Preface

The model of random interlacements was introduced in 2007 by A.-S. Sznitman in the seminal paper [41], motivated by questions about the disconnection of discrete cylinders and tori by the trace of simple random walk. In fact, random interlacements is a random subset of $\mathbb{Z}^d$, $d \geq 3$, which on a mesoscopic scale does appear as the limiting distribution of the trace of simple random walk on a large torus when it runs up to times proportional to the volume. It serves as a model for corrosion and in addition gives rise to interesting and challenging percolation problems.

Random interlacements can be constructed via a Poisson point process of labeled doubly infinite random walk trajectories in $\mathbb{Z}^d$. In fact, there is a one-parameter family of random interlacements. For $u > 0$, the *random interlacements at level* u, denoted by $\mathscr{I}^u$, is the random subset of $\mathbb{Z}^d$ obtained as the union of the ranges of all the trajectories with labels at most u. Thus, the bigger the u, the more trajectories enter into the picture, the bigger the $\mathscr{I}^u$. The law of $\mathscr{I}^u$ has nice properties such as invariance and ergodicity with respect to lattice shifts. It also exhibits long-range correlations, which leads to interesting challenges in its investigation.

By construction, the graph induced by $\mathscr{I}^u$ consists of only infinite connected components. In fact, it is almost surely connected for any level u. In contrast, the complement of $\mathscr{I}^u$, the so-called vacant set $\mathscr{V}^u = \mathbb{Z}^d \setminus \mathscr{I}^u$, exhibits a percolation phase transition. Namely, there exists $u_* \in (0, \infty)$ such that

- for all $u > u_*$, the graph induced by $\mathscr{V}^u$ consists almost surely of only finite connected components and
- for all $u < u_*$, this graph contains an almost surely unique infinite connected component.

The intensive research that has been conducted on this model during the last years has led to the development of powerful techniques (such as various decoupling inequalities), which have also found their applications to other percolation models with long-range correlations, such as the level sets of the Gaussian free field.

These lecture notes grew out of a graduate class "Selected topics in probability: random interlacements" which was held by the authors during the spring semester

2012 at ETH Zürich. Our aim is to give an introduction to the model of random interlacements which is self-contained and is accessible for graduate students in probability theory.

We will now provide a short outline of the structure of these lecture notes.

In Chap. 1 we introduce some notation and basic facts of simple random walk on $\mathbb{Z}^d$, $d \geq 3$. We will also review some potential theory, since the notion of the capacity $\mathrm{cap}(K)$ of a finite subset K of $\mathbb{Z}^d$ will play a central role when we deal with random interlacements.

In Chap. 2 we give an elementary definition of random interlacements $\mathscr{I}^u$ at level u as a random subset of $\mathbb{Z}^d$, the law of which is characterized by the equations

$$\mathbb{P}[\mathscr{I}^u \cap K = \emptyset] = e^{-u \cdot \mathrm{cap}(K)} \quad \text{for any finite subset } K \text{ of } \mathbb{Z}^d. \tag{0.0.1}$$

The above equations provide the shortest definition of random interlacements $\mathscr{I}^u$ at level u, which will be shown to be equivalent to a more constructive definition that we give in later chapters. In Chap. 2 we deduce some of the basic properties of $\mathscr{I}^u$ from the definition (0.0.1). In particular, we show that the law of $\mathscr{I}^u$ is invariant and ergodic with respect to lattice shifts. We also point out some evidence that the random set $\mathscr{I}^u$ behaves rather differently from classical percolation: we show that $\mathscr{I}^u$ exhibits polynomial decay of spatial correlations and that $\mathscr{I}^u$ and Bernoulli site percolation with density p do not stochastically dominate each other for any value of $u > 0$ and $0 < p < 1$.

In Chap. 3 we prove that if we run a simple random walk with a uniform starting point on the d-dimensional torus $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$ with side length N for $\lfloor uN^d \rfloor$ steps, then the trace $\mathscr{I}^{u,N}$ of this random walk converges locally in distribution to $\mathscr{I}^u$ as $N \to \infty$.

In Chap. 4 we give a short introduction to the notion of a Poisson point process (PPP) on a general measurable space as well as the basic operations with PPPs.

In Chap. 5 we introduce the random interlacement point process as a PPP on the space of equivalence classes modulo time-shift of bi-infinite nearest neighbor paths labeled with positive real numbers, which we call interlacement trajectories. This way we get a more hands-on definition of random interlacement $\mathscr{I}^u$ at level u as the trace of interlacement trajectories with label less than u.

