Chapter 2
Banach Algebras

In this chapter we present the basic concepts on Banach algebras and C^*-algebras, which are needed to understand many of the further topics in this book. In particular, we shall treat the basics of the Gelfand theory for commutative Banach algebras, and we shall give a proof of the Gelfand-Naimark theorem, which asserts that a commutative C^*-algebra is naturally isomorphic to the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space.

2.1 Banach Algebras

Recall the notion of an *algebra* from Sect. 1.6. A *Banach algebra* is an algebra $\mathcal{A}$ over the complex numbers together with a norm $\|\cdot\|$, in which $\mathcal{A}$ is complete, i.e., $\mathcal{A}$ is a Banach space, such that the norm is submultiplicative, i.e., the inequality

$$\|a \cdot b\| \leq \|a\| \|b\|$$

holds for all $a, b \in \mathcal{A}$. Note that this inequality in particular implies that the multiplication on $\mathcal{A}$ is a continuous map from $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which means that if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences in $\mathcal{A}$ converging to a and b, respectively, then the product sequence $a_n b_n$ converges to ab. This follows from the estimate

$$\begin{aligned}\|a_n b_n - ab\| &= \|a_n b_n - a_n b + a_n b - ab\| \\ &\leq \|a_n\| \|b_n - b\| + \|b\| \|a_n - a\|,\end{aligned}$$

as the latter term tends to zero as $n \to \infty$.

Examples 2.1.1

- The algebra $\mathrm{M}_n(\mathbb{C})$ equipped with the norm

  $$\|a\| = \sum_{i,j=1}^{n} |a_{i,j}|$$

  is a Banach algebra.

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- For a topological space X let $C(X)$ denote the vector space of continuous functions $f : X \to \mathbb{C}$. If X is compact, the space $C(X)$ becomes a commutative Banach algebra if it is equipped with the sup-norm $\|f\|_X = \sup_{x\in X} |f(x)|$.
- If G is a locally compact group, then $L^1(G)$ equipped with $\|\cdot\|_1$ and the convolution product $(f, g) \mapsto f * g$ is a Banach algebra by Theorem 1.6.2, which is commutative if and only if G is abelian by Theorem 1.6.4.
- Let V be a Banach space. For a linear operator $T : V \to V$ define the *operator norm* by

$$\|T\|_{op} \stackrel{\text{def}}{=} \sup_{v\neq 0} \frac{\|Tv\|}{\|v\|}.$$

  The operator T is called a *bounded operator* if $\|T\|_{\text{op}} < \infty$. By Lemma C.1.2 an operator is bounded if and only if it is continuous. The set $\mathcal{B}(V)$ of all bounded linear operators on V is a Banach algebra with the operator norm (see Exercise 2.1 below).

Definition An algebra $\mathcal{A}$ is *unital* if there exists an element $1_{\mathcal{A}} \in \mathcal{A}$ such that

$$1_{\mathcal{A}} a = a 1_{\mathcal{A}} = a \quad \text{for every } a \in \mathcal{A}.$$

The element $1_{\mathcal{A}}$ is then called the *unit* of $\mathcal{A}$. It is uniquely determined, for if $1'_{\mathcal{A}}$ is a second unit, one has $1_{\mathcal{A}} = 1_{\mathcal{A}} 1'_{\mathcal{A}} = 1'_{\mathcal{A}}$. We shall often write 1 for $1_{\mathcal{A}}$ if no confusion can arise.

Recall that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a complex vector space V are called *equivalent norms* if there is $C > 0$ with

$$\frac{1}{C}\|\cdot\| \;\le\; \|\cdot\|' \;\le\; C\|\cdot\|.$$

In that case, V is complete in the norm $\|\cdot\|$ if and only if it is complete in the norm $\|\cdot\|'$ and both norms define the same topology on V.

Lemma 2.1.2 *Let $\mathcal{A}$ be a unital Banach algebra with unit* 1. *Then* $\|1\| \ge 1$ *and there is an equivalent norm* $\|\cdot\|'$ *such that* $(\mathcal{A}, \|\cdot\|')$ *is again a Banach algebra with* $\|1\|' = 1$.

With this lemma in mind, we will, when talking about a unital Banach algebra, always assume that the unit element is of norm one.

