

Chapter 2

Methods from the Calculus

2.1 Introduction

In this chapter we revisit some facts from mathematical analysis and show how these may be used to establish important inequalities. We begin by reviewing convergence of real number sequences and continuity of real functions of a single variable.

2.2 Limits and Continuity

Convergent Sequences of Real Numbers

We say that a real sequence $\{x_n\}$ is bounded above if there exists M such that $x_n \leq M$ for all n . It is bounded below if there exists m such that $x_n \geq m$ for all n , and it is *bounded* if it is bounded above and bounded below (i.e., if there exists B such that $|x_n| \leq B$ for all n). Although unbounded sequences can be fascinating, our main interest will be in bounded sequences.

We say that $\{x_n\}$ is a Cauchy sequence if for every positive number ε there exists a positive integer N (dependent on ε) such that

$$|x_n - x_m| < \varepsilon \quad \text{whenever } m, n > N .$$

A sequence $\{x_n\}$ is convergent and has limit x if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \tag{2.1}$$

whenever $n > N$. In this case we write

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{or} \quad x_n \rightarrow x \quad \text{as } n \rightarrow \infty .$$

The limit of a convergent sequence is unique.

Remark 2.1. We have $x_n \rightarrow x$ if and only if for each $\varepsilon > 0$, the solution set with respect to n of the inequality (2.1) contains an interval of the form $(N(\varepsilon), \infty)$. $\square$

We say that $\{x_n\}$ is increasing if $x_{n+1} \geq x_n$ for all n , or decreasing if $x_{n+1} \leq x_n$ for all n . If strict inequality holds we use the terms strictly increasing or decreasing, respectively. An increasing sequence of real numbers is convergent if and only if it is bounded above, and a decreasing sequence is convergent if and only if it is bounded below. Let $\{x_n\}$ be a bounded sequence. Define

$$a_n = \inf_{k \geq n} x_k, \quad b_n = \sup_{k \geq n} x_k.$$

Then $\{a_n\}$ is increasing and bounded above, while $\{b_n\}$ is decreasing and bounded below. The two numbers defined respectively by

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n, \quad \overline{\lim}_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} b_n,$$

are the *limit inferior* and the *limit superior* of $\{x_n\}$. Although we have

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_3 \leq b_2 \leq b_1,$$

we are not guaranteed that $\{x_n\}$ is convergent. The sequence $\{(-1)^k\}$, for example, oscillates between -1 (its limit inferior) and $+1$ (its limit superior). However, if the limit inferior and the limit superior of a bounded sequence happen to coincide as a number x , then the sequence has limit x .

Sequences receive extensive coverage in any standard calculus text. There are many useful results in the subject (e.g., the various tests for convergence and divergence) and a number of these serve as interesting applications of inequalities (e.g., the comparison tests). We will assume a working knowledge of the basic theorems on sequence limits (the limit of a sum is the sum of the limits, etc.). The following two results, however, are central to our purposes.

Lemma 2.1 (Limit Passage). *If $\{x_n\}$ and $\{y_n\}$ are real sequences such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$ with $x_n \leq y_n$ for all n , then $x \leq y$.*

Proof. Let $\varepsilon > 0$ be given and choose N_1 and N_2 so that $n \geq \max(N_1, N_2)$ implies $x - \varepsilon/2 < x_n$ and $y_n < y + \varepsilon/2$. The inequality $x - \varepsilon/2 < x_n \leq y_n < y + \varepsilon/2$ shows that $x - y < \varepsilon$, and since $\varepsilon > 0$ is arbitrary we have $x - y \leq 0$. Hence $x \leq y$. $\square$

Note that, in general, an inequality may be blunted by a limit passage. That is, we may have $x_n < y_n$ for all n but $x \leq y$. Consider $x_n = 0$ and $y_n = 1/n$, for example.

Lemma 2.2 (Squeeze Principle). *If $a_n \rightarrow L$ and $c_n \rightarrow L$ as $n \rightarrow \infty$ and there exists N such that $a_n \leq b_n \leq c_n$ for all $n > N$, then $b_n \rightarrow L$ as $n \rightarrow \infty$.*

Proof. Let $\varepsilon > 0$ be given. There exists M such that $n > M$ implies

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon.$$

Hence $|b_n - L| \leq \varepsilon$ for all $n > M$. $\square$

Example 2.1. In Russia, the squeeze principle is commonly called the *policemen theorem*: $\{a_n\}$ and $\{c_n\}$ are described as “policeman” sequences who funnel “criminal” sequence $\{b_n\}$ toward a police station. Let us combine the squeeze principle with a nonreversible transformation. We can discard almost all the terms from the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \quad (2.2)$$

and write, for instance,

$$(1+x)^n > \frac{n(n-1)}{2}x^2 \quad (x > 0, n \in \mathbb{N}, n > 1). \quad (2.3)$$

To show that

$$\lim_{n \rightarrow \infty} \frac{n}{a^n} = 0 \quad (a > 1), \quad (2.4)$$

we can use (2.3) to write

$$\frac{n}{a^n} = \frac{n}{[1 + (a-1)]^n} < \frac{n}{\frac{n(n-1)}{2}(a-1)^2} = \frac{2}{(n-1)(a-1)^2}.$$

Then

$$0 < \frac{n}{a^n} < \frac{2}{(n-1)(a-1)^2} \quad (n > 1)$$

and Lemma 2.2 gives (2.4). $\square$

Limits and Continuity for Real Functions of a Single Variable

Let $f = f(x)$ be a real-valued function of the real variable x . We say that f has limit L as $x \rightarrow x_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ (dependent on ε) such that $|f(x) - L| < \varepsilon$ whenever $|x - x_0| < \delta$. We assume a working knowledge of the basic limit theorems for functions (the limit of a product is the product of the limits, and so on). Lemmas 2.1 and 2.2 have their counterparts for functions. For example, if $g(x) \rightarrow L$ and $h(x) \rightarrow L$ as $x \rightarrow x_0$, with $g(x) \leq f(x) \leq h(x)$, then $f(x) \rightarrow L$ as $x \rightarrow x_0$. To be complete, however, we would have to state several additional cases for functions. For instance, f has limit L as $x \rightarrow +\infty$ if for every $\varepsilon > 0$ there exists N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. The squeeze principle could be rephrased accordingly.

