Chapter 1
Introduction

We will now give an elementary presentation, based on simple examples, of our general approach to the fulfillment of our commitments, exposed in the preface. For references and historical background on the notions introduced here we refer to the heart of the book, and to the end of this introduction for a (non-exhaustive) list of the author's original contributions.

1.1 Variational Calculus and Parametrized Geometry

The notion of field is very general. From the mathematician's viewpoint, it means a function

$$x : M \longrightarrow C$$

between two spaces that represents the motion of a physical object in a configuration space C, with parameter space M. Here the notion of space must be taken in a widely generalized sense, because even electrons have a mathematical formalization as functions between spaces of a non-classical kind, i.e., which are not modeled on subsets of the real affine space $\mathbb{R}^n$.

One may then base classical mechanics on variational calculus, following Lagrange and Hamilton. This is done by using Maupertuis' principle of least action, which says that the true physical trajectories of a system may be chosen among fields as those that minimize (or extremize) a given functional $S(x)$, called the action functional.

Let us first illustrate this very general notion with the simple example of Newtonian mechanics in the plane. Let $M = [0, 1]$ be a time parameter interval, and $C = \mathbb{R}^2$ be the configuration space. In this case, a field is simply a function

$$x : [0, 1] \longrightarrow \mathbb{R}^2$$

that represents the motion of a particle (i.e., a punctual object) in the plane (see Fig. 1.1).

F. Paugam, *Towards the Mathematics of Quantum Field Theory*,
Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern
Surveys in Mathematics 59, DOI 10.1007/978-3-319-04564-1_1,

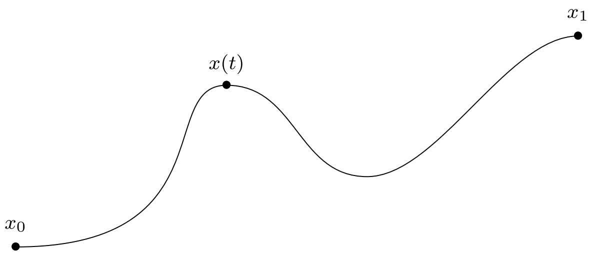


Fig. 1.1 Motion of a particle in the plane

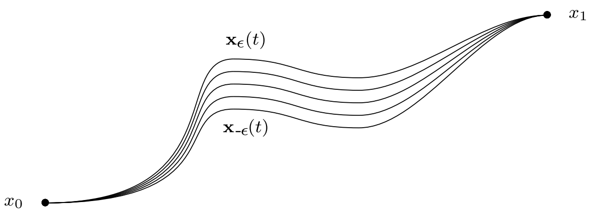


Fig. 1.2 Path in the space of trajectories with fixed end-points

The action functional of the particle in free motion is given by

$$S(x) := \int_M \frac{1}{2} m \left\| \partial_t x(t) \right\|^2 dt,$$

where m is a real number encoding the mass of the particle. We will suppose given the additional datum of some starting and ending points $x_0, x_1 \in C$ for the trajectories, and of the corresponding speed vectors $\mathbf{v}_0$ and $\mathbf{v}_1$.

With this additional datum, one may compute the physical trajectories by computing the infinitesimal variation of the action functional S along a small path

$$\begin{aligned} \mathbf{x} :]-\epsilon, \epsilon[\times [0,1] &\to \mathbb{R}^2 \\ (\lambda, t) &\mapsto \mathbf{x}_\lambda(t) \end{aligned}$$

in the space of fields at a field $x = \mathbf{x}_0$, with fixed starting and ending points (see Fig. 1.2), by the formula

$$\frac{\delta S}{\delta \mathbf{x}}(x) := \partial_\lambda S(\mathbf{x}_\lambda)|_{\lambda=0}.$$

By applying an integration by parts, one gets

$$\frac{\delta S}{\delta \mathbf{x}}(x) = \int_M \left\langle -m\partial_t^2 x, \partial_\lambda \mathbf{x} \right\rangle dt,$$

and this expression is zero for all variations $\mathbf{x}$ of x if and only if Newton's law for the motion of a free particle

$$m\partial_t^2 x = 0$$

is satisfied. This equation may be solved using the given initial condition $(x_0, \mathbf{v}_0)$, or the given final condition $(x_1, \mathbf{v}_1)$. In both cases, the model tells us, with no surprise,

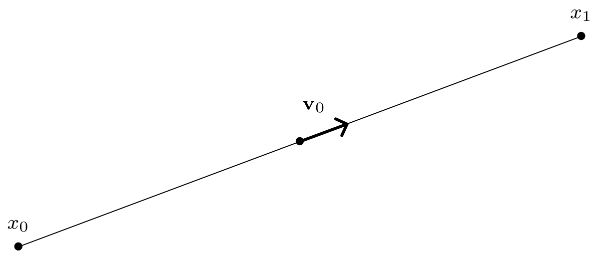


Fig. 1.3 The free particle moving on a *straight line* at constant speed

that the free particle is simply moving on a straight line at constant speed, starting at the given point, with the given fixed speed (see Fig. 1.3). We may identify the space of trajectories with the space of pairs $(x_0, \mathbf{v}_0)$ forming the initial conditions for the corresponding differential equation. This space of pairs is the cotangent space

$$T^*C \cong \mathbb{R}^2 \times \mathbb{R}^2$$

of the configuration space.

We remark that the above computation of the equations of motion for the given action functional does not really involve the datum of a topology on the space H of fields with fixed starting and ending point. One only needs to know the value of the action $S : H \to \mathbb{R}$ on a family $\{\mathbf{x}_\lambda\}$ of fields parametrized by the small interval $]-\epsilon, \epsilon[$. One may even easily define the value of S on such a family, whenever λ is a point in an open subset U of $\mathbb{R}^n$ for some n. The awkward idea of replacing the simple parameter interval $U =]-\epsilon, \epsilon[$ by any such open set will allow us to think of the space H of fields as a space of the same nature as the spaces M and C of parameters and configurations. For $U \subset \mathbb{R}^n$, let us denote by $M(U)$, $C(U)$ and $\mathbb{R}(U)$ the sets of smooth maps from U to M, C and $\mathbb{R}$. One may think of these sets as sets of points of M, C and $\mathbb{R}$ parametrized by U. Let us also define the space of histories by setting

$$\begin{aligned} H(U) := \Big\{ & x : U \times M \to C \text{ smooth, such that} \\ & x(u, 0) = x_0 \text{ and } x(u, 1) = x_1 \text{ for all } u \in U \Big\}. \end{aligned}$$

These are points of the space of fields parametrized by U, with given fixed starting and ending points. We give an example with parameter λ in the open unit disc $U = D(0, 1) \subset \mathbb{R}^2$ in Fig. 1.4. Note that these sets are all compatible with smooth changes of parameters $V \to U$, because of the stability of smoothness by composition. One may then define a smooth functional

$$S : H \to \mathbb{R}$$

as the datum of a family of set functions

$$S_U : H(U) \to \mathbb{R}(U),$$

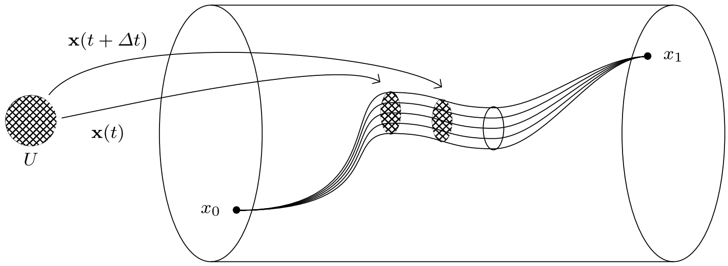


Fig. 1.4 A point in the space of trajectories parametrized by the unit disc

which are compatible with smooth changes of parameters. In modern mathematical language, M, C, H and $\mathbb{R}$ defined as above are functors from the category of smooth open subsets in real affine spaces to sets, and S is a natural transformation.

The main advantage of the above interpretation of variational calculus is that it allows us to work with a much more general notion of space, and these generalized kinds of spaces are necessary to understand the computations of quantum field theory from a completely geometrical viewpoint, generalizing what we explained above. One may call this geometry

parametrized geometry.

The combination of parametrized geometry with the usual differential geometric tools, such as manifolds, involves a refinement of the functorial notion of parametrized space by the notion of a sheaf, which allows us to define a smooth manifold as the parametrized space obtained by pasting some basic open subset $U \subset \mathbb{R}^n$, viewed as basic parametrized spaces, along smooth maps.

