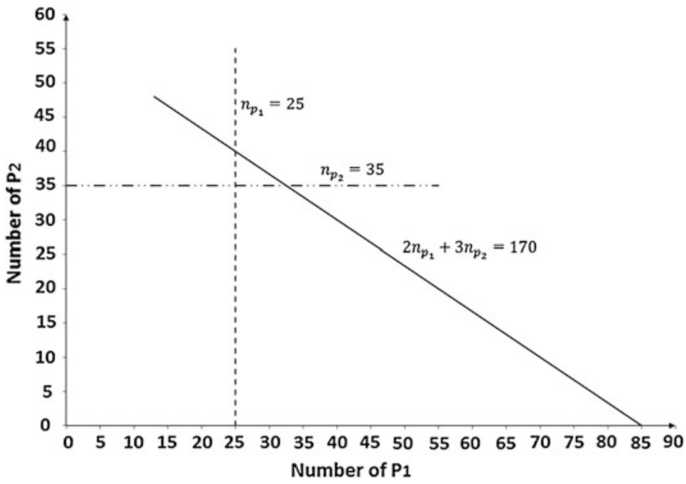


Fig. 2.8 The feasible production combinations for new labor constraint

Table 2.6 The possible combination of piezometer productions with new labor constraint

n_{p2}	$n_{p1} = (170 - 3n_{p2})/2$	$R(\$)$	n_{p2}	$n_{p1} = (170 - 3n_{p2})/2$	$R(\$)$
0	85	1,700	24	49	1,580
2	82	1,690	26	46	1,570
4	79	1,680	28	43	1,560
6	76	1,670	30	40	1,550
8	73	1,660	32	37	1,540
10	70	1,650	34	34	1,530
12	67	1,640	36	31	1,520
14	64	1,630	38	28	1,510
16	61	1,620	40	25	1,500
18	58	1,610	42	22	1,490
20	55	1,600	44	19	1,480
22	52	1,590	46	16	1,470

function can be increased or decreased to infinity without leaving the feasible region. For instance if we don't consider any constraint for n_{p1} (ignore constraints 1 and 3) its value can be varied from zero to infinity, and hence, the profit (R) simultaneously rises and approaches infinity (Fig. 2.10).

4. If the profit for each P_2 piezometer is changed from 25\$ to \$30, the profit function will be changed as follow:

$$R(\$) = \left(n_{p_1} \times 20\$\right) + \left(n_{p_2} \times 30\$\right)$$

In this case, there may be more than one combination of producing P_1 and P_2 piezometers to reach the maximum profit. As can be seen in Table 2.7, the

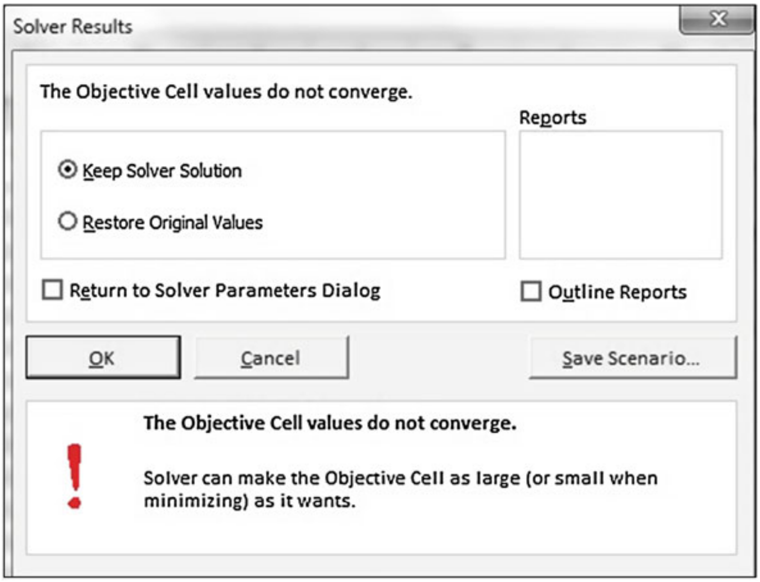


Fig. 2.9 The solver results

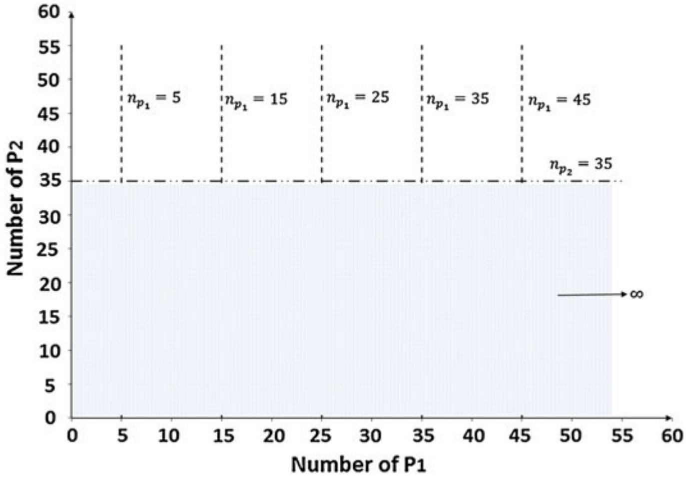


Fig. 2.10 The unbounded condition

production of different numbers of P_1 and P_2 piezometers resulted in same profit for Caspian Company. In other words, it is possible to produce several different combination of P_1 piezometer (e.g., 19, 22, or 25) or P_2 piezometer (e.g., 34, 32, or 30), while the profit is still constant and it is 1,400\$. Hence, there are multiple optimal solutions to produce n_{p2} and n_{p1} in comparison to a unique optimal solution discussed above.

5. It is important to note that addition or subtraction of a positive constant value to or from the objective function will not change the optimum solution of the problem. This rule also is confirmed for multiplication or division of objective function by a positive constant value. In this section, the Caspian Company's profits are calculated for different profit functions as follows, and then the optimum solutions are determined in each case.

$$
\begin{aligned}
R(\$) &= \left[\left(n_{p_1} \times 20\right) + \left(n_{p_2} \times 25\right)\right] + 35 \\
R(\$) &= \left[\left(n_{p_1} \times 20\right) + \left(n_{p_2} \times 25\right)\right] - 20 \\
R(\$) &= \left[\left(n_{p_1} \times 20\right) + \left(n_{p_2} \times 25\right)\right] \times 1.5 \\
R(\$) &= \left[\left(n_{p_1} \times 20\right) + \left(n_{p_2} \times 25\right)\right] / 5
\end{aligned}
$$

The results show that the optimum solutions in all cases are constant as $n_{p1} = 25$, and $n_{p2} = 30$ and they are not changed by changing the profit function with a positive constant value (Table 2.8).

