Scattering Relations for a Multi-Layered Chiral Scatterer in an Achiral Environment

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Abstract In this work we study scattering of a plane electromagnetic wave by a multi-layered chiral body in free space. In the interior of the scatterer exists a core which is either a perfect conductor or a dielectric. We obtain integral representations of the scattered fields which consist of a chiral and an achiral counterpart incorporating the boundary and transmission conditions. We introduce a dimensionless version of the scattering problem and we prove the reciprocity principle and a general scattering theorem for the far-field patterns. Finally, we define Herglotz functions and we state the general scattering theorem in terms of the far-field operator which expresses the superposition of the far-field pattern.

Introduction

This paper is concerned with the reciprocity principle and general scattering theorem for the far-field patterns corresponding to the scattering of time-harmonic electromagnetic plane waves upon a multi-layered chiral scatterer with either a perfectly conducting core or a dielectric. This type of scatterer is consisted of a finite number of layers with a homogeneous isotropic chiral medium. On the surfaces of this nested body, transmission conditions are imposed which express the continuity of the medium and the balance of forces acting on it.

A chiral object is a body that cannot be brought into congruence with its mirror by translation and rotation. Chirality is common in a variety of naturally occurring and manmade objects. DNA in a molecular scale, helices, medicine drugs and air defence industry are some examples in which chirality appears. From a technical point of view, chirality is introduced into the classical Maxwell equations via the

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Drude-Born-Fedorov constitutive relations in which the electric and magnetic field are connected through a new material parameter, the well known chirality measure. A large part of the work on scattering problems in chiral media exists in the book of Lakhtakia [20] as well as in [21, 22].

Twersky had made a major contribution to this research area with his papers [23–25] where he proved reciprocity, scattering and optical theorems for acoustic and electromagnetic scattering. Solvability and reciprocity principle for conductive boundary value problems and for the far-field patterns corresponding to an impedance boundary value problem have been proved in [10] and [11] respectively. Reciprocity relation and scattering theorems when the incident wave is a Herglotz function have been proved in [15, 16] for acoustic and electromagnetic scattering. DeFacio developed the impedance theory for electromagnetic scattering by an obstacle with a finite non-intersecting boundaries in [19]. In [5, 6] acoustic scattering amplitudes were constructed for a multi-layered scatterer, and scattering theorems were proved for time-harmonic electromagnetic waves in a piecewise homogeneous medium respectively. Multi-layered ellipsoidal scatterers with sound-soft and -hard core were used in [3] and [4] in which the first three low-frequency coefficients were obtained under ellipsoidal geometry. Dassios et al. proved reciprocity relations and general scattering theorems for far-field patterns in spherical coordinates in elasticity in [18]. Scattering relations for a homogeneous chiral obstacle have been proved in [8] while low-frequency electromagnetic scattering theory for a multi-layered chiral obstacle was developed in [2], where the scattering problem was reduced to an iterative sequence of problems in potential theory and the leading term approximation of the electric far-field pattern was constructed.

In section "Formulation", we proceed by formulating the corresponding scattering problems for the electric field. In section "The Electric Far-Field Pattern", we construct the electric far-field patterns and we determine their chiral and achiral part. In section "Scattering Relations", we restate the problem in terms of a dimensionless version and we prove the reciprocity, general scattering theorem and optical theorem. Finally in section "Herglotz Functions", we introduce Herglotz functions and the far-field operator and we restate a general scattering theorem in terms of Herglotz functions.

Formulation

Let D be a multi-layered chiral scatterer which is a bounded, closed, convex subset of $\mathbb{R}^3$ with a C^2-boundary S_0. The interior of D is divided by means of closed and non-intersecting C^2-surfaces into layers D_j with $S_j = \partial D_j \cap \partial D_{j+1}, j = 1, 2, \ldots, N$. There is one normal unit vector $\hat{\mathbf{n}}(\mathbf{r})$ at each point of any surface S_j pointing at D_j while the surface S_{j-1} surrounds S_j. Each of the layers, D_j, is occupied by a homogeneous,isotropic,chiral medium with electric permittivity ϵ_j, magnetic permeability μ_j and chirality measure β_j for $j = 1, 2, \cdots N$ and vanishing conductivity. The layer D_{N+1} is the core of the scatterer D, within which is the origin and its surface, S_N satisfies the perfect conductor boundary condition or the transmission conditions. The exterior D_0 of the scatterer is an infinite homogeneous isotropic achiral medium with electric permittivity ε_0, magnetic permeability μ_0 and vanishing conductivity. We assume that all physical parameters are real numbers.

We will consider the scattering of time-harmonic electromagnetic plane waves by a multi-layered chiral scatterer D. Let $(\mathbf{E}^i, \mathbf{H}^i)$ be a time-harmonic incident electromagnetic plane wave and $(\mathbf{E}^s, \mathbf{H}^s)$ be the corresponding scattered field. The total electromagnetic field $(\mathbf{E}_0, \mathbf{H}_0)$ in D_0 is given by

$$\mathbf{E}_0 = \mathbf{E}^i + \mathbf{E}^s, \tag{1}$$

$$\mathbf{H}_0 = \mathbf{H}^i + \mathbf{H}^s. \tag{2}$$

The scattered field $(\mathbf{E}^s, \mathbf{H}^s)$ satisfies the Silver-Muller radiation condition [15] while the total electromagnetic exterior field $(\mathbf{E}_0, \mathbf{H}_0)$ satisfies the Maxwell equations in D_0,

