Chapter 2
Discrete Time Renewal Processes

Abstract In this chapter, we give a review of discrete renewal theory and prove the basic theorems for renewal sequences. We provide two different proofs of the theorem of Erdös-Feller-Pollard. Using extensions of a theorem of Wiener, we also obtain several rate of convergence results in discrete renewal theory.

Keywords Discrete time · Renewal sequence · Erdös-Feller-Pollard theorem · Lifetime processes · Rate of convergence

2.1 Introduction

In Chap. 1 we see that in the formulations of the main theorems one has to consider separately the cases of lattice and nonlattice distributions of the interarrival times. In the present chapter, we will consider discrete time renewal processes assuming that the interarrival times T_n have a lattice distribution with span $a = 1$. In other words, we assume that

$$T,\ T_1,\ T_2,\ \ldots,\ T_n,\ \ldots$$

are i.i.d. integer valued, nonnegative r.v.'s with distribution

$$\Pr\{T_n = t\} = p_t \geq 0, \quad t = 0, 1, 2, \ldots,$$

such that $GCD\{t : p_t > 0\} = 1$, i.e. $\{p_t\}_{t=0}^{\infty}$ is nonperiodic.

Let us denote by

$$f(z) := \mathbf{E}[z^T] = \sum_{t=0}^{\infty} p_t z^t, \quad |z| \leq 1$$

K. V. Mitov and E. Omey, *Renewal Processes*, SpringerBriefs in Statistics,
DOI: 10.1007/978-3-319-05855-9_2,

the probability generating function (p.g.f.) of the r.v. T. Evidently, under these assumptions the renewal sequence

$$S_0 = 0, \quad S_n = T_1 + T_2 + \ldots + T_n, \quad n = 1, 2, \ldots$$

takes values in nonnegative integers. Hence, the renewal counting processes

$$N(t) = \sup\{n \geq 0 : S_n \leq t\} \text{ and } M(t) = \inf\{n \geq 1 : S_n > t\}$$

can increase only at the integer time points $t = 0, 1, 2, \ldots$. So, they are discrete time stochastic processes. Denote by:

$$p_t(n) := \Pr\{S_n = t\} = p_t^{n*}, t = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots$$

and

$$f_n(z) := \mathbf{E}[z^{S_n}] = \sum_{t=0}^{\infty} p_t(n) z^t, \quad |z| \leq 1, n = 1, 2, \ldots.$$

Let us recall that $p_t^{1*} = p_t, \quad t = 0, 1, 2, \ldots$ and $p_t^{n*} = \sum_{k=0}^{t} p_{t-k}^{(n-1)*} p_k^{1*}$ for $t = 0, 1, 2 \ldots$ and $n = 2, 3, \ldots$

Clearly, $f_n(z) = f(z)^n, n = 1, 2, 3, \ldots$.

Let us recall that the path properties of the process $N(t)$ do not depend on the fact that the distribution of interarrival times is lattice or not. So we will restrict ourselves only to represent the discrete time versions of the main limit theorems.

2.2 Theorem of Erdös, Feller and Pollard

Similarly to the continuous time case, we are interested in the behavior or the renewal function $U(t)$ as $t \to \infty$. Let us note that now the renewal events can occur only in the integer time points $t = 0, 1, 2, \ldots$. Therefore, it makes sense to study the measure $U(\{t\})$, that is the mean number of the renewals at time t. In other words, if we define the random variable $I_t = 1$, if there is a renewal event at time t and $I_t = 0$ otherwise then $U(\{t\}) = \mathbf{E}[I_t]$. Let us denote

$$u_t = \Pr\{I_t = 1\}, \quad 1 - u_t = \Pr\{I_t = 0\}$$

for $t = 0, 1, 2, \ldots$ then

$$U(\{t\}) = \mathbf{E}[I_t] = u_t.1 + (1 - u_t).0 = u_t, \; t = 0, 1, 2, \ldots. \tag{2.1}$$

