Chapter 2
Introduction to the Chapman–Enskog Method: Linear Models with Migrations

Abstract In this chapter we provide a gentle introduction of the Chapman–Enskog-type asymptotic expansion and of the basic techniques of proving its convergence. To make the presentation not too technical, it is illustrated on systems of linear ordinary differential equations. The chapter begins with a survey of necessary results from linear algebra and theory of finite-dimensional dynamical systems and it is concluded with a detailed analysis of linear population models with geographical structure in which the migration between geographical patches is much faster than the demographic processes.

Keywords Chapman–Enskog asymptotic procedure • Linear dynamical systems

The Hilbert expansion method, discussed in the previous chapter, is the most natural approach to asymptotic expansions. However, in applications we are most often interested in the limit equation describing the evolution occurring in the hydrodynamic space, which represents the aggregated, macroscopic model. For instance, in the original application of the asymptotic analysis to the Boltzmann equation, the macro-model consists of the fluid dynamics equations which are of utmost importance in applications. Thus, working with the Hilbert expansion, we often use (1.69) for $\mathbf{u}_0$. This, however, indicates a drawback of the Hilbert approach. Indeed, in general, in the expansion (1.65), each term may have a part living in the hydrodynamic space and, only focusing on $\mathbf{u}_0$, we may lose the hydrodynamic information contained in the other terms. This drawback is addressed in the so-called Chapman–Enskog procedure which in a modified abstract version, introduced in [167,168], is presented below. Roughly speaking, in this method first we decompose the solution to (1.64) into the hydrodynamic and the kinetic parts and, in the asymptotic expansion procedure, only the kinetic part is expanded, as in (1.65), while the hydrodynamic part of the solution is not expanded. Thanks to this, the whole information carried by the hydrodynamic part is kept together. We reiterate

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that this is in contrast with the Hilbert expansions, where only the zeroth-order term of the expansion of the hydrodynamic part is recovered from the limit equation.

We note that the introduction to more general asymptotic methods by first presenting them for linear systems can be also found in [177, Chap. 2] where, however, the authors use the linear estimates as a basis for an alternative proof of the Tikhonov theorem.

Since the hydrodynamic space is defined as the null-space of $\mathcal{C}$ (see Definition 1.4.1), a systematic way of achieving such a decomposition is by using spectral projections of $\mathcal{C}$ that correspond to the eigenvalue $\lambda = 0$ [or for the linearization of $\mathcal{C}$ corresponding to the solutions to (1.67), [168]]. Hence, before we begin the discussion of the Chapman–Enskog method, we shall provide a brief survey of basic mathematical tools related to spectral projections of matrices which will be needed in this chapter.

We emphasize that for simplicity the discussion here is carried out for linear dynamical systems in finite-dimensional spaces. However, most of the presented ideas can be extended to nonlinear models and/or infinite-dimensional spaces. Such extensions, however, require many technical assumptions and much more involved calculations and usually are constructed for specific applications.

2.1 Basics of Linear Dynamical Systems

Consider the system

$$\frac{\mathrm{d}\,\mathbf{y}}{\mathrm{d}\,t} = \mathcal{A}\mathbf{y}, \tag{2.1}$$

where $\mathbf{y}(t) = (y_1(t), \ldots, y_n(t))$ and $\mathcal{A} = \{a_{ij}\}_{1 \le i,j \le n}$ is an $n \times n$ matrix. System (2.1) is considered together with the initial condition

$$\mathbf{y}(t_0) = \overset{\circ}{\mathbf{y}}. \tag{2.2}$$

We want to study (2.1) as a dynamical system in the state space $\mathbb{R}^n$. To be able to express such concepts as stability, asymptotic stability or the convergence of solutions, we must introduce a way of measuring distance between points of $\mathbb{R}^n$. Typically, the distance is defined by a norm; that is, a functional $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$ satisfying, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \alpha \in \mathbb{R}$,

$$\|\mathbf{x}\| = 0 \quad \text{iff} \quad \mathbf{x} = 0, \qquad \|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|, \qquad \|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|.$$

There is a variety of norms in $\mathbb{R}^n$ (all defining the same topology; see [129, Chap. One]), the most common being the Euclidean norm

$$\|\mathbf{y}\|_2 = \sqrt{\sum_{i=1}^{n} |y_i|^2}. \tag{2.3}$$

However, bearing in mind the applications discussed in this chapter, where typically the solution vector $\mathbf{y}(t) = (y_1(t), \ldots, y_n(t))$ describes the distribution of a population among the states, we see that the most natural norm is

$$\|\mathbf{y}\| = \sum_{i=1}^{n} |y_i| \tag{2.4}$$

which, for $y_i \geq 0, i = 1, \ldots, n$, simplifies to

$$\|\mathbf{y}\| = \sum_{i=1}^{n} y_i \tag{2.5}$$

which is the total population of the ensemble.

Since we want $\mathcal{A}$ to act from $\mathbb{R}^n$ to $\mathbb{R}^n$ with the same way of measuring distances, we should have

$$\|\mathcal{A}\mathbf{x}\| = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sum_{j=1}^{n} |x_j| \sum_{i=1}^{n} |a_{ij}| \leq \|\mathbf{x}\| \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| =: \|\mathcal{A}\| \|\mathbf{x}\|,$$

where

$$\|\mathcal{A}\| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \tag{2.6}$$

is called the norm of the matrix/operator $\mathcal{A}$. It is a norm of an operator in the functional analytic sense (see Sect. 5.1), and though one can define other norms, the one defined above is consistent with the interpretation of the problem and will be used in the notes.

The concept of stability is central in most applications of differential equations, both linear and nonlinear; see, e.g. [21, 56]. Usually it refers to an equilibrium or stationary solutions; that is, to solutions which are independent of time. Without going into formal definitions, we mention that a stationary solution is *stable* if all other solutions that originate from a sufficiently small neighbourhood of it stay close to it for all time. Further, an equilibrium is *asymptotically stable* if it is stable and all other solutions emanating from a sufficiently small neighbourhood of it converge to it as time goes to infinity. An important class of asymptotically stable stationary solutions are *globally asymptotically stable* stationary solutions: they attract all solutions of the system of equations. Otherwise, an asymptotically stable solution is called *locally asymptotically stable*. Sometimes, when it is clear

which stationary solution is referred to, we say that the dynamical system itself is stable, or asymptotically stable.

Finally, we note that in many applications, any solution emanating from nonnegative data must stay nonnegative. For instance, in population theory, the solution typically represents the population size or density and thus should stay at least nonnegative for all times. This requires extension of the concept of positivity to vectors and matrices. We say that a vector $\mathbf{x} = (x_1, \ldots, x_n)$ is nonnegative (resp. positive) if for all $i = 1, \ldots, n$, $x_i \geq 0$ (resp. $x_i > 0$). We denote this as $\mathbf{x} \geq 0$ (resp. $\mathbf{x} > 0$). Similarly, we say that a matrix $\mathcal{A} = (a_{ij})_{1 \leq i,j \leq n}$ is nonnegative (resp. positive) and writes $\mathcal{A} \geq 0$ (resp. $\mathcal{A} > 0$) if $a_{ij} \geq 0$ (resp. $a_{ij} > 0$) for all $i, j = 1, \ldots, n$. Also, for a vector $\mathbf{x}$, or a matrix $\mathcal{A}$, with arbitrary entries, we denote $|\mathbf{x}| = (|x_1|, \ldots, |x_n|)$ and $|\mathcal{A}| = \{|a_{ij}|\}_{1 \leq i,j \leq n}$.

2.1.1 Fundamental Solution Matrix

The solvability and uniqueness of (2.1) and (2.2) follow from the Picard theorem; see, e.g. [56]. We summarize the relevant properties of solutions in the following theorem.

Theorem 2.1.1.

(i) There exists one and only one solution of the initial value problem (2.1) and (2.2), which is defined for all $t \in \mathbb{R}$.

(ii) The set $\mathbf{X}$ of all solutions to (2.1) is a linear space of dimension n.

(iii) Let $\mathbf{y}_1(t), \ldots, \mathbf{y}_k(t)$ be solutions of (2.1) and let $t_0 \in \mathbb{R}$ be an arbitrary number. Then $\{\mathbf{y}_1(t), \ldots, \mathbf{y}_k(t)\}$ form a linearly independent set of functions if and only if $\{\mathbf{y}_1(t_0), \ldots, \mathbf{y}_k(t_0)\}$ is a linearly independent set of vectors in $\mathbb{R}^n$.

An important consequence of (iii) is that solutions starting from linearly independent initial conditions remain linearly independent. Theorem 2.1.1 implies that there is a matrix $\mathcal{E}(t)$ such that the solution $\mathbf{y}(t)$ can be represented as

$$\mathbf{y}(t) = \mathcal{E}(t)\overset{\circ}{\mathbf{y}} \tag{2.7}$$

and satisfies $\mathcal{E}(0) = \mathcal{I}$ (the identity matrix); that is, $\mathcal{E}(t)$ is the matrix whose columns are solutions to (2.1) emanating from the vectors of the canonical base of $\mathbb{R}^n$. The matrix $\mathcal{E}(t)$ is called the *fundamental solution matrix*. To find this matrix, assume that we can find n linearly independent vectors $\mathbf{v}^i$, $i = 1, \ldots, n$, for which $\mathcal{E}(t)\mathbf{v}^i$ can be easily evaluated. Then, for arbitrary $\overset{\circ}{\mathbf{y}} \in \mathbb{R}^n$, we can find constants $c_1, \ldots, c_n$ such that

$$\overset{\circ}{\mathbf{y}} = c_1\mathbf{v}^1 + \ldots + c_n\mathbf{v}^n;$$

that is, denoting $\mathbf{c} = (c_1, \ldots, c_n)$,

$$\mathbf{c} = \mathcal{V}^{-1}\mathbf{x^0}, \tag{2.8}$$

where $\mathcal{V}$ is given by

$$\mathcal{V} = \begin{pmatrix} | & \ldots & | \\ \mathbf{v}^1 & \ldots & \mathbf{v}^n \\ | & \ldots & | \end{pmatrix}. \tag{2.9}$$

Note that $\mathcal{V}$ is invertible as the vectors $\mathbf{v}^i$ are linearly independent.

Thus, for an arbitrary $\overset{\circ}{\mathbf{y}}$, we have

$$\mathcal{E}(t)\overset{\circ}{\mathbf{y}} = \mathcal{E}(t)(c_1\mathbf{v}^1 + \ldots + c_n\mathbf{v}^n) = c_1\mathcal{E}(t)\mathbf{v}^1 + \ldots + c_n\mathcal{E}(t)\mathbf{v}^n. \tag{2.10}$$

Denoting by $\mathcal{E}_{\mathbf{v}}(t)$ the matrix whose columns are vectors $\mathcal{E}(t)\mathbf{v}^1, \ldots, \mathcal{E}(t)\mathbf{v}^n$, we can write

$$\mathcal{E}(t)\overset{\circ}{\mathbf{y}} = \mathcal{E}_{\mathbf{v}}(t)\mathbf{c} = \mathcal{E}_{\mathbf{v}}(t)\mathcal{V}^{-1}\,\overset{\circ}{\mathbf{y}}\,. \tag{2.11}$$

Hence, the problem rests with finding linearly independent vectors $\mathbf{v}^i, i = 1, \ldots, k$, on which $\mathcal{E}(t)$ can be easily evaluated.

It is easy to see that $\mathbf{y}(t) = e^{\lambda t}\mathbf{e}$ is a solution if and only if $\mathbf{e}$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ; that is, $\mathcal{A}\mathbf{e} = \lambda\mathbf{e}$. Moreover, the eigenvectors corresponding to different eigenvalues are linearly independent. Hence, if we happen to have n linearly independent eigenvectors of $\mathcal{A}$, say $\mathbf{e}^1, \ldots, \mathbf{e}^n$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$, then, by Theorem 2.1.1, the solutions $\mathbf{y}^j(t) = e^{\lambda_j t}\mathbf{e}^j$, $j = 1, \ldots, n$, are linearly independent. In such a case, the general solution of (2.1) is of the form

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{e}^1 + \ldots + c_n e^{\lambda_n t}\mathbf{e}^n, \tag{2.12}$$

with $\mathbf{c} = (c_1, \ldots, c_n)$ given by

$$\mathbf{c} = \mathcal{V}^{-1}\,\overset{\circ}{\mathbf{y}}, \tag{2.13}$$

or

$$\mathcal{E}(t)\,\overset{\circ}{\mathbf{y}} = \begin{pmatrix} | & \ldots & | \\ e^{\lambda_1 t}\mathbf{v^1} & \ldots & e^{\lambda_n t}\mathbf{v^n} \\ | & \ldots & | \end{pmatrix}\mathcal{V}^{-1}\,\overset{\circ}{\mathbf{y}}\,. \tag{2.14}$$

Unfortunately, in many cases there is insufficiently many eigenvectors to generate all solutions. To cater for such a situation, we have to recall basic concepts related to eigenvalues and eigenvectors of matrices.

