Chapter 2
Bicomplex Functions and Matrices

Abstract This chapter gives a review of known and new properties of bicomplex holomorphic functions as well as bicomplex analogues of some properties of matrices with real and complex entries.

Keywords Bicomplex holomorphic functions · Complex valued holomorphic functions of two complex variables · Derivative of a bicomplex function · Matrices with bicomplex entries

2.1 Bicomplex Holomorphic Functions

The notion of bicomplex holomorphic functions was introduced a long time ago and we refer the reader to the introduction of the book [1]. The book by itself can serve as a first reading on this subject. For the latest developments see [2] and [3]. In this section we present a summary of the basic facts of the theory that will be used in the last chapter.

The *derivative* $F'(Z_0)$ of the function $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ at a point $Z_0 \in \Omega$ is defined to be the limit, if it exists,

$$F'(Z_0) := \lim_{\mathfrak{S}_0 \not\ni H \to 0} \frac{F(Z_0 + H) - F(Z_0)}{H}, \tag{2.1}$$

such that $H = h_1 + \mathbf{j}h_2$ is an invertible bicomplex number. In this case, the function F is called *derivable* at Z_0.

In analogy with holomorphic functions in one complex variable the Cauchy–Riemann type conditions arise where now the complex, not real, partial derivatives participate. More explicitly, if we consider a bicomplex function $F = f_1 + \mathbf{j}f_2$ derivable at Z_0, then we have that the complex partial derivatives

$$F'_{z_1}(Z_0) = \lim_{h_1 \to 0} \frac{F(Z_0 + h_1) - F(Z_0)}{h_1},$$

D. Alpay et al., *Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis*, SpringerBriefs in Mathematics,
DOI: 10.1007/978-3-319-05110-9_2, © The Author(s) 2014

$$F'_{z_2}(Z_0) = \lim_{h_2 \to 0} \frac{F(Z_0 + \mathbf{j}h_2) - F(Z_0)}{h_2},$$

exist and verify the identity:

$$F'(Z_0) = F'_{z_1}(Z_0) = -\mathbf{j}F'_{z_2}(Z_0)\,, \tag{2.2}$$

which is equivalent to the *complex Cauchy-Riemann system* for F (at Z_0):

$$(f_1)'_{z_1}(Z_0) = (f_2)'_{z_2}(Z_0)\,, \qquad (f_1)'_{z_2}(Z_0) = -(f_2)'_{z_1}(Z_0)\,. \tag{2.3}$$

Of course, if F has bicomplex derivative at each point of Ω, we will say that F is a *bicomplex holomorphic*, or $\mathbb{BC}$-holomorphic, function.

For a $\mathbb{BC}$-holomorphic function F, formulas (2.3) imply that

$$\frac{\partial F}{\partial \bar{z}_1}(Z_0) = \frac{\partial F}{\partial \bar{z}_2}(Z_0) = 0\,, \tag{2.4}$$

i.e.,

$$\frac{\partial f_1}{\partial \bar{z}_1}(Z_0) = \frac{\partial f_1}{\partial \bar{z}_2}(Z_0) = \frac{\partial f_2}{\partial \bar{z}_1}(Z_0) = \frac{\partial f_2}{\partial \bar{z}_2}(Z_0) = 0\,, \tag{2.5}$$

where the symbols $\dfrac{\partial}{\partial \bar{z}_1}$ and $\dfrac{\partial}{\partial \bar{z}_2}$ are the commonly used formal operations on functions of z_1 and z_2.

This means in particular that F is holomorphic with respect to z_1 for any z_2 fixed and F is holomorphic with respect to z_2 for any z_1 fixed. Thus, see for instance [4, pp. 4–5], F is holomorphic in the classical sense of two complex variables. This implies immediately many quite useful properties of F, in particular, it is of class $\mathcal{C}^\infty(\Omega)$.

In one complex variable there exist two mutually conjugate Cauchy–Riemann operators, which characterize the usual holomorphy. Since in $\mathbb{BC}$ there exist three conjugations, it turns out that there exist four bicomplex operators, which characterize in a similar fashion the $\mathbb{BC}$-holomorphy. They are:

$$\begin{aligned}
\frac{\partial}{\partial Z} &:= \frac{1}{2}\left(\frac{\partial}{\partial z_1} - \mathbf{j}\frac{\partial}{\partial z_2}\right), \qquad & \frac{\partial}{\partial Z^\dagger} &:= \frac{1}{2}\left(\frac{\partial}{\partial z_1} + \mathbf{j}\frac{\partial}{\partial z_2}\right), \\
\frac{\partial}{\partial \bar{Z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial \bar{z}_1} - \mathbf{j}\frac{\partial}{\partial \bar{z}_2}\right), \qquad & \frac{\partial}{\partial Z^*} &:= \frac{1}{2}\left(\frac{\partial}{\partial \bar{z}_1} + \mathbf{j}\frac{\partial}{\partial \bar{z}_2}\right).
\end{aligned} \tag{2.6}$$

Theorem 2.1.1. Given $F \in \mathcal{C}^1(\Omega, \mathbb{BC})$, it is $\mathbb{BC}$-holomorphic if and only if

$$\frac{\partial F}{\partial Z^\dagger}(Z) = \frac{\partial F}{\partial \bar{Z}}(Z) = \frac{\partial F}{\partial Z^*}(Z) = 0 \tag{2.7}$$

holds on Ω. Moreover, when it is true one has: $F'(Z) = \dfrac{\partial F}{\partial Z}(Z)$.