1.2 Motion with Interactions and the Geometry of Differential Equations

We continue to consider the motion of a Newtonian particle in the plane, given by a map

$$x : M \to C,$$

with $M = [0, 1]$ and $C = \mathbb{R}^2$, however we now suppose that it is not moving freely, but is interacting with its environment, as described in Fig. 1.5. Since Newton, the mathematical formalization of this environment has been given by a function

$$V : C \to \mathbb{R}$$

called the potential energy density. The associated force field is given by the gradient vector field

$$\mathbf{F} = -\overrightarrow{\mathrm{grad}}(V).$$

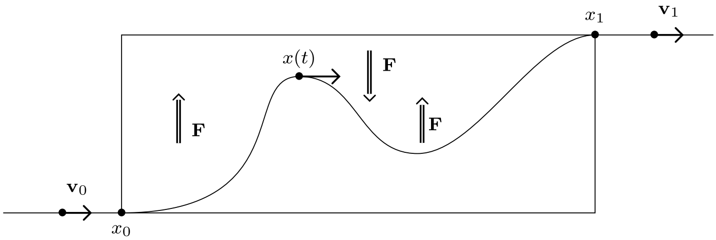


Fig. 1.5 Motion of a particle in a potential

The action functional for this interacting particle is given by a functional

$$S(x) := \int_M L\big(x(t), \partial_t x(t)\big)dt,$$

where the function

$$L(x, \partial_t x) := \frac{1}{2}m\|\partial_t x\|^2 + V(x)$$

is called the Lagrangian density. Functionals S of the above form are called local functionals. The associated equations of motion are again given by Newton's law

$$m\partial_t^2 x = \mathbf{F}(x).$$

As before, we suppose that the initial and final conditions $(x_0, \mathbf{v}_0)$ and $(x_1, \mathbf{v}_1)$ are fixed. We may split the action functional $S = S_{free} + S_{int}$ into a sum of the free (kinetic) action functional

$$S_{free}(x) = \int_M \frac{1}{2}m\left\|\partial_t x(t)\right\|^2 dt,$$

whose equations of motion were already considered in the previous section, and the interaction functional

$$S_{int}(x) = \int_M V\big(x(t)\big)dt.$$

We may then represent the motion of our particle as being free before and after the experiment, and being influenced by the potential within the experiment's spacetime box.

Following Lagrange, it will be convenient, along the way, to develop a more algebraic version of variational calculus corresponding to a differential calculus for local functionals.

In the above example, one simply considers the manifold $\mathrm{Jet}(M, C)$ with coordinates (t, x_i) given by formal derivatives of the variables x in $C = \mathbb{R}^2$, indexed by integers $i \in \mathbb{N}$. One may view the Lagrangian density as a function $L : \mathrm{Jet}(M, C) \to \mathbb{R}$

defined by

$$L(t, x_0, x_1) := \frac{1}{2}m\|x_1\|^2 + V(x_0).$$

The only additional datum needed to formulate variational calculus in purely algebraic terms is the differential relation between the family of formal coordinates x_i, given by

$$\partial_t x_i = x_{i+1}.$$

This universal differential relation is encoded in a connection ∇ on the bundle $\mathrm{Jet}(M, C) \to M$, called the Cartan connection. As the action of ∂_t on the jet bundle functions is given by the total derivative operator $\frac{d}{dt}$, the differential calculus underlying variational calculus is completely encoded by the above differential structure on the jet bundle.

There is also a parametrized geometry underlying this differential calculus that is morally a geometry of formal solution spaces to non-linear partial differential equations, called $\mathcal{D}$-geometry. Its parameter spaces are essentially given by jet bundles $(\mathrm{Jet}(U, U \times V), \nabla)$ of morphisms $\pi : U \times V \to U$ (for $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ open subsets), together with their Cartan connection. This geometry of differential equations is often used by physicists to study classical and quantum systems, and in particular their local symmetries and conservation laws.

1.3 Measurable Densities and Functorial Analysis

We continue to study the motion of a particle $x : [0, 1] \to \mathbb{R}^2$ in the plane, but we shall change the action functional S by replacing it with a parametrization-invariant version, given by the formula

$$S(x) := \int_M \frac{1}{2}m\left\|\partial_t x(t)\right\|dt$$

which involves the norm, i.e., the square root of the metric on $\mathbb{R}^2$. From the viewpoint of parametrized geometry, this expression makes sense when we replace the trajectory x by a smooth family

$$x : [0, 1] \times U \to \mathbb{R}^2$$

parametrized by an arbitrary open set $U \subset \mathbb{R}^n$. However, since the square root is a non-smooth function at 0, the function

$$\begin{aligned} U &\longrightarrow \mathbb{R} \\ \lambda &\longmapsto S(x_\lambda) \end{aligned}$$

will not be smooth in the parameter in general if the particle has zero speed at some point of the trajectory. We thus have to restrict the domain of definition of the functional S to a parametrized subspace $D_S \subset H$, called its domain of definition.

More generally, we could take any Lebesgue integrable density $L : [0, 1] \times T\mathbb{R}^2 \to \mathbb{R}$ and force the formula

$$S(x_\lambda) := \int_M L\big(t, x_\lambda(t), \partial_t x_\lambda(t)\big)dt$$

to define a smooth function of the auxiliary parameter $\lambda \in U$. The domain of definition for this functional is easily computed using Lebesgue's domination criterion for the derivative of an integral with parameters. It is given by the subspace $D_S \subset H$ with parametrized points

$$D_S(U) := \left\{x_\lambda |\text{ locally on } U, \exists g_i \in L^1\big([0, 1]\big),\ \left|\partial^i_\lambda L\big(t, x_\lambda(t), \partial_t x_\lambda(t)\big)\right| \leq g_i(t)\right\}.$$

The above discussion shows that it is necessary to generalize the notion of functional introduced in parametrized geometry by authorizing partially defined functions, thus obtaining more flexibility and a better compatibility with the usual tools of functional analysis. This extension of functional analysis is called

functorial analysis.

1.4 Differential Calculus and Functional Geometry

It was already remarked by Weil that one may formulate differential calculus in an algebraic way, by using algebras with nilpotent elements, such as $\mathcal{C}^\infty(\mathbb{R})/(\epsilon^2)$ for example, where $\epsilon = \mathrm{id} : \mathbb{R} \to \mathbb{R}$ is the standard coordinate function on $\mathbb{R}$. One may think of this algebra as the algebra of coordinate functions on an infinitesimal thickening $\vec{\bullet}$ of the point $\bullet$, defined by adding a universal tangent vector to it. Indeed, if M is a manifold, the choice of a tangent vector $\mathbf{v}$ to M at a point $m \in M$ is equivalent to the choice of a derivation

$$D : \mathcal{C}^\infty(M) \to \mathbb{R}$$

compatible with the evaluation algebra morphism

$$\begin{aligned} \mathrm{ev}_m : \mathcal{C}^\infty(M) &\to \mathbb{R} \\ f &\mapsto f(m), \end{aligned}$$

meaning that for all $f, g \in \mathcal{C}^\infty(M)$, one has

$$D(fg) = f(m)D(g) + D(f)g(m).$$

The datum of such a derivation is also equivalent to that of the algebra morphism

$$\begin{aligned} h_D : \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(\mathbb{R})/\big(\epsilon^2\big) \\ f &\mapsto f(m) + \epsilon D(f). \end{aligned}$$

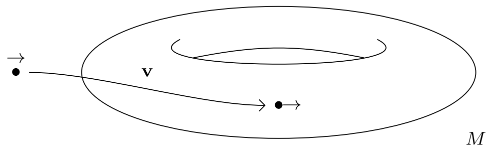


Fig. 1.6 Infinitesimally parametrized points are tangent vectors

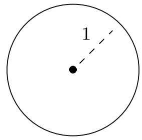


Fig. 1.7 The unit circle

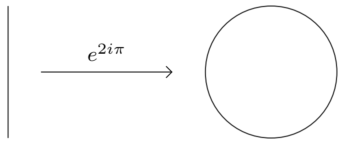


Fig. 1.8 The exponential map

We may interpret this algebra morphism as a morphism $\mathbf{v} : \vec{\bullet} \to M$, i.e., as a point of M together with a tangent vector on it, as described in Fig. 1.6.

This last interpretation of tangent vectors admits a very general formulation which works essentially without change in all of the parametrized settings discussed in this introduction.

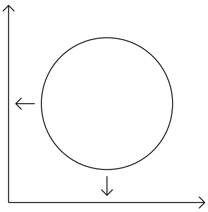
To make this general formulation work, say, in the smooth setting, one first has to replace the parameters $U \subset \mathbb{R}^n$ by their algebras $\mathcal{C}^\infty(U)$ of functions, which makes no essential difference because of the equivalence between the datum of U and the datum of $\mathcal{C}^\infty(U)$.