2.1.2 Eigenvalues, Eigenvectors and Associated Eigenvectors

Let $\mathcal{A}$ be an $n \times n$ matrix. As mentioned above, a number λ (real or complex) is an *eigenvalue* of $\mathcal{A}$ if there exists a nonzero solution of the equation

$$\mathcal{A}\mathbf{e} = \lambda\mathbf{e}. \tag{2.15}$$

Such a solution is called an *eigenvector* of $\mathcal{A}$.

Collection of all eigenvalues of $\mathcal{A}$ is called the *spectrum of* $\mathcal{A}$ and denoted by $\sigma(\mathcal{A})$.

Equation (2.15) is equivalent to the homogeneous system $(\mathcal{A}-\lambda\mathcal{I})\mathbf{v} = \mathbf{0}$, where $\mathcal{I}$ is the identity matrix; therefore λ is an eigenvalue of $\mathcal{A}$ if and only if

$$\det(\mathcal{A}-\lambda\mathcal{I}) = \begin{vmatrix} a_{11}-\lambda & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn}-\lambda \end{vmatrix} = 0. \tag{2.16}$$

Evaluating the determinant we obtain a polynomial in λ of degree n. This polynomial is also called the *characteristic polynomial* of the system (2.1). We shall denote this polynomial by $p(\lambda)$. From algebra, we know that there are exactly n, possibly complex, roots of $p(\lambda)$. Some of them may be multiple, so that in general $p(\lambda)$ factorizes into

$$p(\lambda) = (\lambda_1-\lambda)^{n_1}\cdot\ldots\cdot(\lambda_k-\lambda)^{n_k}, \tag{2.17}$$

with $n_1+\ldots+n_k = n$. It is also worthwhile to note that since the coefficients of the polynomial are real, then complex roots appear always in conjugate pairs; that is, if $\lambda_j = \xi_j + i\omega_j$ is a characteristic root, then so is $\bar{\lambda}_j = \xi_j - i\omega_j$. Thus, eigenvalues are the roots of the characteristic polynomial of $\mathcal{A}$. The exponent n_i appearing in the factorization (2.17) is called the *algebraic multiplicity* of λ_i. For each eigenvalue λ_i there corresponds an eigenvector $\mathbf{e}^i$ and eigenvectors corresponding to distinct eigenvalues are linearly independent. The set of all eigenvectors corresponding to λ_i spans a subspace, called the *eigenspace* corresponding to λ_i, which we will denote by $\tilde{E}_{\lambda_i}$. The dimension of $\tilde{E}_{\lambda_i}$ is called the *geometric multiplicity* of λ_i. In general, algebraic and geometric multiplicities are different with geometric multiplicity being at most equal to the algebraic one. Thus, in particular, if λ_i is a single root of the characteristic polynomial, then the eigenspace corresponding to λ_i is one-dimensional.

If the geometric multiplicities of the eigenvalues add up to n; that is, if we have n linearly independent eigenvectors, then these eigenvectors form a basis for $\mathbb{R}^n$. In particular, this happens if all eigenvalues are single roots of the characteristic polynomial. If this is not the case, then we do not have sufficiently many eigenvectors to span $\mathbb{R}^n$ and hence, if we need a basis for $\mathbb{R}^n$, then we have to find additional linearly independent vectors. A procedure that can be employed here, and that will be very useful in our treatment of systems of differential equations, is to find solutions to equations of the form $(\mathcal{A}-\lambda_i\mathcal{I})^k\mathbf{v}=0$ for $1<k\leq n_i$, where n_i is the algebraic multiplicity of λ_i. Precisely speaking, if λ_i has algebraic multiplicity n_i and if

$$(\mathcal{A}-\lambda_i\mathcal{I})\mathbf{e}=0 \tag{2.18}$$

has only $\nu_i<n_i$ linearly independent solutions, then we consider the equation

$$(\mathcal{A}-\lambda_i\mathcal{I})^2\mathbf{w}=0, \tag{2.19}$$

or, equivalently, the equations

$$(\mathcal{A}-\lambda_i\mathcal{I})\mathbf{w}=\mathbf{e}, \tag{2.20}$$

where $\mathbf{e}$ are eigenvectors. Clearly all solutions of (2.18) (eigenvectors) solve (2.19), but there is at least one more independent solution (see [113]) so that we have at least ν_i+1 independent vectors (note that these new vectors are no longer eigenvectors). If the number of independent solutions is still less than n_i, then we consider

$$(\mathcal{A}-\lambda_i\mathcal{I})^3\mathbf{z}=0, \tag{2.21}$$

or, equivalently,

$$(\mathcal{A}-\lambda_i\mathcal{I})\mathbf{z}=\mathbf{w}, \tag{2.22}$$

where vectors $\mathbf{w}$ solve (2.19), till we get a sufficient number of them. Note that to make sure that in the step j we select solutions that are independent of the solutions obtained in step $j-1$ it is enough to find solutions to $(\mathcal{A}-\lambda_i\mathcal{I})^j\mathbf{v}=0$ that satisfy $(\mathcal{A}-\lambda_i\mathcal{I})^{j-1}\mathbf{v}\neq 0$.

Vectors obtained in this way for a given λ_i are called *generalized* or *associated eigenvectors* corresponding to λ_i and they span an n_i dimensional subspace called a *generalized* or *associated eigenspace* corresponding to λ_i, denoted hereafter by E_{λ_i}.

2.1.3 The Exponential of a Matrix

A systematic construction of a fundamental matrix for (2.1) is offered by the exponential of $\mathcal{A}$. Recall that for a single equation $y' = ay$, where a is a constant, the general solution is given by $y(t) = e^{at}C$, where C is a constant. In a similar way, we would like to say that the general solution to (2.1) is $\mathbf{y} = e^{\mathcal{A}t}\,\mathring{\mathbf{y}}$, where $\mathring{\mathbf{y}}$ is any vector in $\mathbb{R}^n$. If we remember that the exponential of a scalar a is given by the power (Maclaurin) series

$$e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \ldots + \frac{a^k}{k!} + \ldots,$$

where the only involved operations on the argument a are additions, scalar multiplications and taking integer powers, then we can emulate this construction by defining

$$e^{\mathcal{A}} = \mathcal{I} + \mathcal{A} + \frac{1}{2}\mathcal{A}^2 + \frac{1}{3!}\mathcal{A}^3 + \ldots + \frac{1}{k!}\mathcal{A}^k + \ldots. \tag{2.23}$$

It can be proved, similarly to the scalar case, that this series of matrices converges with respect to the norm (2.6) (or any other norm in $\mathbb{R}^{n^2}$), to a matrix denoted hereafter by $e^{\mathcal{A}}$. To justify algebraic manipulations below, we note that

$$e^{\mathcal{A}+\mathcal{B}} = e^{\mathcal{A}}e^{\mathcal{B}}, \tag{2.24}$$

provided the matrices $\mathcal{A}$ and $\mathcal{B}$ commute: $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$. This yields

$$\left(e^{\mathcal{A}}\right)^{-1} = e^{-\mathcal{A}}.$$

Define a function of t by

$$e^{t\mathcal{A}} = \mathcal{I} + t\mathcal{A} + \frac{t^2}{2}\mathcal{A}^2 + \frac{t^3}{3!}\mathcal{A}^3 + \ldots + \frac{t^k}{k!}\mathcal{A}^k + \ldots\,. \tag{2.25}$$

This function has the same properties as the sum of a convergent power series in the scalar case. In particular, it is differentiable with respect to t and its derivative can be evaluated by termwise differentiation; that is,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}e^{t\mathcal{A}} &= \mathcal{A} + t\mathcal{A}^2 + \frac{t^2}{2!}\mathcal{A}^3 + \ldots + \frac{t^{k-1}}{(k-1)!}\mathcal{A}^k + \ldots \\ &= \mathcal{A}\left(\mathcal{I} + t\mathcal{A} + \frac{t^2}{2!}\mathcal{A}^2 + \ldots + \frac{t^{k-1}}{(k-1)!}\mathcal{A}^{k-1} + \ldots\right) \\ &= \mathcal{A}e^{t\mathcal{A}} = e^{t\mathcal{A}}\mathcal{A}.\end{aligned}$$

This shows that $y(t) = e^{t\mathcal{A}}\,\mathring{\mathbf{y}}$ is the solution of the Cauchy problem (2.1), (2.2) and thus $e^{t\mathcal{A}}$ is a fundamental solution matrix.

Though it is difficult to find directly the explicit form of $e^{t\mathcal{A}}$, we can always find n linearly independent vectors $\mathbf{v}^j$, $j = 1, \ldots, n$, for which the series $e^{t\mathcal{A}}\mathbf{v}^j$ are finite. This is based on the following two observations. Firstly, since $\lambda\mathcal{I}$ and $\mathcal{A} - \lambda\mathcal{I}$ commute, we have, by (2.24),

$$e^{t\mathcal{A}}\mathbf{v} = e^{t(\mathcal{A}-\lambda\mathcal{I})}e^{t\lambda\mathcal{I}}\mathbf{v} = e^{\lambda t}e^{t(\mathcal{A}-\lambda\mathcal{I})}\mathbf{v}.$$

Secondly, if $(\mathcal{A} - \lambda\mathcal{I})^m\mathbf{v} = \mathbf{0}$ for some m, then

$$(\mathcal{A} - \lambda\mathcal{I})^r\mathbf{v} = \mathbf{0}, \tag{2.26}$$

for all $r \geq m$. This follows from

$$(\mathcal{A} - \lambda\mathcal{I})^r\mathbf{v} = (\mathcal{A} - \lambda\mathcal{I})^{r-m}[(\mathcal{A} - \lambda\mathcal{I})^m\mathbf{v}] = \mathbf{0}.$$

Consequently, for such a $\mathbf{v}$

$$e^{t(\mathcal{A}-\lambda\mathcal{I})}\mathbf{v} = \mathbf{v} + t(\mathcal{A} - \lambda\mathcal{I})\mathbf{v} + \ldots + \frac{t^{m-1}}{(m-1)!}(\mathcal{A} - \lambda\mathcal{I})^{m-1}\mathbf{v}$$

and

$$\begin{aligned} e^{t\mathcal{A}}\mathbf{v} &= e^{\lambda t}e^{t(\mathcal{A}-\lambda\mathcal{I})}\mathbf{v} \\ &= e^{\lambda t}\left(\mathbf{v} + t(\mathcal{A} - \lambda\mathcal{I})\mathbf{v} + \ldots + \frac{t^{m-1}}{(m-1)!}(\mathcal{A} - \lambda\mathcal{I})^{m-1}\mathbf{v}\right). \end{aligned} \tag{2.27}$$

From our discussion of eigenvalues and eigenvectors it follows that if λ_i is a multiple eigenvalue of $\mathcal{A}$ of algebraic multiplicity n_i and the geometric multiplicity ν_i is less than n_i; that is, there is less than n_i linearly independent eigenvectors corresponding to λ_i, then the missing independent vectors can be found by solving successively equations $(\mathcal{A} - \lambda_i\mathcal{I})^k\mathbf{v} = \mathbf{0}$ with k running at most up to n_i. In this way, we can evaluate $e^{t\mathcal{A}}$ on n linearly independent eigenvectors and associated eigenvectors and hence on any $\mathbf{x} \in \mathbb{R}^n$.

Calculation of the Matrix Exponential

We will find $e^{t\mathcal{A}}$ for

$$\mathcal{A} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}.$$

To obtain the eigenvalues we calculate the characteristic polynomial

$$p(\lambda) = det(\mathcal{A} - \lambda\mathcal{I}) = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix}$$
$$= -(1+\lambda)(1-\lambda)(2-\lambda) + 12 + 2 - 8(2-\lambda) + (1-\lambda) - 3(1+\lambda)$$
$$= -(1+\lambda)(1-\lambda)(2-\lambda) + 4\lambda - 4 = (1-\lambda)(\lambda-3)(\lambda+2),$$

so that the eigenvalues of $\mathcal{A}$ are $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = -2$. All the eigenvalues have algebraic multiplicity of 1 so that they should give rise to 3 linearly independent eigenvectors.