Let us see now how the idempotent representation of bicomplex numbers becomes crucial in a deeper understanding of the nature of $\mathbb{BC}$-holomorphic functions. Take a bicomplex function $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ with Ω being a domain.

We write all the bicomplex numbers involved in idempotent form, for instance,

$$\begin{aligned} Z &= \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger = (\ell_1 + \mathbf{i} m_1)\mathbf{e} + (\ell_2 + \mathbf{i} m_2)\mathbf{e}^\dagger\,, \\ F(Z) &= G_1(Z)\mathbf{e} + G_2(Z)\mathbf{e}^\dagger\,, \\ H &= \eta_1 \mathbf{e} + \eta_2 \mathbf{e}^\dagger = (u_1 + \mathbf{i} v_1)\mathbf{e} + (u_2 + \mathbf{i} v_2)\mathbf{e}^\dagger\,. \end{aligned}$$

Let us introduce the sets

$$\Omega_1 := \{\beta_1 \,\big|\, \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger \in \Omega\} \subset \mathbb{C}(\mathbf{i})$$

and

$$\Omega_2 := \{\beta_2 \,\big|\, \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger \in \Omega\} \subset \mathbb{C}(\mathbf{i}).$$

It is direct to prove that Ω_1 and Ω_2 are domains in $\mathbb{C}(\mathbf{i})$.

Theorem 2.1.2. A bicomplex function $F = G_1\mathbf{e} + G_2\mathbf{e}^\dagger : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ of class $\mathcal{C}^1$ is $\mathbb{BC}$-holomorphic if and only if the following two conditions hold:

(I) The component G_1, seen as a $\mathbb{C}(\mathbf{i})$-valued function of two complex variables (β_1, β_2) is holomorphic; what is more, it does not depend on the variable β_2 and thus G_1 is a holomorphic function of the variable β_1.

(II) The component G_2, seen as a $\mathbb{C}(\mathbf{i})$-valued function of two complex variables (β_1, β_2) is holomorphic; what is more, it does not depend on the variable β_1 and thus G_2 is a holomorphic function of the variable β_2.

Corollary 2.1.3. Let F be a $\mathbb{BC}$-holomorphic function in Ω, then F is of the form $F(Z) = G_1(\beta_1)\mathbf{e} + G_2(\beta_2)\mathbf{e}^\dagger$ with $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \Omega$ and its derivative is given by

$$F'(Z) = \mathbf{e} \cdot G_1'(\beta_1) + \mathbf{e}^\dagger \cdot G_2'(\beta_2)\,,$$

or equivalently:

$$F'(z_1 + \mathbf{j} z_2) = \mathbf{e} \cdot G_1'(z_1 - \mathbf{i} z_2) + \mathbf{e}^\dagger \cdot G_2'(z_1 + \mathbf{i} z_2)\,.$$

2.2 Bicomplex Matrices

We will denote by $\mathbb{BC}^{m\times n}$ the set of $m \times n$ matrices with bicomplex entries. For any such matrix $A = (a_{\ell j}) \in \mathbb{BC}^{m\times n}$, it is possible to consider both its cartesian and idempotent representation, which are obtained by accordingly decomposing each of its entries so that, for example, we have

$$A = \mathscr{A}_{1,\mathbf{i}}\mathbf{e} + \mathscr{A}_{2,\mathbf{i}}\mathbf{e}^{\dagger} = \mathscr{A}_{1,\mathbf{j}}\mathbf{e} + \mathscr{A}_{2,\mathbf{j}}\mathbf{e}^{\dagger}$$

where $\mathscr{A}_{1,\mathbf{i}}, \mathscr{A}_{2,\mathbf{i}} \in \mathbb{C}^{m\times n}(\mathbf{i})$ and $\mathscr{A}_{1,\mathbf{j}}, \mathscr{A}_{2,\mathbf{j}} \in \mathbb{C}^{m\times n}(\mathbf{j})$.

Of course the set $\mathbb{BC}^{m\times n}$ inherits many structures from $\mathbb{BC}$. It is obviously a $\mathbb{BC}$-module (a concept that will be discussed in detail in the next chapter) such that

$$\begin{aligned}\mathbb{BC}^{m\times n} &= \mathbb{C}^{m\times n}(\mathbf{i}) \cdot \mathbf{e} + \mathbb{C}^{m\times n}(\mathbf{i}) \cdot \mathbf{e}^{\dagger} \\ &= \mathbb{C}^{m\times n}(\mathbf{j}) \cdot \mathbf{e} + \mathbb{C}^{m\times n}(\mathbf{j}) \cdot \mathbf{e}^{\dagger},\end{aligned}$$

where the summands are $\mathbb{BC}$-submodules of $\mathbb{BC}^{m\times n}$. In particular, given $\mathscr{B} \in \mathbb{C}^{m\times n}(\mathbf{i})$ then $\mathscr{B} \cdot \mathbf{e} \in \mathbb{C}^{m\times n}(\mathbf{i}) \cdot \mathbf{e}$ and $\mathbf{e}^{\dagger} \cdot (\mathscr{B} \cdot \mathbf{e}) = 0_{m\times n}$.