Then, one needs to allow more general algebras as parameters. For example, we would like to consider the nilpotent algebra $\mathcal{C}^\infty(\mathbb{R})/(\epsilon^2)$ as an admissible parameter. More generally, algebras of formal solutions to equations, such as $\mathcal{C}^\infty(U)/(f_1, \ldots, f_n)$, may also be useful. Note that using such algebras is also a very natural thing to do in classical differential geometry, where one is interested in varieties defined by equations. For example, the algebra of smooth functions on the real circle S^1, pictured in Fig. 1.7, may be easily defined as the quotient algebra

$$\mathcal{C}^\infty\big(S^1\big) := \mathcal{C}^\infty(\mathbb{C})/\big(|z|^2 - 1\big) \cong \mathcal{C}^\infty\big(\mathbb{R}^2\big)/\big(x^2 + y^2 - 1\big).$$

One may use the exponential map $e^{2i\pi} : \mathbb{R} \to S^1$, drawn in Fig. 1.8, or the implicit function theorem, pictured in Fig. 1.9, to define S^1 as a smooth manifold. However it is less comfortable to work with charts here than to work directly with the above algebra. Indeed, $S^1 \subset \mathbb{C}$ is naturally defined by the closed algebraic condition $|z|^2 = 1$, and most of its properties may be actually derived from the immediate study of this equation. However, if we think of such quotients as the usual real algebras, given by sets, together with compatible multiplication and addition operations,

Fig. 1.9 The implicit function theorem



we run into the following difficulty: the natural product map

$$\mathcal{C}^\infty(U) \otimes \mathcal{C}^\infty(V) \longrightarrow \mathcal{C}^\infty(U \times V)$$

is not an isomorphism because some functions on the right may only be described as infinite converging series of tensors.

This problem may be resolved by using a completed topological tensor product. However, this also has the drawback of introducing unnatural complications, which, furthermore, are incompatible with generalizations to the other kinds of parametrized geometries discussed in this introduction. There is however a very elegant solution to this problem of completing the category of algebras of the form $\mathcal{C}^\infty(U)$, for $U \subset \mathbb{R}^n$, in a way that forces the product maps

$$\mathcal{C}^\infty(U) \otimes \mathcal{C}^\infty(V) \longrightarrow \mathcal{C}^\infty(U \times V)$$

to be isomorphisms. One does that by observing that the functor

$$\underline{\mathcal{C}}^\infty(U) : \mathbb{R}^m \supset V \mapsto \mathcal{C}^\infty(U, V)$$

of smooth functions on U with values in open sets of arbitrary real affine spaces commutes with transversal fiber products of open sets, i.e., satisfies

$$\underline{\mathcal{C}}^\infty\Big(U, V \overset{t}{\underset{W}{\times}} V'\Big) \cong \underline{\mathcal{C}}^\infty(U, V) \underset{\underline{\mathcal{C}}^\infty(U,W)}{\times} \underline{\mathcal{C}}^\infty\big(U, V'\big).$$

One then defines a more general smooth algebra to be a set-valued functor

$$\mathbb{R}^m \supset V \mapsto A(V)$$

satisfying the above condition of commutation with transversal fiber products. This process of generalizing spaces by generalizing functions is called

functional geometry.

When conveniently combined with parametrized geometry, it furnishes a workable definition of a vector field on a space of fields and, more generally, of differential calculus on parametrized spaces.

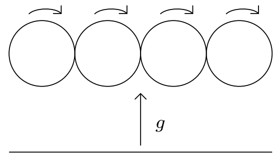


Fig. 1.10 Gauge symmetries with values in S^1

1.5 Motion with Symmetries and Homotopical Geometry

Before further discussing the quantization problem, we have to deal with systems that have some given gauge symmetries, which cause the Cauchy problem for the equations of motion to be ill-posed. To obtain a well defined notion of quantization for such a system, one needs a canonical way of breaking the symmetry, in order to retrieve a well-posed Cauchy problem with nice initial conditions. This method is usually called the gauge fixing procedure.

Here is how things appear in Yang-Mills gauge theory, which is at the heart of modern particle physics. Let M be a manifold (such as the affine space $\mathbb{R}^2$ we used before) equipped with a metric, and let G be a compact Lie group with Lie algebra $\mathfrak{g}$, equipped with an invariant pairing. Let P be a principal G-bundle over M. It is given by a (locally trivial) fiber bundle $P \to M$ together with an action of the group bundle $G_M := G \times M$ that is simply transitive, giving locally an isomorphism $G_M \cong P$.

A principal G-connection on P is morally given by a G-equivariant parallel transport of sections of P along infinitesimally closed points in M. This may also be formalized as a G-invariant differential 1-form

$$A \in \Omega^1(P, \mathfrak{g})^G$$

on P satisfying an additional non-degeneracy condition. The corresponding action functional is then given by

$$S(A) := \int_M \frac{1}{4} \langle F_A \wedge *F_A \rangle,$$

where

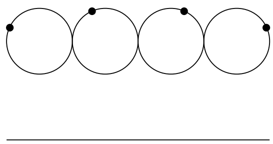
$$F_A := dA + [A \wedge A] \in \Omega^2(P, \mathfrak{g})^G$$

is the curvature of the connection and $*$ is the Hodge $*$-operator on differential forms associated to the given metric g on M. The solutions of the corresponding equations of motion on a compact manifold M contain flat connections.

Let us consider the situation $M = \mathbb{R}$ and P a principal S^1-bundle. The gauge symmetries, pictured in Fig. 1.10, are then given by functions $g : M \to S^1$ and a principal connection A on the trivial bundle $P = S^1 \times \mathbb{R}$ is given by a parallel transport along $\mathbb{R}$ for points in the circle fibers S^1 of P, as described in Fig. 1.11.

There is no reason to choose a particular principal bundle P, and one may also think of the action functional as a function $S(P, A)$ on isomorphism classes of principal bundles with connection. Homotopical geometry is a setting where one may

Fig. 1.11 Parallel transport along a principal connection



naturally interpret the bundle P as a field, i.e., a function

$$P : M \longrightarrow C = BG,$$

where BG is the classifying space of principal G-bundles. Bundles with connections also have a classifying space BG_{conn}, so that pairs (P, A) composed of a principal bundle and a principal G-connection may also be regarded as fields

$$(P, A) : M \longrightarrow C = BG_{conn},$$

with values in this classifying space. It is not possible to define spaces such as BG in the classical parametrized setting: one must enhance the sets of parametrized points to groupoids. Moreover, if we want to work with higher principal bundles (also called gerbes), associated to cohomology classes in $H^n(X, G)$ (where G is now assumed to be a commutative Lie group, like S^1 above), we also need to work with higher groupoids. This general enhancement of equivariant geometry under group actions is given by the setting of homotopical geometry. The space of fields of Yang-Mills theory is given by the space $\underline{\mathrm{Hom}}(M, BG_{conn})$ whose points parametrized by an open subset $U \subset \mathbb{R}^n$ are given by the groupoid of isomorphism classes of pairs (P, A) parametrized by U.

The advantage of formulating gauge theory in this setting is that it allows a clear treatment of gauge symmetries, which are manifest in the above definition of Yang-Mills theory. It also gives a more geometric intuition of the obstructions to quantizing a system with symmetries, called anomalies in physics, and (differential) group and Lie algebra cohomology in mathematics.

1.6 Deformation Theory and Derived Geometry

Many problems of physics can be formulated as deformation problems. As we already explained when we discussed functional geometry, differential calculus itself can be formulated in terms of infinitesimal thickenings and infinitesimal deformations of points, corresponding to tangent vectors. We have also seen that the geometrical formulation of theories with symmetries, like Yang-Mills theory, involve the use of an equivariant geometry, called homotopical geometry. The definition of differential invariants of equivariant spaces also involves the use of derived geometrical parameters. Perturbative quantization, as we will see, is also a deformation problem from classical field theory to quantum field theory.

Derived geometry is a parametrized geometry, where one replaces the usual open parameters $U \subset \mathbb{R}^n$ by some homotopical generalizations $\widetilde{U} \to \mathbb{R}^n$, obtained

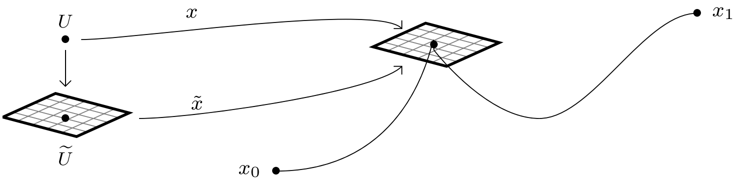


Fig. 1.12 Trajectories parametrized by homotopical coordinates

Fig. 1.13 The singular cross

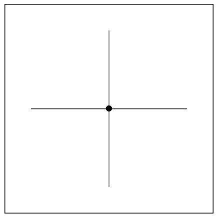
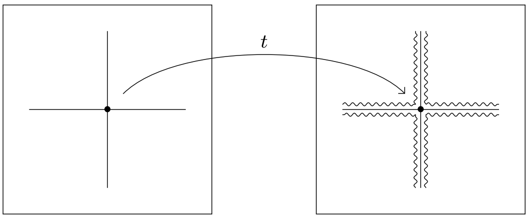


Fig. 1.14 First order infinitesimal thickening of the singular cross



by adding to the real coordinates $u \in U$ higher homotopical coordinates $\tilde{u} \in \widetilde{U}$, as pictured in Fig. 1.12. The coordinate algebra on $\tilde{U}$ is a simplicial smooth algebra $\mathcal{C}^\infty(\tilde{U})$ such that $\pi_0(\mathcal{C}^\infty(\tilde{U})) = \mathcal{C}^\infty(U)$. These parameters are used to measure obstructions to deformations, i.e., non-smoothness properties. To be precise, coordinate functions on these homotopical parameter spaces are obtained by generalizing smooth algebras (used to formalize infinitesimal calculus in functional geometry). This generalization is obtained by considering functors $\mathbb{R}^m \supset U \mapsto A(U)$ on open subsets of arbitrary real affine spaces, with values in simplicial sets, and commuting (homotopically) with transversal fiber products of open sets.