(i) $\lambda_1 = 1$: we seek a nonzero vector $\mathbf{v}$ such that

$$(\mathcal{A} - 1\mathcal{I})\mathbf{v} = \begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ u_4 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus

$$-v_4 + 4v_3 = 0, \qquad 3v_1 + v_2 - v_3 = 0, \qquad 2v_1 + v_2 - 2v_3 = 0$$

and we get $v_2 = 4v_3$ and $v_1 = -v_3$ from the first two equations and the third is automatically satisfied. Thus we obtain the eigenspace corresponding to $\lambda_1 = 1$ containing all the vectors of the form

$$\mathbf{v^1} = C_1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix},$$

where C_1 is any constant and the corresponding solutions

$$\mathbf{y^1}(t) = C_1 e^t \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}.$$

(ii) $\lambda_2 = 3$: we seek a nonzero vector $\mathbf{v}$ such that

$$(\mathcal{A} - 3\mathcal{I})\mathbf{v} = \begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence

$$-2v_1 - v_2 + 4v_3 = 0, \qquad 3v_1 - v_2 - v_3 = 0, \qquad 2v_1 + v_2 - 4v_3 = 0.$$

Solving for v_1 and v_2 in terms of v_3 from the first two equations gives $v_1 = v_3$ and $v_2 = 2v_3$. Consequently, vectors of the form

$$\mathbf{v^2} = C_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

are eigenvectors corresponding to the eigenvalue $\lambda_2 = 3$ and the function

$$\mathbf{y^2}(t) = C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

is the second solution of the system.

(iii) $\lambda_3 = -2$: We have to solve

$$(\mathcal{A} + 2\mathcal{I})\mathbf{v} = \begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus

$$3v_1 - v_2 + 4v_3 = 0, \qquad 3v_1 + 4v_2 - v_3 = 0, \qquad 2v_1 + v_2 + v_3 = 0.$$

Again, solving for v_1 and v_2 in terms of v_3 from the first two equations gives $v_1 = -v_3$ and $v_2 = v_3$ so that each vector

$$\mathbf{v^3} = C_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to the eigenvalue $\lambda_3 = -2$. Consequently, the function

$$\mathbf{y^3}(t) = C_3 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

is the third solution of the system.

As in (2.14),

$$\mathcal{V} = \begin{pmatrix} -1 & 1 & -1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

with

$$\mathcal{V}^{-1} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}$$

and hence

$$e^{t\mathcal{A}} = \begin{pmatrix} -e^{t} & e^{3t} & -e^{-2t} \\ 4e^{t} & 2e^{3t} & e^{-2t} \\ e^{t} & e^{3t} & e^{-2t} \end{pmatrix} \begin{pmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}.$$

2.1.4 Spectral Decomposition

If $\mathbf{e}$ is an eigenvector of a matrix $\mathcal{A}$ corresponding to an eigenvalue λ, then the one-dimensional eigenspace $\tilde{E}_\lambda$ is *invariant* under $\mathcal{A}$ and hence under $e^{t\mathcal{A}}$; that is, if $\mathbf{y} \in \tilde{E}_\lambda$, then $\mathcal{A}\mathbf{y} \in \tilde{E}_\lambda$ (and $e^{t\mathcal{A}}\mathbf{y} \in \tilde{E}_\lambda$ for all $t > 0$). Thus, if $\mathcal{A}$ is diagonalizable, then the evolution governed by $\mathcal{A}$ can be decomposed into n independent scalar evolutions occurring in the eigenspaces of $\mathcal{A}$. The situation is more complicated, when we have multiple eigenvalues, as the one-dimensional spaces spanned by individual generalized eigenvectors are not invariant under $\mathcal{A}$. However, the construction (2.18)–(2.22) gives

Proposition 2.1.2. *Each generalized eigenspace E_λ of $\mathcal{A}$ is invariant under $\mathcal{A}$; that is, for any $\mathbf{v} \in E_\lambda$ we have $\mathcal{A}\mathbf{v} \in E_\lambda$. It is also invariant under $e^{t\mathcal{A}}$, $t \geq 0$.*

This result suggests that the evolution governed by $\mathcal{A}$ can be broken into several simpler and independent processes occurring in each generalized eigenspace. We begin by writing the expansion $\overset{\circ}{\mathbf{x}} = c_1\mathbf{e}^1 + \ldots + c_n\mathbf{e}^n$, where $\mathbf{e}^j$, $j = 1, \ldots, n$ are eigenvectors and generalized eigenvectors and c_j, $j = 1, \ldots, n$ are given by $\mathcal{V}^{-1}\mathbf{x}$ (see (2.8)), as

$$\mathbf{x} = \sum_{\lambda \in \sigma(\mathcal{A})} \mathcal{P}_\lambda \mathbf{x},$$

where $\mathcal{P}_\lambda$ is the so-called *spectral projection* corresponding to λ; that is, the 'part' of $\mathbf{x}$ belonging to the generalized eigenspace E_λ:

$$\mathcal{P}_\lambda \mathbf{x} = \sum_{\mathbf{e}^j \in E_\lambda} c_j \mathbf{e}^j.$$

Thus, we can write

$$\mathcal{A}\mathbf{x} = \sum_{\lambda \in \sigma(A)} \mathcal{A}\mathcal{P}_\lambda \mathbf{x}, \tag{2.28}$$

where, by Proposition 2.1.2, we have $\mathcal{A}\mathcal{P}_\lambda \mathbf{x} \in E_\lambda$. Further, we get $\mathcal{P}_\lambda \mathcal{A}\mathcal{P}_\mu = 0$ for $\lambda, \mu \in \sigma(\mathcal{A}), \lambda \neq \mu$. It also follows that

$$\mathcal{P}_\lambda \mathcal{A}\mathbf{x} = \mathcal{P}_\lambda \mathcal{A}\mathcal{P}_\lambda \mathbf{x} = \mathcal{A}\mathcal{P}_\lambda \mathbf{x}.$$

Thus, (2.28) defines a decomposition of the action of $\mathcal{A}$ into non-overlapping subspaces E_λ, $\lambda \in \sigma(\mathcal{A})$, which is called the *spectral decomposition* of $\mathcal{A}$. Using the fact that the spaces E_λ are also invariant under powers of $\mathcal{A}$, (2.23) and (2.27) give

$$e^{t\mathcal{A}}\mathbf{x} = \sum_{\lambda \in \sigma(\mathcal{A})} e^{\lambda t} \mathcal{P}_\lambda \mathbf{x} = \sum_{\lambda \in \sigma(\mathcal{A})} e^{\lambda t} \mathbf{q}_\lambda(t)\mathbf{x}, \tag{2.29}$$

where $\mathbf{q}_\lambda$ are polynomials in t, of degree strictly smaller than the algebraic multiplicity of λ and with vector coefficients being linear combinations of eigenvectors and associated eigenvectors corresponding to λ.

Thus, we see that in each eigenspace E_λ, the long time behaviour of $(e^{t\mathcal{A}})_{t\geq 0}$ is determined by $e^{t\lambda}$, possibly multiplied by a polynomial of a degree smaller than the algebraic multiplicity of λ. Therefore in the long run $(e^{t\mathcal{A}})_{t\geq 0}$ grows no faster than $e^{t\lambda'}$, where λ' is any number larger than the real part of the largest eigenvalue λ_{max} if $\Re\lambda_{max} > 0$, and decays to zero no slower than $e^{-\lambda' t}$, where $0 < \lambda' < -\Re\lambda_{max}$ if $\Re\lambda_{max} < 0$. In other words

$$\|e^{t\mathcal{A}}\mathbf{x}\| \leq M e^{\lambda' t}\|\mathbf{x}\|,$$

where λ' is any number satisfying $\lambda' > \Re\lambda_{max}$ and M is a constant.

In what follows we will need to explicitly find the spectral decomposition corresponding to a given eigenvalue λ':

$$\mathbf{x} = \mathcal{P}_{\lambda'}\mathbf{x} + \sum_{\lambda \in \sigma(\mathcal{A}), \lambda \neq \lambda'} \mathcal{P}_\lambda \mathbf{x} = \mathcal{P}_{\lambda'}\mathbf{x} + \mathcal{Q}_{\lambda'}\mathbf{x},$$

where $\mathcal{Q}_{\lambda'}$ is called the complementary spectral projection. In other words, we need to find the spectral projections $\mathcal{P}$ and $\mathcal{Q}$ in concrete cases. For this, we will need some new notation. Since the eigenvectors we have worked with so far satisfy $\mathcal{A}\mathbf{e} = \lambda\mathbf{e}$, they are often called *right eigenvectors* (with the same convention

for the associated eigenvectors); correspondingly we will use the notation E^r_λ to denote their span. Similarly, the vectors satisfying $\mathbf{f}\mathcal{A} = \lambda\mathbf{f}$ (or, equivalently, $\mathcal{A}^T\mathbf{f} = \lambda\mathbf{f}$, where $\mathcal{A}^T$ is the transpose of $\mathcal{A}$) are called *left eigenvectors* and the span of left eigenvectors and left associated eigenvectors is denoted by E^l_λ. Since $\sigma(\mathcal{A}) = \sigma(\mathcal{A}^T)$, there is no need to introduce right and left spectrum.

The importance of this distinction follows from the following result.

Proposition 2.1.3. *Let E^r_λ and E^l_μ be generalized right and left eigenspaces of $\mathcal{A}$ corresponding to different eigenvalues: $\lambda \neq \mu$. If $\mathbf{f} \in E^l_\mu$ and $\mathbf{e} \in E^r_\lambda$, then*

$$\mathbf{f} \cdot \mathbf{e} = 0. \tag{2.30}$$

Thus, if λ is a simple eigenvalue of $\mathcal{A}$, with corresponding left and right eigenvectors $\mathbf{e}$ and $\mathbf{f}$, respectively, then by writing $\mathbf{x}$ as the linear combination of right eigenvectors and associated eigenvectors and multiplying both sides with $\mathbf{f}$ we obtain

$$\mathcal{P}_\lambda \mathbf{x} = \frac{\mathbf{f} \cdot \mathbf{x}}{\mathbf{f} \cdot \mathbf{e}} \mathbf{e}. \tag{2.31}$$

Often it is useful to consider left and right eigenvectors normalized so as $\mathbf{f} \cdot \mathbf{e} = 1$. Then (2.31) takes the form

$$\mathcal{P}_\lambda \mathbf{x} = (\mathbf{f} \cdot \mathbf{x})\mathbf{e}. \tag{2.32}$$

However, if λ is not simple, then we can only claim that

$$\mathcal{P}_\lambda \mathbf{x} = c^1 \mathbf{e}_1 + \cdots + c^{n_\lambda} \mathbf{e}_{n_\lambda},$$

where n_λ is the algebraic multiplicity of λ and $\mathbf{e}_1, \cdots, \mathbf{e}_{n_\lambda} \in E^r_\lambda$. Since we can chose arbitrary bases in E^r_λ and E^l_λ, we can chose biorthonormal bases; that is, bases consisting of vectors satisfying

$$\mathbf{e}_i \cdot \mathbf{f}_j = 0, \quad \mathbf{e}_i \cdot \mathbf{f}_j = 1, \qquad \mathbf{e}_i \in E^r_\lambda, \mathbf{f}_j \in E^l_\lambda, \quad i, j = 1, \ldots, n_\lambda,$$

which gives

$$\mathcal{P}_\lambda \mathbf{x} = (\mathbf{x} \cdot \mathbf{f}_1)\mathbf{e}_1 + \cdots + (\mathbf{x} \cdot \mathbf{f}_{n_\lambda})\mathbf{e}_{n_\lambda}. \tag{2.33}$$

The complementary space W is then determined by

$$W = \{\mathbf{y};\ \mathcal{P}_\lambda \mathbf{y} = \mathbf{0}\},$$

but it is not necessary to use the eigenvectors as the basis for it. We can use the simplest basis and find the coefficients of the expansion of $\mathcal{Q}\mathbf{x}$ by solving the resulting system of linear algebraic equations.

In particular, if λ_{max} is a real, simple and dominant eigenvalue; that is, $\lambda_{max} > \Re\lambda$ for any other $\lambda \in \sigma(\mathcal{A})$, then, for any initial condition $\mathbf{x}$ for which $\mathcal{P}_{\lambda_{max}}\mathbf{x} \neq 0$, we have

$$e^{t\mathcal{A}}\mathbf{x} \approx e^{\lambda_{max}t}\frac{< \mathbf{f}_{max}, \mathbf{x} >}{< \mathbf{f}_{max}, \mathbf{e}_{max} >}\mathbf{e}_{max},$$

for large t, where $\mathbf{e}_{max}$ and $\mathbf{f}_{max}$ are right and left eigenvectors of $\mathcal{A}$, corresponding to λ_{max}.

2.1.5 Transition Matrices

Let us have a closer look at the special class of transition, or projection, matrices $\mathcal{C} = \{c_{ij}\}_{1\leq i,j\leq n}$ which will play a role in further considerations. We assume that they are Kolmogorov matrices, which are introduced in Sect. 1.3.3. We recall that then $c_{ij} \geq 0$ for $i \neq j$ and the columns satisfy (1.24). The property (1.24) can be expressed as

$$\mathbf{1}\mathcal{C} = \mathbf{0}$$

which shows that $\lambda = 0$ is an eigenvalue of $\mathcal{C}$ with $\mathbf{1}$ being its left eigenvector.