$$\nabla \times \mathbf{E}_0 = i\omega\mu_0\mathbf{H}_0, \tag{3}$$

$$\nabla \times \mathbf{H}_0 = -i\omega\varepsilon_0\mathbf{E}_0, \tag{4}$$

where ω is the angular frequency. In each layer D_j $j = 1, \cdots N$, the total field satisfies the modified Maxwell equations, in view of Born-Drude-Fedorov [22] constitutive relations

$$\nabla \times \mathbf{E}_j = i\omega\mu_j \frac{\gamma_j^2}{\kappa_j^2}\mathbf{H}_j + \beta_j\gamma_j^2\mathbf{E}_j, \tag{5}$$

$$\nabla \times \mathbf{H}_j = -i\omega\varepsilon_j \frac{\gamma_j^2}{\kappa_j^2}\mathbf{E}_j + \beta_j\gamma_j^2\mathbf{H}_j, \tag{6}$$

where $\gamma_j^2 = \dfrac{\kappa_j^2}{1-\kappa_j^2\beta_j^2}$ and $\kappa_j^2 = \omega^2\varepsilon_j\mu_j$ are real physical parameters [22]. Note that the solutions of (3)–(6) are divergence free.

By eliminating the magnetic field, in (3)–(6) we conclude to the following modified Helmholtz type equation

$$\nabla \times \nabla \times \mathbf{E}_j - 2\beta_j{\gamma_j}^2\nabla \times \mathbf{E}_j - {\gamma_j}^2\mathbf{E}_j = 0 \text{ in } D_j \text{ for } j = 0, \cdots N, \tag{7}$$

where $\beta_0 = 0$ in free space and $\gamma_0 = \kappa_0 = \omega\sqrt{\varepsilon_0\mu_0}$ is the free space wave number in the exterior region D_0 of the scatterer. It is easy to see that the following relation holds valid [9],

$$\kappa_j^2 = \frac{\varepsilon_j \mu_j}{\varepsilon_0 \mu_0} \kappa_0^2, \;\; j = 0, \cdots N. \tag{8}$$

The electric scattered field satisfies the Silver-Muller radiation condition,

$$\lim_{r \to \infty} \left[\mathbf{r} \times \nabla \times \mathbf{E}^s(\mathbf{r}) + i\kappa_0 r \mathbf{E}^s(\mathbf{r}) \right] = \mathbf{0}, \tag{9}$$

uniformly in all directions $\hat{\mathbf{r}} \in S^2$. We introduce the transmission conditions,

$$\hat{\mathbf{n}} \times \mathbf{E}_j = \hat{\mathbf{n}} \times \mathbf{E}_{j+1}, \tag{10}$$

$$\hat{\mathbf{n}} \times \mathbf{H}_j = \hat{\mathbf{n}} \times \mathbf{H}_{j+1}, \tag{11}$$

on $j = 0, 1, \cdots, N-1$. Substituting the magnetic field in (11) from (5) we get

$$\hat{\mathbf{n}} \times \nabla \times \mathbf{E}_j = \frac{\varepsilon_{j+1}}{\varepsilon_j} \frac{\gamma_j^2}{\gamma_{j+1}^2} \hat{\mathbf{n}} \times \nabla \times \mathbf{E}_{j+1} + \gamma_j^2 \left(\beta_j - \frac{\varepsilon_{j+1}}{\varepsilon_j} \beta_{j+1} \right) \hat{\mathbf{n}} \times \mathbf{E}_{j+1}. \tag{12}$$

We assume that the core of the multi-layered scatterer is a perfect conductor with boundary condition,

$$\hat{\mathbf{n}} \times \mathbf{E}_N(r) = \mathbf{0}, \;\; \text{on } S_N, \tag{13}$$

or the core is a dielectric with transmission conditions

$$\hat{\mathbf{n}} \times \mathbf{E}_N = \hat{\mathbf{n}} \times \mathbf{E}_{N+1}, \tag{14}$$

$$\hat{\mathbf{n}} \times \nabla \times \mathbf{E}_N = \frac{\varepsilon_{N+1}}{\varepsilon_N} \frac{\gamma_N^2}{\gamma_{N+1}^2} \hat{\mathbf{n}} \times \nabla \times \mathbf{E}_{N+1} + \gamma_N^2 \left(\beta_N - \frac{\varepsilon_{N+1}}{\varepsilon_N} \beta_{N+1} \right) \hat{\mathbf{n}} \times \mathbf{E}_{N+1}. \tag{15}$$

From now on, the problem that consists of (7), (9),(10), (12) and (13) will be denoted as (P_1) and the problem that consists of (7), (9), (10), (12), (13) and (14), (15) will be denoted as (P_2). Note that the above transmission problem is well posed and has been studied in [1, 7, 14]. The same problem can be, also, studied by eliminating the electric field in (3)–(6) following an analogous procedure as the one of the electric field.

In this work we will focus on the far-field patterns and the proofs of reciprocity and general scattering theorems.

The Electric Far-Field Pattern

The electric far-field pattern $\mathbf{E}^\infty(\hat{\mathbf{r}})$ is related to the scattered electric field $\mathbf{E}^s$ and it is given by the relation [17]

$$\mathbf{E}^s(\mathbf{r}) = \mathbf{E}^\infty(\hat{\mathbf{r}})h(\kappa_0 r) + O\left(\frac{1}{r^2}\right), \; r \to \infty \tag{16}$$

uniformly in all directions where $h(x) = e^{ix}/(ix)$ is the zeroth-order spherical Hankel function of the first kind. In order to obtain the electric far-field pattern we construct an integral representation of the total exterior electric field where the transmission and boundary conditions are incorporated. We have the following integral representation of the scattered field

