

Preface

The ergodicity of the geodesic flow $(g_t)_{t \in \mathbb{R}}$ with respect to the Liouville measure m on the unit tangent bundle T^1S of a compact surface (or of finite area) with curvature -1 was originally proved by G. A. Hedlund and E. Hopf in the 1930s, by a simple and enlightening idea, the *Hopf argument*: any measurable set invariant under the geodesic flow is actually invariant by the stable and unstable distributions of the flow. A different and elegant argument, holding in the more general setting of symmetric spaces, and based on some commutation relations between diagonal and unipotent matrices, was used in the 1950s by F. I. Mautner. However, the dynamical approach initiated by Hopf was so robust to hold not only for the geodesic flow of compact surfaces with negative, constant curvature, but also for the geodesic flow of general n -manifolds with variable negative curvature, and even for a much larger class of dynamical systems, now called *hyperbolic*.¹ The door to more complex situations and questions was then open!

By the ergodicity of $(g_t)_{t \in \mathbb{R}}$, we knew that for any observable $\varphi \in \mathbb{L}^1(m)$, the quantity $\frac{1}{t} \int_0^t \varphi(g_s \cdot v) ds$ converges m -almost surely towards the integral of φ with respect to m ; the question of the optimal normalization of these sums, for observables φ with mean 0, naturally arised. The investigation of the convergence of properly normalized ergodic sums towards a Gaussian law needed the development of new, important tools; by a control of the speed of equirepartition of suitable means towards the Liouville measure, in 1960, Y. G. Sinai proved a Central Limit Theorem for the geodesic flow of compact manifolds M with constant negative curvature, for a wide class of regular observables φ :

¹It is worth to stress that the horocycle flow is far to be *hyperbolic*; nevertheless, the intertwining relations with the geodesic flow lead quite naturally to its unique ergodicity. In this regard, let us cite at least the original works of H. Furstenberg and B. Marcus, and a recent work of Y. Coudène proving, along the lines of Marcus' dynamical proof, the unique ergodicity of the horocycle flow associated to an Anosov flow with one dimensional orientable strong stable distribution. Here again we see an argument *à la Hopf*, based on the local product structure of the unit tangent bundle induced by the stable and unstable foliations of the geodesic flow.

There exists a constant $\sigma := \sigma(\phi) > 0$ such that, for any $A \in \mathbb{R}$

$$\lim_{t \rightarrow +\infty} m \left\{ x : \frac{1}{t} \int_0^t \varphi(g_s \cdot v) ds - \bar{\phi} \leq A \cdot \frac{\sigma}{\sqrt{t}} \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^A e^{-s^2/2} ds$$

where $\bar{\phi} := \int_{T^1 M} \phi(v) m(dv)$.

During the 1960s and 1970s, another approach, using codings given by Markov partitions and tools coming from thermodynamic formalism, allowed several authors to describe the stochastic behavior of a larger class of dynamical systems, the so-called *Anosov systems*. For instance, a Central Limit Theorem was obtained by M. Ratner for special flows built over subshifts of finite type and satisfying an uniform exponential mixing; the method relied on the local expansion of the dominant eigenvalue of a family of operators, the so-called *transfer* or *Ruelle* operators with potential corresponding to the roof function of the special flow. The remarkable point of this approach is again its flexibility, being valid for the geodesic flow of general compact, negatively curved manifolds.

Let us mention here another original approach to the Central Limit Theorem for the geodesic flow, due to Y. Le Jan and J. Franchi, based on the comparison between geodesics and trajectories of the hyperbolic Brownian motion. The basic idea is to notice that the integral of some regular observable φ along a geodesic coincides with the integral of a closed 1-form along the stable leaf defined by this geodesic and the corresponding horocycles at $+\infty$; then, changing the integration path, the geodesic may be replaced by a Brownian motion path on this leaf. This approach can be used to extend the Ratner-Sinai theorem to noncompact manifolds of constant curvature, and does not require any coding (which is difficult to obtain in the finite volume case); on the other hand, it needs some refined results in potential theory, forcing in particular the curvature to be constant, which was not the case in Ratner's approach. Indeed, as for the coding method, an important role in the proof is played by a spectral gap argument, yielding a potential operator for the geodesic flow.

Still in the 1960s, the classical counting problem for closed geodesics on a negative curved manifold was also deeply investigated, through several other approaches, adapted to each context. For instance, for compact surfaces of constant curvature -1 , H. Huber obtained in 1959 the main term of the asymptotic behaviour of the number of closed geodesic with length smaller than t , for $t \rightarrow +\infty$. G. Margulis, S. J. Patterson, D. Sullivan and others proposed quite different techniques to approach this counting problem in negative, variable curvature, for compact and non-compact manifolds. The great challenge was: first, to construct the unique measure of maximal entropy (the so-called *Bowen-Margulis measure*) for the flow $(g_t)_{t \in \mathbb{R}}$ restricted to its non-wandering set; second, to describe the expansion properties of the associated conditional measures with respect to the stable and unstable foliations. A complete answer, in a very general setting, was given in 2003 by T. Roblin in his seminal work in *Bulletin de la SMF*, for any discrete group of isometries Γ of a CAT(-1) space with finite Bowen-Margulis measure, provided that Γ has a non-arithmetic length spectrum:

There exists a constant $C(x, y) > 0$ such that

$$N_\Gamma(x, y|R) := \#\{\gamma \in \Gamma : d(x, \gamma \cdot y) \leq R\} = (C(x, y) + o(R)) e^{\delta_\Gamma R}$$

where δ_Γ is the Poincaré critical exponent of the group.