An important property of solutions of the problem $\mathbf{y}' = \mathcal{C}\mathbf{y}$ is that they are nonnegative for nonnegative initial conditions. In fact, the following, even more general result, is true.

Proposition 2.1.4. *The solution* $\mathbf{y}(t)$ *of*

$$\frac{\mathrm{d}\,\mathbf{y}}{\mathrm{d}\,t} = \mathcal{A}\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{y}_0,$$

satisfies $\mathbf{y}(t) \geq 0$ *for any* $t > 0$ *for arbitrary* $\mathbf{y}_0 \geq 0$ *if and only if* $\mathcal{A}$ *has nonnegative off-diagonal entries.*

Proof. First let us consider $\mathcal{A} \geq 0$. Then, using the representation (2.25) and the fact that the powers of a nonnegative matrix are nonnegative, we see that $e^{t\mathcal{A}} \geq 0$ for $t \geq 0$. Next, we observe that for any real a and $0 \leq \overset{\circ}{\mathbf{y}} \in \mathbb{R}^n$, the function $\mathbf{y}(t) = e^{at}e^{t\mathcal{A}}\mathbf{y}_0 \geq 0$ and satisfies the equation

$$\frac{\mathrm{d}\,\mathbf{y}}{\mathrm{d}\,t} = a\mathbf{y} + \mathcal{A}\mathbf{y} = (a\mathcal{I} + \mathcal{A})\mathbf{y}.$$

Hence, if the entries of $\mathcal{A}$, a_{ii}, are negative, then, denoting $r = \max_{1\leq i\leq n}\{-a_{ii}\}$, we find that $\tilde{\mathcal{A}} = r\mathcal{I} + \mathcal{A} \geq 0$. Using the first part of the proof, we see that

$$e^{t\mathcal{A}} = e^{-rt}e^{t\tilde{\mathcal{A}}} \geq 0. \tag{2.34}$$

To prove the converse, let us write

$$e^{t\mathcal{A}} = \mathcal{E}(t) = \begin{pmatrix} \varepsilon_{11}(t) & \dots & \varepsilon_{1n}(t) \\ \vdots & & \vdots \\ \varepsilon_{n1}(t) & \dots & \varepsilon_{nn}(t) \end{pmatrix},$$

so that $\varepsilon_{ij}(t) \geq 0$ for all $i, j = 1, \dots, n$, and consider $\mathcal{E}(t)\mathbf{e}_i = (\varepsilon_{1i}(t), \dots, \varepsilon_{ni}(t))$. Then

$$\begin{aligned}(a_{1i}, \dots, a_{ii}, \dots, a_{ni}) &= \mathcal{A}\mathcal{E}(t)\mathbf{e}_i|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{E}(t)\mathbf{e}_i\bigg|_{t=0} \\ &= \lim_{h\to 0^+}\left(\frac{\varepsilon_{1i}(h)}{h}, \dots, \frac{\varepsilon_{ii}(h)-1}{h}, \dots, \frac{\varepsilon_{ni}(h)}{h}\right),\end{aligned}$$

so that $a_{ji} \geq 0$ for $j \neq i$. □

Let us return to the system $\mathbf{y}' = \mathcal{C}\mathbf{y}$ with a Kolmogorov matrix $\mathcal{C}$. We shall show that, indeed, it is a conservative model. The total size of the population at any given time t is given by $N(t) = y_1(t) + \ldots + y_n(t)$. If $\mathbf{y}_0 \geq 0$, then, by the previous proposition, $\mathbf{y}(t) \geq 0$ for all $t > 0$ and the rate of change of N is given by

$$\begin{aligned}\frac{\mathrm{d}\,N}{\mathrm{d}\,t} &= \sum_{i=1}^{n}\frac{dy_i(t)}{\mathrm{d}t} = \sum_{i=1}^{n}\left(-y_i(t)\left(\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ji}\right) + \left(\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ij}\,y_j(t)\right)\right) \\ &= -\sum_{i=1}^{n} y_i(t)\left(\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ji}\right) + \sum_{i=1}^{n} y_i(t)\left(\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ji}\right) = 0,\end{aligned} \tag{2.35}$$

where we used the fact that i, j are dummy variables.

The results above show that $\|\mathbf{y}(t)\| = \|\overset{\circ}{\mathbf{y}}\|$ for any t and $\overset{\circ}{\mathbf{y}} \geq 0$. It follows that $\|\mathbf{y}(t)\| \leq \|\overset{\circ}{\mathbf{y}}\|$ for arbitrary $\overset{\circ}{\mathbf{y}}$. More generally, we prove that if solutions of $\mathbf{y}' = \mathcal{A}\mathbf{y}$, where $\mathcal{A}$ is positive off-diagonal, are bounded (or converge to $\mathbf{0}$) for nonnegative initial conditions, they are also bounded (or converge to $\mathbf{0}$) for arbitrary initial conditions. Indeed, let $\overset{\circ}{\mathbf{y}} \in \mathbb{R}^n$ be arbitrary. Since for any vector $\mathbf{x}$ we have $\|\mathbf{x}\| = \||\mathbf{x}|\|$ and $e^{t\mathcal{A}} \geq 0$ (so that $|e^{t\mathcal{A}}| = e^{t\mathcal{A}}$),

$$\|\mathbf{y}(t)\| = \left\|e^{t\mathcal{A}}\,\overset{\circ}{\mathbf{y}}\right\| \leq \left\||e^{t\mathcal{A}}|\,|\,\overset{\circ}{\mathbf{y}}\,|\right\| = \left\|e^{t\mathcal{A}}|\,\overset{\circ}{\mathbf{y}}\,|\right\| \tag{2.36}$$

and we see that the growth rate of a solution emanating from an arbitrary initial condition is controlled by the growth of the solution having the absolute value of this initial condition as its starting point.

This result can be used to show that $\lambda = 0$ is a dominant eigenvalue of a Kolmogorov matrix $\mathcal{C}$. Indeed, assume that there is $\lambda \in \sigma(\mathcal{C})$ with $\Re\lambda > 0$ and that $\mathbf{e}$ is a corresponding eigenvector. Then, (2.36),

$$\|e^{t\mathcal{C}}\mathbf{e}\| \leq \|\mathbf{e}\|. \tag{2.37}$$

However, $|e^{t\mathcal{C}}\mathbf{e}| = e^{\Re\lambda t}|\mathbf{e}|$ which is unbounded. Next, assume that there is an eigenvalue with $\Re\lambda = 0$ and different from $\lambda = 0$; that is, $\lambda = \pm i\omega$. But then there is a solution of the system which is periodic with orbits enclosing $\mathbf{0}$. However, such a solution cannot stay nonnegative.

2.2 The Asymptotic Procedure

Let us consider the singularly perturbed problem (1.64)

$$\begin{aligned} \frac{\mathrm{d}\,\mathbf{u}_\varepsilon}{\mathrm{d}\,t} &= \mathcal{S}\mathbf{u}_\varepsilon + \frac{1}{\varepsilon}\mathcal{C}\mathbf{u}_\varepsilon, \\ \mathbf{u}_\varepsilon(0) &= \overset{\circ}{\mathbf{u}}, \end{aligned} \tag{2.38}$$

where $\mathcal{S}$ and $\mathcal{C}$ are $n \times n$ matrices. According to the discussion in Sect. 1.4.1, the first step of the asymptotic procedure is determining the null-space $\mathbb{V}$ of the dominant transition operator $\mathcal{C}$; then the decomposition is performed using the (spectral) projection $\mathcal{P}$ onto $\mathbb{V}$, by applying $\mathcal{P}$ and the complementary projection $\mathcal{Q} = I - \mathcal{P}$ to (2.38). In this way we obtain a coupled system of evolution equations in the subspaces $\mathbb{V}$ and $\mathbb{W}$. At this point the kinetic part of the solution is expanded in series of ε, but the hydrodynamic part of the solution is left unexpanded. In other words, we keep all orders of approximation of the hydrodynamic part compressed into a single function.

Thus, let $\lambda = 0$ be an eigenvalue of $\mathcal{C}$. We write $\mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon = \mathcal{P}\mathbf{u}_\varepsilon + \mathcal{Q}\mathbf{u}_\varepsilon$, where $\mathcal{P}$ is the spectral projection corresponding $\lambda = 0$ of the state space $\mathbb{X}$ (here $\mathbb{X} = \mathbb{R}^N$) onto the hydrodynamic subspace $\mathbb{V}$ and $\mathcal{Q}$ is the complementary spectral projection onto the kinetic subspace $\mathbb{W}$. By properties of spectral projections, we have $\mathcal{PC} = \mathcal{CP} = 0$. Applying these projections to both sides of (2.38), we get

$$\begin{aligned} \frac{\mathrm{d}\,\mathbf{v}_\varepsilon}{\mathrm{d}\,t} &= \mathcal{PSP}\mathbf{v}_\varepsilon + \mathcal{PSQ}\mathbf{w}_\varepsilon, \\ \varepsilon\frac{\mathrm{d}\,\mathbf{w}_\varepsilon}{\mathrm{d}\,t} &= \varepsilon\mathcal{QSQ}\mathbf{w}_\varepsilon + \varepsilon\mathcal{QSP}\mathbf{v}_\varepsilon + \mathcal{QCQ}\mathbf{w}_\varepsilon, \end{aligned} \tag{2.39}$$

with the initial conditions

$$\mathbf{v}_\varepsilon(0) = \overset{\circ}{\mathbf{v}},\ \mathbf{w}_\varepsilon(0) = \overset{\circ}{\mathbf{w}},$$

where $\overset{\circ}{\mathbf{v}} = \mathcal{P}\overset{\circ}{\mathbf{u}}$, $\overset{\circ}{\mathbf{w}} = \mathcal{Q}\overset{\circ}{\mathbf{u}}$. We have kept the superfluous symbols $\mathcal{P}\mathbf{v} = \mathbf{v}$ and $\mathcal{Q}\mathbf{w} = \mathbf{w}$ for the sake of notational symmetry.

2.2.1 The Bulk Approximation

We begin with the bulk part, supposed to give a good approximation away from $t = 0$. Our main aim is to derive the limit equation (1.69) for the hydrodynamic part of the solution. We apply the Chapman–Enskog procedure and write $\mathbf{w}_\varepsilon = \overline{\mathbf{w}}_0 + \varepsilon\overline{\mathbf{w}}_1 + \dots$, but leave $\mathbf{v}_\varepsilon$ unexpanded. Now, let us insert this expansion into the system above getting

$$\begin{aligned}
\frac{\mathrm{d}\,\mathbf{v}_\varepsilon}{\mathrm{d}\,t} &= \mathcal{P}\mathcal{S}\mathcal{P}\mathbf{v}_\varepsilon + \mathcal{P}\mathcal{S}\mathcal{Q}\overline{\mathbf{w}}_0 + \varepsilon\mathcal{P}\mathcal{S}\mathcal{Q}\overline{\mathbf{w}}_1 + \dots,\\
\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\overline{\mathbf{w}}_0 + \varepsilon\overline{\mathbf{w}}_1 + \dots\right) &= \mathcal{Q}\mathcal{S}\mathcal{P}\mathbf{v}_\varepsilon + \mathcal{Q}\mathcal{S}\mathcal{Q}\overline{\mathbf{w}}_0 + \varepsilon\mathcal{Q}\mathcal{S}\mathcal{Q}\overline{\mathbf{w}}_1 + \dots\\
&\quad + \frac{1}{\varepsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}\overline{\mathbf{w}}_0 + \mathcal{Q}\mathcal{C}\mathcal{Q}\overline{\mathbf{w}}_1 + \dots,\\
\mathbf{v}_\varepsilon(0) &= \overset{\circ}{\mathbf{v}}, \qquad \overline{\mathbf{w}}_0(0) + \varepsilon\overline{\mathbf{w}}_1(0) + \dots = \overset{\circ}{\mathbf{w}}\,.
\end{aligned} \tag{2.40}$$

Here we see the advantage of the Chapman–Enskog procedure: the first equation of (2.40) describes an evolution in $\mathbb{V}$ and no longer is it singularly perturbed. Thus, its formal limit as $\varepsilon \to 0$ is a good candidate for the limit equation (1.69).

