

## Chapter 2

# Random Interlacements: First Definition and Basic Properties

In this chapter we give the first definition of random interlacements at level  $u > 0$  as a random subset of  $\mathbb{Z}^d$ . We then prove that it has polynomially decaying correlations and is invariant and ergodic with respect to the lattice shifts.

Along the way we also collect various basic definitions pertaining to a measurable space of subsets of  $\mathbb{Z}^d$ , which will be used often throughout these notes.

### 2.1 Space of Subsets of $\mathbb{Z}^d$ and Random Interlacements

Consider the space  $\{0, 1\}^{\mathbb{Z}^d}$ ,  $d \geq 3$ . This space is in one-to-one correspondence with the space of subsets of  $\mathbb{Z}^d$ , where for each  $\xi \in \{0, 1\}^{\mathbb{Z}^d}$ , the corresponding subset of  $\mathbb{Z}^d$  is defined by

$$\mathcal{S}(\xi) = \{x \in \mathbb{Z}^d : \xi_x = 1\}.$$

Thus, we can think about the space  $\{0, 1\}^{\mathbb{Z}^d}$  as the space of subsets of  $\mathbb{Z}^d$ . For  $x \in \mathbb{Z}^d$ , we define the function  $\Psi_x : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \{0, 1\}$  by  $\Psi_x(\xi) = \xi_x$  for  $\xi \in \{0, 1\}^{\mathbb{Z}^d}$ . The functions  $(\Psi_x)_{x \in \mathbb{Z}^d}$  are called *coordinate maps*.

**Definition 2.1.** For  $K \subset \mathbb{Z}^d$ , we denote by  $\sigma(\Psi_x, x \in K)$  the sigma-algebra on the space  $\{0, 1\}^{\mathbb{Z}^d}$  generated by the coordinate maps  $\Psi_x$ ,  $x \in K$ , and we define  $\mathcal{F} = \sigma(\Psi_x, x \in \mathbb{Z}^d)$ .

If  $K \subset \mathbb{Z}^d$  and  $A \in \sigma(\Psi_x, x \in K)$ , then we say that  $A$  is a *local event* with support  $K$ .

For any  $K_0 \subseteq K \subset \mathbb{Z}^d$ ,  $K_1 = K \setminus K_0$  we say that

$$\{\forall x \in K_0 : \Psi_x = 0; \forall x \in K_1 : \Psi_x = 1\} = \{\mathcal{S} \cap K = K_1\} \quad (2.1.1)$$

is a *cylinder event* with base  $K$ .

**Remark 2.2.** Every local event is a finite disjoint union of cylinder events. More precisely stated, for any  $K \subset \subset \mathbb{Z}^d$ , the sigma-algebra  $\sigma(\Psi_x, x \in K)$  is atomic and has exactly  $2^{|K|}$  atoms of form (2.1.1).

For  $u > 0$ , we consider the one-parameter family of probability measures  $\mathcal{P}^u$  on  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F})$  satisfying the equations

$$\mathcal{P}^u[\mathcal{S} \cap K = \emptyset] = e^{-u \text{cap}(K)}, \quad K \subset \subset \mathbb{Z}^d. \quad (2.1.2)$$

These equations uniquely determine the measure  $\mathcal{P}^u$  since the events

$$\{\xi \in \{0, 1\}^{\mathbb{Z}^d} : \mathcal{S}(\xi) \cap K = \emptyset\} = \{\xi \in \{0, 1\}^{\mathbb{Z}^d} : \Psi_x(\xi) = 0 \text{ for all } x \in K\}, \quad K \subset \subset \mathbb{Z}^d$$

form a  $\pi$ -system (i.e., a family of sets which is closed under finite intersections) that generates  $\mathcal{F}$ , and Dynkin's  $\pi - \lambda$  lemma (see Theorem 3.2 in [7]) implies the following result.

**Claim 2.3.** If two probability measures on the same measurable space coincide on a  $\pi$ -system, then they coincide on the sigma-algebra generated by that  $\pi$ -system.