$$\begin{aligned}\mathbf{E}^s(\mathbf{r}) = \int_{S_0} & [(\nabla \times \mathbf{E}^s(\mathbf{r}')) \cdot (\hat{\mathbf{n}} \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}')) \\ & - (\hat{\mathbf{n}} \times \mathbf{E}^s(\mathbf{r}')) \cdot (\nabla_{\mathbf{r}'} \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}'))]ds(\mathbf{r}'), \; \mathbf{r} \in D_0\end{aligned} \tag{17}$$

where $\tilde{\Gamma}(\mathbf{r},\mathbf{r}')$ is the free-space dyadic Green's function

$$\tilde{\Gamma}(\mathbf{r},\mathbf{r}') = (\tilde{I} + \kappa_0^{-2}\nabla\nabla)\frac{e^{i\kappa_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{18}$$

with $\tilde{I} = \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3$ is the identity dyadic and $\hat{\mathbf{e}}_j$, $j = 1, 2, 3$ are the cartesian unit vectors. Inserting (1) in (17) and taking into account that $\mathbf{E}^i$ is a solution of the Eq. (7) for $j = 0$ we obtain

$$\begin{aligned}\mathbf{E}_0(\mathbf{r}) = & \mathbf{E}^i(\mathbf{r}) + \int_{S_0} [(\nabla \times \mathbf{E}_0(\mathbf{r}')) \cdot (\hat{\mathbf{n}} \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}')) \\ & - (\hat{\mathbf{n}} \times \mathbf{E}_0(\mathbf{r}')) \cdot (\nabla_{\mathbf{r}'} \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}'))]ds(\mathbf{r}').\end{aligned} \tag{19}$$

Making use of the transmission conditions (10), (12) on S_0, (19) is equal to

$$\begin{aligned}\mathbf{E}_0(\mathbf{r}) = & \mathbf{E}^i(\mathbf{r}) - \frac{\varepsilon_1\gamma_0^2}{\varepsilon_0\gamma_1^2}\int_{S_0} \hat{\mathbf{n}} \cdot [(\nabla \times \mathbf{E}_1(\mathbf{r}')) \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}')]ds(\mathbf{r}') \\ & + \gamma_0^2\frac{\varepsilon_1}{\varepsilon_0}\beta_1\int_{S_0} \hat{\mathbf{n}} \cdot [\mathbf{E}_1(\mathbf{r}') \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}')]ds(\mathbf{r}') \\ & - \int_{S_0} \hat{\mathbf{n}} \cdot [\mathbf{E}_1(\mathbf{r}') \times (\nabla_{\mathbf{r}'} \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}'))]ds(\mathbf{r}').\end{aligned} \tag{20}$$

We apply successively the dyadic form of the divergence theorem, taking into account that $\mathbf{E}_j$ and $\tilde{\Gamma}$ are solutions of (7) in D_j, $j = 1, \cdots N$ and D_0, respectively; we introduce the transmission conditions (10), (12) and we obtain (20).

If the core is a perfect conductor, we use the boundary condition (13) and we obtain that

$$\mathbf{E}_0 = \mathbf{E}^i(\mathbf{r}) + \mathbf{E}^s_c(\mathbf{r}) + \mathbf{E}^s_a(\mathbf{r}) \tag{21}$$

where the indexes in $\mathbf{E}_c^s$ and $\mathbf{E}_a^s$ go for the chiral and the achiral parts of the electric scattered field $\mathbf{E}^s$, respectively and are given by

$$\begin{aligned}\mathbf{E}_c^s(\mathbf{r}) = &-\frac{\varepsilon_N}{\varepsilon_0}\beta_N^2\kappa_0^2\int_{S_N}(\nabla\times\mathbf{E}_N(\mathbf{r}'))\cdot(\hat{\mathbf{n}}\times\tilde{\Gamma}(\mathbf{r},\mathbf{r}'))ds(\mathbf{r}')\\
&-\kappa_0^2\sum_{j=1}^{N}\frac{\varepsilon_j}{\varepsilon_0}\beta_j^2\int_{D_j}(\nabla\times\mathbf{E}_j(\mathbf{r}'))\cdot(\nabla_{\mathbf{r}'}\times\tilde{\Gamma}(\mathbf{r},\mathbf{r}'))dv(\mathbf{r}')\\
&-\kappa_0^2\sum_{j=1}^{N}\frac{\varepsilon_j}{\varepsilon_0}\beta_j\int_{D_j}(\nabla\times\mathbf{E}_j(\mathbf{r}'))\cdot\tilde{\Gamma}(\mathbf{r},\mathbf{r}')dv(\mathbf{r}')\\
&-{\kappa_0}^2\sum_{j=1}^{N}\frac{\varepsilon_j}{\varepsilon_0}\beta_j\int_{D_j}\mathbf{E}_j(\mathbf{r}')\cdot(\nabla_{\mathbf{r}'}\times\tilde{\Gamma}(\mathbf{r},\mathbf{r}'))dv(\mathbf{r}'),\ \mathbf{r}\in D_0 \qquad (22)\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_a^s(\mathbf{r}) = &\frac{\mu_0}{\mu_N}\int_{S_N}(\nabla\times\mathbf{E}_N(\mathbf{r}'))\cdot(\hat{\mathbf{n}}\times\tilde{\Gamma}(\mathbf{r},\mathbf{r}'))ds(\mathbf{r}')\\
&+\kappa_0^2\sum_{j=1}^{N}\left(1-\frac{\varepsilon_j}{\varepsilon_0}\right)\int_{D_j}\mathbf{E}_j(\mathbf{r}')\cdot\tilde{\Gamma}(\mathbf{r},\mathbf{r}')dv(\mathbf{r}')\\
&+\sum_{j=1}^{N}\left(\frac{\mu_0}{\mu_j}-1\right)\int_{D_j}(\nabla\times\mathbf{E}_j(\mathbf{r}'))\cdot(\nabla_{\mathbf{r}'}\times\tilde{\Gamma}(\mathbf{r},\mathbf{r}'))dv(\mathbf{r}')\ \mathbf{r}\in D_0.\end{aligned} \qquad (23)$$

The volume integrals express the contribution of each layer to the exterior field, whereas the surface integrals express the impact of the core.