To illustrate the general relation between deformation theory and derived geometry in simple terms, let us consider the singular cross, drawn in Fig. 1.13, given by the union X of the two main axes in the plane, with polynomial equation $xy = 0$ in $\mathbb{R}[x, y]$. This equation is singular at the origin. This singularity can be explained by the fact that the coordinate $\mathbb{R}$-algebra $\mathbb{R}[x, y]/(xy)$ on X is not smooth, because the universal point $\mathrm{id} : X \to X$ of X cannot be deformed along the infinitesimal thickening $t : X \to T$, with coordinate algebra $\mathbb{R}[x, y]/(xy)^2$, pictured in Fig. 1.14. Such a deformation is defined as a factorization of the map $\mathrm{id} : X \to X$ as a sequence

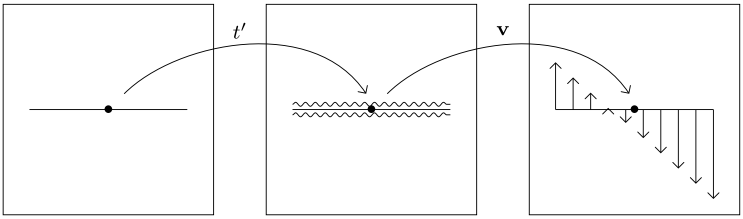


Fig. 1.15 A global vector field as an infinitesimally parametrized point

$X \xrightarrow{t} T \to X$, which would correspond to a factorization

$$\begin{tikzcd} \mathbb{R}[x,y]/(xy) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \mathbb{R}[x,y]/(xy)^2 \arrow[r, "t^*"] & \mathbb{R}[x,y]/(xy) \end{tikzcd}$$

at the level of coordinate algebras. But such a factorization does not exist. The obstruction to this existence problem lies in the cohomology group $\mathrm{Ext}^1(\mathbb{L}_X, \mathcal{I})$, where $\mathcal{I} = (xy)$ is the square zero ideal in $\mathbb{R}[x,y]/(xy)^2$, equipped with its canonical module structure over $\mathcal{O}_X = \mathbb{R}[x,y]/(xy)$, and $\mathbb{L}_X$ is a differential graded module over $\mathcal{O}_X$ called the cotangent complex.

In the case of the simpler equation $y = 0$ of the horizontal axis L, we don't have this problem, because the choice of a deformation of the universal point of L along its order two thickening $t' : L \to T'$ is given by a factorization

$$\begin{tikzcd} \mathbb{R}[x,y]/(y) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \mathbb{R}[x,y]/(y)^2 \arrow[r, "(t')^*"] & \mathbb{R}[x,y]/(y), \end{tikzcd}$$

corresponding to the choice of a global vector field on L (drawn in Fig. 1.15 with a rotation of 90 degrees), given by a morphism $\mathbf{v} : x \mapsto x + \mathbf{v}(x).y$, with $\mathbf{v}(x) \in \mathbb{R}[x]$. The cotangent complex $\mathbb{L}_L$ of this smooth space is concentrated in degree zero, and is equal there to the module of differential forms Ω^1_L. The space of deformations of the identity point $\mathrm{id} : L \to L$ along the thickening $L \to T'$ is then given by the space

$$\mathrm{Hom}_L\big(T', L\big) \cong \mathrm{Ext}^0(\mathbb{L}_L, \mathcal{O}_L)$$

of sections of the thickening map $t' : L \to T'$. So in the smooth case, one can interpret deformations of points in a given space in terms of degree zero cohomology of the cotangent complex.

The aim of derived geometry is also to interpret the space of obstruction classes $[c] \in \mathrm{Ext}^1(\mathbb{L}_X, \mathcal{I})$ to the deformation of the singular cross X as a space of infinitesimal extensions

$$\mathrm{Hom}_L(T, X) \cong \mathrm{Ext}^1(\mathbb{L}_X, \mathcal{I}_X).$$

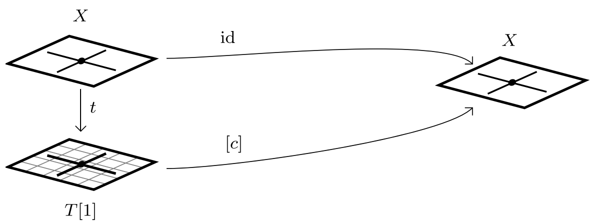


Fig. 1.16 Obstruction to deformations as homotopically parametrized points

One may also think of it as a way to define canonical formal (or more precisely, homotopical) tubular neighborhoods of singular subspaces. This can only be done by using an infinitesimal extension $t : X \to T[1] = X(\mathcal{I}[1])$, pictured in Fig. 1.16, with coordinate ring the nilpotent extension

$$\mathrm{Sym}_{\mathcal{O}_X}\big(\mathcal{I}[1]\big) := \mathbb{R}[x, y]/(xy) \oplus (xy)/(xy)^2[1],$$

with cohomological infinitesimal variable in a module of cohomological degree -1, and zero differential.

Another important motivation for working with derived geometrical spaces is given by the fact that many spaces considered in physics are given by non-transverse intersections of subspaces $L_1, L_2 \subset X$ in some given space X. For example, the space of trajectories for Yang-Mills theory is given by the intersection

$$\mathrm{Im}(dS) \cap H \subset T^*H$$

of the image of the differential of the action functional with the zero section inside the cotangent bundle T^*H. Morally, this intersection is non-transverse, and one would like to take a small generic perturbation of $\mathrm{Im}(dS)$ (by which we mean a kind of formal tubular neighborhood of this subspace inside T^*H) to retrieve a transverse intersection.

Let us illustrate this idea of a small generic perturbation in a simpler setting. Let us consider the intersection, in the plane, of the horizontal axis $y = 0$ and the line $y = \lambda x$ of slope $\lambda \geq 0$. If we vary $\lambda > 0$ the line rotates, as pictured in Fig. 1.17, but the intersection remains the origin, and the dimension of the intersection is zero there, because the coordinate ring on an intersection is given by the tensor product

$$\mathbb{R}[x, y]/(y, y - \lambda x) = \mathbb{R}[x, y]/(y) \otimes_{\mathbb{R}[x,y]} \mathbb{R}[x, y]/(y - \lambda x),$$

and is isomorphic to $\mathbb{R}$, i.e., to the coordinate algebra of functions on a point. We may also compute this dimension of the point by differential methods: its cotangent space is also of dimension zero. However, if λ tends to zero, the two lines coincide, and their intersection is no longer transversal. The intersection is then a line, not a point, and the intersection dimension is 1. We thus have bad permanence properties for the intersection dimension in parametrized families. But the derived tensor product

$$\mathbb{R}[x, y]/(y) \otimes^{\mathbb{L}}_{\mathbb{R}[x,y]} \mathbb{R}[x, y]/(y)$$

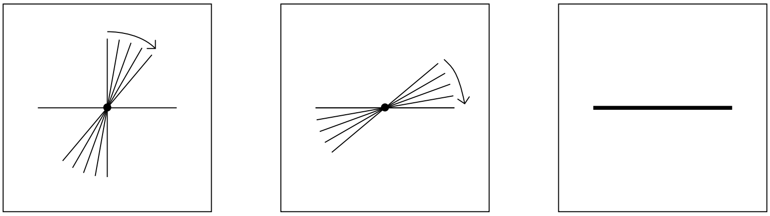


Fig. 1.17 Generic deformation of a line

may be thought of as a coordinate algebra for the derived intersection of the line with itself. This tensor product is computed by replacing one of the two sides of the tensor product by the differential graded $\mathbb{R}[x, y]$-algebra

$$\mathrm{Sym}_{dg-\mathbb{R}[x,y]}\big(\mathbb{R}[x, y][1] \xrightarrow{\cdot y} \mathbb{R}[x, y]\big)$$

that is a cofibrant resolution of the algebra $\mathbb{R}[x, y]/(y)$ (we may omit the symmetric algebra operation, because the corresponding differential graded module is already canonically a differential graded algebra). This results in a differential graded ring over $\mathbb{R}[x, y]/(y)$ that has cohomology only in degrees 0 and 1. The cotangent complex of this space, i.e., its complex of differential forms, has cohomology of dimension 1 in degree zero, and 1 in degree 1. The Euler characteristic of its cohomology thus returns a derived intersection of virtual dimension 0 at the origin. One may think of this derived intersection as a kind of generic perturbation of the non-transverse intersection, obtained by moving one of the lines back to a line of non-zero slope.