As in the scalar case, the operations over the matrices in the idempotent decomposition can be realized component-wise (though keeping in mind the non-commutativity of matrix multiplication).

Proposition 2.2.1. (See Theorem 3.1 in [5].) Let A be an $n \times n$ bicomplex matrix

$$A = \mathscr{A}_{1,\mathbf{i}}\mathbf{e} + \mathscr{A}_{2,\mathbf{i}}\mathbf{e}^{\dagger} = \mathscr{A}_{1,\mathbf{j}}\mathbf{e} + \mathscr{A}_{2,\mathbf{j}}\mathbf{e}^{\dagger}.$$

Then its determinant is given by

$$\det A = \det\mathscr{A}_{1,\mathbf{i}}\,\mathbf{e} + \det\mathscr{A}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger} = \det\mathscr{A}_{1,\mathbf{j}}\,\mathbf{e} + \det\mathscr{A}_{2,\mathbf{j}}\,\mathbf{e}^{\dagger}.$$

Proof:
The proof can be done by induction on n. In the case $n = 2$ this is immediately demonstrated by the following easy calculation:

$$\begin{aligned}\det A = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\ &= \left(a'_{11}\mathbf{e} + a''_{11}\mathbf{e}^{\dagger}\right)\left(a'_{22}\mathbf{e} + a''_{22}\mathbf{e}^{\dagger}\right) \\ &\quad - \left(a'_{12}\mathbf{e} + a''_{12}\mathbf{e}^{\dagger}\right)\left(a'_{21}\mathbf{e} + a''_{21}\mathbf{e}^{\dagger}\right)\end{aligned}$$

$$= \left(a'_{11}a'_{22} - a'_{12}a'_{21}\right)\ \mathbf{e}\ +\ \left(a''_{11}a''_{22} - a''_{12}a''_{21}\right)\ \mathbf{e}^{\dagger}$$

$$= \det\mathscr{A}_1\mathbf{e} + \det\mathscr{A}_2\mathbf{e}^{\dagger}\ ,$$

where the idempotent decompositions may be taken with coefficients either in $\mathbb{C}(\mathbf{i})$ or in $\mathbb{C}(\mathbf{j})$. For the general case, one can simply use the Laplace theorem, that gives the general formula for the determinant of an $n \times n$ matrix in terms of the determinants of suitable $(n-1) \times (n-1)$ matrices. □

An immediate consequence of this result is the Binet theorem for bicomplex matrices; compare with Remark after Theorem 3.1 in [5].

Corollary 2.2.2. Let A and B be two square bicomplex matrices. Then

$$\det(AB) = \det A \cdot \det B.$$

Proof:
By using the previous result, we see that

$$\det(AB) = \det\left(\left(\mathscr{A}_{1,\mathbf{i}}\,\mathbf{e} + \mathscr{A}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}\right)\left(\mathscr{B}_{1,\mathbf{i}}\,\mathbf{e} + \mathscr{B}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}\right)\right)$$

$$= \det\left(\mathscr{A}_{1,\mathbf{i}}\mathscr{B}_{1,\mathbf{i}}\,\mathbf{e} + \mathscr{A}_{2,\mathbf{i}}\mathscr{B}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}\right)$$

$$= \det\left(\mathscr{A}_{1,\mathbf{i}}\mathscr{B}_{1,\mathbf{i}}\right)\ \mathbf{e} + \det\left(\mathscr{A}_{2,\mathbf{i}}\mathscr{B}_{2,\mathbf{i}}\right)\ \mathbf{e}^{\dagger}$$

$$= \det\mathscr{A}_{1,\mathbf{i}} \cdot \det\mathscr{B}_{1,\mathbf{i}}\,\mathbf{e} + \det\mathscr{A}_{2,\mathbf{i}} \cdot \det\mathscr{B}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}$$

$$= \left(\det\mathscr{A}_{1,\mathbf{i}}\,\mathbf{e} + \det\mathscr{A}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}\right)\left(\det\mathscr{B}_{1,\mathbf{i}}\,\mathbf{e} + \det\mathscr{B}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}\right)$$

$$= \det A \cdot \det B\ ,$$

where one can take $\mathscr{A}_{1,\mathbf{j}}$, $\mathscr{A}_{2,\mathbf{j}}$, $\mathscr{B}_{1,\mathbf{j}}$, $\mathscr{B}_{2,\mathbf{j}}$ instead of their $\mathbb{C}(\mathbf{i})$-counterparts. This concludes the proof. □

Analogously, one can use the idempotent representation of a bicomplex matrix to determine its invertibility.