Proof In the situation of the lemma one has $\|1\|^2 \ge \|1^2\| = \|1\|$, so $\|1\| \ge 1$. For $a \in \mathcal{A}$ let $\|a\|'$ be the *operator norm* of the multiplication operator M_a, which sends x to ax, so $\|a\|' = \sup_{x\neq 0} \frac{\|ax\|}{\|x\|}$. Then $\|\cdot\|'$ is a norm with $\|1\|' = 1$. Since $\|ax\| \le \|a\|\|x\|$, it follows that $\|a\|' \le \|a\|$. On the other hand one has $\|a\|' = \sup_{x\neq 0} \frac{\|ax\|}{\|x\|} \ge \frac{\|a\cdot 1\|}{\|1\|} = \frac{\|a\|}{\|1\|}$. This shows that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. The inequality $\|ab\|' \le \|a\|'\|b\|'$ is easy to show (See Exercise 2.1). $\square$

Proposition 2.1.3 *Let G be a locally compact group. The algebra $\mathcal{A} = L^1(G)$ is unital if and only if G is discrete.*

Proof If G is discrete, then the function $\mathbf{1}_{\{1\}}$ is easily seen to be a unit of $\mathcal{A}$. Conversely, assume that $\mathcal{A} = L^1(G)$ possesses a unit ϕ and G is non-discrete. The latter fact implies that any unit neighborhood U has at least two points. This implies by Urysohn's Lemma (A.8.1) that for every unit-neighborhood U there are two Dirac functions ϕ_U and ψ_U, both with support in U, such that the supports of ϕ_U and ψ_U are disjoint, hence in particular, $\|\phi_U - \psi_U\|_1 = 2$ for every $n \in \mathbb{N}$. The function ϕ being a unit means that we have $\phi * f = f * \phi = f$ for every $f \in L^1(G)$. There exists a unit-neighborhood U, such that one has $\|\phi_U * \phi - \phi\|_1 < 1$ and $\|\psi_U * \phi - \phi\|_1 < 1$. Hence $2 = \|\phi_U - \psi_U\|_1 \le \|\phi_U - \phi\|_1 + \|\phi - \psi_U\|_1 < 2$, a contradiction! Hence the assumption is false and G must be discrete. □

Definition Let $\mathcal{A}, \mathcal{B}$ be Banach algebras. A *homomorphism of Banach algebras* is by definition a continuous algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$. This means that ϕ is continuous, $\mathbb{C}$-linear and multiplicative, i.e., satisfies $\phi(ab) = \phi(a)\phi(b)$. A *topological isomorphism of Banach algebras* is a homomorphism with continuous inverse, and an *isomorphism of Banach algebras* is an isomorphism ϕ, which is an isometry, i.e., which satisfies $\|\phi(a)\| = \|a\|$ for every $a \in \mathcal{A}$. For better distinction we will call an isomorphism of Banach algebras henceforth an *isometric isomorphism of Banach algebras*.

Example 2.1.4 Let $Y \subset X$ be a compact subspace of the compact topological space X. Then the restriction of functions is a homomorphism of Banach algebras from $C(X)$ to $C(Y)$. Note that this includes the special case when $Y = \{x\}$ consists of a single element. In this case $C(Y) \cong \mathbb{C}$, and the restriction is the *evaluation homomorphism* $\delta_x : C(X) \to \mathbb{C}$ mapping f to $f(x)$.

If $\mathcal{A}$ is a unital Banach algebra, we denote by $\mathcal{A}^\times$ the group of invertible elements of $\mathcal{A}$, i.e., the multiplicative group of all a in $\mathcal{A}$, for which there exists some $b \in \mathcal{A}$ with $ab = ba = 1$. This b then is uniquely determined, as for a second such b' one has $b' = b'ab = b$. Therefore it is denoted a^{-1} and called the *inverse* of a.

Recall that for $a \in \mathcal{A}$ we denote by $B_r(a)$ the open ball of radius $r > 0$ around $a \in \mathcal{A}$, in other words, $B_r(a)$ is the set of all $z \in \mathcal{A}$ with $\|a - z\| < r$.