So we will keep in mind that the optimal notion of formal generic perturbation, which has good permanence properties in families, and which makes all reasonable intersections transverse, is given by the derived intersection. In the Yang-Mills setting, this corresponds to a derived mapping space

$$\mathbb{R}\underline{\mathrm{Hom}}(M, BG_{flat})$$

encoding derived principal bundles with flat connections. As mentioned earlier, its parameters are given by homotopical infinitesimal extensions $U \to \tilde{U}$ of open subsets $U \to \mathbb{R}^n$ that encode obstructions to deforming the field variables (P, A). One may show that this derived space is naturally equipped with a local symplectic structure, making it an analog of the space T^*C of initial conditions for the free particle mechanics, which has the advantage of keeping the gauge symmetries of the problem manifest.

The definition of a well-posed Cauchy problem for the equations of motion is usually given by physicists through the additional choice of a so-called Lagrangian subspace L in the (graded) derived critical space, called the gauge fixing condition.

Physicists also give another way to define a gauge fixed Cauchy problem, based on a generalization of Noether's theorem, which roughly states that the relations between the equations of motion, called Noether identities, are in one to one correspondence with local symmetries of the system. This general method, called the Batalin-Vilkovisky formalism, essentially yields, in the Yang-Mills case, the same result as the above alluded to derived critical space of the action functional on the stacky space of gauge field histories (P, A).

1.7 Fermionic Particles and Super-Geometry

One way of mathematically interpreting Dirac's quantization of the electron is to consider that this quantum particle has a classical counterpart whose time parameter anticommutes with itself. Let us first explain how one may give a mathematical description of such a strange object.

We have already roughly defined the notion of space parametrized by smooth open subsets of affine spaces. Since such an open subset $U \subset \mathbb{R}^n$ is determined, up to smooth isomorphism, by its algebra of real valued smooth functions $\mathcal{C}^\infty(U)$, we may define a more general notion of parametrized geometry by using more general types of algebras. Let us consider Grassmann algebras of the form $\mathcal{C}^\infty(U, \wedge^*\mathbb{R}^m)$ for $U \subset \mathbb{R}^n$ smooth. These associative algebras are not commutative but fulfill a super-commutation rule, given by

$$fg = (-1)^{|f|.|g|} gf,$$

where $|f|$ and $|g|$ denote the degree in the anticommuting variable. We will think of these algebras as coordinates $\mathcal{C}^\infty(U^{n|m})$ on some spaces $U^{n|m}$, these being parameters for a new type of geometry called parametrized super-geometry.

This geometry generalizes parametrized smooth geometry in the following sense: we may extend the parametrized spaces $M = [0, 1]$, $C = \mathbb{R}^2$ and H to the category of super-opens $U^{n|m}$ by setting, for example

$$\mathbb{R}^2\big(U^{n|m}\big) = \mathbb{R}^{2|0}\big(U^{n|m}\big) := \mathcal{C}^\infty\big(U, \wedge^{2*}\mathbb{R}^m\big)^2,$$

and more generally

$$M\big(U^{n|m}\big) := \mathrm{Hom}_{\mathrm{ALG}}\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty\big(U^{n|m}\big)\big).$$

We shall not explain here why the above extensions are natural, but will illustrate these new spaces by describing the classical counterpart of the fermionic particle.

The parameter space for the electronic particle is the odd line $M = \mathbb{R}^{0|1}$, with coordinate functions the polynomial algebra $\mathbb{R}[\theta]$ in an anticommuting parameter. So we consider the space of maps

$$x : \mathbb{R}^{0|1} \to C,$$

where $C = \mathbb{R}^2$ is the plane, as before. The natural inclusion $\{*\} \subset \mathbb{R}^{0|1}$ of the point $\{*\} = \mathbb{R}^{0|0}$ into the odd line induces a bijection of sets

$$\mathrm{Hom}\big(\mathbb{R}^{0|1}, C\big) \xrightarrow{\sim} \mathrm{Hom}\big(\{*\}, C\big) = C,$$

so that the electronic particle looks like an inert object, because its trajectories do not depend on the parameter θ.

However, working with the parametrized super-space $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, C)$ of maps from $\mathbb{R}^{0|1}$ to C, whose points parametrized by $U^{n|m}$ are defined by

$$\underline{\mathrm{Hom}}\big(\mathbb{R}^{0|1}, C\big)\big(U^{n|m}\big) := \mathrm{Hom}\big(\mathbb{R}^{0|1} \times U^{n|m}, C\big),$$

one may easily prove that this space is a non-trivial super-space, with algebra of coordinates the superalgebra

$$\Omega^*(C) = \mathcal{C}^\infty\big(\mathbb{R}^2, \wedge^*\big(\mathbb{R}^2\big)^*\big)$$

of differential forms on $C = \mathbb{R}^2$. This superspace is called the odd tangent bundle of C, and denoted by $T[1]C$. The given fixed metric on C identifies it with the odd cotangent bundle of C, denoted by $T^*[1]C$, whose algebra of coordinates is the algebra of multivector fields. We will also think of it as the space of initial conditions for the equations of motion.

1.8 Quantization of the Free Classical and Fermionic Particles

For the two classical systems described above let us now define a simplified notion of quantization, that is very close to the one originally used by Dirac.

Recall that we have defined two kinds of spaces of trajectories in $C = \mathbb{R}^2$: the system of the classical particle is given by maps $x : [0, 1] \to C$, with space of initial conditions for the equations of motion given by T^*C, and the fermionic particle is given by maps $x : \mathbb{R}^{0|1} \to C$, with space of initial conditions for the equations of motion given by the odd cotangent bundle $T^*[1]C$. Both spaces are equipped with a symplectic structure, allowing us to give a static formulation of the mechanics of the system, called Hamiltonian mechanics.

A canonical quantization of a system with space of initial conditions P (chosen among T^*C and $T^*[1]C$) is given by a filtered algebra (A, F^*) (meaning an algebra A with a collection of imbricated subspaces $F^0 \subset \cdots \subset F^n \subset \cdots \subset A$ compatible with its product operation), such that there is an isomorphism of the associated graded algebra

$$\mathrm{gr}^F A := \oplus_{n \geq 0} F^{n+1}/F^n \xrightarrow{\sim} \mathcal{C}^\infty(P).$$

It is easy to see that the so-called Weyl algebra of differential operators

$$\mathcal{D}_C = \left\{\sum_n a_n(t)\partial_t,\ a_n \text{ smooth on C}\right\}$$

on $C = \mathbb{R}^2$, with its filtration by the degree in ∂_t, is a canonical quantization of the algebra of functions $\mathcal{C}^\infty_p(T^*C)$ on T^*C that are polynomial in the fiber coordinate. This algebra has a natural representation on the space $\mathcal{C}^\infty(C, \mathbb{R})$ of real valued functions on C.

The canonical quantization of the algebra $\mathcal{C}^\infty(T^*[1]C)$ is given by the Clifford algebra Cliff(T^*C, g), where g is (opposite to) the standard metric on $\mathbb{R}^2$. This algebra has a natural representation on the odd Hilbert space $H = L^2(C, S)$, where S is the spinorial representation of the Clifford algebra, regarded as a super-vector space lying in odd degree.

One may refine the above notion of canonical quantization to obtain in addition operators $(\Delta + m^2)$ and $(\not{D} + m)$ on the above super-Hilbert spaces, which give information on the associated quantum equations of motion.

The models of modern particle physics are based on a second quantization procedure which views the above quantum operators as the classical equations of motion for some new field configurations, with new parameter space M' the original configuration space C of punctual mechanics, and new configuration spaces C' the supermanifolds $\mathbb{R}$ and S. The spaces of field configurations are thus given by parametrized spaces of maps

$$\varphi : M' \to \mathbb{R} \quad \text{and} \quad \psi : M' \to S,$$

called bosonic and fermionic fields. These constructions show that it is very helpful to have a formalism for classical and quantum field theory that allows a uniform treatment of all these examples.

1.9 Quantizing Interactions: The Functional Integral

We continue to discuss the basic example of the motion of a particle

$$x : M \to C,$$

with time parameter in $M = [0, 1]$, configuration space the affine plane $C = \mathbb{R}^2$, and interacting Lagrangian density

$$L(x, \partial_t x) := \frac{1}{2} m \|\partial_t x\|^2 + V(x).$$

Recall that the equations of motion are given by

$$m\partial_t^2 x = -\overrightarrow{\text{grad}}(V)(x).$$

The Cauchy theorem for differential equations tells us that the space of solutions to the above equation is, again, isomorphic to the cotangent space T^*C which encodes the initial conditions. One may very well quantize this space as above, but the associated quantum equations of motion are not as simple as the operator $\Delta + m^2$ that we previously used: one must take into account interactions. This is where the perturbative renormalization method comes in: one will treat the theory as depending on a small (or formal) coefficient λ, called the coupling constant, that one may put in front of the potential, yielding a term $\lambda V(x)$ in the Lagrangian density.

