Preface

This book is an applied pure mathematics textbook on quantum field theory. Its aim is to introduce mathematicians (and, in particular, graduate students) to the mathematical methods of theoretical and experimental quantum field theory, with an emphasis on

coordinate-free presentations

of the mathematical objects in play, but also of the mathematical theories underlying those mathematical objects. The main objective of this presentation is to promote interactions between mathematicians and physicists, by supplying a common and flexible language, hopefully for the good of both communities, even if the mathematical one is our primary target.

We propose here a reference book that gives a coherent and systematic mathematical toolbox for the coordinate-free treatment of classical and quantum field theories. Giving such a complete presentation, made accessible to graduate students and researchers through a compact formalism, is an original contribution to the mathematical literature, despite the fact that there already exist good specialized references on some parts of the subject, cited in the bulk of our book.

We don't pretend to exhaustivity, because it is unattainable for such a wide subject. We have aimed to produce a book that attains the optimal unity and generality that could be hoped for within our working capacities.

Moreover, we have always tried, with the aim of teaching in mind, to favor general methods that apply to all of the numerous examples treated in this book. This allows us to avoid unworthy repetitions, which would have prevented us from covering so much material in a single volume. The mathematician's approach of going from the general to the particular case also allows us to give optimal and precise hypotheses and results on theories that are usually treated in the physics literature through the study of particular examples.

We also describe a large class of examples of variational problems from classical, quantum and theoretical physics, directly in our language. Finally, we give an overview of various modern quantization and renormalization procedures, together with a mathematical formalization of them that is coherent with our general formalism.

The main problem that we had to face when we started this project was the vast zoology of types of structures that appear in the mathematical formalization of quantum field theory. Here are the specifications that we used to arrive at a locally finitely presented book.

Our viewpoint of physics is very naive and reductionist: a physical theory is given by a family of experiments and a model, i.e.

a mathematical machinery,

that explains these experiments. The main constraint on such a theory is that one must be able to discuss it with colleagues and students, so that they test it, correct it and improve it with later developments. We will mostly be interested in models, and refer to the physics literature for the description of the corresponding experiments. Some of the models we present are theoretical extrapolations of existing physical models. They are either hoped to be tested at some point, or only serve to uncover and prove nice mathematical theorems.

There are various ways to describe a mathematical machinery, and all of them are based on some common metamathematical (i.e. natural) language.

In the set theoretical language, one has the *axiom of choice* which provides a tool to prove

existence results.

In the categorical language (see Lawvere [Law05, Law06] and Ehresmann [Ehr81]), one has *Yoneda's lemma*, which allows us to prove general

unicity statements.

The rough idea of categorical mathematics is to define general mathematical objects and theories not as sets with additional structures, but as objects of a (possibly higher) category (in a metamathematical, i.e., linguistic sense), with a particular universal mapping property, or equivalently, as representing objects for a given (generalized) functor.

To describe all the constructions needed in this book in a unified and compact setting, we decided to identify (higher) categorical logic and (higher) category theory.

It is known that distilled axioms are pretty indigestible. We strongly suggest the reader follow the complicated alembics of Chaps. 1 and 2 simultaneously with respective sections of more wholesome following chapters.[1]

We don't aim at a complete account of all these theories, but we will present enough of each of them to make our description of physical models mathematically clear and consistent. There are many other ways to approach the mathematics of quantum field theory. All of them have advantages on those used here. We have chosen these approaches simply because we are more comfortable with them, and because we don't see another way of covering all of the material presented in this book.

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[1] Exercise: find another occurrence of this paragraph in the mathematical literature.

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