Figure 2.11 shows any addition, subtraction, multiplication, and division of a constant value to or from desired objective function doesn't change the optimum solution for n_{p1}.

In the following section the application of simplex method for solving three types of common optimization problems in the field of water resources engineering are presented. The problems cover the process of linear programming for water distribution networks with and without pump stations, confined aquifer with one-dimensional steady-state flow, and confined aquifer with two-dimensional steady-state flow.

2.3.1 Optimization of Water Distribution Networks

A water distribution network is a major urban infrastructure which distributes water supply to residential, industrial, and commercial customers under various demand conditions at adequate pressures and flows. In general, water distribution systems are composed of pipes, pumps, distribution storages like reservoirs, and other hydraulic components. In addition to design and analysis of a water distribution system form a hydraulic point of view, a designer needs to determine the minimum cost of a distribution system to meet demands for all users at required pressure level. The overall cost of a water distribution system includes:

1. Cost of piping and appurtenances such as pumps, valves, flush hydrants, reservoirs, tanks, etc.,
2. Cost of energy for pumping the water to desired network connections to provide the minimum required pressure head elevation, and

Table 2.7 The possible combination of piezometer productions with multiple solutions

n_{p2}	$n_{p1} = (140 - 3n_{p2})/2$	R($)	n_{p2}	$n_{p1} = (140 - 3n_{p2})/2$	R($)
0	70	1,400	24	34	1,400
2	67	1,400	26	31	1,400
4	64	1,400	28	28	1,400
6	61	1,400	30	25	1,400
8	58	1,400	32	22	1,400
10	55	1,400	34	19	1,400
12	52	1,400	36	16	1,400
14	49	1,400	38	13	1,400
16	46	1,400	40	10	1,400
18	43	1,400	42	7	1,400
20	40	1,400	44	4	1,400
22	37	1,400	46	1	1,400

Table 2.8 The values of different profit function for Caspian Company

n_{p1}	n_{p2}	R($)	R + 35 ($)	R − 20 ($)	R × 1.5 ($)	R/ 5 ($)
25	30	1,250	1,285	1,230	1,875	250
22	32	1,240	1,275	1,220	1,860	248
19	34	1,230	1,265	1,210	1,845	246

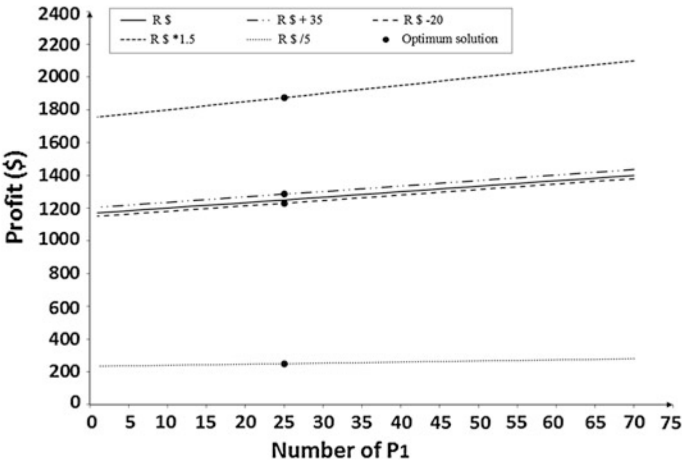


Fig. 2.11 Variation of profits versus n_{p1}

3. Operation and maintenance costs that includes administration and management of personnel, flushing the system at a particular time interval, repairing pipes, servicing of the pumps, billing customers, etc.

In this problem, we will optimize the piping cost (pipe length with a specific diameter of a simple branched water distribution system) with and without considering a pump station in the system. In other words, the main objective function here is obtaining the length of pipe with specific diameter (d) in each reach ($L_{i,j}$) between two connections of desired water distribution networks to minimize the cost of piping. A water distribution system tends to use a pipe or a combination of pipes in each reach to have minimum piping cost, while it has to satisfy the water demands and pressure requirements at all nodes as well as hydraulic constraints. The optimization process in the form of linear programming (LP) can be written as follow:

$$\min\ Z = \sum_{i,j} \sum_{d_{i,j}} C_{i,j,d} l_{i,j,d} + \sum_{n} C_{p_n} H_{p_n} \tag{2.9}$$

where, $C_{i,j,d}$ and $l_{i,j,d}$ are the cost per unit length, and the length of pipe between nodes i and j with diameter d, respectively: H_{pn} is the pumping head, n is the total number of pumps in the system, and C_{pn} is the cost per unit of pumping head. It is important to note that when there is no pump station in a water distribution system, the second term of Eq. (2.9) should be omitted. The main constraints for a water distribution network are energy and length constraints in conjunction with non-negativity of all pipes length and pumping head elevation. The constraints in this case can be written as:

1. Energy constraint:

$$H_{min,k} \le H_c + \sum_{n} H_{p_n} - \sum_{i,j} \sum_{d_{i,j}} I_{i,j,d} l_{i,j,d} \le H_{max,k} \tag{2.10}$$

where, $H_{min,k}$ and $H_{max,k}$ are the minimum and maximum required head at the demand point k, respectively: k is the total number of demand points, H_c is the constant elevation of piping system, and $I_{i,j,d}$ is the hydraulic gradient or gradient between two hydraulic head measurements over the length of the flow path. The energy loss for water flow in a pipe can be estimated using the Darcy-Weisbach equation as:

$$H_L = I \times l = \frac{8\mathrm{f}Q^2}{\pi^2 g d^5} l \tag{2.11}$$

where, f is Darcy-Weisbach friction factor, Q is flow rate (cfs), and d is diameter of pipe (ft).

2. Length constraint:

$$\sum_{i,j,d} l_{i,j,d} = L_{i,j} \tag{2.12}$$

where, $L_{i,j}$ is the total reach length between each two connections that is a known variable in these types of problems. In other words, the total length of pipe in every reach, which can be a combination of pipes with different diameters, must be equaled to the total reach length between two connections.

3. Non-negativity conditions:

$$\begin{cases} l_{i,j,d} \geq 0 \\ H_{p_n} \geq 0 \end{cases} \tag{2.13}$$

Example 2.4 According to the above statements, determine the minimum cost of pipe and pump (head elevation) in various demand nodes in the following conditions:

1. When there is no pump in the system (Fig. 2.12), and
2. When there is a pump station in the system (Fig. 2.13).

Other necessary information includes:

(a) The unit cost of pipe for two standard diameters (Table 2.9),
(b) The minimum required pressure head elevations for all determined users (A, B, and C) are 550 (ft) and the demand discharges are presented in Table 2.10.
(c) Darcy-Weisbach friction factor is 0.02,
(d) The unit cost of pumping head is 220$,
(e) The total length of pipe between each connection is 1,000 ft.
(f) The constant elevation of piping system is assumed 650 ft when there is no pump in the system, and 555 ft when a pump is considered in the system.