The statement that f is *continuous* at $x = x_0$ means that for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. We say that f is continuous on an interval I if f is continuous at every $x \in I$ (with suitable modifications made for continuity at endpoints of closed intervals). Two useful facts about continuity are the following.

Lemma 2.3 (Persistence of Sign). *Suppose f is continuous at $x = x_0$ and $f(x_0)$ is nonzero. Then there is an open interval containing x_0 such that $f(x)$ is nonzero at every point of the interval.*

Proof. Assume $f(x_0) > 0$ (otherwise replace f by $-f$). Let $\varepsilon = f(x_0)$. There exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$, so if $x \in (x_0 - \delta, x_0 + \delta)$ then $-\varepsilon < f(x) - f(x_0) < \varepsilon$. Hence $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$ or, since $f(x_0) = \varepsilon$, we have $0 < f(x)$. $\square$

Theorem 2.1 (Sequential Continuity). *A function f is continuous at x_0 if and only if $f(x_n) \rightarrow f(x_0)$ whenever $x_n \rightarrow x_0$.*

Proof. Suppose f is continuous at x_0 and $x_n \rightarrow x_0$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Now suppose $x_n \rightarrow x_0$. Choose N such that $n > N$ implies $|x_n - x_0| < \delta$. For this N , $n > N$ implies $|f(x_n) - f(x_0)| < \varepsilon$ and therefore $f(x_n) \rightarrow f(x_0)$. Conversely, suppose $f(x_n) \rightarrow f(x_0)$ whenever $x_n \rightarrow x_0$. To show that f is continuous at x_0 , we suppose f is not continuous at x_0 and seek a contradiction. There exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists some x with $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon$. In particular we may choose a sequence $\delta_i = 1/i$ and x_i with $|x_i - x_0| < \delta_i$ but $|f(x_i) - f(x_0)| \geq \varepsilon$ for all $i \in \mathbb{N}$. Then $x_i \rightarrow x_0$, but it is false that $f(x_i) \rightarrow f(x_0)$. $\square$

Theorem 2.1, sometimes called *Heine's theorem*, provides a notion of continuity equivalent to the less intuitive ε - δ definition. However, the ε - δ definition can be more convenient in proving continuity of a particular function, as it reduces to solving the inequality $|f(x) - f(x_0)| < \varepsilon$ and demonstrating that the solution contains the interval $|x - x_0| < \delta$ for some small δ . Note that to prove that f is *not* continuous at x_0 , it suffices to exhibit a sequence $x_k \rightarrow x_0$ such that $f(x_k) \not\rightarrow f(x_0)$ as $k \rightarrow \infty$.

The consequences of continuity on a closed interval are particularly important. We state the following without proof, referring the reader to any standard calculus text for details. One of these consequences is known as the *intermediate value property*.

Theorem 2.2. *If f is continuous on $[a, b]$, then $f(x)$ assumes every value between $f(a)$ and $f(b)$, $f(x)$ is bounded on $[a, b]$, and $f(x)$ takes on its supremum and its infimum in $[a, b]$.*

Finally, we review concepts relating to monotonicity and extrema. A function f is increasing on I if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$ for all $x_1, x_2 \in I$. Similarly, f is decreasing if $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$. If strict inequality holds we use the terms strictly increasing or decreasing, respectively. Let $x_0 \in I$. If $f(x_0) \geq f(x)$ for all $x \in I$, then f has a maximum on I equal to $f(x_0)$. The definition of minimum is analogous.

2.3 Basic Results for Integrals

The formal definition of the Riemann integral appears in Problem 2.2. It is helpful to keep in mind that $f = f(x)$ is integrable on $[a, b]$ if it is continuous or monotonic on $[a, b]$.

Several useful inequalities for integrals can be established by forming Riemann sums. Given an integral

$$\int_a^b f(x) dx ,$$

we use the notation $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$ for $i = 0, \dots, n$, and write the corresponding Riemann sum as

$$\sum_{i=1}^n f(x_i) \Delta x .$$

Once an inequality is established for such a sum, we may let $n \rightarrow \infty$ and, being assured of convergence of the Riemann sums to the integral, apply Lemma 2.1 in order to obtain an inequality involving the integral.

Theorem 2.3. *If f and g are integrable on $[a, b]$ with $f(x) \leq g(x)$, then*

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx .$$

Proof. With the notation described above, we form Riemann sums for the integrals. It is seen that

$$\sum_{i=1}^n f(x_i) \Delta x \leq \sum_{i=1}^n g(x_i) \Delta x .$$

The result follows as $n \rightarrow \infty$ by Lemma 2.1. □

Example 2.2. A simple observation shows that

$$\int_t^\infty \frac{e^{-x^2}}{x^{2n}} dx = \int_t^\infty \frac{xe^{-x^2}}{x^{2n+1}} dx \leq \int_t^\infty \frac{xe^{-x^2}}{t^{2n+1}} dx = \frac{e^{-t^2}}{2t^{2n+1}} .$$

We used the fact that $1/x^{2n+1} \leq 1/t^{2n+1}$ for $x \in [t, \infty)$. □

Corollary 2.1 (Simple Estimate). *If f is integrable on $[a, b]$ with $m \leq f(x) \leq M$, then*

$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a) . \quad (2.5)$$

Consequently, the average value of f over $[a, b]$ lies between m and M .