The existence of a probability measure  $\mathcal{P}^u$  satisfying (2.1.2) is not immediate, but it will follow from Definition 5.7 and Remark 5.8. The measure  $\mathcal{P}^u$  also arises as the local limit of the trace of the first  $\lfloor uN^d \rfloor$  steps of simple random walk with a uniform starting point on the  $d$ -dimensional torus  $(\mathbb{Z}/N\mathbb{Z})^d$ ; see Theorem 3.1 and Exercise 3.2.

The random subset  $\mathcal{S}$  of  $\mathbb{Z}^d$  in  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F}, \mathcal{P}^u)$  is called *random interlacements* at level  $u$ . The reason behind the use of “interlacements” in the name will become clear in Chap. 5; see Definition 5.7, where we define random interlacements at level  $u$  as the range of the (interlacing) SRW trajectories in the support of a certain Poisson point process.

By the inclusion-exclusion formula, we obtain from (2.1.2) the following explicit expressions for the probabilities of cylinder events: for any  $K_0 \subseteq K \subset \subset \mathbb{Z}^d$ ,  $K_1 = K \setminus K_0$ ,

$$\mathcal{P}^u[\Psi|_{K_0} \equiv 0, \Psi|_{K_1} \equiv 1] = \mathcal{P}^u[\mathcal{S} \cap K = K_1] = \sum_{K' \subseteq K_1} (-1)^{|K'|} e^{-u \text{cap}(K_0 \cup K')}. \quad (2.1.3)$$

**Exercise 2.4.** Show (2.1.3).

In the remaining part of this chapter we prove some basic properties of random interlacements at level  $u$  using Eqs. (2.1.2) and (2.1.3).

## 2.2 Correlations, Shift-Invariance, and Ergodicity

In this section we prove that random interlacements at level  $u$  has polynomially decaying correlations and is invariant and ergodic with respect to the lattice shifts. We begin with computing the asymptotic behavior of the covariances of  $\mathcal{S}$  under  $\mathcal{P}^u$ .

**Claim 2.5.** For any  $u > 0$ ,

$$\text{Cov}_{\mathcal{P}^u}(\Psi_x, \Psi_y) \sim \frac{2u}{g(0)^2} g(y-x) \exp\left\{-\frac{2u}{g(0)}\right\}, \quad |x-y| \rightarrow \infty. \quad (2.2.1)$$

*Remark 2.6.* By (2.2.1) and the Green function estimate (1.2.8), for any  $u > 0$  and  $x, y \in \mathbb{Z}^d$ ,

$$c \cdot (|y-x|+1)^{2-d} \leq \text{Cov}_{\mathcal{P}^u}(\mathbb{1}_{\{x \in \mathcal{S}\}}, \mathbb{1}_{\{y \in \mathcal{S}\}}) \leq C \cdot (|y-x|+1)^{2-d},$$

for some constants  $0 < c \leq C < \infty$  depending on  $u$ . We say that random interlacements at level  $u$  exhibits *polynomial decay of correlations*.

*Proof (Proof of Claim 2.5).* We compute

$$\begin{aligned} \text{Cov}_{\mathcal{P}^u}(\Psi_x, \Psi_y) &= \text{Cov}_{\mathcal{P}^u}(1 - \Psi_x, 1 - \Psi_y) = \mathcal{P}^u[\Psi_x = \Psi_y = 0] - \mathcal{P}^u[\Psi_x = 0] \mathcal{P}^u[\Psi_y = 0] \\ &\stackrel{(2.1.2)}{=} \exp\{-u\text{cap}(\{x, y\})\} - \exp\{-u\text{cap}(\{x\})\} \cdot \exp\{-u\text{cap}(\{y\})\} \\ &\stackrel{(1.3.7), (1.3.8)}{=} \exp\left\{-\frac{2u}{g(0) + g(y-x)}\right\} - \exp\left\{-\frac{2u}{g(0)}\right\} \\ &= \exp\left\{-\frac{2u}{g(0)}\right\} \left(\exp\left\{\frac{2ug(x-y)}{g(0)(g(0) + g(x-y))}\right\} - 1\right) \\ &\sim \exp\left\{-\frac{2u}{g(0)}\right\} \frac{2ug(x-y)}{g(0)^2}. \end{aligned}$$

**Definition 2.7.** If  $Q$  is a probability measure on  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F})$ , then a measure-preserving transformation  $T$  on  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F}, Q)$  is an  $\mathcal{F}$ -measurable map  $T : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \{0, 1\}^{\mathbb{Z}^d}$ , such that

$$Q[T^{-1}(A)] = Q[A] \quad \text{for all } A \in \mathcal{F}.$$

Such a measure-preserving transformation is called *ergodic* if all  $T$ -invariant events, i.e., all  $A \in \mathcal{F}$  for which  $T^{-1}(A) = A$ , have  $Q$ -probability 0 or 1.