Proposition 2.2.3. Let $A = \mathscr{A}_{1,\mathbf{i}}\,\mathbf{e} + \mathscr{A}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger} = \mathscr{A}_{1,\mathbf{j}}\,\mathbf{e} + \mathscr{A}_{2,\mathbf{j}}\,\mathbf{e}^{\dagger} \in \mathbb{BC}^{n\times n}$, $\mathscr{A}_{1,\mathbf{i}}$, $\mathscr{A}_{2,\mathbf{i}} \in \mathbb{C}^{n\times n}(\mathbf{i})$, $\mathscr{A}_{1,\mathbf{j}}$, $\mathscr{A}_{2,\mathbf{j}} \in \mathbb{C}^{n\times n}(\mathbf{j})$ be a bicomplex matrix. Then A is invertible if and only if $\mathscr{A}_{1,\mathbf{i}}$ and $\mathscr{A}_{2,\mathbf{i}}$ are invertible in $\mathbb{C}^{n\times n}(\mathbf{i})$ and $\mathscr{A}_{1,\mathbf{j}}$, $\mathscr{A}_{2,\mathbf{j}}$ are invertible in $\mathbb{C}^{n\times n}(\mathbf{j})$.

Proof:
A matrix A is invertible if and only if there exists $B = \mathscr{B}_{1,\mathbf{i}}\,\mathbf{e} + \mathscr{B}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger} \in \mathbb{BC}^{n\times n}$ such that $AB = BA = I$. This is equivalent to

$$I_{\mathbb{C}^{n\times n}(\mathbf{i})}\,\mathbf{e}+I_{\mathbb{C}^{n\times n}(\mathbf{i})}\,\mathbf{e}^{\dagger}=I_{\mathbb{BC}^{n\times n}}=\ \mathscr{A}_{1,\mathbf{i}}\,\mathscr{B}_{1,\mathbf{i}}\,\mathbf{e}+\mathscr{A}_{2,\mathbf{i}}\,\mathscr{B}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}$$

$$=\mathscr{B}_{1,\mathbf{i}}\,\mathscr{A}_{1,\mathbf{i}}\,\mathbf{e}+\mathscr{B}_{2,\mathbf{i}}\,\mathscr{A}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}$$

which is equivalent to $\mathscr{A}_{1,\mathbf{i}}\,\mathscr{B}_{1,\mathbf{i}}=I_{\mathbb{C}^{n\times n}(\mathbf{i})}$ and $\mathscr{A}_{2,\mathbf{i}}\,\mathscr{B}_{2,\mathbf{i}}=I_{\mathbb{C}^{n\times n}(\mathbf{i})}$. The same when one considers the invertibility of $\mathscr{A}_{1,\mathbf{j}}$ and $\mathscr{A}_{2,\mathbf{j}}$. □

The following result is immediate.

Corollary 2.2.4. A matrix $A\in\mathbb{BC}^{n\times n}$ is invertible if and only if $\det A\notin\mathfrak{S}\cup\{0\}$.

One can compare Proposition 2.2.3 and Corollary 2.2.4 with Theorem 3.3 in [5].

We can naturally introduce three conjugations on each bicomplex matrix $A=\left(a_{\ell\,j}\right)\in\mathbb{BC}^{m\times n}$, as follows:

$$A^{\dagger}:=\left(a^{\dagger}_{\ell\,j}\right)\,,\qquad \overline{A}:=\left(\overline{a}_{\ell\,j}\right)\,,\qquad A^{*}:=\left(a^{*}_{\ell\,j}\right)\,.$$

It is immediate to prove that all the conjugations are multiplicative operations, that is, each of them applied to the product of two matrices becomes the product of the conjugate matrices. Obviously these conjugations are additive operations also.

As usual, A^t denotes the transposed matrix:

$$A^t=\left(a_{j\ell}\right),\quad 1\le j\le n,\quad 1\le\ell\le m,$$

and we correspondingly have three adjoint matrices:

$$A^{t\,\dagger}:=\left(A^t\right)^{\dagger}=\left(A^{\dagger}\right)^{t}=\left(a^{\dagger}_{j\ell}\right);$$

$$A^{t\,bar}:=\overline{\left(A^t\right)}=\left(\,\overline{A}\,\right)^{t}=\left(\overline{a}_{j\ell}\right);$$

$$A^{t\,*}:=\left(A^t\right)^{*}=\left(A^{*}\right)^{t}=\left(a^{*}_{j\ell}\right).$$

Since for any two compatible matrices one has that $(AB)^t=B^tA^t$, then $(AB)^{t\,\dagger}=B^{t\,\dagger}A^{t\,\dagger}$, $(AB)^{t\,bar}=B^{t\,bar}A^{t\,bar}$, $(AB)^{t\,*}=B^{t\,*}A^{t\,*}$.

The idempotent representations

$$A=\mathscr{A}_{1,\mathbf{i}}\,\mathbf{e}+\mathscr{A}_{2,\mathbf{i}}\,\mathbf{e}^{\dagger}=\mathscr{A}_{1,\mathbf{j}}\,\mathbf{e}+\mathscr{A}_{2,\mathbf{j}}\,\mathbf{e}^{\dagger}$$

give:

$$A^{\dagger}=\mathscr{A}_{2,\mathbf{i}}\,\mathbf{e}+\mathscr{A}_{1,\mathbf{i}}\,\mathbf{e}^{\dagger}=\mathscr{A}^{*}_{2,\mathbf{j}}\,\mathbf{e}+\mathscr{A}^{*}_{1,\mathbf{j}}\,\mathbf{e}^{\dagger}\,,$$
$$\overline{A}=\overline{\mathscr{A}}_{2,\mathbf{i}}\,\mathbf{e}+\overline{\mathscr{A}}_{1,\mathbf{i}}\,\mathbf{e}^{\dagger}=\mathscr{A}_{2,\mathbf{j}}\,\mathbf{e}+\mathscr{A}_{1,\mathbf{j}}\,\mathbf{e}^{\dagger}\,,$$