Example 2.3. We have

$$(b-a)\sqrt{ca^3+d} < \int_a^b \sqrt{cx^3+d} dx < (b-a)\sqrt{cb^3+d}$$

for any positive constants a, b, c, d with $a < b$. □

Corollary 2.2 (Modulus Inequality). *If f is integrable on $[a, b]$, then*

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx. \quad (2.6)$$

This follows from the inequalities $-|f(x)| \leq f(x) \leq |f(x)|$ and plays the role of the triangle inequality for integrals.

If continuity is assumed in the integrand function f , the persistence of sign property leads to the next result.

Lemma 2.4. *Let f be continuous on $[a, b]$ and suppose $f(x) \geq 0$ on $[a, b]$ with $f(x_0) > 0$ for some $x_0 \in [a, b]$. Then*

$$\int_a^b f(x) dx > 0.$$

Proof. If $x_0 \in (a, b)$, there is an open interval about x_0 where $f(x) > 0$. Choose a smaller closed interval where $f(x) > 0$, say $I = [x_0 - \delta, x_0 + \delta]$. Let m be the minimum value of $f(x)$ in I . Then

$$\int_a^b f(x) dx \geq m(2\delta) > 0.$$

If x_0 is an endpoint, then $f(x)$ is also positive at an interior point so the argument still applies. □

A class of results known as mean value theorems are also useful. We give two of these and refer the reader to Problem 2.5 for other examples.

Theorem 2.4 (Second Mean Value Theorem for Integrals). *If f is continuous on $[a, b]$, and g is integrable and never changes sign on $[a, b]$, then for some $\xi \in [a, b]$*

$$\int_a^b f(x)g(x) dx = f(\xi) \int_a^b g(x) dx. \quad (2.7)$$

Proof. Assume that $g(x) \geq 0$ on $[a, b]$; otherwise, replace $g(x)$ by $-g(x)$. Let M and m denote the maximum and minimum values, respectively, of $f(x)$ on $[a, b]$. Then

$$mg(x) \leq f(x)g(x) \leq Mg(x)$$

for all x , hence

$$m \int_a^b g(x) dx \leq \int_a^b f(x)g(x) dx \leq M \int_a^b g(x) dx .$$

If $\int_a^b g(x) dx = 0$ then any choice of ξ will do. Otherwise

$$m \leq \frac{\int_a^b f(x)g(x) dx}{\int_a^b g(x) dx} \leq M .$$

By the intermediate value property applied to f ,

$$f(\xi) = \frac{\int_a^b f(x)g(x) dx}{\int_a^b g(x) dx}$$

for some $\xi \in [a, b]$, and (2.7) follows.

Corollary 2.3 (First Mean Value Theorem for Integrals). *If f is continuous on $[a, b]$, then for some $\xi \in [a, b]$*

$$\int_a^b f(x) dx = f(\xi)(b - a) .$$

Hence $f(\xi)$ equals the average value of $f(x)$ on $[a, b]$.

Example 2.4. Consider the integral

$$I = \int_0^1 \frac{x^5}{(x+25)^{1/2}} dx .$$

On the interval $[0, 1]$ we have $x^5 \geq 0$; hence by Theorem 2.4 there exists $\xi \in [0, 1]$ such that

$$I = \frac{1}{6(\xi + 25)^{1/2}} .$$

Therefore $1/(6\sqrt{26}) \leq I \leq 1/30$. □

Through a process reminiscent of the integral test for series, we can obtain other inequalities involving integrals.

Example 2.5. The function $f(x) = x^p$ ($-1 < p < 0$) is strictly decreasing on $(0, \infty)$. From Fig. 2.1 it is apparent that

$$\int_1^{n+1} x^p dx < \sum_{k=1}^n k^p < \int_0^n x^p dx .$$

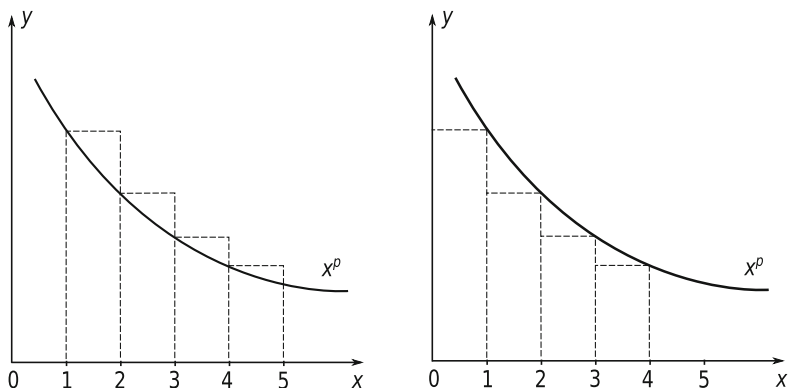


Fig. 2.1 Comparing a sum to two integrals (Example 2.5 with $n = 4$)

Hence, after carrying out the integrations, we get

$$\frac{(n+1)^{p+1} - 1}{p+1} < \sum_{k=1}^n k^p < \frac{n^{p+1}}{p+1}.$$

□

Integration along a contour in the complex plane follows many rules analogous to those for real integration, with little modification. In particular, Corollary 2.2 extends to the complex case: if $g(z)$ is integrable on contour C , then

$$\left| \int_C g(z) dz \right| \leq \int_C |g(z)| |dz|.$$

Example 2.6. Suppose C is of finite length L . If there is a number $M > 0$ such that $|g(z)| < M$ for all $z \in C$, then

$$\left| \int_C g(z) dz \right| \leq \int_C |g(z)| |dz| < \int_C M |dz| = M \int_C |dz| = ML. \quad (2.8)$$

This is sometimes called the *Darboux inequality*.