Solution:

1. As noted above, the objective function is to minimize the cost, of pipes (smaller diameter) and it can be defined as follow:

$$\begin{aligned} \min Z &= \left(C_{0,1,1} l_{0,1,1} + C_{0,1,2} l_{0,1,2}\right) + \left(C_{1,2,1} l_{1,2,1} + C_{1,2,2} l_{1,2,2}\right) + \left(C_{2,3,1} l_{2,3,1} + C_{2,3,2} l_{2,3,2}\right) \\ &\quad + \left(C_{2,4,1} l_{2,4,1} + C_{2,4,2} l_{2,4,2}\right) + \left(C_{1,5,1} l_{1,5,1} + C_{1,5,2} l_{1,5,2}\right) \\ &= \left(10 \times l_{0,1,1} + 15 \times l_{0,1,2}\right) + \left(10 \times l_{1,2,1} + 15 \times l_{1,2,2}\right) + \left(10 \times l_{2,3,1} + 15 \times l_{2,3,2}\right) \\ &\quad + \left(10 \times l_{2,4,1} + 15 \times l_{2,4,2}\right) + \left(10 \times l_{1,5,1} + 15 \times l_{1,5,2}\right) \end{aligned}$$

Subject to

(a) The length constraints as:

$$\begin{aligned} l_{0,1,1} + l_{0,1,2} &= 1{,}000 \text{ ft} \\ l_{1,2,1} + l_{1,2,2} &= 1{,}000 \text{ ft} \\ l_{2,3,1} + l_{2,3,2} &= 1{,}000 \text{ ft} \\ l_{2,4,1} + l_{2,4,2} &= 1{,}000 \text{ ft} \\ l_{1,5,1} + l_{1,5,2} &= 1{,}000 \text{ ft} \end{aligned}$$

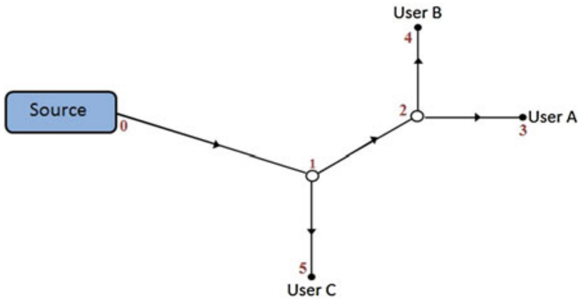


Fig. 2.12 The water distribution system without pumping station

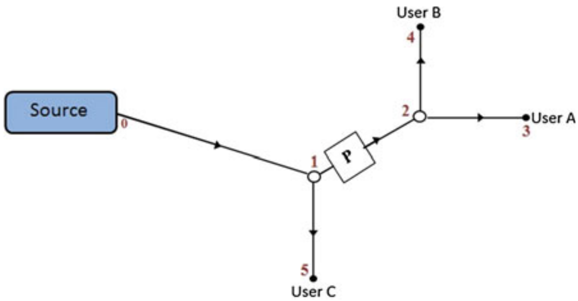


Fig. 2.13 The water distribution system with pumping station

Table 2.9 The cost information of pipes based on diameters

Diameter (*in*)	Diameter (ft)	Cost ($/ft)
21.0	1.75	10.0
24.0	2.00	15.0

Table 2.10 Demand discharges information

	Demand discharges					
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
User A	$Q_A = 10$	$Q_A = 14$	$Q_A = 18$	$Q_A = 15$	$Q_A = 18$	$Q_A = 20$
User B	$Q_B = 12$	$Q_B = 18$	$Q_B = 20$	$Q_B = 17$	$Q_B = 18$	$Q_B = 21$
User C	$Q_C = 14$	$Q_C = 20$	$Q_C = 24$	$Q_C = 36$	$Q_C = 31$	$Q_C = 23$

(b) The energy constraints for all users is as follows:

For user A:

$$650 - \left(I_{0,1,1} \times l_{0,1,1} + I_{0,1,2} l_{0,1,2}\right) - \left(I_{1,2,1} l_{1,2,1} + I_{1,2,2} l_{1,2,2}\right) \\ - \left(I_{2,3,1} l_{2,3,1} + I_{2,3,2} l_{2,3,2}\right) \geq 550$$

where,

$$I_{0,1,1} = \frac{8 f Q^2}{\pi^2 g d^5} = \frac{8 \times 0.02 \times (Q_A + Q_B + Q_C)^2}{\pi^2 \times 32.2 \times (1.75)^5}$$

The hydraulic gradient for Q_1 is:

$$I_{0,1,1} = \frac{8 \times 0.02 \times (36)^2}{\pi^2 \times 32.2 \times (1.75)^5} = 0.0398\,\mathrm{ft/ft}$$

and for $I_{2,4,2}$ and Q_3 the hydraulic gradient is:

$$I_{2,4,2} = \frac{8 \times 0.02 \times (20)^2}{\pi^2 \times 32.2 \times (2)^5} = 0.0063\,\mathrm{ft/ft}$$

Then, the hydraulic constraint for user A in Q_1 will be calculated as:

$$\begin{aligned}650 &- \left(0.0398 \times l_{0,1,1} + 0.0204 \times l_{0,1,2}\right) - \left(0.0149 \times l_{1,2,1} + 0.0076 \times l_{1,2,2}\right) \\ &- \left(0.0031 \times l_{2,3,1} + 0.0016 \times l_{2,3,2}\right) \geq 550\end{aligned}$$

and for user B in Q_1 is:

$$\begin{aligned}650 &- \left(I_{0,1,1} \times l_{0,1,1} + I_{0,1,2} l_{0,1,2}\right) - \left(I_{1,2,1} l_{1,2,1} + I_{1,2,2} l_{1,2,2}\right) - \left(I_{2,4,1} l_{2,4,1} + I_{2,4,2} l_{2,4,2}\right) \\ &= 650 - \left(0.0398 \times l_{0,1,1} + 0.0204 \times l_{0,1,2}\right) - \left(0.0149 \times l_{1,2,1} + 0.0076 \times l_{1,2,2}\right) \\ &\quad - \left(0.0044 \times l_{2,4,1} + 0.0023 \times l_{2,4,2}\right) \geq 550\end{aligned}$$

and finally for user C in Q_1 is:

$$\begin{aligned}650 &- \left(I_{0,1,1} \times l_{0,1,1} + I_{0,1,2} l_{0,1,2}\right) - \left(I_{1,5,1} l_{1,5,1} + I_{1,5,2} l_{1,5,2}\right) \\ &= 650 - \left(0.0398 \times l_{0,1,1} + 0.0204 \times l_{0,1,2}\right) - \left(0.0060 \times l_{1,5,1} + 0.0031 \times l_{1,5,2}\right) \geq 550\end{aligned}$$

The noted procedure above should be repeated for existing users in different demand points. The values of hydraulic gradient for all reaches and various demand discharges are shown in the Table 2.11

In the next step, the optimum values of pipes length are calculated by minimizing the objective function, Z. The simplex LP method (Excel|Data|Solver) is applied for optimization analysis and the results are presented in the Table 2.12.