In the rest of the notes we develop some methods that will allow us to study the percolative properties of the vacant set $\mathscr{V}^u = \mathbb{Z}^d \setminus \mathscr{I}^u$.

In Chap. 6 we formally define the percolation threshold u_* and give an elementary proof of the fact that $u_* > 0$ (i.e., the existence of a nontrivial percolating regime of intensities u for the vacant set $\mathscr{V}^u$) in high dimensions. In fact we will use a Peierls-type argument to show that the intersection of the d-dimensional vacant set $\mathscr{V}^u$ and the plane $\mathbb{Z}^2 \times \{0\}^{d-2}$ contains an infinite connected component if $d \geq d_0$ and $u \leq u_1(d)$ for some $u_1(d) > 0$.

In order to explore the percolation of $\mathscr{V}^u$ in lower dimensions, we will have to obtain a better understanding of the strong correlations occurring in random interlacements. This is the purpose of the following chapters.

In Chap. 7 we take a closer look at the spatial dependencies of $\mathscr{I}^u$ and argue that the correlations between two locally defined events with disjoint spatial supports are caused by the PPP of interlacement trajectories that hit both of the supports. The way to achieve decorrelation is to use a clever coupling to dominate the effect of these trajectories on our events by the PPP of trajectories that hit the support of only one event. This trick is referred to as sprinkling in the literature and it allows us to compare the probability of the joint occurrence of two monotone events under the law of $\mathscr{I}^u$ with the product of their probabilities under the law of $\mathscr{I}^{u'}$, where u' is a small perturbation of u. The difference $u'-u$ and the error term of this comparison will sensitively depend on the efficiency of the coupling and the choice of the support of our events.

In Chap. 8 we provide a setup where the decorrelation result of the previous chapter can be effectively implemented to a family of events which are hierarchically defined on a geometrically growing sequence of scales. This renormalization scheme involves events that are spatially well separated on each scale, and this property guarantees a decoupling with a small error term and a manageable amount of sprinkling. We state (but not yet prove) the basic decorrelation inequality (one-step renormalization) in this setting and use iteration to derive the decoupling inequalities that are the main results of the chapter. As a corollary we deduce that if the density of a certain "pattern" of locally defined, monotone, shift invariant events observed in $\mathscr{I}^u$ is reasonably small, then the probability to observe big connected islands of that pattern in $\mathscr{I}^{u'}$ (where $|u-u'|$ is small) decays very rapidly with the size of the island.

In Chap. 9 we apply the abovementioned corollary to prove the nontriviality of the phase transition of the vacant set $\mathscr{V}^u$ for any $d \geq 3$. Otherwise stated, we prove that there exists some $u_* \in (0,\infty)$ such that for all $u < u_*$ the set $\mathscr{V}^u$ almost surely contains a connected component, but for all $u > u_*$ the set $\mathscr{V}^u$ only consists of finite connected components. We also define the threshold $u_{**} \in [u_*,\infty)$ of local subcriticality and show that for any $u > u_{**}$ the probability that the diameter of the vacant component of the origin is greater than k decays stretched exponentially in k.

In Chap. 10 we complete the proof of the basic decorrelation inequality stated in Chap. 8 by constructing the coupling of two collections of interlacement trajectories, as required by the method of Chap. 7. The proof combines potential theoretic estimates with PPP techniques to achieve an error term of decorrelation which decays much faster than one would naively expect in a model with polynomial decay of correlations.

The main goal of these lecture notes is to provide a self-contained treatise of the percolation phase transition of the vacant set of random interlacements using decoupling inequalities. A significant part of the material covered in these notes is an adaptation of results of [41, 44]. Since the body of work on random interlacements is already quite vast (and rapidly growing), there are some interesting topics that are not at all covered in these notes. To compensate for this, we collect relevant bibliographic notes at the end of most of the chapters. It is also a good moment to point out two other lecture notes covering various aspects of random interlacements. The lecture notes [47] give a self-contained introduction to Poisson gases of Markovian loops and their relation to random interlacements and Gaussian free fields. The lecture notes [9] give an introduction to random interlacements with an emphasis on the sharp percolation phase transition of the vacant set of (a) random interlacements on trees and (b) the trace of random walk on locally treelike mean field random graphs.

Let us now state our convention about constants. Throughout these notes, we will denote various positive and finite constants by c and C. These constants may change from place to place, even within a single chain of inequalities. If the constants only depend on the dimension d, this dependence will be omitted from the notation. Dependence on other parameters will be emphasized, but usually just at the first time the constant is introduced.

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