The idea of asymptotic expansions consists in the ansatz that both sides in (2.40) are treated as polynomials or power series in ε and thus the equality requires that the coefficients multiplying like powers of ε on both sides must be equal. For approximation purposes, usually we are satisfied with just few first powers. Thus, to finalize the construction, we compare like powers of ε in the second equation. At the level ε^{-1} we get $\mathcal{Q}\mathcal{C}\mathcal{Q}\overline{\mathbf{w}}_0 = 0$, which yields $\overline{\mathbf{w}}_0 = 0$ (since $\mathcal{Q}\mathcal{C}\mathcal{Q}$ is invertible on $\mathbb{W}$). Then, retaining only ε^0-order terms in the first equation of (2.40) and using the first initial condition gives a closed (limit) equation for an approximation of the hydrodynamic part $\mathbf{v}_\varepsilon$, denoted by $\overline{\mathbf{v}}$:

$$\begin{aligned}
\frac{\mathrm{d}\,\overline{\mathbf{v}}}{\mathrm{d}\,t} &= \mathcal{P}\mathcal{S}\mathcal{P}\overline{\mathbf{v}},\\
\overline{\mathbf{v}}(0) &= \overset{\circ}{\mathbf{v}}\,.
\end{aligned} \tag{2.41}$$

This is a (formal) limit equation (1.69) constructed using the Chapman–Enskog expansion method. Certainly, we have not proved yet that its solution satisfies $\lim_{\varepsilon\to 0+} \mathbf{v}_\varepsilon = \overline{\mathbf{v}}$. Also, we need to look at the kinetic part. Even if we are only interested in the zeroth-order terms of the expansion, it is advantageous to consider some higher-order terms. Thus, at the $\varepsilon^0 = 1$ level of the second equation in (2.40), we obtain

$$\mathcal{QSP}\mathbf{v}_\varepsilon + \mathcal{QSQ}\overline{\mathbf{w}}_0 + \mathcal{QCQ}\overline{\mathbf{w}}_1 = 0$$

and, upon replacing $\mathbf{v}_\varepsilon$ by its formal approximation $\overline{\mathbf{v}}$,

$$\overline{\mathbf{w}}_1 = (\mathcal{QCQ})^{-1}\mathcal{QSP}\overline{\mathbf{v}}. \tag{2.42}$$

Using $\overline{\mathbf{w}}_1$ we can construct an ε level approximating equation [see (1.68)] for $\mathbf{v}_\varepsilon$ by retaining the ε order term $\varepsilon\mathcal{PSQ}\overline{\mathbf{w}}_1$ in the first equation of (2.40). This yields

$$\frac{\mathrm{d}\,\overline{\mathbf{v}}}{\mathrm{d}\,t} = \mathcal{PSP}\overline{\mathbf{v}} + \varepsilon\mathcal{PSQ}(\mathcal{QCQ})^{-1}\mathcal{QSP}\overline{\mathbf{v}}. \tag{2.43}$$

In concrete applications, (2.43) describes a much richer dynamics than (2.41). For instance, in the nonlinear Boltzmann equation, (2.41) is the system of Euler equations while (2.43) translates into the Navier–Stokes system [68]. In the linear Boltzmann equation, (2.41) is just the free streaming equation and (2.43) is a much more complex drift-diffusion equation; see Chap. 6 and, for a more comprehensive theory, [168]. A similar problem is also considered in Sect. 4.3. In this introductory discussion, however, we shall focus on the zeroth-order approximation.

Though in many fields of the applied sciences the formal asymptotic analysis, such as described above, seems to be sufficient, from the mathematical point of view, the procedure is valid only if we can prove that the incurred error; that is, the difference between the exact solution and the constructed approximation, tends to 0 if $\varepsilon \to 0$. In other words, we have to prove the validity of (1.66).

The error incurred by only considering the constructed terms of the expansion is given by the pair

$$\begin{aligned}\mathbf{E}(t) &= (\mathbf{e}(t), \mathbf{f}(t)) \\ &:= (\mathbf{v}_\varepsilon(t) - \overline{\mathbf{v}}(t), \mathbf{w}_\varepsilon(t) - \varepsilon\overline{\mathbf{w}}_1(t)).\end{aligned} \tag{2.44}$$

One can note an apparently inconsistent inclusion of an ε order term $\varepsilon\overline{\mathbf{w}}_1$ into the approximation which should only contain the zeroth-order terms. As it turns out, the inclusion of this term simplifies the calculations below. To find the equation of the error, we substitute the above equation into (2.38). Thus

$$\begin{aligned}\frac{\mathrm{d}\,\mathbf{e}}{\mathrm{d}\,t} &= \frac{\mathrm{d}\,\mathbf{v}_\varepsilon}{\mathrm{d}\,t} - \frac{\mathrm{d}\,\overline{\mathbf{v}}}{\mathrm{d}\,t} = \mathcal{PSP}\mathbf{v} + \mathcal{PSQ}\mathbf{w} - \frac{\mathrm{d}\,\overline{\mathbf{v}}}{\mathrm{d}\,t} \\ &= \mathcal{PSP}\mathbf{e} + \mathcal{PSQ}\mathbf{f} + \varepsilon\mathcal{PSQ}\overline{\mathbf{w}}_1 + \mathcal{PSP}\overline{\mathbf{v}} - \frac{\mathrm{d}\,\overline{\mathbf{v}}}{\mathrm{d}\,t} \\ &= \mathcal{PSP}\mathbf{e} + \mathcal{PSQ}\mathbf{f} + \varepsilon\mathcal{PSQ}\overline{\mathbf{w}}_1, \end{aligned} \tag{2.45}$$

where we used the first equation of (2.41). Next

$$\begin{aligned}\frac{\mathrm{d}\,\mathbf{f}}{\mathrm{d}\,t} &= \frac{\mathrm{d}\,\mathbf{w}_\varepsilon}{\mathrm{d}\,t} - \varepsilon\frac{\mathrm{d}\,\overline{\mathbf{w}}_1}{\mathrm{d}\,t} = \mathcal{QSQ}\mathbf{w} + \mathcal{QSP}\mathbf{v} + \frac{1}{\varepsilon}\mathcal{QCQ}\mathbf{w} - \varepsilon\frac{\mathrm{d}\,\overline{\mathbf{w}}_1}{\mathrm{d}\,t} \\ &= \mathcal{QSQ}\mathbf{f} + \mathcal{QSP}\mathbf{e} + \mathcal{QSP}\overline{\mathbf{v}} + \varepsilon\mathcal{QSQ}\overline{\mathbf{w}}_1 + \frac{1}{\varepsilon}\mathcal{QCQ}\mathbf{f} + \mathcal{QCQ}\overline{\mathbf{w}}_1 - \varepsilon\frac{\mathrm{d}\,\overline{\mathbf{w}}_1}{\mathrm{d}\,t} \\ &= \mathcal{QSQ}\mathbf{f} + \frac{1}{\varepsilon}\mathcal{QCQ}\mathbf{f} - \varepsilon\frac{\mathrm{d}\,\overline{\mathbf{w}}_1}{\mathrm{d}\,t}, \end{aligned} \tag{2.46}$$

where we used (2.42). The error satisfies the following initial conditions

$$\mathbf{e}(0) = 0, \qquad \mathbf{f}(0) = \overset{\circ}{\mathbf{w}} - \varepsilon\overline{\mathbf{w}}_1(0) = \overset{\circ}{\mathbf{w}} + \varepsilon(\mathcal{QCQ})^{-1}\mathcal{QSP}\overset{\circ}{\mathbf{w}}.$$

Combining the above, we see that the error $\mathbf{E}$ is the solution of the problem

$$\begin{aligned}\frac{\mathrm{d}\,\mathbf{E}}{\mathrm{d}\,t} &= \mathcal{S}\mathbf{E} + \frac{1}{\varepsilon}\mathcal{C}\mathbf{E} + \varepsilon\mathbf{R}_1, \\ \mathbf{E}(0) &= \mathbf{R}_2 + \varepsilon\mathbf{R}_3, \end{aligned} \tag{2.47}$$

where

$$\mathbf{R}_1 = \begin{pmatrix} \mathcal{PSQ}\overline{\mathbf{w}}_1 \\ -\frac{\mathrm{d}}{\mathrm{d}\,t}\overline{\mathbf{w}}_{1,t} \end{pmatrix}, \quad \mathbf{R}_2 = \begin{pmatrix} 0 \\ \overset{\circ}{\mathbf{w}} \end{pmatrix}, \quad \mathbf{R}_3 = \begin{pmatrix} 0 \\ (\mathcal{QCQ})^{-1}\mathcal{QSP}\overset{\circ}{\mathbf{v}} \end{pmatrix}.$$

We observe that the problem for the error has the same structure as the original one, but it is inhomogeneous. It is natural to accept (as will be indeed demonstrated below) that the ε order contributions to the inhomogeneities, both in the equation and in the initial condition, result in the ε order contributions to the errors. Even then, unfortunately, (2.47) does not give us good estimates for the error. This is due to the contribution $\mathbf{R}_2$ in the initial condition which is not of ε order.

There are two natural ways to deal with this problem. One is to adopt appropriate assumptions which will eliminate the troublesome term. For instance, we see that if the initial condition satisfies $\overset{\circ}{\mathbf{u}}_\varepsilon = (\overset{\circ}{\mathbf{v}}, 0)$; that is, if we start from the hydrodynamic subspace, only the ε order terms will be present in the initial condition.

Under this assumption the equation for the error takes the following form:

$$\begin{aligned} \frac{\mathrm{d}\,\mathbf{E}}{\mathrm{d}\,t} &= \mathcal{S}\mathbf{E} + \frac{1}{\varepsilon}\mathcal{C}\mathbf{E} + \varepsilon\mathbf{R}_1, \\ \mathbf{E}(0) &= \varepsilon\mathbf{R}_3. \end{aligned} \tag{2.48}$$

Using the variation of constants formula, we have

$$\mathbf{E}(t) = \varepsilon e^{t\mathcal{K}_\varepsilon}\mathbf{R}_3 + \varepsilon \int_0^t e^{(t-s)\mathcal{K}_\varepsilon}\mathbf{R}_1(s)\mathrm{d}s. \tag{2.49}$$

Now, we have to prove that the matrix exponential $(e^{t\mathcal{K}_\varepsilon})_{t\geq 0}$, where $\mathcal{K}_\varepsilon := \mathcal{S} + \frac{1}{\varepsilon}\mathcal{C}$, is bounded independently of ε. For this we use (2.39) written as

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\begin{pmatrix}\mathbf{v}_\varepsilon(t)\\ \mathbf{w}_\varepsilon(t)\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{1}{\varepsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}\end{pmatrix}\begin{pmatrix}\mathbf{v}_\varepsilon(t)\\ \mathbf{w}_\varepsilon(t)\end{pmatrix} + \begin{pmatrix}\mathcal{P}\mathcal{S}\mathcal{P} & \mathcal{P}\mathcal{S}\mathcal{Q}\\ \mathcal{Q}\mathcal{S}\mathcal{P} & \mathcal{Q}\mathcal{S}\mathcal{Q}\end{pmatrix}\begin{pmatrix}\mathbf{v}_\varepsilon(t)\\ \mathbf{w}_\varepsilon(t)\end{pmatrix}. \tag{2.50}$$

Then the variation of constants formula gives

$$\begin{aligned} e^{t\mathcal{K}_\varepsilon}\begin{pmatrix}\overset{\circ}{\mathbf{v}}\\ \overset{\circ}{\mathbf{w}}\end{pmatrix} &= \begin{pmatrix}\mathbf{v}_\varepsilon(t)\\ \mathbf{w}_\varepsilon(t)\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & e^{\frac{t}{\varepsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}}\end{pmatrix}\begin{pmatrix}\overset{\circ}{\mathbf{v}}\\ \overset{\circ}{\mathbf{w}}\end{pmatrix} \\ &+ \int_0^t \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & e^{\frac{t-s}{\varepsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}}\end{pmatrix}\begin{pmatrix}\mathcal{P}\mathcal{S}\mathcal{P} & \mathcal{P}\mathcal{S}\mathcal{Q}\\ \mathcal{Q}\mathcal{S}\mathcal{P} & \mathcal{Q}\mathcal{S}\mathcal{Q}\end{pmatrix}\begin{pmatrix}\mathbf{v}_\varepsilon(s)\\ \mathbf{w}_\varepsilon(s)\end{pmatrix}\mathrm{d}s. \end{aligned} \tag{2.51}$$

Now, since all eigenvalues of $\mathcal{Q}\mathcal{C}\mathcal{Q}$ have negative real parts, $\left(e^{\frac{t}{\varepsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}}\right)_{t\geq 0}$ is bounded for all $t \geq 0$. Hence, we can continue the above estimates as

$$\left\|\begin{pmatrix}\mathbf{v}_\varepsilon(t)\\ \mathbf{w}_\varepsilon(t)\end{pmatrix}\right\| \leq M\left\|\begin{pmatrix}\overset{\circ}{\mathbf{v}}\\ \overset{\circ}{\mathbf{w}}\end{pmatrix}\right\| + M\|\mathcal{S}\|\int_0^t \left\|\begin{pmatrix}\mathbf{v}_\varepsilon(s)\\ \mathbf{w}_\varepsilon(s)\end{pmatrix}\right\|\mathrm{d}s$$

and, by the Gronwall lemma,

$$\left\|\begin{pmatrix}\mathbf{v}_\varepsilon(t)\\ \mathbf{w}_\varepsilon(t)\end{pmatrix}\right\| \leq M\left\|\begin{pmatrix}\overset{\circ}{\mathbf{v}}\\ \overset{\circ}{\mathbf{w}}\end{pmatrix}\right\| e^{\|S\|Mt}. \tag{2.52}$$

Thus, the error $\mathbf{E}$ can be estimated as

$$\|\mathbf{E}(t)\| \leq \varepsilon M_1 e^{\omega T}\|\overset{\circ}{\mathbf{v}}\| + \varepsilon M_2 e^{\omega T} T \sup_{0\leq t\leq T}\|\overline{\mathbf{w}}_1(t)\| \leq \varepsilon M_3(T), \tag{2.53}$$

where $\omega = M\|\mathcal{S}\|$, M_1, M_2 depend on the norms of $\mathcal{PSQ}, \mathcal{PSP}, \mathcal{QSP}, \mathcal{QSQ}$, $(\mathcal{QSQ})^{-1}$ and M_3 combines all other constants and clearly depends on T. This gives the convergence, uniform on any finite time interval $[0, T]$.