We now define the measure-preserving transformations we will have a look at. For  $x \in \mathbb{Z}^d$  we introduce the canonical shift

$$t_x : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \{0, 1\}^{\mathbb{Z}^d}, \quad \Psi_y(t_x(\xi)) = \Psi_{y+x}(\xi), \quad y \in \mathbb{Z}^d, \xi \in \{0, 1\}^{\mathbb{Z}^d}.$$

Also, for  $K \subseteq \mathbb{Z}^d$ , we define  $K + x = \{y + x : y \in K\}$ .

**Lemma 2.8.** *For any  $x \in \mathbb{Z}^d$  and any  $u > 0$  the transformation  $t_x$  preserves the measure  $\mathcal{P}^u$ .*

*Proof.* Let  $x \in \mathbb{Z}^d$ . We want to prove that the pushforward of  $\mathcal{P}^u$  by  $t_x$  coincides with  $\mathcal{P}^u$ , i.e.,  $(t_x \circ \mathcal{P}^u)[A] = \mathcal{P}^u[A]$  for all  $A \in \mathcal{F}$ . By Claim 2.3 it suffices to show that  $t_x \circ \mathcal{P}^u$  satisfies (2.1.2).

Let  $K \subset \subset \mathbb{Z}^d$ . We compute

$$(t_x \circ \mathcal{P}^u)[\mathcal{S} \cap K = \emptyset] = \mathcal{P}^u[\mathcal{S} \cap (K - x) = \emptyset] \stackrel{(2.1.2)}{=} e^{-u \text{cap}(K-x)} = e^{-u \text{cap}(K)}.$$

Before we state the next property of random interlacements, we recall the following classical approximation result (see, e.g., [7, Theorem 11.4]). For  $A, B \in \mathcal{F}$ , we denote by  $A \Delta B \in \mathcal{F}$  the symmetric difference between  $A$  and  $B$ , i.e.,

$$A \Delta B = (A \setminus B) \cup (B \setminus A). \quad (2.2.2)$$

**Exercise 2.9.** Let  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F}, Q)$  be a probability space, and take  $B \in \mathcal{F}$ . Prove that

$$\text{for any } \varepsilon > 0 \text{ there exist } K \subset \subset \mathbb{Z}^d \text{ and } B_\varepsilon \in \sigma(\Psi_x, x \in K) \text{ such that } Q[B_\varepsilon \Delta B] \leq \varepsilon. \quad (2.2.3)$$

Hint: it is enough to show that the family of sets  $B \in \mathcal{F}$  that satisfy (2.2.3) is a sigma-algebra that contains the local events.

The next result states that random interlacements is ergodic with respect to the lattice shifts.

**Theorem 2.10 ([41], Theorem 2.1).** *For any  $u \geq 0$  and  $0 \neq x \in \mathbb{Z}^d$ , the measure-preserving transformation  $t_x$  is ergodic on  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F}, \mathcal{P}^u)$ .*

*Proof.* Let us fix  $0 \neq x \in \mathbb{Z}^d$ . In order to prove that  $t_x$  is ergodic on  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F}, \mathcal{P}^u)$ , it is enough to show that for any  $K \subset \subset \mathbb{Z}^d$  and  $B_\varepsilon \in \sigma(\Psi_x, x \in K)$  we have

$$\lim_{n \rightarrow \infty} \mathcal{P}^u[B_\varepsilon \cap t_x^n(B_\varepsilon)] = \mathcal{P}^u[B_\varepsilon]^2. \quad (2.2.4)$$

Indeed, let  $B \in \mathcal{F}$  be such that  $t_x(B) = B$ . Note that for any integer  $n$ ,  $t_x^n(B) = B$ .