2. In this section, the optimization problem is formulated for desired water distribution network by considering a pump station in the system (Fig. 2.13). Therefore, the objective function in this case can be written as:

$$\begin{aligned}\min Z = &\left(C_p H_p\right) + \left(C_{0,1,1} l_{0,1,1} + C_{0,1,2} l_{0,1,2}\right) + \left(C_{1,2,1} l_{1,2,1} + C_{1,2,2} l_{1,2,2}\right) \\ &+ \left(C_{2,3,1} l_{2,3,1} + C_{2,3,2} l_{2,3,2}\right) + \left(C_{2,4,1} l_{2,4,1} + C_{2,4,2} l_{2,4,2}\right) + \left(C_{1,5,1} l_{1,5,1} + C_{1,5,2} l_{1,5,2}\right) \\ = &\left(220 \times H_p\right) + \left(10 \times l_{0,1,1} + 15 \times l_{0,1,2}\right) + \left(10 \times l_{1,2,1} + 15 \times l_{1,2,2}\right) \\ &+ \left(10 \times l_{2,3,1} + 15 \times l_{2,3,2}\right) + \left(10 \times l_{2,4,1} + 15 \times l_{2,4,2}\right) \\ &+ \left(10 \times l_{1,5,1} + 15 \times l_{1,5,2}\right)\end{aligned}$$

The length constraints are the same as presented in the previous section, and the energy constraint for user A can be written as:

Table 2.11 The hydraulic gradient for all reaches and demand points

Demand discharge (*cfs*)						
Hydraulic gradient	$Q_A = 10$ $Q_B = 12$ $Q_C = 14$	$Q_A = 14$ $Q_B = 18$ $Q_C = 20$	$Q_A = 18$ $Q_B = 20$ $Q_C = 24$	$Q_A = 18$ $Q_B = 18$ $Q_C = 31$	$Q_A = 15$ $Q_B = 17$ $Q_C = 36$	$Q_A = 20$ $Q_B = 21$ $Q_C = 23$
$I_{0,1,1}$	0.0398	0.0830	0.1180	0.1378	0.1420	0.1258
$I_{0,1,2}$	0.0204	0.0426	0.0605	0.0707	0.0728	0.0645
$I_{1,2,1}$	0.0149	0.0314	0.0443	0.0398	0.0314	0.0516
$I_{1,2,2}$	0.0076	0.0161	0.0227	0.0204	0.0161	0.0265
$I_{2,3,1}$	0.0031	0.0060	0.0099	0.0099	0.0069	0.0123
$I_{2,3,2}$	0.0016	0.0031	0.0051	0.0051	0.0035	0.0063
$I_{2,4,1}$	0.0044	0.0099	0.0123	0.0099	0.0089	0.0135
$I_{2,4,2}$	0.0023	0.0051	0.0063	0.0051	0.0046	0.0069
$I_{1,5,1}$	0.0060	0.0123	0.0177	0.0295	0.0398	0.0162
$I_{1,5,2}$	0.0031	0.0063	0.0091	0.0151	0.0204	0.0083

$$555 + H_p - \left(I_{0,1,1} \times l_{0,1,1} + I_{0,1,2} l_{0,1,2}\right) - \left(I_{1,2,1} l_{1,2,1} + I_{1,2,2} l_{1,2,2}\right) - \left(I_{2,3,1} l_{2,3,1} + I_{2,3,2} l_{2,3,2}\right) \geq 550$$

and for user B is:

$$555 + H_p - \left(I_{0,1,1} \times l_{0,1,1} + I_{0,1,2} l_{0,1,2}\right) - \left(I_{1,2,1} l_{1,2,1} + I_{1,2,2} l_{1,2,2}\right) - \left(I_{2,4,1} l_{2,4,1} + I_{2,4,2} l_{2,4,2}\right) \geq 550$$

and finally for user C has following form:

$$555 + H_p - \left(I_{0,1,1} \times l_{0,1,1} + I_{0,1,2} l_{0,1,2}\right) - \left(I_{1,5,1} l_{1,5,1} + I_{1,5,2} l_{1,5,2}\right) \geq 550$$

.

The optimized lengths for different demand points are presented in Table 2.13.

2.3.2 *Optimization of One-Dimensional Confined Aquifers*

The occurrence and movement of water beneath the surface of the Earth is called groundwater flow, and it when occurs in the saturated soil and rock below the water table, it is called saturated flow. Groundwater flow is an important part of the hydrologic cycle where different types of surface water such as reservoirs, rivers, streams, and overland flow from precipitation infiltrate into the earth crust and become subsurface water. A significant part of the subsurface water can be either stored or transmitted through a geological unit called *aquifer*. In other words, an aquifer is an underground water-saturated formation or layer consisting of permeable rock, sediment, or soil that yields usable amounts of water to wells and springs. Wells can be drilled into the aquifers to pump groundwater from the aquifer and deliver it to various demand points such as domestic, industrial, agricultural,

Table 2.12 The optimized lengths when there is no pumping station

Length of pipe (ft)						
Pipe segment	$Q_A = 10$ $Q_B = 12$ $Q_C = 14$	$Q_A = 14$ $Q_B = 18$ $Q_C = 20$	$Q_A = 18$ $Q_B = 20$ $Q_C = 24$	$Q_A = 18$ $Q_B = 18$ $Q_C = 31$	$Q_A = 15$ $Q_B = 17$ $Q_C = 36$	$Q_A = 20$ $Q_B = 21$ $Q_C = 23$
$l_{0,1,1}$	1,000	396.21	0.00	0.00	0.00	0.00
$l_{0,1,2}$	0.00	603.79	1,000	1,000	1,000	1,000
$l_{1,2,1}$	1,000	1,000	205.42	0.00	142.02	0.00
$l_{1,2,2}$	0.00	0.00	794.57	1,000	857.97	1,000
$l_{2,3,1}$	1,000	1,000	1,000	781.85	1,000	454.29
$l_{2,3,2}$	0.00	0.00	0.00	218.14	0.00	545.70
$l_{2,4,1}$	1,000	1,000	1,000	781.85	1,000	314.15
$l_{2,4,2}$	0.00	0.00	0.00	218.14	0.00	685.84
$l_{1,5,1}$	1,000	1,000	1,000	985.69	349.02	1,000
$l_{1,5,2}$	0.00	0.00	0.00	14.30	650.97	0.00
min Z($)	50,000	53,018.95	58,972.86	62252.99	62,544.721	66,157.75