Under additional conditions, which often are met in practice, this convergence is uniform on $[0, \infty[$. Indeed, if we assume that the dynamical system generated by $\mathcal{PSP}$ is asymptotically stable; that is,

$$\|e^{t\mathcal{PSP}} \overset{\circ}{\mathbf{v}}\| \leq L_2 e^{-\sigma' t} \|\overset{\circ}{\mathbf{v}}\|, \tag{2.54}$$

and $(e^{t\mathcal{K}_\varepsilon})_{t\geq 0}$ is bounded with respect to ε on $[0, \infty)$, then (2.53) can be improved as follows:

$$\begin{aligned}\|\mathbf{E}(t)\| &\leq \varepsilon M_1 \|\overset{\circ}{\mathbf{v}}\| + \varepsilon M_4 \int_0^t \|\overline{\mathbf{w}}_1(s)\| \mathrm{d}s \leq \varepsilon M_1 \|\overset{\circ}{\mathbf{v}}\| + \varepsilon M_5 \|\overset{\circ}{\mathbf{v}}\| \int_0^\infty e^{-\sigma' s} \mathrm{d}s \\ &\leq \varepsilon M_6, \end{aligned} \tag{2.55}$$

uniformly on $[0, \infty[$ where, by (2.42), the asymptotic behaviour of $\overline{\mathbf{w}}_1$ is the same as that of $\overline{\mathbf{v}}$. We note that a related result for nonlinear systems was proved in [116].

The approach discussed above is, however, not completely satisfactory as it requires the system to start from the equilibrium (averaged) initial data and we miss the transient phenomena occurring when the system stabilizes.

To remedy the situation we have to introduce a correction which will take care of the transient phenomena occurring close to $t = 0$. It should not 'spoil' the approximation away from the temporal boundary and thus it should rapidly decrease to zero with increasing t.

2.2.2 *The Initial Layer*

As was mentioned earlier, we blow up the neighbourhood of $t = 0$ by rescaling time as $\tau = \frac{t}{\varepsilon}$ and look for a new approximate solution $\tilde{\mathbf{f}}(\tau) = (\tilde{\mathbf{v}}(\tau), \tilde{\mathbf{w}}(\tau))$. Thanks to the linearity of the problem, we will try to approximate the solution $\mathbf{u}_\varepsilon$ as the sum of the bulk part, obtained above, and the initial layer which we construct now. As here we do not need to keep the hydrodynamic part in one piece, we use the simpler Hilbert expansion; that is, we write

$$\begin{aligned}\tilde{\mathbf{v}}(\tau) &= \tilde{\mathbf{v}}_0(\tau) + \varepsilon \tilde{\mathbf{v}}_1(\tau) + \dots, \\ \tilde{\mathbf{w}}(\tau) &= \tilde{\mathbf{w}}_0(\tau) + \varepsilon \tilde{\mathbf{w}}_1(\tau) + \dots \end{aligned}$$

and insert this expansion into (2.39), getting

$$\varepsilon^{-1}\left(\frac{\mathrm{d}\,\tilde{\mathbf{v}}_0}{\mathrm{d}\,\tau}+\varepsilon\frac{\mathrm{d}\,\tilde{\mathbf{v}}_1}{\mathrm{d}\,\tau}+\ldots\right)=\mathcal{PSP}(\tilde{\mathbf{v}}_0+\varepsilon\tilde{\mathbf{v}}_1+\ldots)+\mathcal{PSQ}(\tilde{\mathbf{w}}_0+\varepsilon\tilde{\mathbf{w}}_1+\ldots),$$

$$\varepsilon^{-1}\left(\frac{\mathrm{d}\,\tilde{\mathbf{w}}_0}{\mathrm{d}\,\tau}+\varepsilon\frac{\mathrm{d}\,\tilde{\mathbf{w}}_1}{\mathrm{d}\,\tau}+\ldots\right)=\mathcal{QSQ}(\tilde{\mathbf{w}}_0+\varepsilon\tilde{\mathbf{w}}_1+\ldots)+\mathcal{QSP}(\tilde{\mathbf{v}}_0+\varepsilon\tilde{\mathbf{v}}_1+\ldots)$$

$$+\frac{1}{\varepsilon}\mathcal{QCQ}(\tilde{\mathbf{w}}_0+\varepsilon\tilde{\mathbf{w}}_1+\ldots),$$

$$\tilde{\mathbf{v}}(0)=0,\qquad \overline{\mathbf{w}}_0(0)+\varepsilon\overline{\mathbf{w}}_1(0)+\ldots=\overset{\circ}{\mathbf{w}},\tag{2.56}$$

where in the initial condition we have taken into account that the bulk hydrodynamic approximation $\overline{\mathbf{v}}$ already satisfies the hydrodynamic part of the initial data for (2.39), but the bulk kinetic part cannot satisfy the kinetic part of the original initial data.

As before, we compare the coefficients at like powers of ε. Beginning with the ε^{-1} level, from the first equation and the first initial condition, we immediately obtain $\tilde{\mathbf{v}}_0 = 0$. Then, at the same ε^{-1} level, from the second equation and the corresponding initial condition, we obtain

$$\frac{\mathrm{d}\,\tilde{\mathbf{w}}_0}{\mathrm{d}\,\tau}=\mathcal{QCQ}\tilde{\mathbf{w}}_0\tag{2.57}$$

which yields

$$\tilde{\mathbf{w}}_0(\tau)=e^{\tau\mathcal{QCQ}}\,\overset{\circ}{\mathbf{w}}\,.$$

We note that due to the assumption that $\lambda = 0$ is the dominant eigenvalue of $\mathcal{C}$ and $\mathcal{Q}$ is the complementary spectral projection corresponding to $\lambda = 0$, $\tilde{\mathbf{w}}_0$ decays to 0 exponentially fast.

Let us modify the approximation taking into account the initial layer:

$$\mathbf{u}_\varepsilon(t)=(\mathbf{v}_\varepsilon(t),\mathbf{w}_\varepsilon(t))\approx(\overline{\mathbf{v}}(t),\varepsilon\overline{\mathbf{w}}_1+\tilde{\mathbf{w}}_0)\tag{2.58}$$

and define the new error

$$\begin{aligned}\tilde{\mathbf{E}}(t)&=(\tilde{\mathbf{e}}(t),\tilde{\mathbf{f}}(t))=(\mathbf{v}_\varepsilon(t)-\overline{\mathbf{v}}(t),\mathbf{w}_\varepsilon(t)-\varepsilon\overline{\mathbf{w}}_1(t)-\tilde{\mathbf{w}}_0(t/\varepsilon))\\&=(\mathbf{e}(t),\mathbf{f}(t)-\tilde{\mathbf{w}}_0(t/\varepsilon)).\end{aligned}\tag{2.59}$$

Using the fact that the problem is linear we get, from (2.45) and (2.46),

$$\frac{\mathrm{d}\,\tilde{\mathbf{e}}}{\mathrm{d}\,t}=\frac{\mathrm{d}\,\mathbf{e}}{\mathrm{d}\,t}=\mathcal{PSP}\mathbf{e}+\mathcal{PSQ}\mathbf{f}+\varepsilon\mathcal{PSQ}\overline{\mathbf{w}}_1=\mathcal{PSP}\tilde{\mathbf{e}}+\mathcal{PSQ}\tilde{\mathbf{f}}+\mathcal{PSQ}\tilde{\mathbf{w}}_0+\varepsilon\mathcal{PSQ}\overline{\mathbf{w}}_1$$

and, using (2.57) and $\frac{\mathrm{d}}{\mathrm{d}t} = \varepsilon^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}$,

$$\begin{aligned}\frac{\mathrm{d}\tilde{\mathbf{f}}}{\mathrm{d}t} &= \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} - \frac{1}{\varepsilon}\frac{\mathrm{d}\tilde{\mathbf{w}}_0}{\mathrm{d}\tau} = \mathcal{QSP}\mathbf{e} + \mathcal{QSQ}\mathbf{f} + \frac{1}{\varepsilon}\mathcal{QCQ}\mathbf{f} - \varepsilon\frac{\mathrm{d}\overline{\mathbf{w}}_1}{\mathrm{d}t} - \frac{1}{\varepsilon}\frac{\mathrm{d}\tilde{\mathbf{w}}_0}{\mathrm{d}\tau}\\ &= \mathcal{QSP}\tilde{\mathbf{e}} + \mathcal{QSQ}\tilde{\mathbf{f}} + \frac{1}{\varepsilon}\mathcal{QCQ}\tilde{\mathbf{f}} + \mathcal{QSQ}\tilde{\mathbf{w}}_0 - \varepsilon\frac{\mathrm{d}\overline{\mathbf{w}}_1}{\mathrm{d}t} + \frac{1}{\varepsilon}\mathcal{QCQ}\tilde{\mathbf{w}}_0 - \frac{1}{\varepsilon}\frac{\mathrm{d}\tilde{\mathbf{w}}_0}{\mathrm{d}\tau}\\ &= \mathcal{QSP}\tilde{\mathbf{e}} + \mathcal{QSQ}\tilde{\mathbf{f}} + \frac{1}{\varepsilon}\mathcal{QCQ}\tilde{\mathbf{f}} + \mathcal{QSQ}\tilde{\mathbf{w}}_0 - \varepsilon\frac{\mathrm{d}\overline{\mathbf{w}}_1}{\mathrm{d}t}.\end{aligned}$$

Similarly, we get

$$\tilde{\mathbf{e}}(0) = 0, \qquad \tilde{\mathbf{f}}(0) = -\varepsilon\overline{\mathbf{w}}_1(0) = \varepsilon(\mathcal{QCQ})^{-1}\mathcal{QSP}\,\mathring{\mathbf{v}}\,.$$

Applying the variation of constants (Duhamel) formula to $(\mathbf{e}(t), \mathbf{f}(t))$ and using the fact that the inhomogeneity is the sum of the terms present in (2.53) and of the initial layer contribution, we obtain

$$\begin{aligned}\|\tilde{\mathbf{E}}(t)\| &\le \varepsilon M_1\|\,\mathring{\mathbf{v}}\,\| + \varepsilon M_2 T \sup_{0\le t\le T}\|\overline{\mathbf{w}}_1(t)\| + M_7\int_0^t e^{-\frac{\eta s}{\varepsilon}}\mathrm{d}s\\ &\le \varepsilon M_3(T) + \varepsilon M_7\int_0^{t/\varepsilon} e^{-\eta r}\mathrm{d}r = \varepsilon M_8(T),\end{aligned} \tag{2.60}$$

where $-\eta < 0$ is any number greater than the real part of all eigenvalues of $\mathcal{QCQ}$. Since the estimate for the initial layer term does not depend on T, we see that the error is uniform on $[0,\infty[$ under the same assumptions as before: that $\mathcal{PSP}$ generates an asymptotically stable dynamical system and $(e^{t\mathcal{K}_\varepsilon})_{t\ge 0}$ is equibounded in ε on $[0,\infty[$.