For any  $\varepsilon > 0$ , let  $B_\varepsilon \in \mathcal{F}$  be a local event satisfying (2.2.3) with  $Q = \mathcal{P}^u$ , i.e.,  $\mathcal{P}^u[B_\varepsilon \Delta B] \leq \varepsilon$ . Note that

$$\mathcal{P}^u[t_x^n(B_\varepsilon) \Delta B] = \mathcal{P}^u[t_x^n(B_\varepsilon) \Delta t_x^n(B)] = \mathcal{P}^u[t_x^n(B_\varepsilon \Delta B)] = \mathcal{P}^u[B_\varepsilon \Delta B] \leq \varepsilon,$$

where in the last equality we used Lemma 2.8. Therefore, for all  $n$ , we have

$$|\mathcal{P}^u[B_\varepsilon \cap t_x^n(B_\varepsilon)] - \mathcal{P}^u[B]| \leq \mathcal{P}^u[(B_\varepsilon \cap t_x^n(B_\varepsilon))\Delta B] \leq 2\varepsilon,$$

and we conclude that

$$\mathcal{P}^u[B] = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \mathcal{P}^u[B_\varepsilon \cap t_x^n(B_\varepsilon)] \stackrel{(2.2.4)}{=} \lim_{\varepsilon \rightarrow 0} \mathcal{P}^u[B_\varepsilon]^2 = \mathcal{P}^u[B]^2,$$

which implies that  $\mathcal{P}^u[B] \in \{0, 1\}$ . This proves the ergodicity of  $t_x$  on  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F}, \mathcal{P}^u)$  given the mixing property (2.2.4).

We begin the proof of (2.2.4) by showing that for any  $K_1, K_2 \subset \subset \mathbb{Z}^d$ ,

$$\lim_{|y| \rightarrow \infty} \text{cap}(K_1 \cup (K_2 + y)) = \text{cap}(K_1) + \text{cap}(K_2). \quad (2.2.5)$$

Let  $K_y = K_1 \cup (K_2 + y)$ . By the definition (1.3.2) of capacity, we only need to show that

$$\forall z \in K_1 : \lim_{|y| \rightarrow \infty} e_{K_y}(z) = e_{K_1}(z), \quad (2.2.6)$$

$$\forall z \in K_2 : \lim_{|y| \rightarrow \infty} e_{K_y}(z + y) = e_{K_2}(z) \quad (2.2.7)$$

in order to conclude (2.2.5). We only prove (2.2.6). For any  $z \in K_1$  we have

$$\begin{aligned} 0 \leq e_{K_1}(z) - e_{K_y}(z) &= P_z[\tilde{H}_{K_1} = \infty, \tilde{H}_{K_y} < \infty] \leq P_z[H_{K_2+y} < \infty] \\ &\leq \sum_{v \in K_2} P_z[H_{\{v+y\}} < \infty] \stackrel{(1.3.7)}{\leq} \sum_{v \in K_2} g(z, v+y) \stackrel{(1.2.8)}{\leq} C_g \sum_{v \in K_2} |v+y-z|^{2-d} \rightarrow 0, \quad |y| \rightarrow \infty, \end{aligned}$$

thus we obtain (2.2.6). The proof of (2.2.7) is analogous and we omit it.

We first prove (2.2.4) if  $A$  is a cylinder event of form (2.1.1).

If  $n$  is big enough, then  $K \cap (K + nx) = \emptyset$ ; therefore, we have

$$\begin{aligned} \mathcal{P}^u[A \cap t_x^n(A)] &= \mathcal{P}^u[\mathcal{S} \cap (K \cup (K + nx)) = K_1 \cup (K_1 + nx)] \stackrel{(2.1.3)}{=} \\ &\sum_{K'' \subseteq K_1} \sum_{K' \subseteq K_1} (-1)^{|K''|+|K'|} \exp\left(-u \text{cap}((K_0 \cup K'') \cup ((K_0 \cup K') + nx))\right). \end{aligned}$$