Table 2.13 The optimized lengths when there is a pumping station

Length of pipe (ft)						
Pipe segment	$Q_A = 10$ $Q_B = 12$ $Q_C = 14$	$Q_A = 14$ $Q_B = 18$ $Q_C = 20$	$Q_A = 18$ $Q_B = 20$ $Q_C = 24$	$Q_A = 18$ $Q_B = 18$ $Q_C = 31$	$Q_A = 15$ $Q_B = 17$ $Q_C = 36$	$Q_A = 20$ $Q_B = 21$ $Q_C = 23$
$l_{0,1,1}$	1,000	0.00	0.00	0.00	0.00	0.00
$l_{0,1,2}$	0.00	1,000	1,000	1,000	1,000	1,000
$l_{1,2,1}$	1,000	1,000	1,000	1,000	1,000	0.00
$l_{1,2,2}$	0.00	0.00	0.00	0.00	0.00	1,000
$l_{2,3,1}$	1,000	1,000	1,000	1,000	1,000	1,000
$l_{2,3,2}$	0.00	0.00	0.00	0.00	0.00	0.00
$l_{2,4,1}$	1,000	1,000	1,000	1,000	1,000	1,000
$l_{2,4,2}$	0.00	0.00	0.00	0.00	0.00	0.00
$l_{1,5,1}$	1,000	1,000	1,000	1,000	1,000	1,000
$l_{1,5,2}$	0.00	0.00	0.00	0.00	0.00	0.00
Optimum of Hp	54.077	78.977	112.160	115.441	108.140	99.524
min Z($)	61,897.00	72,374.83	79,675.28	80,396.92	78,790.82	81,895.20

or environmental segments. Based on the physical characteristics of aquifers, they can be categorized into confined and unconfined aquifers. The confined or artesian aquifer is one in which the groundwater is sandwiched between two layers with low permeability and it is under pressure greater than atmospheric. On the other hand, the unconfined aquifers contain a water table instead of impermeable layer above the saturation zone (Fig. 2.14). It is important to note that when a well is drilled into confined aquifers, the groundwater rises above the upper boundary of aquifer and may even flow from the well onto the land surface, while, the water level in wells will be at the same elevation as the water table in an unconfined aquifer.

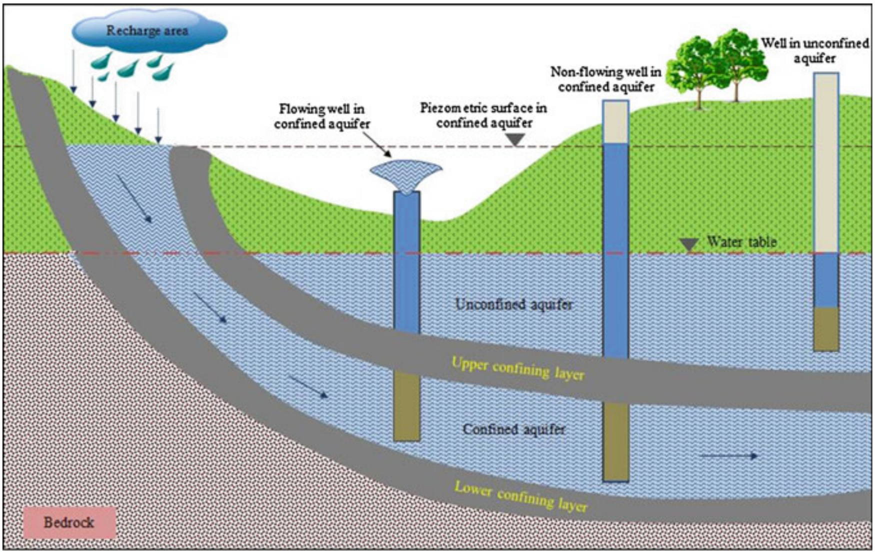


Fig. 2.14 Confined and unconfined aquifers

The general form of a two-dimensional diffusion equation for flow through a heterogeneous anisotropic media considering recharge or discharge (for example from a well) can be written as:

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} = S \frac{\partial h}{\partial t} + W \tag{2.14}$$

where $T(L^2/T)$ is transmissivity, $h(L)$ is hydraulic head, S is the storage coefficient (dimensionless), t is time, and $W(L/T)$ is a sink term. The transmissivity is an important hydraulic property of aquifers which shows the capability of aquifer to transmit water through its whole saturated thickness. In other words, transmissivity can be defined as the rate of water flow through a cross-sectional area of an aquifer with a unit width and thickness b under unit hydraulic gradient. This parameter in a confined aquifer is calculated, as follow:

$$T = K.b \tag{2.15}$$

where, $K(L/T)$ and $b(L)$ are the hydraulic conductivity and the saturated thickness of aquifer respectively. For unconfined aquifers the saturated thickness can be replaced with the hydraulic head h as:

$$T = K.h \tag{2.16}$$

The other variable in Eq. (2.14) is the dimensionless factor storage coefficient or storativity that is defined as volume of water the aquifer will store or release per unit surface area and per unit decrease or increase in hydraulic head. This factor shows the ability of aquifer to store water and can be computed as:

$$S = b.S_s \tag{2.17}$$

where, S_s (L^{-1}) is the specific storage and it is defined as amount of water that an aquifer releases from storage per unit volume of saturated area per unit decline or raise in hydraulic head while remaining fully saturated. It is important to know that the values of storage coefficient in confined aquifers are less than 5×10^{-3} and more than 5×10^{-5} (Todd 1980). For unconfined aquifer, the storage coefficient varies from 0.01 to 0.30. The sink term W is the net discharge (e.g., withdrawal from a well) or recharge (q) from the control volume and is equal to:

$$W_{i,j} = \frac{q_{i,j}}{\Delta x_i \Delta y_j} \tag{2.18}$$

The positive and negative values of q represents pumping and recharge, respectively.

Example 2.5 Consider a confined aquifer with one-dimensional steady-state flow and fixed hydraulic heads along the boundaries, as is shown in Fig. 2.15. Develop an LP model to maximize the hydraulic heads for various pumping rates and determine the optimum head in each well for the following conditions:

1. The minimum value of the total desired discharge (W_{min}) from all wells is equal to 4 ft/day,
2. The minimum value of the desired discharge (W_{min}) from each well is equal to 4 ft/day.

The necessary information to solve this problem are: $W_{min} = 4$ ft/day, $\Delta x = 100$ ft, $T = 10{,}000$ ft^2/day, $h_0 = 125$ ft, $h_4 = 100$ ft.