Lemma 2.1.5 (Neumann series). *Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$ with $\|a\| < 1$. Then $1 - a$ is invertible with inverse*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

The unit group $\mathcal{A}^\times$ is an open subset of $\mathcal{A}$. With the subspace topology, $\mathcal{A}^\times$ is a topological group.

Proof Since $\|a\| < 1$ one has $\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n < \infty$, so the series $b = \sum_{n=0}^{\infty} a^n$ converges absolutely in $\mathcal{A}$, and we get the first assertion by computing $(1-a)b = (1-a)\sum_{n=0}^{\infty} a^n = \sum_{n=0}^{\infty} a^n - \sum_{n=0}^{\infty} a^{n+1} = 1$, and likewise $b(1-a) = 1$.

For the second assertion let $y \in \mathcal{A}^\times$. As the multiplication on $\mathcal{A}$ is continuous, the map $x \mapsto yx$ is a homeomorphism. This implies that $yB_1(1) \subset \mathcal{A}^\times$ is an open neighborhood of y, so $\mathcal{A}^\times$ is indeed open.

To show that $\mathcal{A}^\times$ is a topological group, it remains to show that the inversion is continuous on $\mathcal{A}^\times$. Note that the map $a \mapsto \sum_{n=0}^{\infty} a^n = (1-a)^{-1}$ is continuous on $B_1(0)$, which implies that inversion is continuous on $B_1(1)$. But then it is continuous on the open neighborhood $yB_1(1) \subset \mathcal{A}^\times$ of any $y \in \mathcal{A}^\times$. □

Examples 2.1.6

- Let $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$. Then the unit group $\mathcal{A}^\times$ is the group of invertible matrices, i.e., of those matrices $a \in \mathcal{A}$ with $\det(a) \neq 0$. The continuity of the determinant function in this case gives another proof that $\mathcal{A}^\times$ is open.
- Let $\mathcal{A} = C(X)$ for a compact Hausdorff space X. Then the unit group $\mathcal{A}^\times$ consists of all $f \in C(X)$ with $f(x) \neq 0$ for every $x \in X$.

2.2 The Spectrum $\sigma_{\mathcal{A}}(a)$

Let $\mathcal{A}$ be a unital Banach algebra. For $a \in \mathcal{A}$ we denote by

$$\mathrm{Res}\,(a) \stackrel{\mathrm{def}}{=} \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is invertible}\}$$

the *resolvent set* of $a \in \mathcal{A}$. Its complement,

$$\sigma_{\mathcal{A}}(a) \stackrel{\mathrm{def}}{=} \mathbb{C} \smallsetminus \mathrm{Res}(a)$$

is called the *spectrum* of a. Since $\mathcal{A}^\times$ is open in $\mathcal{A}$ by Lemma 2.1.5 and since $\lambda \mapsto (\lambda 1 - a)$ is continuous, we see that $\mathrm{Res}(a)$ is open, and $\sigma_{\mathcal{A}}(a)$ is closed in $\mathbb{C}$.

Examples 2.2.1

- Let $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$. Then for $a \in \mathcal{A}$ the spectrum $\sigma(a)$ equals the set of eigenvalues of a.
- Let X be a compact topological space, and let $\mathcal{A} = C(X)$. For $f \in \mathcal{A}$ the spectrum $\sigma(f)$ equals the image of the map $f : X \to \mathbb{C}$.

Lemma 2.2.2 *Let $\mathcal{A}$ be a unital Banach algebra. Then for every $a \in \mathcal{A}$ the spectrum $\sigma(a)$ is a closed subset of the closed ball $\bar{B}_{\|a\|}(0)$ around zero of radius $\|a\|$, so in particular, $\sigma(a)$ is compact.*

Proof As the resolvent set is open, the spectrum is closed. Let $a \in \mathcal{A}$, and let $\lambda \in \mathbb{C}$ with $|\lambda| > \|a\|$. We have to show that $\lambda 1 - a$ is invertible. As $\|\lambda^{-1}a\| < 1$, by Lemma 2.1.5 one has $1 - \lambda^{-1}a \in \mathcal{A}^\times$; it follows that $\lambda \cdot 1 - a = \lambda(1 - \lambda^{-1}a) \in \mathcal{A}^\times$, so $\lambda \in \operatorname{Res}(a)$. □

