Preface

Natural processes usually are driven by mechanisms widely differing from each other by the time or space scale at which they operate. Thus, they should be described by appropriate multiscale models. However, looking at all such scales simultaneously often is infeasible and costly and provides information which is redundant for particular applications. Hence, there has been a growing interest in providing a more focused description of multiscale processes by aggregating variables in a way that is relevant for a particular purpose and that preserves the salient features of the dynamics and many ad hoc methods for this have been devised in the applied sciences. The aim of this book is to describe some tools which provide a systematic way of deriving the so-called limit equations for such aggregated variables and ensuring that the coefficients of these equations encapsulate the relevant information from the discarded levels of description. Since any approximation is only valid if an estimate of the incurred error is available, the tools we describe allow for proving that the solutions to the original multiscale family of equations converge to the solution of the limit equation if the relevant parameter converges to its critical value.

All problems discussed in this book belong to the class of singularly perturbed problems; that is, problems in which the structure of the limit equation is significantly different from that of the multiscale model. Such problems appear in all areas of science and can be approached by many techniques. In this book we present the classical asymptotic analysis based on the expansion of the solution in a series of powers of the parameter and, particularly, for the finite dimensional models, we explore the full power of the Tikhonov–Vasilyeva theory. The applications mostly are drawn from mathematical biology and epidemiology, but we discuss also some classical problems in other applied sciences. It is important, however, to realize that the approach to singularly perturbed problems presented in the book is by no means unique. There is a similar comprehensive theory based on the centre manifold theorem, called the geometric singular perturbation theory (see, e.g. [92, 124]), and the application of which to singularly perturbed nonlinear systems of ordinary differential equations modelling biological phenomena has been explored in many papers; see [16, 18, 110, 187] and references therein. In our opinion, however, the asymptotic expansion method, being possibly less elegant, is nevertheless more intuitive and more flexible and requires less theoretical background.

The book is organized as follows. In Chap. 1 we introduce basic ideas of asymptotic analysis and present a number of models which describe complex processes, the components of which occur at significantly different rates. Such models in a natural way contain a small parameter which is the ratio of the slow and the fast rates, thus lending themselves to asymptotic analysis. We discuss, among others, classical models of fluid dynamics and kinetic theory, population problems with fast migrations, epidemiological problems concerning diseases with quick turnover, models of enzyme kinetics and Brownian motion with fast direction changes. We also discuss initial and boundary layer phenomena using a simplified fluid dynamics equation as an example. In the conclusion of the chapter we discuss a model of enzyme kinetics and show in detail the application of the Hilbert expansion method to derive (formally) the Michaelis–Menten model; a rigorous derivation is referred to Chap. 3.

The following chapters mostly are devoted to analysis of the models introduced in Chap. 1. We arranged them according to the mathematical complexity of the analysis, from systems of ordinary linear differential equations, through nonlinear ordinary differential equations, to linear and nonlinear partial differential equations. Usually each chapter begins with a survey of mathematical techniques needed for the analysis; the exceptions are Chap. 4, which is based on the theory developed in Chaps. 3 and 8, in which an overview of asymptotic relationship between the three main scales of description of natural phenomena; that is, between the micro-, the meso- and the macro-scale, is presented.

Chapter 2 is designed as a gentle introduction of the Chapman–Enskog-type asymptotic expansion and of the basic techniques of proving its convergence. To make the presentation not too technical, it is illustrated on systems of linear ordinary differential equations. The chapter begins with a survey of necessary results from linear algebra and theory of finite-dimensional dynamical systems and it is concluded with a detailed analysis of linear population models with geographical structure in which the migration between geographical patches is much faster than the demographic processes.