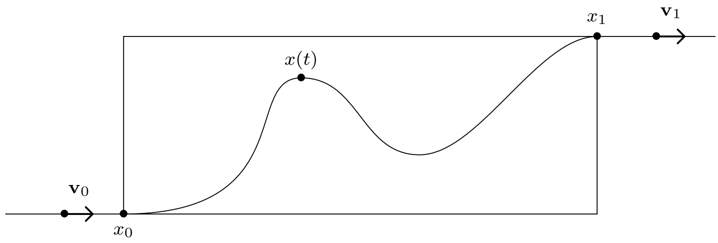
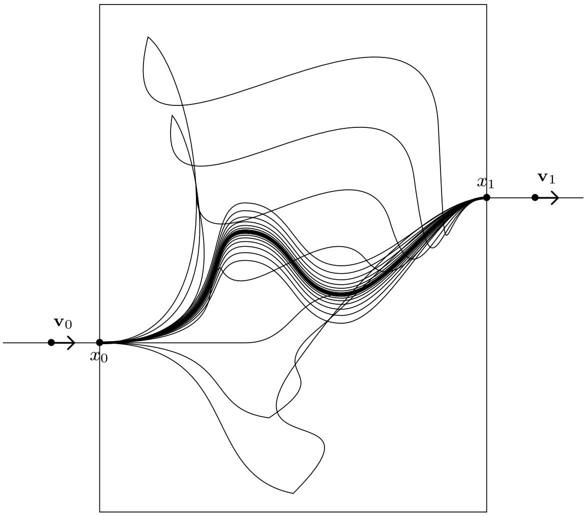


Fig. 1.18 Motion of a particle in a Newtonian potential

Fig. 1.19 Quantum trajectories concentrate around classical solutions



One may give a very intuitive presentation of this perturbative quantization method by using Feynman's functional integral approach to quantization. Recall from Fig. 1.18 the shape of the classical motion of a particle in a Newtonian potential V. Feynman's interpretation of a quantum process is that the quantum particle actually takes all possible paths between the incoming and the outgoing configurations, but that the probability of observing it on a given path concentrates around the classical solutions to the equations of motion, as pictured in Fig. 1.19.

Note that the widely spread idea that Feynman's functional integral does not exist is false. What is true, is that this mathematical device is not an integral in the naive sense of functional analysis, meaning a continuous positive functional

$$\int_H : \mathcal{C}^0(H) \to \mathbb{R},$$

where $H \subset \mathrm{Hom}(M, C)$ is the topological space of field histories. This would not even make any sense in the case of electrons, which are fermionic particles, because of their anti-commuting coordinates.

In all known cases, though, one may use functorial analysis to describe a space $\mathcal{O}(H)$ of (possibly formal) functionals on H together with a well defined functional

$$\langle - \rangle_H : \mathcal{O}(H) \to \mathbb{R}[[\hbar]]$$

which formally satisfies properties similar to the average value of a normalized Gaussian integration measure on H of the form

$$d\mu_S = \frac{e^{\frac{i}{\hbar}S(x)} d\mu(x)}{\int_H e^{\frac{i}{\hbar}S(x)} d\mu(x)}.$$

The definition of this integral is even harder to give if one is working with the higher dimensional parameter space $M = [0, 1] \times \mathbb{R}^3$ given by Lorentzian space-time, and with configuration space, say, $C = \mathbb{R}$. Field histories are then given by real valued functions

$$\varphi : M \to \mathbb{R},$$

and, M being non-compact, one is forced to work with a very big infinite dimensional analog of a Gaussian integral.

A very simple way to develop an intuitive understanding of the renormalization problem is to formulate, following Feynman's thesis, the problem of defining the functional integral as the limit of a family of finite dimensional Lebesgue integrals, by choosing a compact box $B \subset M$, which represents the laboratory, and a lattice $\Lambda \subset M$, which represents the grid of the experimental device. The intersection $M' = B \cap \Lambda \subset M$ is a finite set, so that the mean value $\langle - \rangle_H$ is well defined on the space of restricted field configurations $\varphi_{|M'} : M' \to \mathbb{R}$ which is identified with the finite dimensional real vector space $\mathbb{R}^{M'}$. The renormalization problem then comes from the fact that, if $f \in \mathcal{O}(H)$ is a reasonable functional with finite dimensional mean value

$$\langle f \rangle_H(B, \Lambda) := \langle f \rangle_{\mathbb{R}^{M'}},$$

the limit

$$\langle f \rangle_H := \lim_{\substack{B \to M \\ \Lambda \to M}} \langle f \rangle_H(B, \Lambda)$$

does not exist when the box size increases up to M and the lattice step tends to zero. The whole point of renormalization theory is to give a refined way of taking this limit, by eliminating, inductively on the degree of the formal power series involved, all the infinities that appear in it.

1.10 The Geometry of Perturbative Quantization

The main advantage of formulating classical field theory in terms of parametrized geometry is that it allows to view the various perturbative quantization methods that

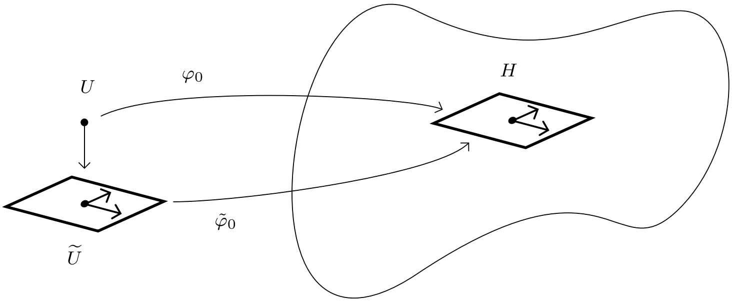


Fig. 1.20 A two dimensional infinitesimal point

appear in physics geometrically, working directly on the space of field configurations. This way of formulating quantization, which is close to DeWitt's covariant field theory, has the advantage of allowing a general treatment of theories with symmetries.

The basic idea is to consider a given solution $\varphi_0 : M \to C$ of the equations of motion, called the background field, and to work with the formal completion $\widehat{\mathcal{O}}_{\varphi_0}(H)$ of the algebra of functionals on the space H of field histories (one may also work with the derived critical space of the action functional here) around this background field. To get an idea of the type of algebra that one may obtain by this formal completion process, consider a vectorial field theory on a manifold M, with configuration space $C = E \to M$ a vector bundle and space of histories $H = \underline{\Gamma}(M, E)$ its space of sections. If φ_0 is the zero field, the formal completion $\widehat{\mathcal{O}}_{\varphi_0}(H)$ may be described explicitly by using distributional symmetric formal power series:

$$\widehat{\mathcal{O}}_{\varphi_0}(H) = \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{R}}\big(\Gamma\big(M^n, E^{\boxtimes n}\big), \mathbb{R}\big)^{S_n}.$$

One may think of these formal power series as forming the algebra of functions on the restriction of the space $\underline{\Gamma}(M, E)$ to infinitesimal thickenings $\bullet \to \widetilde{T}$ of the point $\varphi_0 = 0$ in H, given by algebras of the form $\mathcal{C}^\infty(\mathbb{R}^n)/(x_1, \dots, x_n)^m$. A drawing for the case $n = 2$ and $m = 2$ is given in Fig. 1.20.

In the case of a theory with symmetries, e.g., Yang-Mills theory, we take the formal completion

$$\widehat{\mathrm{Crit}^h(S_{YM})}_{(P_0,A_0)}$$

of the (flat part of the) derived critical space $\mathrm{Crit}^h(S_{YM}) = \mathbb{R}\mathrm{Hom}(M, BG_{flat})$ at a given background flat gauge field (P_0, A_0). This formal completion, pictured in Fig. 1.21, is defined as the restriction of the derived critical space to the category of nilpotent homotopical thickenings $\tilde{T}$ of the point (P_0, A_0). The formal derived space is completely determined by its tangent (homotopical) local Lie algebra $\mathfrak{g}$, so that the formal completion of the functionals of the classical field theory is completely

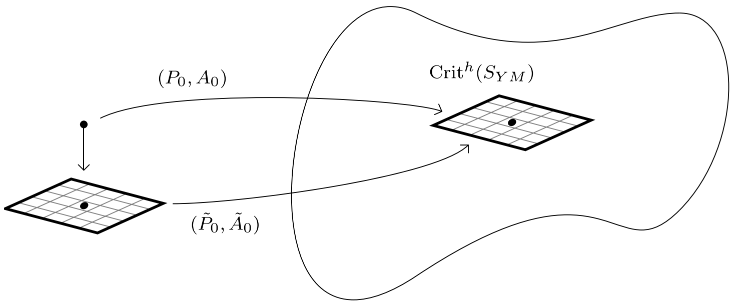


Fig. 1.21 The formal completion of the derived critical space at a given homotopical point

determined as the Chevalley-Eilenberg algebra

$$\mathcal{O}\big(\widehat{\mathrm{Crit}^h(S_{YM})}_{(P_0,A_0)}\big) = \mathrm{CE}^*(\mathfrak{g})$$

of this Lie algebra (formulated in the $\mathcal{D}$-geometrical setting, to make $\mathfrak{g}$ finite dimensional in the differential sense). There is a local (i.e., differentially structured) Poisson bracket on the derived critical space $\mathrm{Crit}^h(S_{YM})$, and this induces an invariant pairing on $\mathfrak{g}$, i.e., a Poisson structure on $\mathrm{CE}^*(\mathfrak{g})$.