□

2.4 Results from the Differential Calculus

A function f is said to be n times continuously differentiable on an interval I if the first n derivatives of f exist and are continuous on I . We first recall the Fundamental Theorem of Calculus. A proof may be found in any standard calculus text.

Theorem 2.5. If f is continuous on $[a, b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

The next result is a source of series expansions that are useful in approximating functions and, as we shall see, in generating inequalities.

Theorem 2.6 (Taylor's Theorem). *Let $x > a$, suppose f is n times continuously differentiable on $[a, x]$, and suppose $f^{(n+1)}(x)$ exists on (a, x) . Then*

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some ξ strictly between a and x .

Proof. The first $n+1$ terms constitute the Taylor polynomial of degree n for $f(x)$ about the point a ; the last term is called the remainder term. To simplify the proof, assume f is $n+1$ times continuously differentiable on $[a, b]$. By Theorem 2.5,

$$f(x) - f(a) = \int_a^x f'(t) dt.$$

Integrate by parts with $u = f'(t)$, $du = f''(t) dt$, $v = -(x-t)$, and $dv = dt$; then

$$\int_a^x f'(t) dt = f'(a)(x-a) + \int_a^x f''(t)(x-t) dt.$$

Repeat with $u = f''(t)$, $du = f'''(t) dt$, $v = -(x-t)^2/2$, $dv = (x-t) dt$, and continue the process until

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

Because $(x-t)^n$ never changes sign in the interval with endpoints a and x , by (2.7) the remainder term can be rewritten

$$\frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt = \frac{f^{(n+1)}(\xi)}{n!} \int_a^x (x-t)^n dt = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some ξ between a and x . □

We can sometimes establish inequalities through inspection of series expansions.

Example 2.7. From the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we see that

$$e^x > 1 + x + \frac{1}{2}x^2 \quad (x > 0).$$

Even more simply we have $e^x > 1 + x$, but we can replace x by x/n to get the less obvious result

$$e^x > \left(1 + \frac{x}{n}\right)^n \quad (x > 0, n \in \mathbb{N}).$$

If $z \in \mathbb{C}$, relation (1.17) yields

$$|\sin z| = \left| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} \right| \leq \sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} \right| = \sum_{n=1}^{\infty} \frac{|z|^{2n-1}}{(2n-1)!}$$

and we have $|\sin z| \leq \sinh |z|$. $\square$

The next two results, although important in their own right, can be viewed as immediate consequences of Taylor's theorem.

Corollary 2.4 (Mean Value Theorem). *Suppose f is continuous on $[a, b]$ and differentiable in (a, b) . Then there exists $\xi \in (a, b)$ such that*

$$f(b) - f(a) = f'(\xi)(b - a). \quad (2.9)$$

Intuitively, there is a point in (a, b) such that the slope of the line tangent to $f(x)$ at that point equals the slope of the secant line connecting the function values at the endpoints of $[a, b]$. See Fig. 2.2.

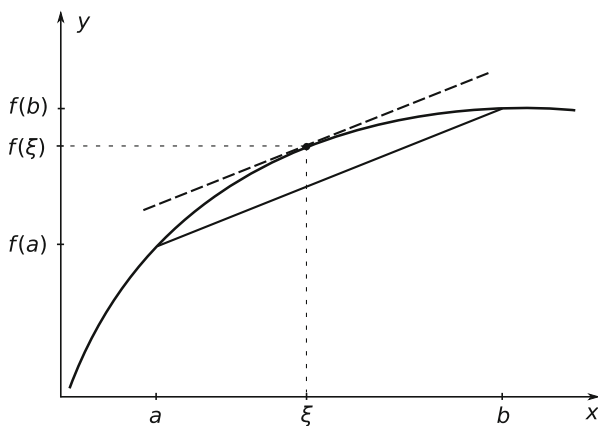


Fig. 2.2 Mean value theorem. The heavier dashed line is tangent to $y = f(x)$ at $x = \xi$

Example 2.8. We can verify the inequality

$$\tan x > x \quad (0 < x < \pi/2) \quad (2.10)$$

by applying Corollary 2.4 with $f(x) = \tan x$, $a = 0$, and $b = x < \pi/2$; i.e., by asserting that

$$\tan x - \tan 0 = \frac{1}{\cos^2 \xi}(x - 0) \quad \text{for some } \xi \in (0, x)$$

and noting that $0 < \cos^2 \xi < 1$. Similarly, we have $\sin x < x$ for $x > 0$ so that

$$\sin x < x < \tan x \quad (0 < x < \pi/2) .$$

Applying Corollary 2.4 to the natural log, we obtain

$$\ln(1+x) - \ln 1 = \frac{1}{\xi}[(1+x) - 1] \text{ for some } \xi \in (1, 1+x) .$$

Therefore

$$\frac{x}{1+x} < \ln(1+x) < x \quad (x > 0) . \quad (2.11)$$

This is the *logarithmic inequality*. It also holds for $-1 < x < 0$. $\square$

Corollary 2.5 (Rolle's Theorem). *If f is continuous on $[a, b]$ and differentiable in (a, b) with $f(b) = f(a) = 0$, then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Rolle's theorem indicates that between every two zeros of a continuous function the derivative has at least one zero.