The techniques introduced in this chapter can be also used for nonlinear systems of ordinary differential equations and for partial differential, or integro-differential, equations, but in most cases, the proofs of the convergence have to be tailor-made for each application. An exception are systems of nonlinear ordinary differential equations for which there exists a comprehensive theory, based on the Tikhonov theorem which is introduced in Chap. 3. The Tikhonov theorem is the main workhorse of the singular perturbation theory. It describes how to approximate solutions of complex first-order nonlinear ordinary differential equations, in which the small parameter multiplies some of the derivatives, by solutions of simpler equations which do not contain the small parameter; it also provides conditions under such an approximation is valid. The chapter is devoted to the detailed discussion of assumptions of the theory and to the proofs of the Tikhonov theorem and of the Vasilyeva theorem. The latter provides constructive estimates of the error of approximation obtained in the Tikhonov theorem and, in full strength, it gives error estimates for a general asymptotic expansion of the solution.

The applications of the Tikhonov theorem are discussed in Chap. 4. Here we provide a rigorous derivation of the Allee model from a system of mass-action type population equations and discuss an SIS epidemiological model with vital processes in which the latter act on a much slower timescale than the disease: think of a common cold or flu in human population, which both have turnover of days while the vital processes occur on the timescale of years. The chapter is concluded with an analysis of a predator–prey model with prey being able to move between geographical patches at a fast rate.

In Chap. 5 we generalize the analysis of the examples from Chap. 2 by allowing for a continuous age structure of the population. This leads to the McKendrick model, first introduced in Chap. 1, which is a system of partial differential equations with nonlocal boundary conditions. Thus, the Tikhonov theorem cannot be applied here and we have to return to the asymptotic expansion introduced in Chap. 2. As noted before, the proof of the convergence of the approximation must be adopted to this specific model and becomes quite complex, involving the analysis of initial, boundary and corner layers. To carry it out, we need some sophisticated tools from functional analysis and semigroups of operators theory, the rudiments of which are presented in the introductory sections of the chapter.

In Chap. 6 we return to the example of correlated and uncorrelated random walks, first discussed in Chap. 1. We begin with providing the mathematical setting necessary for the analysis of this problem, which include further topics from the semigroup theory and some facts pertaining to Sobolev spaces. The main aim of the chapter is to prove that the probabilistic densities describing correlated random walk, which are solutions of the hyperbolic telegraphers' equation, can be approximated by solutions of a specially constructed diffusion equation which describes uncorrelated random walk (if the coefficients of the equations are constant). If this is the case, we further show that the zeroth, first and second moments of both solutions coincide so that at the level of expectations and variances, the approximating solution is equivalent to the original one. An important result of this chapter is that, in contrast to most previous works, we are able to prove that an uncorrelated random walk is a good approximation of the correlated one under the sole assumption that the reversal rate is very large without imposing any requirements on the velocity of jumps.

Chapter 7 is the last chapter in which we show applications of asymptotic expansions to models describing processes occurring at two different timescales. Here the model describes individuals who may switch the direction of motion according to the prevalent direction of other individuals in their neighbourhood. The small parameter in this model is related to the mean time between the changes of the direction of motion. The main result of the chapter is that if this time becomes small, the population can be approximately described as a wave travelling in the direction in which the majority of the initial population moved. This result provides a new approach to the phenomena of swarming.

Chapter 8 has a slightly different character than the rest of the book. It can be considered as a general overview of multiscale descriptions of natural phenomena and, in contrast to the previous chapters, spans all three scales, from the micro- to the macro-scale. It begins with the microscopic, the so-called individually based, models in which each individual in the population (agent) is characterized by certain properties. The models at this level are represented by (large) systems of linear integro-differential equations describing appropriate jump Markov processes. The passage to the meso-scale is accomplished by means of an asymptotic limit when a small parameter, which here is related to the (inverse) of the size of the population, tends to 0 (i.e. the size of the population tends to infinity). In the resulting limit the population is described by a distribution function which is a solution of a bilinear, Boltzmann-like, integro-differential equation. Finally, the micro-scale description of the population is provided by a diffusion-type equation obtained in the asymptotic limit of the mesoscopic bilinear equation, when the range of the interactions tends to 0. The chapter also contains an extensive survey of models fitting into the framework of the theory and of their properties.

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