$$A^* = \overline{\mathscr{A}}_{1,\mathbf{i}}\,\mathbf{e} + \overline{\mathscr{A}}_{2,\mathbf{i}}\,\mathbf{e}^\dagger = \mathscr{A}^*_{1,\mathbf{j}}\,\mathbf{e} + \mathscr{A}^*_{2,\mathbf{j}}\,\mathbf{e}^\dagger\,,$$
$$A^t = \mathscr{A}^t_{1,\mathbf{i}}\,\mathbf{e} + \mathscr{A}^t_{2,\mathbf{i}}\,\mathbf{e}^\dagger = \mathscr{A}^t_{1,\mathbf{j}}\,\mathbf{e} + \mathscr{A}^t_{2,\mathbf{j}}\,\mathbf{e}^\dagger,$$

and hence

$$A^{t\,\dagger} = \mathscr{A}^t_{2,\mathbf{i}}\,\mathbf{e} + \mathscr{A}^t_{1,\mathbf{i}}\,\mathbf{e}^\dagger = \mathscr{A}^{t\,*}_{2,\mathbf{j}}\,\mathbf{e} + \mathscr{A}^{t\,*}_{1,\mathbf{j}}\,\mathbf{e}^\dagger\,,$$
$$A^{t\,bar} = \overline{\mathscr{A}}^{\,t}_{2,\mathbf{i}}\,\mathbf{e} + \overline{\mathscr{A}}^{\,t}_{1,\mathbf{i}}\,\mathbf{e}^\dagger = \mathscr{A}^t_{2,\mathbf{j}}\,\mathbf{e} + \mathscr{A}^t_{1,\mathbf{j}}\,\mathbf{e}^\dagger\,,$$
$$A^{t\,*} = \overline{\mathscr{A}}^{\,t}_{1,\mathbf{i}}\,\mathbf{e} + \overline{\mathscr{A}}^{\,t}_{2,\mathbf{i}}\,\mathbf{e}^\dagger = \mathscr{A}^{t\,*}_{1,\mathbf{j}}\,\mathbf{e} + \mathscr{A}^{t\,*}_{2,\mathbf{j}}\,\mathbf{e}^\dagger\,.$$

Similarly, we can define the notion of self-adjoint matrix with respect to each one of the conjugations introduced above; specifically, we will say that a matrix A is self-adjoint if, respectively, it satisfies one of the following equalities:

$$A = A^{t\,\dagger}, \quad A = A^{t\,bar}, \quad A = A^{t\,*}.$$

We can express these definitions in terms of idempotent components.

(a) The matrix A is †-self-adjoint, or †-Hermitian, if and only if

$$\mathscr{A}_{1,\mathbf{i}} = \mathscr{A}^t_{2,\mathbf{i}}, \quad \text{if and only if} \quad \mathscr{A}_{1,\mathbf{j}} = \mathscr{A}^{t\,*}_{2,\mathbf{j}},$$

which is true if and only if
$$A^t = A^\dagger;$$

thus all †-self-adjoint matrices are of the form

$$A = \mathscr{A}_{1,\mathbf{i}}\,\mathbf{e} + \mathscr{A}^t_{1,\mathbf{i}}\,\mathbf{e}^\dagger = \mathscr{A}_{1,\mathbf{j}}\,\mathbf{e} + \mathscr{A}^{t\,*}_{1,\mathbf{j}}\,\mathbf{e}^\dagger,$$

with $\mathscr{A}_{1,\mathbf{i}}$ an arbitrary matrix in $\mathbb{C}^{n\times n}(\mathbf{i})$, and $\mathscr{A}_{1,\mathbf{j}}$ any matrix in $\mathbb{C}^{n\times n}(\mathbf{j})$. Note that $\mathscr{A}^{t\,*}_{1,\mathbf{j}}$ is the usual $\mathbb{C}(\mathbf{j})$ adjoint of $\mathscr{A}_{1,\mathbf{j}}$.

(b) The matrix A is bar-self-adjoint, or bar-Hermitian, if and only if

$$\mathscr{A}_{1,\mathbf{i}} = \overline{\mathscr{A}}^{\,t}_{2,\mathbf{i}}, \quad \text{if and only if} \quad \mathscr{A}_{1,\mathbf{j}} = \mathscr{A}^t_{2,\mathbf{j}},$$

which is true if and only if
$$A^t = \overline{A};$$

thus all bar-self-adjoint matrices are of the form

$$A = \mathscr{A}_{1,\mathbf{i}}\,\mathbf{e} + \overline{\mathscr{A}}^{\,t}_{1,\mathbf{i}}\,\mathbf{e}^\dagger = \mathscr{A}_{1,\mathbf{j}}\,\mathbf{e} + \mathscr{A}^t_{1,\mathbf{j}}\,\mathbf{e}^\dagger,$$

with $\mathscr{A}_{1,\mathbf{i}}$ an arbitrary matrix in $\mathbb{C}^{n\times n}(\mathbf{i})$, and $\mathscr{A}_{1,\mathbf{j}}$ any matrix in $\mathbb{C}^{n\times n}(\mathbf{j})$. Note that $\overline{\mathscr{A}}^{\,t}_{1,\mathbf{i}}$ is the usual $\mathbb{C}(\mathbf{i})$ adjoint of $\mathscr{A}_{1,\mathbf{i}}$.