Definition Let $D \subset \mathbb{C}$ be an open set, and let $f : D \to V$ be a map, where V is a Banach space. We say that f is *holomorphic* if for every $z \in D$ the limit

$$f'(z) = \lim_{h \to 0} \frac{1}{h}(f(z+h) - f(z))$$

exists in V. Note that if f is holomorphic and $\alpha : V \to \mathbb{C}$ is a continuous linear functional, then the function $z \mapsto \alpha(f(z))$ is a holomorphic function from D to $\mathbb{C}$. A holomorphic map is continuous (see Exercise 2.2).

Lemma 2.2.3 *Let $a \in \mathcal{A}$, then the map $f : \lambda \mapsto (\lambda - a)^{-1}$ is holomorphic on the resolvent set* $\operatorname{Res}(a)$. *Here we have written* λ for $\lambda 1 \in \mathcal{A}$.

Proof Let $\lambda \in \operatorname{Res}(a)$, and let h be a small complex number. Then $\frac{1}{h}(f(\lambda+h) - f(\lambda))$ equals

$$\begin{aligned}
&\frac{1}{h}\left((\lambda + h - a)^{-1} - (\lambda - a)^{-1}\right) \\
&= \frac{1}{h}\left((\lambda - a) - (\lambda + h - a)\right)(\lambda + h - a)^{-1}(\lambda - a)^{-1} \\
&= -(\lambda + h - a)^{-1}(\lambda - a)^{-1}.
\end{aligned}$$

This map is continuous at $h = 0$, since the inversion is a continuous map on the resolvent set by Lemma 2.1.5. □

Theorem 2.2.4 *Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$. Then $\sigma_{\mathcal{A}}(a) \neq \emptyset$.*

Proof Assume there exists $a \in \mathcal{A}$ with empty spectrum. Let α be a continuous linear functional on $\mathcal{A}$, then the function $f_\alpha : \lambda \mapsto \alpha\left((a - \lambda)^{-1}\right)$ is entire. As α is continuous, hence bounded, there exists $C > 0$ such that $|\alpha(b)| \le C\|b\|$ holds for every $b \in \mathcal{A}$. For $|\lambda| > 2\|a\|$ we get

$$\begin{aligned}
|f_\alpha(\lambda)| &= |\alpha\left((a - \lambda)^{-1}\right)| = \frac{1}{|\lambda|}|\alpha\left((1 - \lambda^{-1}a)^{-1}\right)| \\
&= \frac{1}{|\lambda|}\left|\alpha\left(\sum_{n=0}^{\infty}(\lambda^{-1}a)^n\right)\right| \le \frac{1}{|\lambda|}\sum_{n=0}^{\infty}|\alpha\left((\lambda^{-1}a)^n\right)| \\
&\le \frac{C}{|\lambda|}\sum_{n=0}^{\infty}\|\lambda^{-1}a\|^n < \frac{C}{|\lambda|}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{2\,C}{|\lambda|}.
\end{aligned}$$

It follows that the entire function f_α is constantly zero by the Theorem of Liouville [Rud87]. This holds for every α, so the Hahn-Banach Theorem C.1.3 implies that $f : \lambda \mapsto (\lambda - a)^{-1}$ is the zero map as well, which is a contradiction. □

Corollary 2.2.5 (Gelfand-Mazur). *Let $\mathcal{A}$ be a unital Banach algebra such that all non-zero elements $a \in \mathcal{A}$ are invertible. Then $\mathcal{A} = \mathbb{C}1$.*

Proof If $a \in \mathcal{A} \setminus \mathbb{C}1$ we have $\lambda 1 - a$ invertible for every $\lambda \in \mathbb{C}$. But this means that $\sigma_{\mathcal{A}}(a) = \emptyset$, which contradicts Theorem 2.2.4. □

Definition For an element a of a unital Banach algebra $\mathcal{A}$ we define the *spectral radius* $r(a)$ of a by

$$r(a) \stackrel{\text{def}}{=} sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}.$$

In what follows next we want to prove an important formula for the spectral radius $r(a)$.