One then formulates and solves various different (but essentially equivalent) renormalization problems to define a perturbative functional integral (i.e., a renormalized Gaussian average)

$$\langle - \rangle_H : \widehat{\mathcal{O}}_{\varphi_0}(H) \to \mathbb{R}[\![\hbar]\!].$$

Parametrized geometry is also the natural setting to formalize a non-perturbative renormalization problem, in terms of non-formal functionals in $\mathcal{O}(H)$.

Finally, one may give a very algebraic formulation of the perturbative quantization problem, that is a quantum analog of Lagrange's algebraic formulation of variational calculus. This involves the use of the Ran space $\mathrm{Ran}(M)$ of configurations of points in the parameter space for trajectories, which gives a geometric way of encoding formal power series of elements in $\mathcal{O}(H)$, once for all background fields $\varphi_0 : M \to C$. This space is also a convenient setting for the geometry of multi-jets, which provides a tool for the simultaneous study of Taylor series of functions at different points.

This space may be defined as the parametrized space whose points with values in an open set $U \subset \mathbb{R}^n$ are given by

$$\mathrm{Ran}(M)(U) = \big\{[S] : S \times U \to M \text{ smooth, with } S \text{ a finite set}\big\}.$$

It is morally the union of all finite powers M^S of M along all diagonal maps (and in particular, along symmetries induced by bijections of S). One may also view this space as a useful way of encoding notions that are compatible with the collapsing of families of points in M to one given point (see Fig. 1.22).

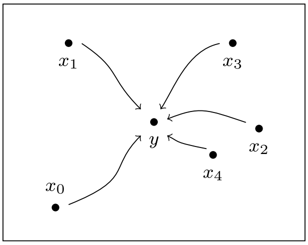


Fig. 1.22 The collapsing of a family of points in a manifold

The corresponding parametrized geometry is given by the $\mathcal{D}$-geometry of factorization spaces on the Ran space. The quantization problem may be formalized in the following way: if X is a derived $\mathcal{D}$-space over M (for example, the derived critical space $\mathrm{Crit}^h(S)$ of a given action functional, given by the Euler-Lagrange system of partial differential equations), one may associate to it a derived $\mathcal{D}$-space $X^{(Ran)}$ (by a kind of homotopical multi-jet construction) over $\mathrm{Ran}(M)$, with an additional factorization property. Its algebra of functions is called a commutative factorization algebra. One may try to deform this commutative structure to a non-commutative factorization algebra. In most cases, this is not possible in the non-derived setting, as one can see in the example of quantum group deformations: there are no such deformations in a strict set theoretical context, because of the classical Heckman-Hilton argument of algebraic topology, which says that two compatible monoid operations on a set are equal and commutative. However, if one works with a homotopical version of factorization spaces, whose algebras of functions are commutative homotopical algebras, the quantization problem is better behaved. This forces us to view the Ran space as a derived space, whose points are parametrized by homotopical smooth algebras. The most natural thing to do is to work directly with the differential graded category of modules over the space $X^{(Ran)}$, which forms a factorization category, whose deformations are called categorical factorization quantizations. This gives an optimal setting in which one can state a general quantization problem, giving a kind of quantum version of Lagrange's analytic formulation of mechanics. Note that this formulation cannot be obtained without the use of homotopical methods.

1.11 Doctrines and Theories

We now briefly describe the general setting that allowed us to treat all the constructions described in this introduction in a unified setting, basing our approach on (higher) categorical logic.

1.11.1 The Categorical Approach to Geometry

We first motivate our systematic use of categories by explaining how it lies at the basis of parametrized (see Sect. 1.1) and functional (see Sect. 1.4) geometry.

Suppose given a category LEGOS of simple building blocks for a given geometry (e.g., $U \subset \mathbb{R}^n$ smooth open subset). Usually, this category has the following drawbacks:

- It doesn't have enough limits (e.g., spaces of solutions to equations, fiber products), and not enough colimits (e.g., quotients).
- It doesn't contain "spaces of maps", i.e., functional spaces $\underline{\mathrm{Hom}}(U, V)$.

The solution to this problem chosen by Grothendieck, Lawvere and Ehresmann is to use the Yoneda embeddings

$$\mathrm{LEGOS} \hookrightarrow \underline{\mathrm{Hom}}\big(\mathrm{LEGOS}^{op}, \mathrm{SETS}\big), \qquad \mathrm{LEGOS}^{op} \hookrightarrow \underline{\mathrm{Hom}}(\mathrm{LEGOS}, \mathrm{SETS})$$

and to restrict the class of functors via the condition of commutation to limits (e.g., sheaves in parametrized geometry, sketches in functional geometry).

If LEGOS is the category of smooth open subsets $U \subset \mathbb{R}^n$, we may view

1. *functional geometry* as the definition of generalized algebras, regarded as *functors of functions*

$$A : \mathrm{LEGOS} \to \mathrm{SETS}$$

   that commute with prescribed limits (e.g., transversal pullbacks), e.g.

$$\underline{\mathcal{C}}^{\infty}(M) : U \mapsto \mathcal{C}^{\infty}(M, U).$$

2. *parametrized geometry* as the definition of generalized spaces, regarded as *functors of points*

$$X : \mathrm{LEGOS}^{op} \to \mathrm{SETS}$$

   that commute with prescribed limits (nerves of coverings), e.g.,

$$\underline{M} : U \mapsto \mathcal{C}^{\infty}(U, M).$$

One then has, by definition, the right tensor product for generalized algebras

$$\underline{\mathcal{C}}^{\infty}(U \times V) \cong \underline{\mathcal{C}}^{\infty}(U) \otimes \underline{\mathcal{C}}^{\infty}(V),$$

and the right pasting for generalized spaces

$$\underline{U \coprod_{U \cap V} V} \cong \underline{U} \coprod_{\underline{U \cap V}} \underline{V}.$$

The main interest of this categorical approach is that it generalizes easily to the setting of higher categories, which we have chosen to express obstruction theoretic statements using generalized universal properties. The category SETS is the category $0 - \text{CAT}$ of 0-categories, which one can replace by the ∞-category ${}^{\infty}\text{GRPD}$ of ∞-groupoids (for homotopical geometrical generalizations) or by other higher categories, e.g., the 2-category CAT of (1-)categories. This allows a smooth treatment of obstruction theories (homology, homotopy). It also allows us to give natural definition of higher categorical algebraic objects (e.g., monoidal categories).

1.11.2 Higher Categories and Categorical Logic

Suppose given a notion of n-category for $n \geq 0$. All theories, types of theories and their models (also called semantics) that will be considered in this book can be classified, in a coordinate-free fashion, by the following definition of a doctrine.

Definition 1.11.1 *Let $n \geq 1$ be an integer. A* doctrine *is an $(n+1)$-category $\mathcal{D}$. A* theory *of type $\mathcal{D}$ is an object $\mathcal{C}$ of $\mathcal{D}$. A* model *for a theory of type $\mathcal{D}$ in another theory of the same type is an object*

$$M : \mathcal{C}_1 \to \mathcal{C}_2$$

of the n-category $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)$.

The coordinate definition of a given theory is usually called its *syntax*, and the study of its models is called its *semantics*. We will often describe theories using their syntax, but work with their semantics in a coordinate-free, higher categorical fashion.

Note that the doctrines we use are usually given by $(n+1)$-categories of n-categories with additional *structures* and *properties*. One may define additional structures via a (possibly inductive) sketch-like construction (see Chaps. 2 and 3), and additional properties via the existence of representing objects for some functors. We decided to use the above general identification between higher category theory and categorical logic in order to obtain the optimal pedagogical presentation, but the proper formulation of a categorical analog of the axiom of choice (an existence axiom for models of certain theories), called the *doctrinal axiom of choice* would necessarily involve a refined language, necessary to define the finiteness notion of the *coherence* of a theory. However, as this book is applications-oriented, we didn't want to burry its results under six feet of formalism.[1]

The types of doctrines used in our presentation of quantum field theory are given, for example, by:

- The 2-category of categories with finite products (or more generally finite limits) with product preserving functors between them and natural transformations, whose theories are called algebraic theories. These will be used to formalize infinitesimals in smooth differential calculus.

[1] Did we?