On the other hand, the counting problem was also investigated for Anosov flows, for instance by D. Ruelle, R. Bowen, M. Pollicott et al., via transfer operators $(\mathcal{L}_s)_{s \in \mathbb{R}}$ and the corresponding dynamical Zeta functions. The fact that the dominating eigenvalue of $\mathcal{L}_s$, restricted to some suitable functional space, is an isolated point of the spectrum (the one of maximal modulus) is a key argument of all these works,² and is closely related to the mixing property of the Anosov flow. This is the approach which led Ratner to the Central Limit Theorem; using techniques from renewal theory for Markov random walks, this method also yields precise estimates of the main asymptotic term of the counting functions considered above, thus giving another illustration of the stochastic behavior of the geodesic flow in negative curvature.

While the situation concerning the asymptotic dominant term of the counting function of closed geodesics (and closed paths) in negative curvature is now well known, the question of their asymptotic expansions is always quite open. During the 1970s, for hyperbolic surfaces $S = \Gamma \backslash \mathbb{H}^2$ of finite area, Patterson obtained such expansions, with control of the error term, by a method based on the Selberg trace formula related to the hyperbolic Laplacian:

Let $0 = \lambda_0 > \lambda_1 \geq \dots \geq \lambda_q > -\frac{1}{4}$ be the eigenvalues greater than $-\frac{1}{4}$ of the hyperbolic Laplacian, with corresponding eigenfunctions $\phi_1, \dots, \phi_q$; then, for all $x, y \in \mathbb{H}^2$

$$N_\Gamma(x, y|R) = \frac{\pi}{\text{area}(S)} e^R + \sqrt{\pi} \sum_{k=1}^q \frac{(s_k - 3/2)!}{s_k!} \phi_k(x) \phi_k(y) e^{s_k R} + O(e^{\frac{3}{4}R})$$

where $s_k := \frac{1}{2} + \sqrt{\frac{1}{4} + \lambda_k}$.

Further extensions of this result have followed, first of all for finite volume hyperbolic manifolds; but the question is much more complex in variable curvature, or when the volume is infinite, even for geometrically finite manifolds. This question is closely related to the control of the speed of mixing for the flow; D. Dolgopyat's work on decay of correlation in Anosov flows, at the end of the 1990s, led several authors to an upper bound for the error term in the asymptotic expansion. For instance, for any compact, negatively curved surface S with fundamental group Γ , M. Pollicott and R. Sharp showed:

²Whatever method we adopt, a spectral gap property appears somewhere! In the transfer operator approach, the general paradigm is: first, proving the existence of a unique $s_0 \in \mathbb{R}$ such that the spectral radius of $\mathcal{L}_{s_0}$ is equal to 1; then, deducing the formulas for the local expansion at s_0 of the dominant eigenvalue of the $\mathcal{L}_s$.

For all x, y in the universal covering $\hat{S}$, there exists a constant $C(x, y) > 0$ such that

$$N_F(x, y|R) = C(x, y)e^{\delta_F R} + O(e^{\lambda R}).$$

for some $0 < \lambda < \delta_F$ which does not depend on x, y .

However, if we look for an analogous of Patterson's result, we need a more flexible operator, whose infinitesimal generator replaces the hyperbolic Laplacian and whose spectrum can be controlled. This domain is presently very active and many progresses have been obtained in the last years.

The two texts presented in this book are independent and concern directly these two topics: the Central Limit Theorem (CLT) and the counting problem.

Martingale methods for hyperbolic dynamical systems, by Stéphane Le Borgne, focuses on the CLT for a large class of hyperbolic systems, via martingales theory. Based on Gordin's decomposition, this method requires some information about the speed of equirepartition of some means on the corresponding unstable foliation. In the case of the geodesic flow, it corresponds to a quantitative control of the ergodicity of the unstable horocycle flow.³ The approach by martingales presented by Leborgne can be used in several situations, even for weakly hyperbolic flows: e.g., diagonal flows on compact quotients of $SL(d; \mathbb{R})$.

The second text of the book, *A Semiclassical Approach for the Ruelle-Pollicott Spectrum of Hyperbolic Dynamics*, by Frédéric Faure and Masato Tsujii, is related to the counting problem via thermodynamic formalism. In this situation, the transfer operator associated to the dynamic replaces the hyperbolic Laplacian for the geodesic flow in constant curvature -1 , and the question is to describe the structure of its spectrum. For instance, when the dynamical system is a contact Anosov flow, the *Ruelle-Pollicott spectrum* of its generator has a structure in vertical bands, and the trace formula of Atiyah-Bott then leads to an asymptotic expansion of the counting function of periodic orbits; one may also define a Zeta function which generalizes the Selberg Zeta function in the case of constant curvature and which has similar properties: i.e. its zeroes lie, asymptotically, on a vertical line. In this text, the purpose of the authors is not to analyze the asymptotic development of N_F ; however, they show how to recover the dominant term, and their accurate description of the spectrum will certainly lead, in further study, to more refined counting formulas for the periodic orbits of the flow.

Rennes, France
Tours, France
Roma, Italy
December 2013

Françoise Dal'Bo
Marc Peigné
Andrea Sambusetti

³We notice that also Sinai's original approach to the CLT needed such a quantitative information, as mentioned before.

Analytic and Probabilistic Approaches to Dynamics in
Negative Curvature

Dal'Bo, F.; Peigné, M.; Sambusetti, A. (Eds.)

2014, XI, 138 p. 32 illus., 12 illus. in color., Hardcover

ISBN: 978-3-319-04806-2