Solution: The governing equation for the one-dimensional steady-state flow in(only x - direction) considering the pumping wells in confined aquifer can be derived from Eq. (2.14) as follows:

$$T_x \frac{\partial^2 h}{\partial x^2} + \underbrace{T_y \frac{\partial^2 h}{\partial y^2} = S\frac{\partial h}{}}_{\text{This term becomes 0}} + W \Rightarrow \frac{\partial^2 h}{\partial x^2} = \frac{W}{T_x} \tag{2.19}$$

The implementation form of the Eq. (2.19) based on the central finite difference technique has the following form:

$$\frac{h_{i+1} - 2h_i + h_{i-1}}{(\Delta x)^2} = \frac{W_i}{T_x} \tag{2.20}$$

The objective function to maximize the hydraulic heads for various pumping rates can be written as (Aguado et al. 1974):

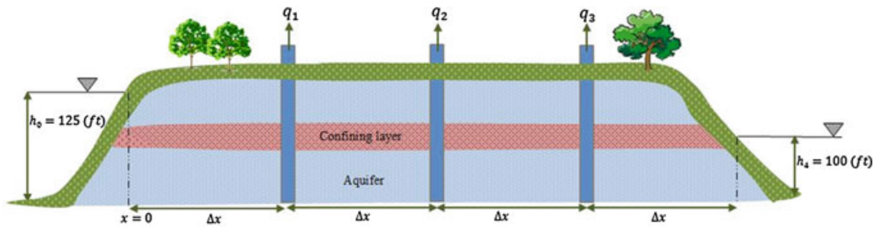


Fig. 2.15 A confined aquifer with one-dimensional steady-state flow and fixed hydraulic heads

$$\max Z = \sum_{i=1}^{n} h_i \tag{2.21}$$

where, n is the number of wells (in this example $n = 3$), and h_i is the hydraulic head in each well. The constraints that should be applied in this problem are:

$$\text{Subject to} \begin{cases} \dfrac{h_{i+1} - 2h_i + h_{i-1}}{(\Delta x)^2} = \dfrac{W_i}{T_x} \\ W_i \geq 0 \quad i = 1 \text{to} n \\ h_i \geq 0 \end{cases} \tag{2.22}$$

It should be noted that to provide a minimum specific pumping rates for all wells together, the following constraint also must be considered:

$$\sum_{i=1}^{n} W_i \geq W_{min} \tag{2.23}$$

in which, W_{min} is the minimum value of the total desired discharge from wells. The following supplementary constraint is useful for finding the optimum hydraulic heads in wells:

$$h_i \geq h_{i+1} \quad i = 0 \text{ to } n \tag{2.24}$$

The developed LP model for having minimum value of 4 (ft/day) discharge from all wells together, can be written as:

$$\max Z = h_1 + h_2 + h_3$$

Subject to the following constraints as:

$$\begin{cases} i = 1 \rightarrow \dfrac{h_2 - 2h_1 + h_0}{(\Delta x)^2} = \dfrac{W_1}{T_x} \\ i = 2 \rightarrow \dfrac{h_3 - 2h_2 + h_1}{(\Delta x)^2} = \dfrac{W_2}{T_x} \\ i = 3 \rightarrow \dfrac{h_4 - 2h_3 + h_2}{(\Delta x)^2} = \dfrac{W_3}{T_x} \end{cases}$$

Which can be summarized as:

$$\begin{cases} i = 1 \rightarrow h_0 = 125 = (2h_1 - h_2) + \left(\dfrac{W_1 \times \Delta x^2}{T_x}\right) \\ i = 2 \rightarrow 0 = (2h_2 - h_1 - h_3) + \left(\dfrac{W_2 \times \Delta x^2}{T_x}\right) \\ i = 3 \rightarrow h_4 = 100 = (2h_3 - h_2) + \left(\dfrac{W_3 \times \Delta x^2}{T_x}\right) \end{cases}$$

and the other constraints are:

$$\begin{aligned} & W_1 + W_2 + W_3 \geq W_{\min} \\ & h_1, h_2, h_3 \geq 0 \\ & W_1, W_2, W_3 \geq 0 \\ & h_0 \geq h_1 \geq h_2 \geq h_3 \geq h_4 \end{aligned}$$

Therefore, the unknowns in this problem are h_1, h_2, h_3 and W_1, W_2, W_3 for all wells completed in the confined aquifer. The essential point here is considering negligible values for well losses and well diameters in this optimization analysis. This problem can be solved simply by using Excel (Data|Solver) and applying the simplex method. The achieved results are presented in the following Table 2.14

The developed LP model for having the minimum value of 4 (ft/day) discharge from each well is almost the same as the previous section except Eq. (2.23) which should be applied as:

$$W_i \geq W_{min}; \quad i = 1 \text{ to } n \tag{2.25}$$

The estimated hydraulic heads and discharge rates for all wells based on the new constraint are presented in Table 2.15.

2.3.3 Optimization of Two-Dimensional Confined Aquifers

The governing equation for steady-state flow in two-dimensional (x and y—directions) considering the pumping wells in the confined aquifer can be derived from Eq. (2.19) as follows:

$$\underbrace{T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2}}_{T_x = T_y = T} = \underbrace{S \frac{\partial h}{\partial t}}_{0} + W \Rightarrow \tfrac{\partial^2 h}{\partial x^2} + \tfrac{\partial^2 h}{\partial y^2} = \tfrac{W}{T} \tag{2.26}$$

The implementation form of Eq. (2.26) using the central finite difference technique can be written as:

Table 2.14 Hydraulic heads and discharge rates with a minimum specific discharge for all wells

Z(ft)	Hydraulic head (ft)					Discharge rate (ft/day)		
	h_0	h_1	h_2	h_3	h_4	W_1	W_2	W_3
331.5	125.0	117.75	110.5	103.25	100.0	0.0	0.0	4.0

Table 2.15 Hydraulic heads and discharge rates with a minimum specific discharge in each well

Z(ft)	Hydraulic head (ft)					Discharge rate (ft/day)		
	h_0	h_1	h_2	h_3	h_4	W_1	W_2	W_3
317.5	125.0	112.75	104.5	100.25	100.0	4.0	4.0	4.0

$$\frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j}}{(\Delta x)^2} + \frac{h_{i,j+1} - 2h_{i,j} + h_{i,j-1}}{(\Delta y)^2} = \frac{W_{i,j}}{T} \tag{2.27}$$

With the assumption of $\Delta x = \Delta y$, Eq. (2.27) can be presented as:

$$h_{i+1,j} - 4h_{i,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} = \frac{(\Delta x)^2}{T} W_{i,j} \tag{2.28}$$

The following problem shows the application of these equations in solving an optimization problem.