(c) The matrix A is $*$-self-adjoint, or $*$-Hermitian, if and only if

$$\mathscr{A}_{1,\mathbf{i}} = \overline{\mathscr{A}}^{\,t}_{1,\mathbf{i}}, \quad \mathscr{A}_{2,\mathbf{i}} = \overline{\mathscr{A}}^{\,t}_{2,\mathbf{i}},$$

if and only if

$$\mathscr{A}_{1,\mathbf{j}} = \mathscr{A}^{t\,*}_{1,\mathbf{j}}, \quad \mathscr{A}_{2,\mathbf{j}} = \mathscr{A}^{t\,*}_{2,\mathbf{i}};$$

that is, both $\mathscr{A}_{1,\mathbf{i}}$ and $\mathscr{A}_{2,\mathbf{i}}$ are usual $\mathbb{C}(\mathbf{i})$ self-adjoint matrices, and both $\mathscr{A}_{1,\mathbf{j}}$, $\mathscr{A}_{2,\mathbf{j}}$ are usual $\mathbb{C}(\mathbf{j})$ self-adjoint matrices.

In what follows we are interested in $*$-self-adjointness since this property implies a form of hyperbolic "positiveness" of bicomplex matrices which we will find quite useful.

Definition 2.2.6. A $*$-self-adjoint matrix $A \in \mathbb{BC}^{n\times n}$ is called *hyperbolic positive*, if for every column $c \in \mathbb{BC}^n$,

$$c^{*\,t} \cdot A \cdot c \in \mathbb{D}^+. \tag{2.8}$$

In this case we write $A \gtrdot 0$. Given two bicomplex matrices A, B, we say that $A \gtrdot B$ if and only if $A - B \gtrdot 0$.

Proposition 2.2.7. Let

$$A = A_1 + \mathbf{j}A_2 = \mathscr{A}_1\mathbf{e} + \mathscr{A}_2\mathbf{e}^\dagger, \tag{2.9}$$

be an element of $\mathbb{BC}^n$, with A_1, A_2, $\mathscr{A}_1$ and $\mathscr{A}_2$ in $\mathbb{C}^{n\times n}(\mathbf{i})$. Then, the following are equivalent:

(a) $A \gtrdot 0$.
(b) Both $\mathscr{A}_1$ and $\mathscr{A}_2$ are complex positive matrices.
(c) $A_1 \geq 0$, the matrix A_2 is skew self adjoint, that is, $A_2 + \overline{A}_2^{\,t} = 0$, and

$$-A_1 \leq \mathbf{i}A_2 \leq A_1. \tag{2.10}$$

Proof:
Assume (a). Take a column $c \in \mathbb{BC}^n$ with representations

$$c = c_1 + \mathbf{j}c_2 = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^\dagger,$$

where the various columns are in $\mathbb{C}^n(\mathbf{i})$. Then,

$$c^{*\,t}Ac = \overline{\zeta}_1^{\,t}\,\mathscr{A}_1\,\zeta_1\mathbf{e} + \overline{\zeta}_2^{\,t}\,\mathscr{A}_2\,\zeta_2\mathbf{e}^\dagger. \tag{2.11}$$

Thus, by definition, $A \succcurlyeq 0$ implies that both $\mathscr{A}_1$ and $\mathscr{A}_2$ are $\mathbb{C}(\mathbf{i})$ positive, and (b) holds.

Assume (b). Since

$$\mathscr{A}_1 = A_1 - \mathbf{i}A_2 \quad \text{and} \quad \mathscr{A}_2 = A_1 + \mathbf{i}A_2,$$

then

$$A_1 = \frac{1}{2}\left(\mathscr{A}_1 + \mathscr{A}_2\right) \geq 0;$$

moreover, $\mathbf{i}A_2 = A_1 - \mathscr{A}_1$ is self-adjoint, and hence A_2 is skew self-adjoint. Furthermore, still in view of (b),

$$A_1 - \mathbf{i}A_2 \geq 0 \quad \text{and} \quad A_1 + \mathbf{i}A_2 \geq 0,$$

and thus we obtain (2.10) and (c) holds. Finally when (c) holds, both the matrices $\mathscr{A}_1$ and $\mathscr{A}_2$ are positive, and thus (a) holds as well in view of (2.11). □

Proposition 2.2.8. Let $A \in \mathbb{BC}^{n\times n}$. The following are equivalent:

(1) A is hyperbolic positive.
(2) $A = B^{*t} \cdot B$ where $B \in \mathbb{BC}^{m\times n}$ for some $m \in \mathbb{N}$.
(3) $A = C^2$ where the matrix C is hyperbolic positive.