Theorem 2.7 (Conditions for Monotonicity). *If f is continuous on $[a, b]$ and differentiable in (a, b) with $f'(x) \geq 0$, then f is increasing on $[a, b]$. If $f'(x) > 0$, then f is strictly increasing. Corresponding statements hold for decreasing functions, for which $f'(x) \leq 0$.*

Proof. We prove only the first part of the theorem, and leave the rest for the reader. Suppose $a \leq x_1 < x_2 \leq b$. By Corollary 2.4, there is a number $\xi \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$. But $f'(\xi) \geq 0$ by hypothesis and $x_2 - x_1 > 0$, so $f(x_2) - f(x_1) \geq 0$. Hence $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$ on $[a, b]$. $\square$

Example 2.9. The average of a positive, increasing function is increasing. For let $f(x)$ be increasing on $[0, a]$. Then for every $x \in (0, a]$ we have

$$f(x) \geq \max_{u \in [0, x]} f(u) = \max_{u \in [0, x]} f(u) \cdot \frac{1}{x} \int_0^x du \geq \frac{1}{x} \int_0^x f(u) du .$$

Hence

$$f(x) - \frac{1}{x} \int_0^x f(u) du \geq 0 \quad \text{so that} \quad \frac{1}{x^2} \left(x f(x) - \int_0^x f(u) du \right) \geq 0 .$$

By the quotient rule for differentiation,

$$\frac{d}{dx} \left(\frac{1}{x} \int_0^x f(u) du \right) \geq 0$$

as required. $\square$

Theorem 2.8 (Cauchy's Mean Value Theorem). Suppose f, g are continuous on $[a, b]$ and differentiable in (a, b) . Then there exists $\xi \in (a, b)$ such that

$$[f(b) - f(a)] g'(\xi) = [g(b) - g(a)] f'(\xi) .$$

Proof. Call $A = f(b) - f(a)$, $B = -[g(b) - g(a)]$, $C = -Bf(a) - Ag(a)$, and apply Rolle's theorem to $h(x) = Ag(x) + Bf(x) + C$. $\square$

The following is useful for establishing the monotonicity of the ratio of two functions.

Theorem 2.9 (l'Hôpital's Monotone Rule). Suppose f, g are continuous on $[a, b]$ and differentiable in (a, b) with $g'(x) \neq 0$ in (a, b) . Let $f'(x)/g'(x)$ be increasing (or decreasing) on (a, b) . Then the functions

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)} \quad (2.12)$$

are also increasing (or decreasing) on (a, b) . If $f'(x)/g'(x)$ is strictly increasing (or decreasing) so are the functions (2.12).

Proof. (See [3]). We may assume $g'(x) > 0$ on (a, b) . (If not, multiply f and g by -1 .) By Theorem 2.8, for $x \in (a, b)$ there exists $y \in (a, x)$ with

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(y)}{g'(y)} \leq \frac{f'(x)}{g'(x)} , \quad \text{so} \quad f'(x) \geq g'(x) \frac{f(x) - f(a)}{g(x) - g(a)} .$$

Now use the quotient rule and the last expression to deduce that the derivative of $[f(x) - f(a)]/[g(x) - g(a)]$ is nonnegative, hence Theorem 2.7 applies. $\square$

By l'Hôpital's rule, to evaluate a ratio of the indeterminate form $0/0$ we differentiate both numerator and denominator and try to evaluate again. Theorem 2.9 is almost as easily remembered. To establish that a ratio is monotone on an interval using Theorem 2.9, we verify that we get $0/0$ at one of the endpoints, then differentiate numerator and denominator and check that the resulting quotient is monotone (making sure the new denominator is nonzero on the open interval).

Theorem 2.10 (Second Derivative Test). Assume f is twice continuously differentiable in (a, b) . Let $x_0 \in (a, b)$, and suppose that $f'(x_0) = 0$ and $f''(x_0) > 0$. Then f has a local minimum at x_0 . That is, there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $f(x) > f(x_0)$.

Proof. Because $f''(x_0) > 0$, by Lemma 2.3 there exists $\delta > 0$ such that $f''(x) > 0$ if $|x - x_0| < \delta$. Now let $0 < |\Delta x| < \delta$. By Theorem 2.6 there exists some ξ strictly between x_0 and $x_0 + \Delta x$ such that

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0) \Delta x + \frac{1}{2} f''(\xi) (\Delta x)^2 . \quad (2.13)$$

Since $f'(x_0) = 0$ and $f''(\xi) > 0$ the result follows by inspection. Note that if we assume $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a local maximum at x_0 . We will state and prove the theorem for n variables later. $\square$

Differentiation is a handy device for checking many proposed inequalities. One plan is as follows. Suppose the proposed inequality is of the general form

$$g(x) < h(x) \quad (x > x_0), \quad (2.14)$$

where $g(x_0) = h(x_0)$ and the functions $g(x)$ and $h(x)$ have known derivatives. Defining $f(x) = h(x) - g(x)$, we have $f(x_0) = 0$. If we can further show that $f'(x) > 0$ for $x > x_0$, then (2.14) is established.

Example 2.10. We can prove that for $x > 0$ we have

$$x^r \leq rx + (1 - r) \quad (0 < r < 1). \quad (2.15)$$

Defining $f(x) = (1 - r) + rx - x^r$, we find $f(1) = 0$ and

$$f'(x) = r - rx^{r-1} = r \left(1 - \frac{1}{x^{1-r}} \right).$$

For $x > 1$ we have $f'(x) > 0$; for $0 < x < 1$ we have $f'(x) < 0$. Hence (2.15) holds, with equality if and only if $x = 1$. Similarly, for $x > 0$ we have

$$x^r \geq rx + (1 - r) \quad (r < 0 \text{ or } r > 1). \quad (2.16)$$

Beckenbach and Bellman [7] call these inequalities “remarkable” as they can be used to derive the AM–GM, Hölder, and Minkowski inequalities of Chap. 3. $\square$