Example 2.6 Consider the plan view of steady-state flow in a two-dimensional (x. and y—directions) confined aquifer shown in Fig. 2.16. Develop a LP model to determine the maximum hydraulic heads of wells located at nodes (2, 1), (1, 2), and (2, 3) that are shown as solid red circle, and one well located at any one of nodes (1, 1), (2, 2), and (1, 3) which are shown as hashed circles on the Fig. 2.16. The boundaries (dark hexagon nodes) are considered as fixed hydraulic heads to prevent any drawdown in wells and dewatering of aquifer that can be resulted in aquifer deformation and soil layer compression/consolidation.

The necessary information for this problem are: $W_{min} = 0.5$–2 ft/day, $\Delta x = \Delta y = 500$ ft, $T = 10{,}000$ ft^2/day, and $h_{(0,1)} = h_{(0,2)} = h_{(0,3)} = h_{(1,0)} = h_{(1,4)} = h_{(2,0)} = h_{(2,4)} = h_{(3,1)} = h_{(3,2)} = h_{(3,3)} = 25$ ft.

In this problem, it is assumed that the aquifer is homogeneous, and so, its hydraulic properties are the same at any point of aquifer ($T_x = T_y$). It is good to know that the terms homogeneous and heterogeneous are related to hydraulic conductivity of the aquifer at different locations. If the hydraulic conductivity remains constant, the aquifer is homogeneous, while, the aquifer is heterogeneous (or non-homogeneous), if hydraulic conductivity varies throughout the aquifer.

Solution: The objective function for this problem is (Aguado et al. 1974):

$$\max Z = h_{(1,2)} + h_{(2,1)} + h_{(2,3)} + h_{(1,1)} + h_{(2,2)} + h_{(1,3)}$$

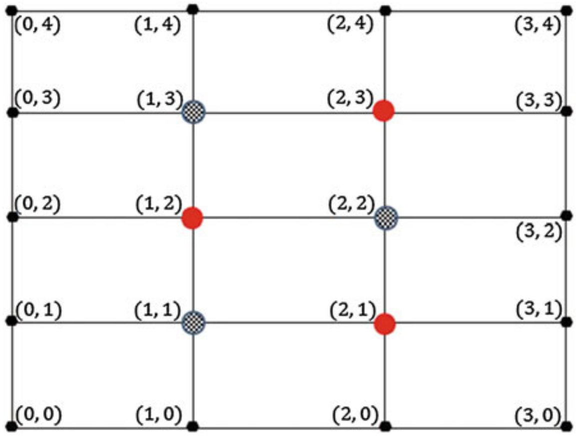


Fig. 2.16 Plan view a two-dimensional confined aquifer

The constraints at desired nodes can be written as
For node (1, 1):

$$h_{2,1} - 4h_{1,1} + h_{0,1} + h_{1,2} + h_{1,0} = \frac{(\Delta x)^2}{T} W_{1,1}$$

$$\Rightarrow 4h_{1,1} - h_{1,2} - h_{2,1} + \frac{(\Delta x)^2}{T} W_{1,1} = h_{0,1} + h_{1,0}$$

For node (2, 1):

$$h_{3,1} - 4h_{2,1} + h_{1,1} + h_{2,2} + h_{2,0} = \frac{(\Delta x)^2}{T} W_{2,1}$$

$$\Rightarrow 4h_{2,1} - h_{1,1} - h_{2,2} + \frac{(\Delta x)^2}{T} W_{2,1} = h_{3,1} + h_{2,0}$$

For node (1, 2):

$$h_{2,2} - 4h_{1,2} + h_{0,2} + h_{1,3} + h_{1,1} = \frac{(\Delta x)^2}{T} W_{1,2}$$

$$\Rightarrow 4h_{1,2} - h_{2,2} - h_{1,3} - h_{1,1} + \frac{(\Delta x)^2}{T} W_{1,2} = h_{0,2}$$

For node (2, 2):

$$h_{3,2} - 4h_{2,2} + h_{1,2} + h_{2,3} + h_{2,1} = \frac{(\Delta x)^2}{T} W_{2,2}$$

$$\Rightarrow 4h_{2,2} - h_{1,2} - h_{2,3} - h_{2,1} + \frac{(\Delta x)^2}{T} W_{2,2} = h_{3,2}$$

For node (1, 3):

$$h_{2,3} - 4h_{1,3} + h_{0,3} + h_{1,4} + h_{1,2} = \frac{(\Delta x)^2}{T} W_{1,3}$$
$$\Rightarrow 4h_{1,3} - h_{1,2} - h_{2,3} + \frac{(\Delta x)^2}{T} W_{1,3} = h_{0,3} + h_{1,4}$$

For node (2, 3):

$$\text{Node } (2,3) \rightarrow h_{3,3} - 4h_{2,3} + h_{1,3} + h_{2,4} + h_{2,2} = \frac{(\Delta x)^2}{T} W_{2,3}$$
$$\Rightarrow 4h_{2,3} - h_{2,2} - h_{1,3} + \frac{(\Delta x)^2}{T} W_{2,3} = h_{3,3} + h_{2,4}$$

These constraints also can be written in the form of matrix as follow:

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & -1 & -1 & 4 \end{bmatrix} \times \begin{bmatrix} h_{1,1} \\ h_{2,1} \\ h_{1,2} \\ h_{2,2} \\ h_{1,3} \\ h_{2,3} \end{bmatrix} + \frac{(\Delta x)^2}{T} \begin{bmatrix} w_{1,1} \\ w_{2,1} \\ w_{1,2} \\ w_{2,2} \\ w_{1,3} \\ w_{2,3} \end{bmatrix}$$
$$= \begin{bmatrix} h_{0,1} + h_{1,0} = 50 \\ h_{3,1} + h_{2,0} = 50 \\ h_{0,2} = 25 \\ h_{3,2} = 25 \\ h_{0,3} + h_{1,4} = 50 \\ h_{3,3} + h_{2,4} = 50 \end{bmatrix}$$

Additional constraints for solving this problem are:

$$\begin{cases} W_{1,1} + W_{2,2} + W_{1,3} \geq W_{min} \\ W_{1,2} \geq W_{min} \\ W_{2,1} \geq W_{min} \\ W_{2,3} \geq W_{min} \\ h_{i,j} \geq 0 \end{cases}$$

The following table shows the hydraulic heads and discharge rates at desired nodes of two-dimensional confined aquifer for different minimum value of the total discharge from internal wells. As it can be seen from the Table 2.16, when the minimum discharge from the well reaches 2 (ft/day) wells cannot meet the requirement, and so, the LP problem is infeasible. To find the optimum value of

Table 2.16 The Hydraulic heads and discharge rates at all internal nodes in various W_{min}