If the core is dielectric, then in relations (22), (23) the surface integrals on S_N disappear and the volume integrals D_j have an extra term for $j = N + 1$.

Using the asymptotic relations

$$|\mathbf{r}-\mathbf{r}'| = r - \hat{\mathbf{r}}\cdot\mathbf{r}' + O\left(\frac{1}{r}\right),\ r\to\infty, \qquad (24)$$

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \hat{\mathbf{r}} + O\left(\frac{1}{r}\right),\ r\to\infty, \qquad (25)$$

in (18) we get the following asymptotic forms

$$\tilde{\Gamma}(\mathbf{r},\mathbf{r}') = \frac{i\kappa_0}{4\pi}(\tilde{I}-\hat{\mathbf{r}}\hat{\mathbf{r}})h(\kappa_0 r)e^{-i\kappa_0\hat{\mathbf{r}}\cdot\mathbf{r}'} + O\left(\frac{1}{r^2}\right),\ r\to\infty, \qquad (26)$$

$$\nabla_{\mathbf{r}'}\times\tilde{\Gamma}(\mathbf{r},\mathbf{r}') = \frac{\kappa_0^2}{4\pi}(\tilde{I}\times\hat{\mathbf{r}})h(\kappa_0 r)e^{-i\kappa_0\hat{\mathbf{r}}\cdot\mathbf{r}'} + O\left(\frac{1}{r^2}\right),\ r\to\infty. \qquad (27)$$

If the core is a perfect conductor then substituting (26)–(27) into (22)–(23) we obtain

$$\mathbf{E}^{\infty}(\hat{\mathbf{r}}) = \mathbf{E}_c^{\infty}(\hat{\mathbf{r}}) + \mathbf{E}_a^{\infty}(\hat{\mathbf{r}}) \tag{28}$$

where

$$\begin{aligned}4\pi \mathbf{E}_c^{\infty}(\hat{\mathbf{r}}) =& i\kappa_0^3 \frac{\varepsilon_N}{\varepsilon_0}\beta_N^2 \int_{S_N} (\nabla \times \mathbf{E}_N(\mathbf{r}')) \cdot (\hat{\mathbf{n}} \times (\tilde{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}))e^{-i\kappa_0 \hat{\mathbf{r}}\cdot\mathbf{r}'} ds(\mathbf{r}') \\ &+ \kappa_0^4 \sum_{j=1}^{N} \frac{\varepsilon_j}{\varepsilon_0}\beta_j^2 \int_{D_j} (\nabla \times \mathbf{E}_j(\mathbf{r}')) \cdot (\tilde{I} \times \hat{\mathbf{r}})e^{-i\kappa_0 \hat{\mathbf{r}}\cdot\mathbf{r}'} dv(\mathbf{r}') \\ &+ i\kappa_0^3 \sum_{j=1}^{N} \frac{\varepsilon_j}{\varepsilon_0}\beta_j \int_{D_j} (\nabla \times \mathbf{E}_j(\mathbf{r}')) \cdot (\tilde{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})e^{-i\kappa_0 \hat{\mathbf{r}}\cdot\mathbf{r}'} dv(\mathbf{r}') \\ &+ \kappa_0^4 \sum_{j=1}^{N} \frac{\varepsilon_j}{\varepsilon_0}\beta_j^2 \int_{D_j} \mathbf{E}_j(\mathbf{r}') \cdot (\tilde{I} \times \hat{\mathbf{r}})e^{-i\kappa_0 \hat{\mathbf{r}}\cdot\mathbf{r}'} dv(\mathbf{r}'),\end{aligned} \tag{29}$$

and

$$\begin{aligned}4\pi \mathbf{E}_a^{\infty}(\hat{\mathbf{r}}) =& -i\kappa_0 \frac{\mu_0}{\mu_N} \int_{S_N} (\nabla \times \mathbf{E}_N(\mathbf{r}')) \cdot (\hat{\mathbf{n}} \times (\tilde{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}))e^{-i\kappa_0 \hat{\mathbf{r}}\cdot\mathbf{r}'} ds(\mathbf{r}') \\ &- i\kappa_0^3 \sum_{j=1}^{N} \left(1 - \frac{\varepsilon_j}{\varepsilon_0}\right) \int_{D_j} \mathbf{E}_j(\mathbf{r}') \cdot (\tilde{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})e^{-i\kappa_0 \hat{\mathbf{r}}\cdot\mathbf{r}'} dv(\mathbf{r}') \\ &- \kappa_0^2 \sum_{j=1}^{N} \left(\frac{\mu_0}{\mu_j} - 1\right) \int_{D_j} (\nabla \times \mathbf{E}_j(\mathbf{r}')) \cdot (\tilde{I} \times \hat{\mathbf{r}})e^{-i\kappa_0 \hat{\mathbf{r}}\cdot\mathbf{r}'} dv(\mathbf{r}').\end{aligned} \tag{30}$$

If the core is dielectric, then the far-field patterns $\mathbf{E}_c^{\infty}$, $\mathbf{E}_a^{\infty}$ are given by (29), (30) where again the surface integrals do not exist and the volume integrals sum up to $j = N+1$. Summarizing the previous results we conclude to the following theorem.