We also observe that the term $\varepsilon\overline{\mathbf{w}}_1$ in (2.58) is of order ε (and uniformly so on $[0,\infty[$ if $\mathcal{PSP}$ is asymptotically stable) so that it can be moved to the error term and the error of the simplified expansion

$$\tilde{\mathbf{E}}^{\#}(t) = (\mathbf{v}(t) - \overline{\mathbf{v}}(t), \mathbf{w}(t) - \tilde{\mathbf{w}}_0(t/\varepsilon))$$

satisfies

$$\|\tilde{\mathbf{E}}^{\#}(t)\| \le \|\tilde{\mathbf{E}}(t)\| + \varepsilon\|\overline{\mathbf{w}}_1(t)\| \le M(T)\varepsilon \tag{2.61}$$

which again is uniform on $[0,\infty[$ if $(e^{t\mathcal{PSP}})_{t\ge 0}$ is asymptotically stable and $(e^{t\mathcal{K}_\varepsilon})_{t\ge 0}$ is equibounded in ε.

2.3 Interacting Populations with Space Structure

To illustrate the theory we discuss a version of the migration problem (1.28). First consider a simplified linear model of interactions between two species,

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t} &= aU + bV, \\ \frac{\mathrm{d}V}{\mathrm{d}t} &= fU + lV. \end{aligned} \tag{2.62}$$

Here a and l are the net reproduction rates and the coefficients b and f describe the influence the species have on each other. For instance, if $b > 0$ and $f < 0$, then U has a detrimental effect on V, but itself benefits from it (like in a prey–predator model).

We note that the solutions of (2.62) are positive for nonnegative initial data only if $b, f \geq 0$ (see Proposition 2.1.4). Thus, in general, we cannot interpret (2.62) as a population model. However, linear problems like this often are obtained as linearizations of nonlinear problems and in such cases U and V are interpreted as deviations from some reference, or equilibrium, levels and thus can take arbitrary sign. We shall adapt this convention and thus we will analyse the problem in the whole space.

Next we assume that each species can reside in one of two patches and they can migrate between them. Accordingly, we subdivide the species as $U = (u_1, u_2)$ and $V = (u_3, u_4)$. Similarly, we introduce the patch-specific growth and influence coefficients. Since the migration is an intraspecies process (you cannot change your species moving between patches), the suitable system can be written as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} &= \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ f_1 & 0 & l_1 & 0 \\ 0 & f_2 & 0 & l_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \\ &\quad + \begin{pmatrix} -d_1 & d_2 & 0 & 0 \\ d_1 & -d_2 & 0 & 0 \\ 0 & 0 & -g_1 & g_2 \\ 0 & 0 & g_1 & -g_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \end{aligned} \tag{2.63}$$

or, in short,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{F} & \mathcal{L} \end{pmatrix}\begin{pmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{pmatrix} + \begin{pmatrix} \mathcal{D} & \mathbf{0} \\ \mathbf{0} & \mathcal{G} \end{pmatrix}\begin{pmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{pmatrix}, \tag{2.64}$$

with the initial condition $(\mathring{\mathbf{u}}^1, \mathring{\mathbf{u}}^2) = (\mathring{u}_1, \mathring{u}_2, \mathring{u}_3, \mathring{u}_4)$.

An interesting variant of this example is offered by the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\\ \mathbf{u}^3\end{pmatrix} = \begin{pmatrix}\mathcal{A} & \mathcal{B}^1 & \mathcal{B}^2\\ \mathcal{F} & \mathcal{L}^1 & \mathcal{L}^2\\ \mathbf{0} & \mathbf{0} & \mathcal{B}^3\end{pmatrix}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\\ \mathbf{u}^3\end{pmatrix} + \begin{pmatrix}\mathcal{D} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathcal{G}^1 & \mathcal{H}\\ \mathbf{0} & \mathbf{0} & \mathcal{G}^2\end{pmatrix}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\\ \mathbf{u}^3\end{pmatrix}, \tag{2.65}$$

which could describe a situation similar to the previous one but in which there is a safe place for the species denoted by $(\mathbf{u}^2, \mathbf{u}^3)$, which is inaccessible to predators $\mathbf{u}^1$ (but also to prey normally living elsewhere). One could think about safe breeding area with resident breeding females (queens) supplying the other areas. Note that the matrices in the system above need not be square, or of the same dimension, but clearly their dimensions must be such that the operations above can be carried out.

We note that (2.63) could be considered as a variant of the problem (1.28), considered in Sect. 1.3.3, to which we added interactions between the species. Thus, it is natural to consider a similar perturbed problem, in which the migrations between geographical locations occur at a much faster rate than the demographic processes or the interactions between species. Thus, we consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\end{pmatrix} = \begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{F} & \mathcal{L}\end{pmatrix}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix}\mathcal{D} & \mathbf{0}\\ \mathbf{0} & \mathcal{G}\end{pmatrix}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\end{pmatrix}, \tag{2.66}$$

where the small parameter ε represents the ratio of the rates of the demographic and the migratory processes. We assume that d_1, d_2, g_1, g_2 are strictly positive. Then the migration matrix is a reducible Kolmogorov matrix with a two-dimensional left and right eigenspaces corresponding to the eigenvalue $\lambda = 0$. In this case it is easy to see this directly. Following the asymptotic procedure described above, we look for solutions to

$$\mathcal{C}\begin{pmatrix}\mathbf{e}^1\\ \mathbf{e}^2\end{pmatrix} = \begin{pmatrix}\mathcal{D} & \mathbf{0}\\ \mathbf{0} & \mathcal{G}\end{pmatrix}\begin{pmatrix}\mathbf{e}^1\\ \mathbf{e}^2\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}$$

which splits into two independent systems

$$\mathcal{D}\tilde{\mathbf{e}}^1 = \mathbf{0}, \qquad \mathcal{G}\tilde{\mathbf{e}}^2 = \mathbf{0}.$$

Thus, we have, up to a scalar multiple,

$$\tilde{\mathbf{e}}^1 = (d_2, d_1), \qquad \tilde{\mathbf{e}}^2 = (g_2, g_1).$$

Since both $\mathcal{C}$ and $\mathcal{G}$ are Kolmogorov matrices, their left eigenvectors are

$$\mathbf{f}^1 = (1, 1), \qquad \mathbf{f}^2 = (1, 1).$$

We normalize $\tilde{\mathbf{e}}^i$ to $\mathbf{e}^i, i = 1, 2$, so that

$$\mathbf{e}^i \cdot \mathbf{f}^i = 1,$$

thus

$$\mathbf{e}^1 = (e_1^1, e_2^1) = \left(\frac{d_2}{d_2 + d_1}, \frac{d_1}{d_2 + d_1}\right), \qquad \mathbf{e}^2 = (e_1^2, e_2^2) = \left(\frac{g_2}{g_2 + g_1}, \frac{g_1}{g_2 + g_1}\right).$$

The left and right eigenspaces corresponding to 0 are given, respectively, by

$$E_0^l = Span\{(\mathbf{f}^1, \mathbf{0}), (\mathbf{0}, \mathbf{f}^2)\}, \quad E_0^r = Span\{(\mathbf{e}^1, \mathbf{0}), (\mathbf{0}, \mathbf{e}^2)\},$$

and the bases are biorthonormal.

Let $\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$. Then, following the procedure of Sect. 2.1.4 and, in particular, (2.33), we obtain the spectral projection $\mathcal{P}$ as

$$\begin{aligned} \mathcal{P}\mathbf{u} &= ((\mathbf{f}^1, \mathbf{0}) \cdot \mathbf{u})(\mathbf{e}^1, \mathbf{0}) + ((\mathbf{0}, \mathbf{f}^2) \cdot \mathbf{u})(\mathbf{0}, \mathbf{e}^2) \\ &= (u_1 + u_2)(\mathbf{e}^1, \mathbf{0}) + (u_3 + u_4)(\mathbf{0}, \mathbf{e}^2). \end{aligned} \tag{2.67}$$

To find the basis for the complementary space, we find $\mathbf{w}$ for which $\mathcal{P}\mathbf{w} = \mathbf{0}$. Thus

$$\mathbf{0} = \mathcal{P}\mathbf{w} = (u_1 + u_2)(\mathbf{e}^1, \mathbf{0}) + (u_3 + u_4)(\mathbf{0}, \mathbf{e}^2)$$

which yields $u_1 = -u_2$ and $u_3 = -u_4$. Hence, the spectral decomposition of $\mathbf{u}$ is

$$\begin{aligned} \mathbf{u} &= \mathcal{P}\mathbf{u} + \mathcal{Q}\mathbf{u} \\ &= (u_1 + u_2)(\mathbf{e}^1, \mathbf{0}) + (u_3 + u_4)(\mathbf{0}, \mathbf{e}^2) + w_1(\mathbf{h}, \mathbf{0}) + w_2(\mathbf{0}, \mathbf{h}), \end{aligned} \tag{2.68}$$

where $\mathbf{h} = (1, -1)$ and w_1, w_2 are to be determined. Denoting $v_1 = u_1 + u_2$ and $v_2 = u_3 + u_4$, we find

$$\begin{aligned} u_1 &= \frac{v_1 c_2}{d_1 + d_2} + w_1, \qquad u_2 = \frac{v_1 c_1}{d_1 + d_2} - w_1, \\ u_3 &= \frac{v_2 g_2}{g_1 + g_2} + w_2, \qquad u_4 = \frac{v_2 g_1}{g_1 + g_2} - w_2 \end{aligned}$$

and, inverting,

$$w_1 = \frac{d_1 u_1 - d_2 u_2}{d_1 + d_2}, \qquad w_2 = \frac{g_1 u_3 - g_2 u_4}{g_1 + g_2}.$$

Now we can write the operators involved in the projected form (2.39) of (2.66). First,

$$\mathcal{SP}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\end{pmatrix}=\begin{pmatrix}\mathcal{A}\ \mathcal{B}\\ \mathcal{F}\ \mathcal{L}\end{pmatrix}\left(v_1\begin{pmatrix}\mathbf{e}^1\\ \mathbf{0}\end{pmatrix}+v_2\begin{pmatrix}\mathbf{0}\\ \mathbf{e}^2\end{pmatrix}\right)=v_1\begin{pmatrix}\mathcal{A}\mathbf{e}^1\\ \mathcal{F}\mathbf{e}^1\end{pmatrix}+v_2\begin{pmatrix}\mathcal{B}\mathbf{e}^2\\ \mathcal{L}\mathbf{e}^2\end{pmatrix}.$$

Thus,

$$\begin{aligned}\mathcal{PSP}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\end{pmatrix}&=\left(v_1(\mathbf{1}\cdot\mathcal{A}\mathbf{e}^1)+v_2(\mathbf{1}\cdot\mathcal{B}\mathbf{e}^2)\right)\begin{pmatrix}\mathbf{e}^1\\ \mathbf{0}\end{pmatrix}\\&\quad+\left(v_1(\mathbf{1}\cdot\mathcal{F}\mathbf{e}^1)+v_2(\mathbf{1}\cdot\mathcal{L}\mathbf{e}^2)\right)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}^2\end{pmatrix},\end{aligned}$$

with

$$\begin{aligned}\mathbf{1}\cdot\mathcal{A}\mathbf{e}^1&=\frac{a_1d_2+a_2d_1}{d_1+d_2},\qquad \mathbf{1}\cdot\mathcal{B}\mathbf{e}^2=\frac{b_1g_2+b_2g_1}{g_1+g_2},\\ \mathbf{1}\cdot\mathcal{F}\mathbf{e}^1&=\frac{f_1d_2+f_2d_1}{d_1+d_2},\qquad \mathbf{1}\cdot\mathcal{E}\mathbf{e}^2=\frac{l_1g_2+l_2g_1}{g_1+g_2}.\end{aligned}$$

Now we can write the equation for the hydrodynamic approximation (2.41),

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{P}\begin{pmatrix}\bar{\mathbf{u}}^1\\ \bar{\mathbf{u}}^2\end{pmatrix}=\mathcal{PSP}\begin{pmatrix}\bar{\mathbf{u}}^1\\ \bar{\mathbf{u}}^2\end{pmatrix},$$

as a system of scalar equations. Since

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\mathcal{P}\begin{pmatrix}\bar{\mathbf{u}}^1\\ \bar{\mathbf{u}}^2\end{pmatrix}\right)=\frac{\mathrm{d}\,\bar{v}_1}{\mathrm{d}\,t}\begin{pmatrix}\mathbf{e}^1\\ \mathbf{0}\end{pmatrix}+\frac{\mathrm{d}\,\bar{v}_2}{\mathrm{d}\,t}\begin{pmatrix}\mathbf{0}\\ \mathbf{e}^2\end{pmatrix},$$

comparing the coefficients at the basis vectors $(\mathbf{e}^1,\mathbf{0})$ and $(\mathbf{0},\mathbf{e}^2)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}\,\bar{v}_1}{\mathrm{d}\,t}&=\bar{v}_1\frac{a_1d_2+a_2d_1}{d_1+d_2}+\bar{v}_2\frac{b_1g_2+b_2g_1}{g_1+g_2},\\ \frac{\mathrm{d}\,\bar{v}_2}{\mathrm{d}\,t}&=\bar{v}_1\frac{f_1d_2+f_2d_1}{d_1+d_2}+\bar{v}_2\frac{l_1g_2+l_2g_1}{g_1+g_2},\end{aligned}\tag{2.69}$$

with the initial conditions $v_1(0)=\mathring{u}_1+\mathring{u}_2$ and $v_2(0)=\mathring{u}_3+\mathring{u}_4$. In a similar way we obtain

$$\mathcal{SQ}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\end{pmatrix}=\begin{pmatrix}\mathcal{A}\ \mathcal{B}\\ \mathcal{F}\ \mathcal{L}\end{pmatrix}\left(w_1\begin{pmatrix}\mathbf{h}\\ \mathbf{0}\end{pmatrix}+w_2\begin{pmatrix}\mathbf{0}\\ \mathbf{h}\end{pmatrix}\right)=w_1\begin{pmatrix}\mathcal{A}\mathbf{h}\\ \mathcal{F}\mathbf{h}\end{pmatrix}+w_2\begin{pmatrix}\mathcal{B}\mathbf{h}\\ \mathcal{L}\mathbf{h}\end{pmatrix}.$$