Example 2.11. For $0 < x < \pi/2$, relation (2.10) yields

$$\frac{d}{dx} \left(\frac{\sin x}{x} \right) = \cos x \left(\frac{x - \tan x}{x^2} \right) < 0$$

so $\sin x/x$ is strictly decreasing. Because $\sin x/x \rightarrow 2/\pi$ as $x \rightarrow \pi/2$, we conclude that

$$\sin x > 2x/\pi \quad (0 < x < \pi/2). \quad (2.17)$$

This is *Jordan's inequality* (Fig. 2.3). The role of concavity suggested here will be exploited further in Sect. 3.9. $\square$

Inequalities are often obtained by solving constrained optimization problems via the Lagrange multiplier technique. The main idea is as follows. To prove an inequality of the form

$$f(x, y) \leq g(x, y), \quad (2.18)$$

we can try to maximize the function $f = f(x, y)$ subject to the condition $g(x, y) = k$ where k is a constant. If for any permissible k the constrained maximum value of $f(x, y)$ is $f_{\max}$ and if $f_{\max} \leq k$, then (2.18) is proved.

Alternatively, we could try to minimize the right member $g(x, y)$ subject to the condition $f(x, y) = c$ with c a constant. If for any permissible c the constrained minimum value of $g(x, y)$ is $g_{\min}$ and if $g_{\min} \geq c$, then (2.18) is likewise established. Let us carry out this procedure to prove a standard inequality obtained (by a different

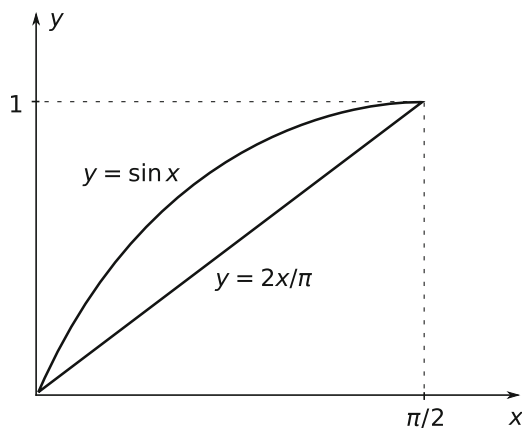


Fig. 2.3 Jordan's inequality (2.17)

approach) in the next chapter. The desired result is a special case of *Young's inequality*. It states that if p and q are positive numbers for which $p^{-1} + q^{-1} = 1$, then the inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \quad (2.19)$$

holds for all nonnegative numbers x and y .

We will minimize the right member

$$g(x, y) = \frac{x^p}{p} + \frac{y^q}{q} \quad (2.20)$$

subject to the constraint that the left member

$$f(x, y) = xy = c, \text{ a constant.} \quad (2.21)$$

Because (2.19) holds trivially when either x or y is zero, we can assume $c \neq 0$. Carrying out the usual Lagrange multiplier technique, we form the Lagrangian function with a multiplier $-\lambda$ (negative sign arbitrary but taken for convenience) as

$$F(x, y) = \frac{x^p}{p} + \frac{y^q}{q} - \lambda xy$$

and differentiate this function with respect to x and y , respectively, to obtain the equations

$$x^{p-1} - \lambda y = 0, \quad (2.22)$$

$$y^{q-1} - \lambda x = 0. \quad (2.23)$$

We then solve the system consisting of (2.22)–(2.23) and the constraint (2.21). The solution is straightforward; we find a constrained stationary point for $g(x, y)$ at

$$(x_s, y_s) = (c^{1/p}, c^{1/q}) .$$

Furthermore, $g(x_s, y_s) = c$. Because $g(x, y)$ in (2.20) is not bounded above when subject to a constraint of the form (2.21), it is clear that (x_s, y_s) is actually the constrained minimum of $g(x, y)$ corresponding to a given value of c .

Finally, any point (x, y) with $x > 0$ and $y > 0$ has a hyperbola of the form $xy = c$ passing through it. The relation

$$g(x, y) = \frac{x^p}{p} + \frac{y^q}{q} \geq g_{\min} = c = xy$$

gives Young's inequality (2.19).

2.5 Problems

2.1. The following exercises involve monotonicity.

(a) Show that if $n \in \mathbb{N}$ then

$$\ln(n+1) > \frac{1}{n} \sum_{k=1}^n \ln k .$$

(b) Show that if ϕ , ψ , and f are increasing functions with $\phi(x) \leq f(x) \leq \psi(x)$, then

$$\phi(\phi(x)) \leq f(f(x)) \leq \psi(\psi(x)) .$$

(c) [56, 65] Show that if f is increasing on $[a, b]$, then

$$\frac{1}{x-a} \int_a^x f(u) du \leq \frac{1}{b-a} \int_a^b f(u) du \leq \frac{1}{b-x} \int_x^b f(u) du \quad (2.24)$$

for any $x \in (a, b)$.

2.2. The following exercises involve the definition of integration. Recall from calculus that f is integrable on $[a, b]$ and

$$\int_a^b f(x) dx = I$$

means that given $\varepsilon > 0$ there exists some $\delta > 0$ such that if

$$a = x_0 < x_1 < \cdots < x_n = b$$

and if $x_i - x_{i-1} < \delta$ for $i = 1, \dots, n$ and if $\xi_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$, then

$$\left| \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) - I \right| < \delta .$$

Note as a special case that if f is integrable on $[a, b]$, then

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(a + i\Delta x) \Delta x = \int_a^b f(x) dx \quad \text{where } \Delta x = (b - a)/n.$$

- (a) Show that if f is integrable on $[a, b]$, then f is bounded on $[a, b]$.
 (b) Show that if f is integrable on $[a, b]$, then F given by

$$F(x) = \int_a^x f(t) dt$$

is continuous on $[a, b]$.