Theorem 1. *The electric far-field patterns of the scattering problems* (P_1) *and* (P_2) *are given by* (28) *where* $\boldsymbol{E}_c^{\infty}$ *and* $\boldsymbol{E}_a^{\infty}$ *are the chiral and achiral counterparts of the corresponding far-field patterns.*

Scattering Relations

In this section we will prove scattering theorems for the scattering problems (P_1) and (P_2). In order to do so, it is more convenient to restate the problem considering a dimensionless version [8] scaling all lengths using α, a typical length scale for the chiral scatterer. Therefore the scattering problems (P_1) and (P_2) take the following form [8]

$$\nabla \times \mathbf{E}_0 - i(\kappa_0\alpha)\mathbf{H}_0 = \mathbf{0} \quad , \nabla \times \mathbf{H}_0 + i(\kappa_0\alpha)\mathbf{E}_0 = \mathbf{0}, \text{ in } D_0 \tag{31}$$

while in the layers $D_j,\ j = 1, 2, \cdots N$

$$\begin{aligned} \nabla \times \mathbf{E}_j - i(\kappa_j\alpha)\left(\frac{\gamma_j}{\kappa_j}\right)^2 \mathbf{H}_j - \beta_j\alpha\gamma_j^2\mathbf{E}_j &= \mathbf{0}, \\ \nabla \times \mathbf{H}_j + i(\kappa_j\alpha)\left(\frac{\gamma_j}{\kappa_j}\right)^2 \mathbf{E}_j - \beta_j\alpha\gamma_j^2\mathbf{H}_j &= \mathbf{0}, \end{aligned} \tag{32}$$

for $j = 1, \cdots N$. The transmission conditions become

$$\begin{aligned} \rho_j \hat{\mathbf{n}} \times \mathbf{E}_j &= \hat{\mathbf{n}} \times \mathbf{E}_{j+1}, \\ \delta_j \hat{\mathbf{n}} \times \mathbf{H}_j &= \hat{\mathbf{n}} \times \mathbf{H}_{j+1} \end{aligned} \tag{33}$$

on $S_j\ j = 0, 1, \cdots N-1$, where $\kappa_0 = \omega\sqrt{\mu_0\varepsilon_0}$, $\rho_j = \sqrt{\dfrac{\mu_j}{\mu_{j+1}}}$ and $\delta_j = \sqrt{\dfrac{\varepsilon_j}{\varepsilon_{j+1}}}$ are real. If the core is dielectric, the Eq. (32) are valid for $j = N+1$ as well while the transmission conditions (33) are also valid for $j = N$ as well.

The scattered field is now a pair $(\mathbf{E}^s, \mathbf{H}^s)$ and satisfy the Silver-Muller radiation condition

$$\hat{\mathbf{r}} \times \mathbf{H}^s + \mathbf{E}^s = o(\frac{1}{r}),\ r \to \infty. \tag{34}$$

We can consider that all lengths have been scaled using α, and thus we can set $\alpha = 1$. It is also more helpful to introduce the fields $\mathbf{U}$ and $\mathbf{U}'$ where $\mathbf{U}'$ is the dual of $\mathbf{U}$. More precisely,

$$\text{if } \mathbf{U} = \mathbf{E} \text{ then } \mathbf{U}' = i\mathbf{H} \tag{35}$$

$$\text{if } \mathbf{U} = \mathbf{H} \text{ then } \mathbf{U}' = -i\mathbf{E}. \tag{36}$$

Therefore, the relation (32) can be rewritten as follows

$$\nabla \times \mathbf{U}_j - \gamma_j^2\beta_j\mathbf{U}_j - \frac{\gamma_j^2}{\kappa_j}\mathbf{U}_j' = \mathbf{0},\ j = 0, \cdots N. \tag{37}$$

and Eq. (31) for the exterior region D_0 becomes

$$\begin{aligned} \nabla \times \mathbf{U} &= \kappa_0\mathbf{U}', \\ \nabla \times \mathbf{U}' &= \kappa_0\mathbf{U}, \end{aligned} \tag{38}$$

where $(\mathbf{U}')' = \mathbf{U}$. The corresponding far-field pattern $(\mathbf{E}^\infty, \mathbf{H}^\infty)$ is given by

$$\begin{aligned}\mathbf{E}^s(\mathbf{r}) &= \mathbf{E}^\infty(\hat{\mathbf{r}})h(\kappa_0 r) + O\left(\frac{1}{r^2}\right), \ r \to \infty,\\ \mathbf{H}^s(\mathbf{r}) &= \mathbf{H}^\infty(\hat{\mathbf{r}})h(\kappa_0 r) + O\left(\frac{1}{r^2}\right), \ r \to \infty,\end{aligned} \tag{39}$$

uniformly in all directions $\hat{\mathbf{r}} \in S^2$ in the unit sphere. We assume that the incident electromagnetic wave is also dimensionless and it has the form [15]

$$\begin{aligned}\mathbf{E}^i(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}) &= i(\kappa_0\alpha)\mathbf{p}\, e^{i\kappa_0\hat{\mathbf{d}}\cdot\mathbf{r}},\\ \mathbf{H}^i(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}) &= \hat{\mathbf{d}} \times \mathbf{E}^i(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}),\end{aligned} \tag{40}$$

setting $\alpha = 1$. The unit vector $\hat{\mathbf{d}}$ describes the direction of propagation, the vector $\mathbf{p}$ the polarization and they connect with the relation $\hat{\mathbf{d}} \cdot \mathbf{p} = 0$. Henceforth, the dependence of the total, scattered and far-field patterns on the direction of propagation and polarization will be denoted by $(\mathbf{E}_0(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}), \mathbf{H}_0(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}))$, $(\mathbf{E}^s(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}), \mathbf{H}^s(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}))$ and $(\mathbf{E}^\infty(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}), \mathbf{H}^\infty(\mathbf{r}; \hat{\mathbf{d}}, \mathbf{p}))$, respectively.