Proof:
Let $A \in \mathbb{BC}^{n\times n}$ be represented as in (2.9), and assume that (1) holds. Then, by the preceding theorem, we have $\mathscr{A}_1 \geq 0$ and $\mathscr{A}_2 \geq 0$, and thus we can write

$$\mathscr{A}_1 = \overline{U}^t U \quad \text{and} \quad \mathscr{A}_2 = \overline{V}^t V, \tag{2.12}$$

where U and V are matrices in $\mathbb{C}^{n\times n}(\mathbf{i})$. Thus $A = B^{*t}B$ with $B := U\mathbf{e} + V\mathbf{e}^\dagger$, so that (2) holds with $m = n$. Assume now that (2) holds with $B \in \mathbb{BC}^{m\times n}$ for some $m \in \mathbb{N}$. Writing $B = U\mathbf{e} + V\mathbf{e}^\dagger$, where now U and V belong to $\mathbb{C}^{m\times n}(\mathbf{i})$ we have

$$A = \overline{U}^t U\mathbf{e} + \overline{V}^t V\mathbf{e}^\dagger,$$

and so (1) holds. The equivalence with (3) stems from the fact that U and V in (2.12) can be chosen positive. □

In the case of a complex matrix, it is equivalent to say that a positive matrix A is Hermitian and to say that its eigenvalues are positive. We now give the corresponding result in the setting of $\mathbb{BC}$. We note that the existence of zero divisors creates problems in an attempt to classify eigenvalues in general. For instance, take two non zero elements a and b such that $ab = 0$. Then every $c \in \mathbb{BC}$ such that $bc = 0$ is an eigenvalue of the matrix

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix} \quad \text{with eigenvector} \quad \begin{pmatrix} b \\ b \end{pmatrix},$$

since

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} b \\ b \end{pmatrix} = c \begin{pmatrix} b \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There is however a relation between the eigenvalues and eigenvectors of a bicomplex matrix A and those of its idempotent components $\mathscr{A}_1$ and $\mathscr{A}_2$. Indeed, let $\lambda = \gamma_1\, \mathbf{e} + \gamma_2\, \mathbf{e}^\dagger \in \mathbb{BC} \setminus \{0\}$ be an eigenvalue of A with a corresponding eigenvector $u = v_1\, \mathbf{e} + v_2\, \mathbf{e}^\dagger$, then

$$Au = \lambda u,$$

which is equivalent to

$$\begin{cases} \mathscr{A}_1\, v_1 = \gamma_1\, v_1, \\ \\ \mathscr{A}_2\, v_2 = \gamma_2\, v_2. \end{cases}$$

If λ is not a zero divisor and $v_1 \neq 0$, $v_2 \neq 0$ then λ is an eigenvalue of A if and only if γ_1 is an eigenvalue of $\mathscr{A}_1$ and γ_2 is an eigenvalue of $\mathscr{A}_2$.

Theorem 2.2.9. A matrix $A \in \mathbb{BC}^{n\times n}$ is hyperbolic positive if and only if

1. A is $*$-Hermitian;
2. none of its eigenvalues is a zero divisor in $\mathbb{D}^+$.

Proof:
That A is hyperbolic positive is equivalent to the complex matrices $\mathscr{A}_1$, $\mathscr{A}_2$ being positive, which is in turn equivalent to stating that $\mathscr{A}_1$ and $\mathscr{A}_2$ are complex Hermitian and any of their eigenvalues are a positive number. Finally, this is equivalent to say that A is $*$-Hermitian and that none of its eigenvalues is a zero divisor in $\mathbb{D}^+$. □

Corollary 2.2.10. A matrix $A \in \mathbb{BC}^{n\times n}$ is hyperbolic positive if and only if:

1. A is $*$–Hermitian;
2. if $\lambda = \lambda_1 + \mathbf{j}\,\lambda_2$ is an eigenvalue for A, then $\lambda_1 > 0$, $\mathbf{i}\,\lambda_2 \in \mathbb{R}$ and

$$-\lambda_1 < \mathbf{i}\,\lambda_2 < \lambda_1\,;$$

Proof:
This is because $\lambda = \lambda_1 + \mathbf{j}\lambda_2 = \gamma_1\, \mathbf{e} + \gamma_2\, \mathbf{e}^\dagger$ with $\gamma_1 = \lambda_1 - \mathbf{i}\,\lambda_2$ and $\gamma_2 = \lambda_1 + \mathbf{i}\,\lambda_2$. □

Remark 2.2.11. The last inequality in the statement of this corollary is equivalent to the system

$$\begin{cases} \lambda_1 - \mathbf{i}\,\lambda_2 > 0, \\ \\ \lambda_1 + \mathbf{i}\,\lambda_2 > 0\,. \end{cases}$$

It is known that in the case of complex matrices every eigenvector corresponds to only one eigenvalue. This is not the case for bicomplex matrices. Specifically, a bicomplex eigenvector can correspond to an infinite family of bicomplex eigenvalues.