From the above identity and (2.2.5) we deduce

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{P}^u[A \cap t_x^n(A)] &= \sum_{K'' \subseteq K_1} \sum_{K' \subseteq K_1} (-1)^{|K''|+|K'|} \exp\left(-u(\text{cap}(K_0 \cup K'') + \text{cap}(K_0 \cup K'))\right) \\ &= \sum_{K'' \subseteq K_1} (-1)^{|K''|} e^{-u \text{cap}(K_0 \cup K'')} \sum_{K' \subseteq K_1} (-1)^{|K'|} e^{-u \text{cap}(K_0 \cup K')} \stackrel{(2.1.3)}{=} \mathcal{P}^u[A]^2, \end{aligned}$$

thus (2.2.4) holds for cylinder events. Now by Remark 2.2 the mixing result (2.2.4) can be routinely extended to any local event. The proof of Theorem 2.10 is complete.

**Exercise 2.11.** Show that the asymptotic independence result (2.2.4) can indeed be extended from the case when  $A$  is a cylinder set of form (2.1.1) to the case when  $B_\varepsilon \in \sigma(\Psi_x, x \in K)$  for some  $K \subset \mathbb{Z}^d$ .

## 2.3 Increasing and Decreasing Events, Stochastic Domination

In this section we introduce increasing and decreasing events, which will play an important role in the sequel. We also define stochastic domination of probability measures and use it to compare the law of random interlacements with that of the classical Bernoulli percolation.

There is a natural partial order on the space  $\{0, 1\}^{\mathbb{Z}^d}$ : we say that  $\xi \leq \xi'$  for  $\xi, \xi' \in \{0, 1\}^{\mathbb{Z}^d}$ , if for all  $x \in \mathbb{Z}^d$ ,  $\xi_x \leq \xi'_x$ .

**Definition 2.12.** An event  $G \in \mathcal{F}$  is called *increasing* (resp., *decreasing*), if for all  $\xi, \xi' \in \{0, 1\}^{\mathbb{Z}^d}$  with  $\xi \leq \xi'$ ,  $\xi \in G$  implies  $\xi' \in G$  (resp.,  $\xi' \in G$  implies  $\xi \in G$ ).

It is immediate that if  $G$  is increasing, then  $G^c$  is decreasing and that the union or intersection of increasing events is again an increasing event.

**Definition 2.13.** If  $P$  and  $Q$  are probability measures on the measurable space  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F})$ , then we say that  $P$  stochastically dominates  $Q$  if for every increasing event  $G \in \mathcal{F}$ ,  $Q[G] \leq P[G]$ .

Random interlacements at level  $u$  is a random subset of  $\mathbb{Z}^d$ , so it is natural to try to compare it to a classical random subset of  $\mathbb{Z}^d$ , namely the Bernoulli site percolation with density  $p$ . It turns out that the laws of the two random subsets do not stochastically dominate one another.

We first define Bernoulli percolation. For  $p \in [0, 1]$ , we consider the probability measure  $\mathcal{Q}^p$  on  $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F})$  such that under  $\mathcal{Q}^p$ , the coordinate maps  $(\Psi_x)_{x \in \mathbb{Z}^d}$  are independent and each is distributed as a Bernoulli random variable with parameter  $p$ , i.e.,

$$\mathcal{Q}^p[\Psi_x = 1] = 1 - \mathcal{Q}^p[\Psi_x = 0] = p.$$

While  $\mathcal{P}^u$  exhibits long-range correlations (see Remark 2.6)  $\mathcal{Q}^p$  is a product measure. For applications, it is often helpful if there is a stochastic domination by (of) a product measure. Unfortunately, this is not the case with  $\mathcal{P}^u$ , as we will now see in Claims 2.14 and 2.15.

**Claim 2.14.** For any  $u > 0$  and  $p \in (0, 1)$ ,  $\mathcal{P}^u$  does not stochastically dominate  $\mathcal{Q}^p$ .