Moreover, we have the following notation for the total fields $\mathbf{E}_{m,n}$, $\mathbf{H}_{m,n}$ for $m = 1, 2$ and $n = 0, \cdots N$. The index m goes for the first or the second total field and the index n for the layer S_j of the multi-layered scatterer. Instead of the electromagnetic pair $(\mathbf{E}, \mathbf{H})$, we shall use the fields $\mathbf{U}$ and $\mathbf{U}'$ as defined in (35)–(36) for the total, incident, scattered and far-field pattern. In addition, we introduce the Twersky [25] notation

$$\{\mathbf{U}_1, \mathbf{U}_2\}_{S_0} := \int_S [(\hat{\mathbf{n}} \times \mathbf{U}_1) \cdot \mathbf{U}_2' - (\hat{\mathbf{n}} \times \mathbf{U}_2) \cdot \mathbf{U}_1'] ds. \tag{41}$$

We will proceed by stating and proving the reciprocity principle.

Theorem 1. *The far-field pattern $\boldsymbol{U}^\infty$ satisfies the reciprocity principle*

$$\mathbf{q} \cdot \mathbf{U}^\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \mathbf{p}) = \mathbf{p} \cdot \mathbf{U}^\infty(-\hat{\mathbf{d}}; -\hat{\mathbf{r}}, \mathbf{q}) \tag{42}$$

for all $\hat{\mathbf{d}}, \hat{\mathbf{r}} \in S^2$ *and* $\mathbf{p}, \mathbf{q} \in \mathbb{C}^3$ *with* $\mathbf{p} \cdot \hat{\mathbf{d}} = \mathbf{q} \cdot \hat{\mathbf{r}} = 0$.

Proof. In view of bilinearity of the form (41) we get

$$\begin{aligned}\{\mathbf{U}_{1,0}, \mathbf{U}_{2,0}\}_{S_0} =& \{\mathbf{U}_1^i, \mathbf{U}_2^i\}_{S_0} + \{\mathbf{U}_1^i, \mathbf{U}_2^s\}_{S_0}\\ &+ \{\mathbf{U}_1^s, \mathbf{U}_2^i\}_{S_0} + \{\mathbf{U}_1^s, \mathbf{U}_2^s\}_{S_0}.\end{aligned} \tag{43}$$

Using the transmission conditions and applying successively the Gauss' theorem we take

$$\{\mathbf{U}_{1,0}, \mathbf{U}_{2,0}\}_{S_0} = \sqrt{\frac{\mu_N}{\mu_0}\frac{\varepsilon_N}{\varepsilon_0}}\{\mathbf{U}_{1,N}, \mathbf{U}_{2,N}\}_{S_N}. \tag{44}$$

For the problem (P_1) taking into account the boundary condition of the core which is now transformed into

$$\hat{\mathbf{n}} \times \mathbf{U}_N = \mathbf{0} \text{ on } S_N, \tag{45}$$

we have

$$\{\mathbf{U}_{1,0}, \mathbf{U}_{2,0}\}_{S_0} = 0. \tag{46}$$

If the core is dielectric, we apply again Gauss' theorem for $\mathbf{U}_{1,N}, \mathbf{U}_{2,N}$ in D_{N+1} and we conclude to (46). Applying again the Gauss' divergence theorem on the exterior region and taking into account the Maxwell equations for the exterior domain (31), setting $\alpha = 1$, we take for the incident part

$$\{\mathbf{U}_1^i, \mathbf{U}_2^i\}_{S_0} = 0. \tag{47}$$

We consider a sphere S_r centered at the origin with radius r large enough to include the scatterer in its interior. Applying the Gauss theorem in the exterior domain we have

$$\{\mathbf{U}_1^s, \mathbf{U}_2^s\}_{S_0} = \{\mathbf{U}_1^s, \mathbf{U}_2^s\}_{S_r}. \tag{48}$$

Letting $R \to \infty$ we pass to the radiation zone and using the radiating condition (34) the surface integral on S_0 becomes zero,

$$\{\mathbf{U}_1^s, \mathbf{U}_2^s\}_{S_0} = 0. \tag{49}$$

Substituting the previous relations in (43) we get

$$\{\mathbf{U}_1^i, \mathbf{U}_2^s\}_{S_0} = -\{\mathbf{U}_1^s, \mathbf{U}_2^i\}_{S_0} \tag{50}$$

which is equal to

$$\{\mathbf{U}_1^i, \mathbf{U}_2^s\}_{S_0} = \{\mathbf{U}_2^i, \mathbf{U}_1^s\}_{S_0}. \tag{51}$$

Taking into account the integral representations of the electric and magnetic fields as in (6.24) in [15] and that $\mathbf{q} \cdot \hat{\mathbf{r}}$ we get

$$\mathbf{q} \cdot \mathbf{U}^{\infty}(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \mathbf{p}) = \frac{-i}{4\pi}\{\mathbf{U}^s(\cdot; \hat{\mathbf{d}}, \mathbf{p}),\ \mathbf{U}^i(\cdot; -\hat{\mathbf{r}}, \mathbf{q})\}_{S_0} \tag{52}$$