Theorem 2.2.6 (Spectral radius formula). *Let $\mathcal{A}$ be a unital Banach algebra. Then $r(a) \le \|a\|$ and*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Proof As $\|a^n\| \le \|a\|^n$ the first assertion follows from the second. We shall show the inequalities

$$r(a) \le \liminf \|a^n\|^{\frac{1}{n}} \le \limsup \|a^n\|^{\frac{1}{n}} \le r(a),$$

which clearly implies the theorem.

For $\lambda \in \sigma_{\mathcal{A}}(a)$, the equation $\lambda^n 1 - a^n = (\lambda 1 - a)\sum_{j=0}^{n-1} \lambda^j a^{n-1-j}$ implies that $\lambda^n \in \sigma_{\mathcal{A}}(a^n)$ and hence that $|\lambda|^n \le \|a^n\|$ for every $n \in \mathbb{N}$. Thus $r(a) \le \|a^n\|^{\frac{1}{n}}$ for every $n \in \mathbb{N}$, which gives the first inequality.

To see that $\limsup \|a^n\|^{\frac{1}{n}} \le r(a)$ recall $(\lambda 1 - a)^{-1} = \lambda^{-1}(1 - \frac{a}{\lambda})^{-1} = \sum_{n=0}^{\infty} a^n \frac{1}{\lambda^{n+1}}$ for $|\lambda| > \|a\|$, and hence, as the function is holomorphic there, the series converges in the norm-topology for every $|\lambda| > r(a)$, as we derive from Corollary B.6.7 applied to $z = \frac{1}{\lambda}$.

For a fixed $|\lambda| > r(a)$, it follows that the sequence $a^n \frac{1}{\lambda^{n+1}}$ is bounded in $\mathcal{A}$, so that there exists a constant $C \ge 0$ such that $\|a^n\| \le C|\lambda|^{n+1}$ for every $n \in \mathbb{N}$. Taking n-th roots on both sides and then applying lim sup shows that $\limsup \|a^n\|^{\frac{1}{n}} \le |\lambda|$. Since this holds for every $|\lambda| > r(a)$ we get $\limsup \|a^n\|^{\frac{1}{n}} \le r(a)$. □

Lemma 2.2.7 *Suppose that $\mathcal{A}$ is a closed subalgebra of the unital Banach algebra $\mathcal{B}$ such that $1 \in \mathcal{A}$. Then*

$$\partial\sigma_{\mathcal{A}}(a) \subset \sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{A}}(a)$$

for every $a \in \mathcal{A}$, where $\partial\sigma_{\mathcal{A}}(a)$ denotes the boundary of $\sigma_{\mathcal{A}}(a) \subset \mathbb{C}$.

Proof If $a \in \mathcal{A}$ is invertible in $\mathcal{A}$ it is invertible in $\mathcal{B} \supset \mathcal{A}$. So $\mathrm{Res}_{\mathcal{A}}(a) \subset \mathrm{Res}_{\mathcal{B}}(a)$, which is equivalent to the second inclusion.

To see the first inclusion let $\lambda \in \partial\sigma_{\mathcal{A}}(a) \subset \sigma_{\mathcal{A}}(a)$, and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Res}_{\mathcal{A}}(a)$ with $\lambda_n \to \lambda$. If $\lambda \in \mathrm{Res}_{\mathcal{B}}(a)$, then $\mathcal{A} \ni (\lambda_n 1 - a)^{-1} \to (\lambda 1 - a)^{-1} \in \mathcal{B}$. Since $\mathcal{A}$ is closed in $\mathcal{B}$ we get $(\lambda 1 - a)^{-1} \in \mathcal{A}$, which implies that $\lambda \in \mathrm{Res}_{\mathcal{A}}(a)$. This contradicts $\lambda \in \sigma_{\mathcal{A}}(a)$. □