	$W_{min} = 0.5$ (ft/day)	$W_{min} = 1.0$ (ft/day)	$W_{min} = 1.5$ (ft/day)	$W_{min} = 2.0$ (ft/day)
$h_{(1,1)}$	22.16	19.33	16.49	No feasible solution
$h_{(2,1)}$	20.10	15.20	10.30	No feasible solution
$h_{(1,2)}$	18.56	12.11	5.67	No feasible solution
$h_{(2,2)}$	20.73	16.46	12.19	No feasible solution
$h_{(1,3)}$	18.83	12.66	6.49	No feasible solution
$h_{(2,3)}$	19.27	13.53	7.80	No feasible solution
$W_{(1,1)}$	0.00	0.00	0.00	No feasible solution
$W_{(2,1)}$	0.50	1.00	1.50	No feasible solution
$W_{(1,2)}$	0.50	1.00	1.50	No feasible solution
$W_{(2,2)}$	0.00	0.00	0.00	No feasible solution
$W_{(1,3)}$	0.50	1.00	1.50	No feasible solution
$W_{(2,3)}$	0.50	1.00	1.50	No feasible solution
Z	119.64	89.29	58.93	No feasible solution

Table 2.17 Hydraulic heads and discharge rates at all internal nodes in various Δx

	$\Delta x = 45$ (ft)	$\Delta x = 200$ (ft)	$\Delta x = 500$ (ft)
$h_{(1,1)}$	24.90	24.09	19.33
$h_{(2,1)}$	24.91	23.43	15.20
$h_{(1,2)}$	24.89	22.94	12.11
$h_{(2,2)}$	24.93	23.63	16.46
$h_{(1,3)}$	24.95	23.02	12.66
$h_{(2,3)}$	24.92	23.16	13.53
$W_{(1,1)}$	1.00	0.00	0.00
$W_{(2,1)}$	1.00	1.00	1.00
$W_{(1,2)}$	1.00	1.00	1.00
$W_{(2,2)}$	0.00	0.00	0.00
$W_{(1,3)}$	0.00	1.00	1.00
$W_{(2,3)}$	1.00	1.00	1.00
Z	149.50	140.28	89.29

hydraulic heads at all internal nodes, you can simply use Excel (Data|Solver) and choose the simplex method.

In addition to considering the effect of different values of minimum discharge (W_{min}) on the hydraulic heads (discharge from the wells and desired objective function in $\Delta x = 500$ ft), the effect of decreasing Δx on those parameters also are considered in $W_{min} = 1.0$, and $\Delta x = 45$ and 200 ft. The results of this part of example are shown in the Table 2.17.

2.4 Problems

Problem 2.1 Minimize the function f(x) using graphical method.

$$\mathrm{f(x)} = 6x_1 + 3x_2$$

Subject to the following constraints:

$$\begin{aligned} 3x_1 + 2x_2 &\leq 21 \\ x_1 - x_2 &\leq 4.5 \\ x_1 + 2x_2 &\geq 3 \\ 4x_1 + x_2 &\geq 5.5 \\ x_1 \geq 0 \quad \text{and} \quad x_2 &\geq 0 \end{aligned}$$

Problem 2.2 Convert the following LP problem in standard form.

$$\min \mathrm{f} = x_1 + 3x_2 - 7x_3$$

Subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 5 \\ 5x_1 - 4x_2 &\geq -11 \\ x_2 + x_3 &\geq -2 \end{aligned}$$

Problem 2.3 Maximize the following objective function using the simplex method.

$$\max \ \mathrm{f}(x) = x_1 + 8x_2$$

Subject to the below constraints:

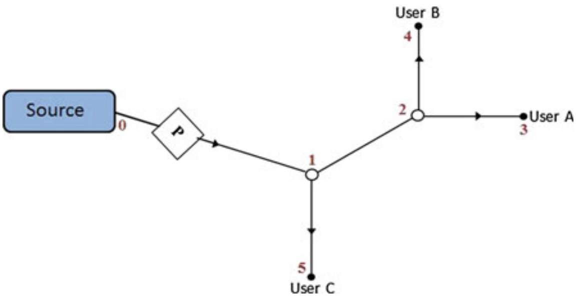
$$\begin{aligned} x_1 - 2x_2 &\leq 11 \\ 2x_1 + 6x_2 &\leq 13 \\ x_1 - x_2 &\geq 6 \\ x_1 \geq 0 \quad \text{and} \quad x_2 &\geq 0 \end{aligned}$$

Problem 2.4 Minimize the following objective function using the simplex method.

$$\min \ \mathrm{f}(x) = 0.35x_1 - x_2 + 2.5x_3$$

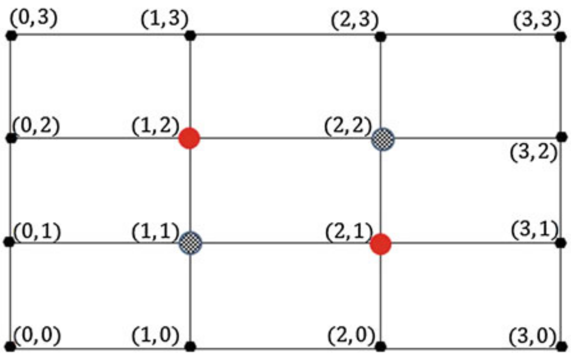
Subject to
$$x_1 - 3x_2 \geq 4$$
$$x_2 + 1.5x_3 \geq 1$$
$$x_1 - x_3 \leq 5$$
$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

Problem 2.5 Determine the minimum cost of pipe and pump (head elevation) in various demand nodes in Example 2.4 by changing the location of pump station from pipe section 1–2 to the pipe section 0–1, and then, compare the optimization results.



Problem 2.6 Determine the optimal pumpage for a confined aquifer with one-dimensional steady-state flow and fixed hydraulic heads along the boundaries in Example 2.5 where $W_{min} = 50$ ft/day, $\Delta x = 25$ ft, $T = 8{,}000$ ft^2/day, $h_0 = 85$ ft, $h_4 = 75$ ft.

Problem 2.7 Consider the plan view of steady-state flow in a two-dimensional (x and y—directions) confined aquifer shown in the following figure. Develop a LP model to determine the maximum hydraulic heads of wells located at nodes (2, 1) and (1, 2), that are shown as solid red circle, and one well located at any one of nodes (1, 1) and (2, 2), which are shown as hashed circles on the figure below. The boundaries (dark hexagon nodes) are considered as fixed hydraulic heads to prevent any drawdown in wells and dewatering of aquifer that can be resulted in aquifer deformation and soil layer compression/consolidation.



The necessary information for this problem are: $W_{min} = 90$ ft/day, $\Delta x = \Delta y = 240$ ft, $T = 10{,}000$ ft^2/day, and $h_{(0,1)} = h_{(1,0)} = h_{(2,0)} = h_{(3,1)} = h_{(0,2)} = h_{(1,3)} = h_{(2,3)} = h_{(3,2)} = 105$ft. In this problem, it is assumed that the aquifer is homogeneous, and so, its hydraulic properties are the same at any point of aquifer ($T_x = T_y$).

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