Combining it with (51) we get

$$\begin{aligned}\mathbf{q}\cdot\mathbf{U}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}},\mathbf{p}) &= -\frac{i}{4\pi}\{\mathbf{U}^{s}(\cdot;-\hat{\mathbf{r}},\mathbf{q}),\mathbf{U}^{i}(\cdot;-\hat{\mathbf{d}},\mathbf{p})\}_{S_0}\\ &= \mathbf{p}\cdot\mathbf{U}^{\infty}(-\hat{\mathbf{d}};-\hat{\mathbf{r}},\mathbf{q}).\end{aligned} \tag{53}$$

which proves the theorem. □

Theorem 2. *The far-field pattern* $\mathbf{U}^{\infty}$ *satisfies the relation*

$$\mathbf{q}\cdot\overline{\mathbf{U}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}},\mathbf{p})}+\bar{\mathbf{p}}\cdot\mathbf{U}^{\infty}(\hat{\mathbf{d}};\hat{\mathbf{r}},\mathbf{q}) = -\frac{1}{2\pi}\int_{S^2}\overline{\mathbf{U}^{\infty}(\hat{\mathbf{r}}';\hat{\mathbf{d}},\mathbf{p})}\cdot\mathbf{U}^{\infty}(\hat{\mathbf{r}}';\hat{\mathbf{r}},\mathbf{q})ds(\mathbf{r}') \tag{54}$$

for all $\hat{\mathbf{d}},\hat{\mathbf{r}}\in S^2$ *and* $\mathbf{p}$, $\mathbf{q}\in\mathbb{C}^3$ *with* $\hat{\mathbf{d}}\cdot\mathbf{p}=\hat{\mathbf{r}}\cdot\mathbf{q}=0$.

Proof. In view again of the bilinearity of (41) we obtain

$$\begin{aligned}\{\overline{\mathbf{U}_0(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}_0(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0} =&\{\overline{\mathbf{U}^{i}(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}^{i}(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0}\\ &+\{\overline{\mathbf{U}^{i}(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}^{s}(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0}\\ &+\{\overline{\mathbf{U}^{s}(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}^{i}(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0}\\ &+\{\overline{\mathbf{U}^{s}(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}^{s}(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0}\end{aligned} \tag{55}$$

The term

$$\{\overline{\mathbf{U}^{i}(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}^{i}(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0} = 0 \tag{56}$$

becomes zero from the divergence theorem and the fact that $\overline{\mathbf{U}^{i}(\mathbf{r}';\hat{\mathbf{d}},\mathbf{p})}$, $\mathbf{U}^{i}(\mathbf{r}';\hat{\mathbf{r}},\mathbf{q})$ are solutions of (37).

The total fields

$$\{\overline{\mathbf{U}_0(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}_0(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0} = 0 \tag{57}$$

following the procedure of Theorem (1) and taking into account the fact that all the physical parameters of the scattering problem are real numbers.

Moreover, from (52) we obtain

$$\{\overline{U_1^{i}(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}_2^{s}(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0} = 4\pi i\,\bar{\mathbf{p}}\cdot\mathbf{U}^{\infty}(\hat{\mathbf{d}};\hat{\mathbf{r}},\mathbf{q}) \tag{58}$$

and

$$\{\overline{\mathbf{U}_1^{s}(\cdot;\hat{\mathbf{d}},\mathbf{p})},\mathbf{U}_2^{i}(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0} = 4\pi i\,\mathbf{q}\cdot\overline{\mathbf{U}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}},\mathbf{p})}. \tag{59}$$

For the scattered fields we consider a sphere S_r centered at the origin with radius r, large enough to include the chiral scatterer in its interior. Applying the Gauss theorem in the region exterior to S_0 and interior to S_r, we obtain

$$\{\overline{\mathbf{U}^s(\cdot;\hat{\mathbf{d}},\mathbf{p})}, \mathbf{U}^s(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0} = \{\overline{\mathbf{U}^s(\cdot;\hat{\mathbf{d}},\mathbf{p})}, \mathbf{U}^s(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_r} \tag{60}$$

Letting $r \to \infty$, we can use the asymptotic forms (39) for the scattered fields. Taking into account that $\hat{\mathbf{n}} \times \mathbf{U}'^{\infty} = i\mathbf{U}^{\infty}$ we conclude that

$$\begin{aligned} \{\overline{\mathbf{U}^s(\cdot;\hat{\mathbf{d}},\mathbf{p})}, \mathbf{U}^s(\cdot;\hat{\mathbf{r}},\mathbf{q})\}_{S_0} &= \int_{S_\infty} |h(\kappa_o r')|^2 (2i\overline{\mathbf{U}^\infty(\hat{\mathbf{r}}';\hat{\mathbf{d}},\mathbf{p})} \cdot \mathbf{U}_2^\infty(\hat{\mathbf{r}}';\hat{\mathbf{r}},\mathbf{q}))ds(\mathbf{r}') \\ &= 2i\int_{S^2} \overline{\mathbf{U}^\infty(\hat{\mathbf{r}}';\hat{\mathbf{d}},\mathbf{p})} \cdot \mathbf{U}^\infty(\hat{\mathbf{r}}';\hat{\mathbf{r}},\mathbf{q})ds(\hat{\mathbf{r}}') \end{aligned} \tag{61}$$

Substituting (56), (57), (58), (59) and (61) into (55) we conclude to (54). □

Theorem 3. *The following relation holds:*

$$\sigma = -4\pi Re(\bar{\mathbf{p}} \cdot \mathbf{U}^\infty(\hat{\mathbf{d}};\hat{\mathbf{d}},\mathbf{p})) \tag{62}$$