We restrict our considerations to hyperbolic positive matrices. Let A be such a matrix, and let $\lambda = \gamma_1 \mathbf{e} + \gamma_2 \mathbf{e}^\dagger$ be one of its eigenvalues (in particular λ is a non zero divisor in $\mathbb{D}^+$). First of all let us show that any such eigenvalue has an eigenvector of the form $u = v_2 \mathbf{e}^\dagger$ with $v_2 \in \mathbb{C}^n(\mathbf{i})$. Since γ_2 is an eigenvalue of $\mathscr{A}_2$, let v_2 be a corresponding eigenvector: $\mathscr{A}_2 v_2 = \gamma_2 v_2$. Consider $u := v_2 \mathbf{e}^\dagger$. Let us show that it is an eigenvector of A corresponding to the above λ. Indeed, $A u = \mathscr{A}_2 v_2 \mathbf{e}^\dagger = \gamma_2 v_2 \mathbf{e}^\dagger$ and $\lambda u = \gamma_2 v_2 \mathbf{e}^\dagger$, thus $A u = \lambda u$.

Now we are in a position to show that this eigenvector corresponds to an infinite family of eigenvalues. For any $r > 0$ set $\lambda_r := r \mathbf{e} + \gamma_2 \mathbf{e}^\dagger$, then $\lambda_r u = \gamma_2 v_2 \mathbf{e}^\dagger = A u$. Hence the whole family $\{ \lambda_r \mid r > 0 \}$ consists of the eigenvalues of A with the same eigenvector $u \in \mathbb{BC}^n_{\mathbf{e}^\dagger}$.

We now study bicomplex $*$-unitary matrices, that is, matrices $U \in \mathbb{BC}^{n\times n}$ such that $UU^{*t} = U^{*t}U = I_n$.

Proposition 2.2.12. Let $U = U_1 + \mathbf{j}U_2 = \mathscr{U}_1 e + \mathscr{U}_2 \mathbf{e}^\dagger \in \mathbb{BC}^{n\times n}$. Then U is unitary if and only if its idempotent components are complex unitary matrices or, equivalently, its cartesian components satisfy

$$U_1 \overline{U}_1^t + U_2 \overline{U}_2^t = \overline{U}_1^t U_1 + \overline{U}_2^t U_2 = I_n$$

and

$$U_2 \overline{U}_1^t = U_1 \overline{U}_2^t, \qquad \overline{U}_1^t U_2 = \overline{U}_2^t U_1.$$

Proof:
Since $U^{*t} = \overline{U}_1^t - \mathbf{j}\,\overline{U}_2^t = \overline{\mathscr{U}}_1^t e + \overline{\mathscr{U}}_2^t \mathbf{e}^\dagger$, then for the idempotent representation we have:

$$\mathscr{U}_1 \overline{\mathscr{U}}_1^t \mathbf{e} + \mathscr{U}_2 \overline{\mathscr{U}}_2^t \mathbf{e}^\dagger = \mathbf{e}\, I_n + \mathbf{e}^\dagger I_n$$

and

$$\overline{\mathscr{U}}_1^t \mathscr{U}_1 \mathbf{e} + \overline{\mathscr{U}}_2^t \mathscr{U}_2 \mathbf{e}^\dagger = \mathbf{e}\, I_n + \mathbf{e}^\dagger I_n.$$

This means that $\mathscr{U}_1$ and $\mathscr{U}_2$ are complex unitary matrices.

Similarly for the cartesian representation we have:

$$U_1 \overline{U}_1^t + U_2 \overline{U}_2^t + \mathbf{j} \left(U_2 \overline{U}_1^t - U_1 \overline{U}_2^t \right) = I_n,$$

$$\overline{U}_1^t U_1 + \overline{U}_2^t U_2 + \mathbf{j} \left(\overline{U}_1^t U_2 - \overline{U}_2^t U_1 \right) = I_n,$$

which means that

$$U_1\,\overline{U}_1^{\,t} + U_2\,\overline{U}_2^{\,t} = \overline{U}_1^{\,t}\,U_1 + \overline{U}_2^{\,t}\,U_2 = I_n$$

and

$$U_2\,\overline{U}_1^{\,t} = U_1\overline{U}_2^{\,t}, \qquad \overline{U}_1^{\,t}\,U_2 = \overline{U}_2^{\,t}\,U_1.$$

The result follows. □

References

1. G.B. Price, *An Introduction to Multicomplex Spaces and Functions* (Marcel Dekker, New York, 1991)
2. M.E. Luna-Elizarrarás, M. Shapiro, D.C. Struppa, A. Vajiac, Complex Laplacian and derivatives of bicomplex functions. Complex Anal. Oper. Theory **7**(5), 1675–1711 (2013)
3. M.E. Luna-Elizarrarás, M. Shapiro, D.C. Struppa, A. Vajiac, Bicomplex numbers and their holomorphic functions. In preparation
4. S.G. Krantz, *Function Theory of Several Complex Variables* (AMS Chelsea Publishing, Providence, Rhode Island, 1992)
5. R. Gervais Lavoie, L. Marchildon, D. Rochon, Finite-dimensional bicomplex Hilbert spaces. Adv. Appl. Clifford Algebras **21**(3), 561–581 (2011)

http://www.springer.com/978-3-319-05109-3

Basics of Functional Analysis with Bicomplex Scalars,
and Bicomplex Schur Analysis
Alpay, D.; Luna-Elizarrarás, M.E.; Shapiro, M.; Struppa,
D.C.
2014, XV, 95 p. 3 illus. in color., Softcover
ISBN: 978-3-319-05109-3