*Proof.* Fix  $u > 0$  and  $p \in (0, 1)$ . For  $R \geq 1$ , let  $G_R = \{\mathcal{S} \cap B(R) = \emptyset\} \in \mathcal{F}$  be the event that the box  $B(R)$  is completely vacant. The events  $G_R$  are clearly decreasing; therefore, in order to prove Claim 2.14, it is enough to show that for some  $R \geq 1$

$$\mathcal{P}^u[G_R] > \mathcal{Q}^p[G_R].$$

For large enough  $R$ , we have

$$\mathcal{P}^u[G_R] \stackrel{(2.1.2)}{=} e^{-u \text{cap}(B(R))} \stackrel{(*)}{>} (1-p)^{|B(R)|} = \mathcal{Q}^p[G_R],$$

where the inequality marked by  $(*)$  indeed holds for large enough  $R$ , because

$$\text{cap}(B(R)) \stackrel{(1.3.14)}{\asymp} R^{d-2} \quad \text{and} \quad |B(R)| \asymp R^d,$$

thus  $e^{-u \text{cap}(B(R))}$  decays to zero slower than  $(1-p)^{|B(R)|}$  as  $R \rightarrow \infty$ . The proof is complete.

The proof of the next claim is an adaptation of [9, Lemma 4.7].

**Claim 2.15.** For any  $u > 0$  and  $p \in (0, 1)$ ,  $\mathcal{Q}^p$  does not stochastically dominate  $\mathcal{P}^u$ .

*Proof.* Fix  $u > 0$  and  $p \in (0, 1)$ . For  $R \geq 1$ , let  $G'_R = \{\mathcal{S} \cap B(R) = B(R)\} \in \mathcal{F}$  be the event that  $B(R)$  is completely occupied. The event  $G'_R$  is clearly increasing; therefore, in order to prove Claim 2.15, it suffices to show that for some  $R \geq 1$ ,

$$\mathcal{P}^u[G'_R] > \mathcal{Q}^p[G'_R]. \quad (2.3.1)$$

On the one hand,  $\mathcal{Q}^p[G'_R] = p^{|B(R)|}$ . On the other hand, we will prove in Claim 5.10 of Sect. 5.3 using the more constructive definition of random interlacements that there exists  $R_0 = R_0(u) < \infty$  such that

$$\forall R \geq R_0 : \mathcal{P}^u[G'_R] \geq \frac{1}{2} \exp\left(-\ln(R)^2 R^{d-2}\right). \quad (2.3.2)$$

Since  $|B(R)| \asymp R^d$ , (2.3.1) holds for large enough  $R$ , and the proof of Claim 2.15 is complete.

## 2.4 Notes

The results of Sect. 2.2 are proved in [41] using Definition 5.7. The latter definition allows to deduce many other interesting properties of random interlacements. For instance, for any  $u > 0$ , the subgraph of  $\mathbb{Z}^d$  induced by random interlacements at level  $u$  is almost surely infinite and connected (see [41, Corollary 2.3]).

For any  $u > 0$ , the measure  $\mathcal{P}^u$  satisfies the so-called FKG inequality, i.e., for any increasing events  $A_1, A_2 \in \mathcal{F}$ ,

$$\mathcal{P}^u[A_1 \cap A_2] \geq \mathcal{P}^u[A_1] \cdot \mathcal{P}^u[A_2],$$

see [50].

Despite the results of Claims 2.14 and 2.15, there are many similarities in geometric properties of the subgraphs of  $\mathbb{Z}^d$  induced by random interlacements at level  $u$  and Bernoulli percolation with parameter  $p > p_c$ , where  $p_c \in (0, 1)$  is the critical threshold for the existence of an (unique) infinite connected component in the resulting subgraph (see [13]). For instance, both the infinite connected components of random interlacements and of Bernoulli percolation are almost surely transient graphs (see [14, 30]); their graph distances are comparable with the graph distance in  $\mathbb{Z}^d$  (see [3, 8, 12]), and simple random walk on each of them satisfies a quenched invariance principle (see [6, 23, 28, 36]).

It is worth mentioning that if  $d$  is high enough and if we restrict our attention to a subspace  $V$  of  $\mathbb{Z}^d$  with large enough co-dimension, then the law of the restriction of random interlacements to  $V$  does stochastically dominate Bernoulli percolation on  $V$  with small enough  $p = p(u)$ , see the Appendix of [8].



An Introduction to Random Interlacements

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2014, X, 120 p. 8 illus., Softcover

ISBN: 978-3-319-05851-1