Proof. Since D_0 is achiral, we can follow the same procedure as in [25] where we can see that

$$\sigma = \int_{S^2} |\mathbf{U}^\infty(\hat{\mathbf{r}},\hat{\mathbf{d}},\mathbf{p})|^2 ds(\hat{\mathbf{r}}) \tag{63}$$

Using Theorem 2 and substituting $\hat{\mathbf{r}} = \hat{\mathbf{d}}$ and $\mathbf{p} = \mathbf{q}$ we conclude to relation (62).□

Herglotz Functions

Next, we will prove a general scattering theorem when the incident field is a Herglotz pair with kernel $\mathbf{g}$. Such a pair is of the form

$$\begin{aligned} \mathbf{U}_{\mathbf{g}}(\mathbf{r}) &= i\kappa_0 \int_{S^2} \mathbf{g}(\hat{\mathbf{q}})\, e^{i\kappa_0 \hat{\mathbf{q}}\cdot\mathbf{r}} ds(\hat{\mathbf{q}}), \\ \mathbf{U}'_{\mathbf{g}}(\mathbf{r}) &= -\kappa_0 \int_{S^2} \hat{\mathbf{q}} \times \mathbf{g}(\hat{\mathbf{q}})\, e^{i\kappa_0 \hat{\mathbf{q}}\cdot\mathbf{r}} ds(\hat{\mathbf{q}}), \end{aligned} \tag{64}$$

where $\mathbf{g} \in L^2(S^2)$ and $\mathbf{g} \cdot \hat{\mathbf{q}} = 0$. We note that these functions given by (64) are solutions of (38) [15, 16]. When the incident field is a Herglotz pair of the form,

$$\mathbf{U}^i_{\mathbf{g}}(\mathbf{r}) = \int_{S^2} \mathbf{U}^i(\mathbf{r};\hat{\mathbf{q}},\mathbf{g}(\hat{\mathbf{q}}))ds(\hat{\mathbf{q}}), \tag{65}$$

then the corresponding scattered field and far-field pattern are given by

$$\begin{aligned} \mathbf{U}^s_{\mathbf{g}}(\mathbf{r}) &= \int_{S^2} \mathbf{U}^s(\mathbf{r}; \hat{\mathbf{q}}, \mathbf{g}(\hat{\mathbf{q}}))ds(\hat{\mathbf{q}}), \\ \mathbf{U}^\infty_{\mathbf{g}}(\hat{\mathbf{r}}) &= \int_{S^2} \mathbf{U}^\infty(\hat{\mathbf{r}}; \hat{\mathbf{q}}, \mathbf{g}(\hat{\mathbf{q}}))ds(\hat{\mathbf{q}}). \end{aligned} \tag{66}$$

Theorem 1. *For the problems* (P_1) *and* (P_2) *the following scattering relations are valid*

$$\{\mathbf{U}^s_{\mathbf{g}}, \overline{\mathbf{U}^i_{\mathbf{h}}}\}_{S_0} = -4\pi i < \mathbf{U}^\infty_{\mathbf{g}}, \mathbf{h} >, \tag{67}$$

$$\{\mathbf{U}^s_{\mathbf{g}}, \overline{\mathbf{U}^s_{\mathbf{h}}}\}_{S_0} = -2i < \mathbf{U}^\infty_{\mathbf{g}}, \mathbf{U}^\infty_{\mathbf{h}} > \tag{68}$$

where $<,>$ *denotes the inner product in* $L^2(S^2)$.

Proof. Relation (67) comes from (52) while relation (68) comes from (61) when the incident field is $\mathbf{U}^i_{\mathbf{h}}$.

Next, we define the far-field operator that corresponds to the far-field pattern $\mathbf{U}^\infty$ as follows $F : L^2(S^2) \to L^2(S^2)$,

$$(F\mathbf{g})(\hat{\mathbf{r}}) = \int_{S^2} \mathbf{U}^\infty(\hat{\mathbf{r}}; \hat{\mathbf{q}}, \mathbf{g}(\hat{\mathbf{q}}))ds(\hat{\mathbf{q}}) \tag{69}$$

with far-field equation $F\mathbf{g} = \mathbf{U}^\infty_{\mathbf{g}}$ [15]. The far-field operator is very important in solving inverse scattering problems. Many methods have been developed in this direction as the dual space method [14], linear sampling method [12] and factorization of the far-field operator [13]. The following corollary derives from the general scattering Theorem 2 considering superpositions of the incident and scattered fields on the unit sphere. □

Corollary 1. *The electric far-field operator* F *corresponding to the problems* (P_1), (P_2) *satisfies the following relation*

$$< \mathbf{Fg}, \mathbf{h} > + < \mathbf{g}, \mathbf{Fh} >= -\frac{1}{2\pi} < \mathbf{Fg}, \mathbf{Fh} > \tag{70}$$

Proof. In order to arrive at this result we apply theorem (2) for $\mathbf{p} = \mathbf{g}(\hat{\mathbf{d}})$ and $\mathbf{q} = \mathbf{h}(\hat{\mathbf{r}})$; and we integrate over the unit sphere twice following an analogous procedure as in [8, 16]. □

Conclusions

The above study can lead to results for simple scatterers (perfect conductor or a dielectric) when the physical parameters satisfy $\varepsilon_j = \varepsilon_{j+1}$, $\mu_j = \mu_{j+1}$, $\beta_j = \beta_{j+1}$ for $j = 0, \cdots, N-1$ [3]. Moreover, if the physical parameters were complex numbers then, in general scattering Theorem 2 and in Corollary 1, there would be an extra term that would derive from the total fields. In optical Theorem 3 the extra term describes the absorbing cross section [8]. We should also note that if chirality measure is zero, $\beta = 0$, then we conclude to already known results for the achiral case.

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