- The 2-category of (nerves of) sites (categories with Grothendieck topology), whose models are called sheaves, or spaces. These will be used to formalize differential geometry on spaces of fields.
- The 2-category of monoidal (resp. symmetric monoidal, resp. multi-) categories with monoidal functors and monoidal natural transformations between them, whose theories are called PROs (resp. PROPs, resp. multicategories). These will be used to formalize fermionic differential calculus, local functional calculus and homotopical Poisson reduction of general gauge theories.
- The 2-category of Quillen's model categories with Quillen adjunctions between them. These will be used to formalize geometrically homotopical Poisson reduction of general gauge theories and higher gauge field theories.
- The ${}^{\infty}2$-category of ∞-categories, which are useful to formalize deformation quantization of gauge theories; the difficulties of passing from ${}^{\infty}1$- to ${}^{\infty}2$-categories are not greater than those that have to be overcome to pass to ${}^{\infty}n$-categories, and the general theory is very useful for making dévissages and classifying theories in the spirit of the Baez and Dolan's categorification program.
- We will also encounter n-doctrines related to topological quantum field theories.
- There is no good reason to restrict to doctrines for $n = 1$, because we may also be interested in studying the relations between doctrines, their geometries, and so on. These may only be studied in higher doctrines.

1.12 What's New in this Book?

We now give a (non-exhaustive) list of some original tools, ideas and results that are developed in this book.

We may roughly say that the bundle composed of Chaps. 2, 3, 4, 12, 13, 20, 22 and 24 (see also Sect. 9.11) contains substantial contributions of the author to the mathematical formalization of classical and quantum field theory. The other chapters contain results that are gathered in the mathematical and physical literature, but which were often in a form that made them inaccessible to graduate students in mathematics, so that our presentation is at least pedagogically original. An important aspect of the pedagogical and bibliographical content of this work was the cautious choice (and the concise but exhaustive description) of the optimal mathematical tools required for the presentation of our results.

Here is a more precise description of some of our main scientific contributions:

- We describe in Sect. 2.1 the main tools of higher category theory in a purely abstract setting, independent of the model chosen to describe higher categories. This allows us to shorten the usual presentation of these tools, and to concentrate on their applications. Some of the notions we introduced here are not yet in the literature.
- To give a conceptual presentation of the various structures used in this book, we introduced in Sect. 1.2 the general doctrinal techniques for higher categorical logic, and developed a systematical categorification process, called the doctrine

machine (see Definition 9.11.2), which automatically gives the right definitions for both higher categorical structures and algebraic and geometric structures, in both classical (see Chap. 3) and homotopical (see Chap. 10) settings.

- In particular, our definition of monoidal and symmetric monoidal higher categories in Sects. 2.2 and 2.3 is obtained by directly generalizing the categorical logical interpretation of monoids and commutative monoids. This is equivalent to other approaches (in particular, to those of Mac Lane and Segal), but looks much more natural. It also has the advantage of having a direct generalization to the higher categorification of other sketch-like theories (e.g., finitely presented or finite limit theories).
- The idea of parametrized and functional geometry (see Chap. 3), was already present in the literature, for example in the work of Grothendieck and Souriau's schools, and in the synthetic geometry community, grounded by Lawvere's categorical approach to dynamics. However, its systematic use in the formalization of the known physical approaches to quantum field theory is new. Our constructions are directly adapted to spaces with boundary and corners (see Example 3.2.4), in contrast to the above references, which mostly work with closed manifolds. We also treat in the same setting super-symmetric field theories (see Sect. 3.3.3), and the geometry of non-algebraic partial differential equations (see Sect. 12.5).
- The proper generalization of classical functional analysis to parametrized and functional geometry, called functorial analysis (see Chap. 4), is also a new idea that is a particularly fruitful way of combining homotopical and super-manifold methods with more classical methods of analysis. There is not really another way to solve this important problem of the formalization of (non-perturbative) quantum field theory computations.
- The systematic definition of differential invariants using categorical methods, grounded in Quillen's work, has been extended in our work (see Sect. 2.5). By adopting this approach, we don't have to repeat the definition of differential invariants in the various geometrical settings we use. The application of these methods to define jet spaces with boundaries and jet spaces in smooth super-geometry seems to be new.
- We have introduced the notion of monoidally enriched differential geometry (see Sect. 3.3.3) to cover various types of functional geometries present in this book, such as super-geometry, graded geometry and relatively algebraic $\mathcal{D}$-geometry.
- We have developed the new notion of a smooth algebra with corners (see Example 3.2.4), that gives, via the associated differential and jet calculus, a natural setting for the study of non-topological higher dimensional quantum field theories with boundaries. This also opens the door to interesting research directions in the differential geometry of higher and derived stacks with corners, and to their factorization quantizations (see Chap. 24).
- We have given a non-abelian cohomological interpretation of Cartan's formalism in general relativity (see Sects. 7.4 and 14.4.2). This opens the possibility to study global properties of moduli stacks of Cartan geometries.
- We have formalized supersymmetric local differential calculus purely in terms of $\mathcal{D}$-modules (see Chap. 12), which gives a better way to interpret its analogy with classical local differential calculus.

- Our formalization of $\mathcal{D}$-geometry over the Ran space (see Sects. 12.3, 12.5, and 24.4) in smooth and analytic geometry is also new.
- Our formalization of the classical Batalin-Vilkovisky formalism and gauge fixing procedure (see Chap. 13), presented in a coordinate-free language, which also works for supersymmetric theories, with optimal finiteness hypotheses, is also a new mathematical result. This part of the theory, which forms the technical heart of our contributions, expands and improves on the two publications [Pau11a] and [Pau11b], by adding a new and precise geometric formalization of the gauge fixing procedure for local field theories.
- The mathematical formalization of non-perturbative quantum field theory (see Chap. 20) using functorial analysis and the Wilson-Wetterich non-perturbative renormalization group, compatible with our formalization of gauge theories, is also new.
- Our formalization of the Epstein-Glaser causal renormalization method (see Chap. 22) in terms of functional geometry and functorial analysis improves on the purely functional analytic approach used in the literature, by giving a clear meaning to the notion of a space of graded or differential graded fields adapted to this method.
- Our general formalization of the categorical quantization problem for factorization spaces in Sect. 24.6 is also new.
- The use of the analytic derived Ran space, and of microlocal algebraic analysis in the geometric study of factorization algebras (see Remark 24.2.3), is also an original contribution which opens the door to a causal treatment, à la Epstein-Glaser, of the quantization problem for factorization algebras on Lorentzian spacetimes, which is an interesting issue of the theory, already studied in the physics literature in particular cases, for example by Hollands and Wald.

1.13 What's Not in this Book?

First, we have omitted a lot of the many applications of quantum field theoretic tools in mathematics.

Moreover, this books essentially treats the formalization of

quantitative aspects

of quantum field theories, the main aim being to give mathematicians direct access to the standard physics textbooks in this subject.

A very interesting mathematical research direction that we have not treated at all in this book is the study of

qualitative aspects

of quantum field theories, that one may putatively call the theory of quantum dynamical systems.

The starting point for the theory of dynamical systems may be said to be Poincaré's paper on the three body problem: he transformed a mistake into interesting mathematical developments. The entrance into the theory of differential dynamical systems that is the closest to the spirit of our book would be Thom's transversality theory (grounded in [Tho56]), expanded to multijets by Mather [Mat70] (see also [Cha01] for a presentation of the subject and [GG73], Sect. II.4 for proofs). Note that these transversality methods are based on codimension considerations for subspaces of infinite dimensional spaces of functions (called spaces of fields in our book) defined by partial differential equations. The better permanence properties of transversality in derived geometrical spaces may give new tools to mathematicians to refine their knowledge of classical transversality methods in dynamics.

The qualitative study of non-perturbative quantum field theory, through the Wilson-Wetterich non-perturbative (also sometimes called functional) renormalization group method, is another interesting open mathematical problem that has, to the author's knowledge, not yet been studied in any depth by pure mathematicians.

Our precise mathematical formalization of the theory via the methods of functorial analysis, which complete, but are strictly compatible with the functional analytic methods, may be useful to interested researchers.

References

[Cha01] Chaperon, M.: Jets, transersalité, singularités:petite introduction aux grandes idées de René Thom (2001, preprint)

[GG73] Golubitsky, M., Guillemin, V.: Stable Mappings and Their Singularities. Graduate Texts in Mathematics, vol. 14, p. 209. Springer, New York (1973)

[Mat70] Mather, J.N.: Stability of C^∞ mappings. V. Transversality. Adv. Math. **4**, 301–336 (1970)

[Pau11a] Paugam, F.: Histories and observables in covariant field theory. J. Geom. Phys. **61**(9), 1675–1702 (2011). doi:10.1016/j.geomphys.2010.11.002

[Pau11b] Paugam, F.: Homotopical Poisson reduction of gauge theories. In: Mathematical Foundations of Quantum Field Theory and Perturbative String Theory. Proc. Sympos. Pure Math., vol. 83, pp. 131–158. Am. Math. Soc., Providence (2011)

[Tho56] Thom, R.: Un lemme sur les applications différentiables. Bol. Soc. Mat. Mexicana **1**, 59–71 (1956)

Springer

http://www.springer.com/978-3-319-04563-4

Towards the Mathematics of Quantum Field Theory
Paugam, F.
2014, XVI, 487 p. 77 illus., 1 illus. in color., Hardcover
ISBN: 978-3-319-04563-4