Thus

$$\mathcal{PSQ}\begin{pmatrix}\mathbf{u}^1\\ \mathbf{u}^2\end{pmatrix} = (w_1(\mathbf{1}\cdot\mathcal{A}\mathbf{h}) + w_2(\mathbf{1}\cdot\mathcal{B}\mathbf{h}))\begin{pmatrix}\mathbf{h}\\ \mathbf{0}\end{pmatrix} + (w_1(\mathbf{1}\cdot\mathcal{F}\mathbf{h}) + w_2(\mathbf{1}\cdot\mathcal{E}\mathbf{h}))\begin{pmatrix}\mathbf{0}\\ \mathbf{h}\end{pmatrix},$$

where, e.g. $\mathbf{1}\cdot\mathcal{A}\mathbf{h} = a_1 - a_2$. The operators $\mathcal{QSP}$ and $\mathcal{QSQ}$ can be calculated in the same way. Further,

$$\mathcal{QCQ} = \begin{pmatrix}-(d_1+d_2) & 0\\ 0 & -(g_1+g_2)\end{pmatrix}.$$

Now, we can write the explicit form of (2.39) in the spectral coordinates. We begin with the hydrodynamic part; that is, with the equations for v_1 and v_2, obtaining

$$\begin{aligned}\frac{\mathrm{d}\,v_1}{\mathrm{d}\,t} &= v_1\frac{a_1d_2 + a_2d_1}{d_1+d_2} + v_2\frac{b_1g_2 + b_2g_1}{g_1+g_2} + w_1(a_1-a_2) + w_2(b_1-b_2),\\ \frac{\mathrm{d}\,v_2}{\mathrm{d}\,t} &= v_1\frac{f_1d_2 + f_2d_1}{d_1+d_2} + v_2\frac{l_1g_2 + l_2g_1}{g_1+g_2} + w_1(f_1-f_2) + w_2(f_1-f_2).\end{aligned} \tag{2.70}$$

Next, the kinetic part is given by

$$\begin{aligned}\frac{\mathrm{d}\,w_1}{\mathrm{d}\,t} &= v_1\frac{d_1d_2(a_1-a_2)}{(d_1+d_2)^2} + w_1\frac{d_1a_1 + d_2a_2}{d_1+d_2} + v_2\frac{d_1b_1g_2 - d_2b_2g_1}{(g_1+g_2)(d_1+d_2)}\\ &\quad + w_2\frac{d_1b_1 + d_2b_2}{d_2+d_1} - \frac{1}{\varepsilon}(d_1+d_2)w_1,\\ \frac{\mathrm{d}\,w_2}{\mathrm{d}\,t} &= v_1\frac{g_1f_1d_2 - g_2f_2d_1}{(d_1+d_2)(g_1+g_2)} + w_1\frac{g_1f_1 + g_2f_2}{g_1+g_2} + v_2\frac{g_1g_2(l_1-l_2)}{(g_1+g_2)^2}\\ &\quad + w_2\frac{g_1l_1 + g_2l_2}{g_1+g_2} - \frac{1}{\varepsilon}(g_1+g_2)w_2.\end{aligned} \tag{2.71}$$

The initial layer equation (2.57) is given by

$$\begin{aligned}\frac{\mathrm{d}\,\tilde{w}_1}{\mathrm{d}\,\tau} &= -(d_1+d_2)\tilde{w}_1,\\ \frac{\mathrm{d}\,\tilde{w}_2}{\mathrm{d}\,\tau} &= -(g_1+g_2)\tilde{w}_2,\end{aligned} \tag{2.72}$$

with the initial data

$$\tilde{w}_1(0) = \frac{d_1 \overset{\circ}{u}_1 - d_2 \overset{\circ}{u}_2}{d_1 + d_2}, \qquad \tilde{w}_2(0) = \frac{g_1 \overset{\circ}{u}_3 - g_2 \overset{\circ}{u}_4}{g_1 + g_2}.$$

Then (2.61) yields

$$\begin{aligned}(v_1(t), v_2(t), w_1(t) - \tilde{w}_1(0)e^{-\frac{(d_1+d_2)t}{\varepsilon}}, w_2(t) - \tilde{w}_2(0)e^{-\frac{(g_1+g_2)t}{\varepsilon}}) \\ \to (\bar{v}_1(t), \bar{v}_2(t), 0, 0)\end{aligned} \tag{2.73}$$

on $[0, T]$. Using (2.68), we can rewrite the above formula in terms of the original variables as

$$\begin{aligned}
u_1(t) - \tilde{w}_1(0)e^{-\frac{(d_1+d_2)t}{\varepsilon}} &= v_1(t)\frac{d_2}{d_2+d_1} + (w_1(t) - \tilde{w}_1(0))e^{-\frac{(d_1+d_2)t}{\varepsilon}} \\
&\to \bar{v}_1(t)\frac{d_2}{d_2+d_1}, \\
u_2(t) + \tilde{w}_1(0)e^{-\frac{(d_1+d_2)t}{\varepsilon}} &= v_1(t)\frac{d_1}{d_2+d_1} - (w_1(t) - \tilde{w}_1(0))e^{-\frac{(d_1+d_2)t}{\varepsilon}} \\
&\to \bar{v}_1(t)\frac{d_1}{d_2+d_1}, \\
u_3(t) - \tilde{w}_2(0)e^{-\frac{(g_1+g_2)t}{\varepsilon}} &= v_2(t)\frac{g_2}{g_2+g_1} + (w_2(t) - \tilde{w}_2(0)e^{-\frac{(g_1+g_2)t}{\varepsilon}}) \\
&\to \bar{v}_2(t)\frac{g_2}{g_2+g_1}, \\
u_4(t) + \tilde{w}_2(0)e^{-\frac{(g_1+g_2)t}{\varepsilon}} &= v_2(t)\frac{g_1}{g_2+g_1} - (w_2(t) - \tilde{w}_2(0)e^{-\frac{(g_1+g_2)t}{\varepsilon}}) \\
&\to \bar{v}_2(t)\frac{g_1}{g_2+g_1}
\end{aligned} \tag{2.74}$$

on $[0, T]$. If we only are interested in the convergence away from zero, we may drop the initial layer terms and write (2.74) in the compact form

$$\begin{aligned}
\mathbf{u}^1(t) = (u_1(t), u_2(t)) &= v_1(t)\mathbf{e}^1 + w_1(t)(1, -1) \to \bar{v}_1(t)\mathbf{e}^1, \\
\mathbf{u}^2(t) = (u_3(t), u_4(t)) &= v_2(t)\mathbf{e}^2 + w_2(t)(1, -1) \to \bar{v}_2(t)\mathbf{e}^2,
\end{aligned}$$

on $]0, T]$.

We see that for small ε (large migration rates between patches) each population quickly stabilizes between the patches, with the fractions occupying each patch given by the components of the stable distribution vectors for each population. Thus, asymptotically, the evolution of each population is one-dimensional. However, the

effect of the populations on each other can be seen in the scalar multipliers in the coupling terms in (2.69). Again, these multipliers are the averages of the influences of the original populations of (2.63) taken with respect to the stable distribution of the population.

2.3.1 Emerging Properties

Despite its simplicity, the model has a number of interesting properties. In particular, one can observe the emergence of new properties in the limit equation induced by migrations. This means that the qualitative behaviour of the limit system may be different from that of the subsystems without migrations.

As an example, we consider a case without migration where, in each patch, the two populations occupying this patch display a periodic behaviour; that is, both (u_1, u_3) and (u_2, u_4) have periodic orbits. For this, it is enough to have

$$0 < a_1 = -l_1, -b_1 f_1 > a_1^2, \quad \text{and} \quad 0 < a_2 = -l_2, -b_2 f_2 > a_2^2. \tag{2.75}$$

Now, let us introduce migration between patches. For simplicity, let us assume that $d_1 = d_2 = 1$. Disregarding the multiplier $1/(d_1 + d_2)(g_1 + g_2)$, the trace of the matrix in (2.69) is given by $a_1 + a_2 - (a_1 g_2 + a_2 g_1)$. Clearly, changing g_1 and g_2, we can make the trace positive and negative. By (2.75) and $d_1, d_2, g_1, g_2 > 0$, the determinant of the matrix in (2.69) satisfies

$$\begin{aligned} &-(a_1 + a_2)(a_1 g_2 + a_2 g_1) - (b_1 g_2 + b_2 g_1)(f_1 + f_2) \\ &\quad = -a_1^2 g_2 - a_1 a_2 (g_1 + g_2) - a_2^2 g_1 - b_1 f_1 g_2 - b_1 g_2 f_2 - b_2 g_1 f_1 - b_2 g_1 f_2 \\ &\quad > -a_1 a_2 (g_1 + g_2) - b_1 g_2 f_2 - b_2 g_1 f_1. \end{aligned}$$

Thus, taking $a_1 = a_2 = 1$ to fix attention, we see that if $g_1 + g_2 < 2$, then the trace is positive, yielding instability of the limit equation. On the other hand, let us take $g_1 = 1, g_2 = 2$. Then the trace is negative and the determinant is greater than

$$-3 - 2b_1 f_2 - b_2 f_1,$$

which is positive for, e.g. $b_1 = -2$, $f_2 = 2, b_2 = -1$ and $f_1 = 1$. This choice clearly satisfies (2.75) and produces a stable limit system.

2.3.2 Asymptotics on $[0, \infty[$

We emphasize that, in general, the convergence in (2.74) occurs on finite (albeit arbitrary large) time intervals. This is important in practical applications as the rate

of convergence can be very slow if T is large. It is clearly visible in (2.53), where the estimating constant grows exponentially with T. Also, in principle, the asymptotic limit may not capture the essential large term dynamics of the original system. To avoid this, by (2.55), it is sufficient to assume that the original dynamics is at least stable independently of ε and that the operator $\mathcal{PSP}$ in the hydrodynamic equation is asymptotically stable.

Without pretences to full generality we just indicate a large class of problems which have the required properties. This occurs if the coefficient matrix is the sub-Kolmogorov matrix; see Sect. 1.3.3. By Proposition 2.1.4, exponentials of such matrices are nonnegative (i.e. solutions emanating from nonnegative data are nonnegative) and stable; see (2.36). We shall demonstrate this for the example at hand.

Hence, let the first matrix in (2.63) be sub-Kolmogorov; that is, let $f_i, b_i \geq 0,\ i = 1, 2$, and

$$a_1 + f_1 \leq 0, a_2 + f_2 \leq 0, b_1 + l_1 \leq 0 \text{ and } b_2 + l_2 \leq 0. \tag{2.76}$$

Then, since the migration matrix is a Kolmogorov matrix, the full coefficient matrix of (2.66) is sub-Kolmogorov for any ε and hence the solutions of (2.66) originating from nonnegative data are nonnegative. For such solutions we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\mathbf{u}(t)\| &= \frac{\mathrm{d}}{\mathrm{d}\,t}(u_1(t) + u_2(t) + u_3(t) + u_4(t)) \\ &= (a_1 + f_1)u_1(t) + (a_2 + f_2)u_2(t) + (b_1 + l_1)u_3(t) + (b_2 + l_2)u_4(t) \leq 0.\end{aligned}$$

This yields $\|\mathbf{u}(t)\| \leq \|\overset{\circ}{\mathbf{u}}\|$ and hence the nonnegative solutions are bounded. By (2.36), this inequality can be extended to solutions emanating from arbitrary initial conditions and thus system (2.66) is stable.

Since $d_i > 0, g_i > 0, i = 1, 2$, we see that also the matrix of coefficients in (2.69) is sub-Kolmogorov. Hence, for its asymptotic stability, it suffices that the inequalities in (2.76) be strict.

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