Example 2.2.8 Let $\mathbb{D} \subset \mathbb{C}$ be the closed disk of radius 1 around zero, and let $\mathring{\mathbb{D}}$ be its interior. The *disk-algebra* $\mathcal{A}$ is by definition the subalgebra of $C(\mathbb{D})$ consisting of all functions that are holomorphic inside $\mathring{\mathbb{D}}$. Since uniform limits of holomorphic functions are again holomorphic, the disk-algebra is a closed subalgebra of $C(\mathbb{D})$ and hence a Banach algebra. Let $\mathbb{T} = \partial\mathbb{D}$ be the circle group. By the maximum principle for holomorphic functions, every $f \in \mathcal{A}$ takes its maximum on $\mathbb{T}$. Therefore the restriction homomorphism $\mathcal{A} \to C(\mathbb{T})$ mapping $f \in \mathcal{A}$ to its restriction $f|_{\mathbb{T}}$ is an isometry. So $\mathcal{A}$ can be viewed as a sub Banach algebra of $\mathcal{B} = C(\mathbb{T})$. For $f \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ the function $(\lambda - f)^{-1}$ is defined in $\mathcal{A}$ if and only if λ is not in the image of f. Therefore, the spectrum $\sigma_{\mathcal{A}}(f)$ equals the image $f(\mathbb{D})$. Likewise, considered as an element of $\mathcal{B}$, the spectrum of f equals $\sigma_{\mathcal{B}}(f) = f(\mathbb{T})$.

2.3 Adjoining a Unit

The results of the previous section always depended on the existence of a unit in the Banach algebra $\mathcal{A}$. But many important Banach algebras, like $L^1(G)$ for a non-discrete locally compact group G, do not have a unit. We solve this problem by adjoining a unit if needed. Indeed, if $\mathcal{A}$ is any Banach algebra (with or without unit), then the cartesian product,

$$\mathcal{A}^e \stackrel{\text{def}}{=} \mathcal{A} \times \mathbb{C}$$

equipped with the obvious vector space structure and the multiplication

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

becomes an algebra with unit $(0, 1)$. If we define $\|(a, \lambda)\| = \|a\| + |\lambda|$, one easily checks that $\mathcal{A}^e$ becomes a Banach algebra containing $\mathcal{A} \cong \mathcal{A} \times \{0\}$ as a closed subalgebra of codimension 1. We call $\mathcal{A}^e$ the *unitization* of $\mathcal{A}$.

If $\mathcal{A}$ already has a unit $1_{\mathcal{A}}$, then the algebra $\mathcal{A}^e$ is isomorphic to the direct sum $\mathcal{A} \oplus \mathbb{C}$ of the algebras $\mathcal{A}$ and $\mathbb{C}$, where we define multiplication component-wise. The isomorphism from $\mathcal{A}^e$ to $\mathcal{A} \oplus \mathbb{C}$ is given by $(a, \lambda) \mapsto (a + \lambda 1_{\mathcal{A}}) \oplus \lambda$.

If $\mathcal{A}$ is a Banach algebra without unit then we define the spectrum of $a \in \mathcal{A}$ as

$$\sigma_{\mathcal{A}}(a) \stackrel{\text{def}}{=} \sigma_{\mathcal{A}^e}(a),$$

where we identify $\mathcal{A}$ with a subset of $\mathcal{A}^e$ via $a \mapsto (a, 0)$. With this convention, all results from the previous section, in particular the spectral radius formula, have natural analogues in the non-unital case.

A very important class of commutative Banach-Algebras without unit is given as follows:

Definition Suppose that X is a locally compact Hausdorff space. A function $f : X \to \mathbb{C}$ is said to *vanish at infinity* if for every $\varepsilon > 0$ there exists a compact set $K = K_\varepsilon \subset X$ such that $|f(x)| < \varepsilon$ holds for every $x \in X \smallsetminus K$. Let $C_0(X)$ denote the vector space of all continuous functions on X that vanish at infinity. Then $C_0(X)$ is a Banach algebra with point-wise multiplication and the sup-norm $\|f\|_X = \sup_{x \in X} |f(x)|$. Note that $C_0(X)$ is unital if and only if X is compact in which case $C_0(X)$ equals $C(X)$.