- (c) Define $f(x) = x^{-1/2}$ if $0 < x \leq 1$ and $f(0) = 0$. Does $\int_0^1 f(x) dx$ exist?

2.3. The exercises below also involve integration.

- (a) Put simple lower and upper bounds on the family of integrals

$$I(\alpha, \beta) = \int_0^1 \frac{dx}{(x^\beta + 1)^\alpha} \quad (\alpha, \beta \geq 0).$$

- (b) Show that

$$\int_0^{\pi/2} \ln(1/\sin t) dt < \infty.$$

- (c) A function f is of *exponential order* on $[0, \infty)$ if there exist positive numbers b and C such that $|f(t)| \leq Ce^{bt}$ for $t \geq 0$. Show that the Laplace transform of f given by

$$F(s) = \int_0^\infty f(t)e^{-st} dt$$

exists if f is of exponential order.

- (d) Verify that

$$\int_0^{\pi/2} (\sin x)^{2n+1} dx \leq \int_0^{\pi/2} (\sin x)^{2n} dx \leq \int_0^{\pi/2} (\sin x)^{2n-1} dx$$

and establish *Wallis's product*

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2m}{2m-1} \cdot \frac{2m}{2m+1} \cdots.$$

- (e) Show that

$$\lim_{T \rightarrow \infty} \int_0^T \frac{\sin x}{x} dx$$

exists and is between 1 and 3.

- (f) Prove that if g is continuous on $[a, b]$ with $g(x) \geq 0$ and $\int_a^b g(x) dx = 0$, then $g(x) \equiv 0$ on $[a, b]$.
 (g) Let $p \in \mathbb{R}$, $p > 0$. Use the fact that $\ln x = \int_1^x dt/t$ and the squeeze principle to show that

$$\lim_{x \rightarrow \infty} (\ln x)/(x^p) = 0.$$

2.4. Let f and g be functions integrable on (a, b) , with $0 \leq g(t) \leq 1$ and f decreasing on (a, b) . Prove *Steffensen's inequality* [7, 56]

$$\int_{b-\lambda}^b f(t) dt \leq \int_a^b f(t)g(t) dt \leq \int_a^{a+\lambda} f(t) dt \quad \text{where } \lambda = \int_a^b g(t) dt.$$

2.5. Prove the following statements. Parts (a) and (b) are challenging; according to Hobson [37], they were first given by Bonnet circa 1850.

- (a) Let f be a monotonic decreasing, nonnegative function on $[a, b]$, and let g be integrable on $[a, b]$. Then for some ξ with $a \leq \xi \leq b$,

$$\int_a^b f(x)g(x) dx = f(a) \int_a^\xi g(x) dx.$$

- (b) Let f be a monotonic increasing, nonnegative function on $[a, b]$, and let g be integrable on $[a, b]$. Then for some η with $a \leq \eta \leq b$,

$$\int_a^b f(x)g(x) dx = f(b) \int_\eta^b g(x) dx.$$

- (c) Let f be bounded and monotonic on $[a, b]$, and let g be integrable on $[a, b]$. Then for some ξ with $a \leq \xi \leq b$,

$$\int_a^b f(x)g(x) dx = f(a) \int_a^\xi g(x) dx + f(b) \int_\xi^b g(x) dx.$$

This is also called the second mean value theorem for integrals, particularly in older books.

- (d) Let f be a monotonic function integrable on $[a, b]$, and suppose that $f(a)f(b) \geq 0$ and $|f(a)| \geq |f(b)|$. Then, if g is a real function integrable on $[a, b]$,

$$\left| \int_a^b f(x)g(x) dx \right| \leq |f(a)| \cdot \max_{a \leq \xi \leq b} \left| \int_a^\xi g(x) dx \right|.$$

This is *Ostrowski's inequality for integrals*.

2.6. Use graphical approaches to complete the following.

- (a) Show that if f is increasing on $[0, \infty)$, then

$$\int_0^n f(x) dx \leq \sum_{k=1}^n f(k) \leq \int_1^{n+1} f(x) dx.$$

Use this to find upper and lower bounds on $\sum_{k=1}^n k^2$.

- (b) Show that

$$\int_1^n \ln x dx < \ln(n!) < \int_1^{n+1} \ln x dx \quad (n \in \mathbb{N}, n > 1).$$

- (c) Sketch the curve $y = 1/x$ for $x > 0$, and consider the area bounded by this curve, the x -axis, and the lines $x = a$ and $x = b$ ($b > a$). Compare this with the areas of two trapezoids and obtain

$$\frac{2(b-a)}{b+a} < \ln \frac{b}{a} < \frac{b^2 - a^2}{2ab}.$$

- (d) (*Integral test inequality* [13].) Show that if f is decreasing on $[1, \infty)$ then

$$\sum_{k=2}^n f(k) \leq \int_1^n f(x) dx \leq \sum_{k=1}^{n-1} f(k).$$

- (e) Show that if f is increasing on $[1, \infty)$, then

$$\sum_{k=1}^{n-1} f(k) \leq \int_1^n f(x) dx \leq \sum_{k=2}^n f(k).$$

- (f) Show that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{5}{4}.$$

- (g) Euler's constant C is defined by

$$C = \lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{1}{j} - \ln n \right).$$

Verify that C exists and is positive by showing that C_n is strictly decreasing with lower bound $1/2$.