Example 2.3.1 As a crucial example we want to compute the unitization $C_0(X)^e$. We recall the construction of the *one point compactification*, also called *Alexandrov compactification* X^∞, of the space X. Let ∞ denote a new point and $X^\infty = X \cup \{\infty\}$, so X^∞ is just a set that contains X as a subset plus one more element. On X^∞ one introduces the following topology. A set $U \subset X^\infty$ is open if it is either already contained in X and open in the topology of X, or if $\infty \in U$ and the set $X \smallsetminus U$ is a compact subset of X. Note that X being a Hausdorff space implies that compact sets in X are closed in X. Every continuous function in $C(X^\infty)$ defines, by restriction, a continuous function on X. In this way one can identify $C_0(X)$ with the subspace of all continuous functions f on X^∞ with $f(\infty) = 0$. This ultimately justifies the notion "vanishing at infinity".

Lemma 2.3.2 *There is a canonical topological isomorphism of Banach algebras* $C(X^\infty) \cong C_0(X)^e$.

Proof Extending every $f \in C_0(X)$ by zero to X^∞, we consider $C_0(X)$ as a subspace of $C(X^\infty)$. Define $\psi : C_0(X)^e \to C(X^\infty)$ by $\psi(f, \lambda) = f + \lambda e$, where $e(x) = 1$ for every $x \in X^\infty$. Then ψ is an isomorphism of algebras. For the norms one has

$$\|\psi(f, \lambda)\|_{X^\infty} = \sup_{x \in X^\infty} |f(x) + \lambda| \leq \sup_{x \in X} |f(x)| + |\lambda| = \|(f, \lambda)\|.$$

This implies that ψ is continuous. On the other hand, one has

$$\sup_{x \in X^\infty} |f(x) + \lambda| \geq |f(\infty) + \lambda| = |\lambda|.$$

Since $|f(x)| \leq |\lambda| + |f(x) + \lambda|$ for every x, we then get

$$\|(f,\lambda)\| = \sup_{x \in X^\infty} |f(x)| + |\lambda| \ \leq\ 3\|\psi(f,\lambda)\|_{X^\infty}.$$

The lemma is proven. Note that ψ is not an isometry, but restricted to $C_0(X)$, it becomes one. □

2.4 The Gelfand Map

In this section we shall always assume that $\mathcal{A}$ is a *commutative* Banach algebra. In this case we define the *structure space* $\Delta_\mathcal{A}$ to be the set of all non-zero continuous algebra homomorphisms $m : \mathcal{A} \to \mathbb{C}$. This space is often called the *maximal ideal space*, which is justified by Theorem 2.5.2 below, that sets up a bijection between $\Delta_\mathcal{A}$ and the set of maximal ideals of $\mathcal{A}$. The elements of $\Delta_\mathcal{A}$ are also called *multiplicative linear functionals*, which is why we use the letter m to denote them. If $\mathcal{A}$ is unital, it follows automatically that $m(1) = 1$ for every $m \in \Delta_\mathcal{A}$, since $m(1) = m(1^2) = m(1)^2$ implies $m(1) = 0$ or $m(1) = 1$. Now $m(1) = 0$ implies $m = 0$, a case which is excluded.

Examples 2.4.1

- Let $\mathcal{A} = C_0(X)$ for a locally compact Hausdorff space X. For $x \in X$ one gets an element m_x of $\Delta_\mathcal{A}$ defined by $m_x(f) = f(x)$.
- Let $\mathcal{A} = L^1(A)$, where A is an LCA-group. Let $\chi \in \widehat{A}$, then the map $m_\chi : \mathcal{A} \to \mathbb{C}$ defined by
$$m_\chi(f) = \hat{f}(\chi) = \int_A f(x)\overline{\chi(x)}\,dx$$
is an element of $\Delta_\mathcal{A}$ as follows from Lemma 1.7.2.

For a given multiplicative functional $m \in \Delta_\mathcal{A}$, there exists precisely one extension of m to a multiplicative functional on $\mathcal{A}^e$ given by

$$m^e(a,\lambda) = m(a) + \lambda.$$

A multiplicative functional of $\mathcal{A}^e$ that is not extended from $\mathcal{A}$ must vanish on $\mathcal{A}$, and hence it must be equal to the *augmentation functional* m_∞ of $\mathcal{A}^e$ given by $m_\infty(a,\lambda) = \lambda$. Thus we get

$$\Delta_{\mathcal{A}^e} = \{m^e : m \in \Delta_\mathcal{A}\} \ \cup\ \{m_\infty\}.$$

We now want to equip $\Delta_\mathcal{A}$ with a natural topology that makes $\Delta_\mathcal{A}$ into a locally compact space (compact if $\mathcal{A}$ is unital). For a Banach space V let V' be the *dual space* consisting of all continuous linear maps $\alpha : V \to \mathbb{C}$. This is a Banach space with the norm $\|\alpha\| = \sup_{v \in V \setminus \{0\}} \frac{|\alpha(v)|}{\|v\|}$.

Lemma 2.4.2 *Suppose that $\mathcal{A}$ is a commutative Banach algebra. Let $m \in \Delta_{\mathcal{A}}$. Then m is continuous with $\|m\| \le 1$.* If *$\mathcal{A}$ is unital, then $\|m\| = 1$.*

Proof We first consider the case when $\mathcal{A}$ is unital. If $a \in \mathcal{A}$, then $m(a - m(a)1) = 0$, which implies that $a - m(a)1$ is not invertible in $\mathcal{A}$, so that $m(a) \in \sigma(a)$. Since $\sigma(a) \subset B_{\|a\|}(0)$, this implies that $|m(a)| \le \|a\|$ for every $a \in \mathcal{A}$. By $m(1) = 1$ we see that m is continuous with $\|m\| = 1$.

If $\mathcal{A}$ is not unital, the extension $m^e : \mathcal{A}^e \to \mathbb{C}$ is continuous with $\|m^e\| = 1$. But then the restriction $m = m^e|_{\mathcal{A}}$ is also continuous with $\|m\| \le 1$. □

It follows from the lemma that $\Delta_{\mathcal{A}} \subset \bar{B}' \subset \mathcal{A}'$, where $\bar{B}' = \{f \in \mathcal{A}' : \|f\| \le 1\}$ is the closed ball of radius one. Recall that for any normed space V, the *weak-*-topology* on V' is defined as the initial topology on V' defined by the maps $\{\delta_v : v \in V\}$ with $\delta_v : V' \to \mathbb{C};\ \alpha \mapsto \alpha(v)$. It is the topology of point-wise convergence, i.e., a net $(\alpha_j)_j$ in V' converges to $\alpha \in V'$ in the weak-* topology if and only if $\alpha_j(v) \to \alpha(v)$ for all $v \in V$.

We need the following important fact, which, as we shall see, is a consequence of Tychonov's Theorem.

Theorem 2.4.3 (Banach-Alaoglu). *Let V be a normed (complex) vector space. Then the closed unit ball*

$$\bar{B}' \stackrel{\text{def}}{=} \{f \in V' : \|f\| \le 1\} \subset V'$$

equipped with the weak--topology is a compact Hausdorff space.*

Proof Recall $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$. For $\alpha \in \bar{B}'$ and $v \in V$ one has $|\alpha(v)| \le \|v\|$, so $\alpha(v)$ is an element of the compact set $\mathbb{D}\,\|v\|$. So one gets an injective map

$$\bar{B}' \to \prod_{v \in V} \mathbb{D}\|v\|$$

$$\alpha \to (\alpha(v))_v.$$

Note that the product space on the right is Hausdorff and compact by Tychonov's Theorem A.7.1. Since a net in the product space converges if and only if it converges in each component, the weak-*-topology on $\bar{B}'$ coincides with the subspace topology if one views $\bar{B}'$ as a subspace of the product space. Thus, all we need to show is that $\bar{B}'$ is closed in the product space. An element x of the product space lies in $\bar{B}'$ if and only if its coordinates satisfy $x_{v+w} = x_v + x_w$ and $x_{\lambda v} = \lambda x_v$ for all $v, w \in V$ and every $\lambda \in \mathbb{C}$. These conditions define a closed subset of the product. □

If $\mathcal{A}$ is any commutative Banach algebra, we equip the structure space $\Delta_{\mathcal{A}} \subset \mathcal{A}'$ of the algebra $\mathcal{A}$ with the topology induced from the weak-*-topology on $\mathcal{A}'$.

Lemma 2.4.4 *Suppose that $\mathcal{A}$ is a commutative Banach algebra. Then the inclusion map $\Phi : \Delta_{\mathcal{A}} \to \Delta_{\mathcal{A}^e}$ that maps m to m^e is a homeomorphism onto its image.*

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