2.7. Use differentiation to prove the results below [1, 57, 69]. Assume $n \in \mathbb{N}$.

- (a) $\ln x \leq x - 1$ for $x > 0$, with equality if and only if $x = 1$,
- (b) $\ln x \leq n(x^{1/n} - 1)$ for $x > 0$, with equality if and only if $x = 1$,
- (c) $x^n + (n - 1) \geq x$ for $x \geq 0$,
- (d) $2 \ln(\sec x) < \sin x \tan x$ for $0 < x < \pi/2$,
- (e) $\sinh x \geq x$ for $x \geq 0$,
- (f) $|x \ln x| \leq e^{-1}$ for $0 \leq x \leq 1$,
- (g) $e^x < (1 - x)^{-1}$ for $x < 1$ and $x \neq 0$,
- (h) $\pi^e < e^\pi$ (more generally [13], $e^x > x^e$ for any $x \neq e$),
- (i) $(s + t)^a \leq s^a + t^a \leq 2^{1-a}(s + t)^a$ for $s, t > 0$ and $0 < a \leq 1$,
- (j) $2^{1-b}(s + t)^b \leq s^b + t^b \leq (s + t)^b$ for $s, t > 0$ and $b \geq 1$,
- (k) $e^x \geq (ex/a)^a$ for $x > a$ and $a > 0$,
- (l) $x^x \geq e^{x-1}$ for $x > 0$,
- (m) $\cos x \geq 1 - x^2/2$ for $x \geq 0$,
- (n) $\sin x \geq x - x^3/3!$ for $x \geq 0$.

2.8. Use Corollary 2.4 to derive the following inequalities [57, 69]:

- (a) $\sin x < x$ for $x > 0$,
- (b) $x/(1 + x^2) < \tan^{-1} x < x$ for $x > 0$,
- (c) $1 + (x/2)\sqrt{1+x} < \sqrt{1+x} < 1 + x/2$ for $x > 0$,
- (d) $e^x(y - x) < e^y - e^x < e^y(y - x)$ for $y > x$,
- (e) $(1 + x)^a \leq 1 + ax(1 + x)^{a-1}$ for $a > 1$ and $x > -1$, with equality if and only if $x = 0$,
- (f) $1 + x > e^{x/(1+x)}$ for $x > -1$ and $x \neq 0$,
- (g) $(y - x)/\cos^2 x < \tan y - \tan x < (y - x)/\cos^2 y$ for $0 \leq x < y < \pi/2$,
- (h) $ex < (y^y/x^x)^{1/(y-x)} < ey$ for $0 < x < y$.

2.9. Derive a nonstrict version of (2.10) by integration.

2.10. The following are applications of l'Hôpital's monotone rule.

(a) For $a > 1$ and $x > -1$, $x \neq 0$, define

$$h(x) = \frac{(1+x)^a - 1}{x}.$$

Use l'Hôpital's rule to define $h(0) = a$. Use Theorem 2.9 to show that $h(x)$ is increasing on $[-1, \infty)$ and hence that

$$(1+x)^a \geq 1+ax$$

with equality if and only if $x = 0$ (cf., Example 2.10).

(b) Show that

$$h(x) = \frac{\ln \cosh x}{\ln((\sinh x)/x)}$$

is decreasing on $(0, \infty)$.

(c) Prove that for $x \in (0, 1)$,

$$\pi < \frac{\sin \pi x}{x(1-x)} \leq 4.$$

(d) Prove that $1 > \sin x/x > 2/\pi$ on $(0, \pi/2)$ (cf., Example 2.11.)

2.11. Use series expansions to establish the following inequalities:

(a) $|\cos z| \leq \cosh |z|$ for $z \in \mathbb{C}$,

(b) $|\ln(1+x)| \leq -\ln(1-|x|)$ if $|x| < 1$,

(c) $\prod_{n=1}^{\infty} (1+a_n) \leq \exp(\sum_{n=1}^{\infty} a_n)$ if $0 \leq a_n < 1$ for all n ,

(d) $e^x > 1 + x^n/n!$ for $n \in \mathbb{N}$ and $x > 0$,

(e) $x < e^x - 1 < x/(1-x)$ for $x < 1$ and $x \neq 0$.

2.12. Show that if n is an integer greater than 1 and a, b are positive with $a > b$, then

$$b^{n-1} < \frac{a^n - b^n}{n(a-b)} < a^{n-1}.$$

Use this to prove that no positive real number can have more than one positive n th root.

2.13. Prove the following generalized version of Rolle's theorem. Let g be n times continuously differentiable on $[a, b]$, and let $x_0 < x_1 < \cdots < x_n$ be $n+1$ points in $[a, b]$. Suppose $g(x_0) = g(x_1) = \cdots = g(x_n) = 0$. Then there exists $\xi \in [a, b]$ such that $g^{(n)}(\xi) = 0$.

2.14. A set A is said to be *dense* in a set B if every element of B is the limit of a sequence of elements belonging to A . Show that if $f(x)$ and $g(x)$ are continuous on B with $f(x) \leq g(x)$ for every x in some dense subset of B , then $f(x) \leq g(x)$ for all $x \in B$. Explain how this idea could be used to extend to real arguments an inequality proved for rational arguments.

2.15. (A simple caution.) Given a valid inequality between two functions, is it generally possible to obtain another valid inequality by direct differentiation? Is it true, for instance, that $f'(x) > g'(x)$ whenever $f(x) > g(x)$? Note, however, that if $f'(x) > g'(x)$ on $[a, b]$, then we do have $f(b) - f(a) > g(b) - g(a)$.

2.16. Use Lagrange multipliers to show that

$$\frac{x^n + y^n}{2} \geq \left(\frac{x+y}{2}\right)^n$$

for $n \geq 1$ and $x, y \geq 0$.

Inequalities

With Applications to Engineering

Cloud, M.J.; Drachman, B.C.; Lebedev, L.P.

2014, XIII, 239 p. 30 illus., Hardcover

ISBN: 978-3-319-05310-3