Since we count $t = 0$ as a renewal then we will assume that $u_0 = 1$. From the definition of u_t it follows that

$$u_t = \sum_{n=0}^{\infty} \Pr\{S_n = t\} = \sum_{n=0}^{\infty} p_t^{n*}, \tag{2.2}$$

and

$$\begin{aligned} U(t) = U([0,t]) = \mathbf{E}\left[\sum_{k=0}^{t} I_k\right] &= \sum_{k=0}^{t} u_k = \sum_{k=0}^{t}\sum_{n=0}^{\infty} \Pr\{S_n = k\} \\ = \sum_{n=0}^{\infty}\sum_{k=0}^{t} \Pr\{S_n = k\} &= \sum_{n=0}^{\infty} \Pr\{S_n \le t\} = \mathbf{E}[N(t)] + 1 = \mathbf{E}[M(t)]. \end{aligned}$$

Multiplying both sides of the Eq. (2.2) by z^t, ($|z| \le 1$) and summing on $t = 0, 1, 2, \ldots$, we get

$$\begin{aligned} u(z) &= \sum_{t=0}^{\infty}\sum_{k=0}^{\infty} \Pr\{S_k = t\} z^t = \sum_{k=0}^{\infty}\sum_{t=0}^{\infty} \Pr\{S_k = t\} z^t \\ &= \sum_{k=0}^{\infty} f^k(z) = \frac{1}{1 - f(z)}. \end{aligned} \tag{2.3}$$

From here it follows that $u(z) = 1 + u(z) f(z)$ and

$$u_0 = 1, \quad u_t = p_t u_0 + p_{t-1} u_1 + \ldots + p_1 u_{t-1} \text{ for } t = 1, 2, \ldots,$$

which is the discrete time version of the renewal equation (1.6).

Example 2.1 Let T be given by $\Pr\{T = 1\} = p$ and $\Pr\{T = 2\} = q = 1 - p$, where $0 < p < 1$. Then $\mu = \mathbf{E}[T] = p + 2q = 1 + q$. In this case we obtain

$$u_0 = 1, \ u_1 = p, \ u_t = p u_{t-1} + q u_{t-2}, t \ge 2.$$

The generating function of u_t is given by

$$\begin{aligned} u(z) &= \frac{1}{1 - pz - qz^2} = \frac{1}{p(1-z) + q(1-z^2)} = \frac{1}{(1-z)(1+qz)} \\ &= \frac{1}{1+q}\frac{1}{1-z} + \frac{q}{1+q}\frac{1}{1+qz}, \end{aligned}$$

for $|z| \le 1$. From here we find that

$$u_t = \frac{1}{1+q} + \frac{q}{1+q}(-q)^t, \quad t \ge 1.$$

Note that

$$u_t \to \frac{1}{1+q} = \frac{1}{\mu}, \quad t \to \infty.$$

The same limit holds true in the general case. The result was proved by Erdös, Feller, and Pollard at 1949 (see [3]) and then it was generalized by Blackwell for the continuous time.

Theorem 2.1 (Erdös, Feller, and Pollard) *Assume that* $\mu = \mathbf{E}[T] \leq \infty$. *Then*

$$\lim_{t\to\infty} u_t = \frac{1}{\mu},$$

where $1/\mu$ *is defined to be zero if* $\mu = \infty$.

Remark 2.1 Here, we give two proofs only in the case when $\mu < \infty$. The first one is based on a result of Wiener (see e.g. Rudin [7], Theorem 18.21) and is close to the first proof in the original paper of Erdös, Feller and Pollard. The second proof is based on the coupling method.

Lemma 2.1 (Rudin [7], Theorem 18.21) *Suppose that the sequence* $\{x_n\}_{n=0}^{\infty}$ *is such that* $x(z) = \sum_{n=0}^{\infty} x_n z^n$ *is absolutely convergent for* $|z| \leq 1$ *and that* $\sum_{n=0}^{\infty} |x_n| < \infty$. *If* $x(z) \neq 0$ *for every* z, $|z| \leq 1$, *then*

$$\frac{1}{x(z)} = \sum_{n=0}^{\infty} y_n z^n$$

is absolutely convergent for $|z| \leq 1$ *and*

$$\sum_{n=0}^{\infty} |y_n| < \infty.$$

Proof I of Theorem 2.1. Define the sequence $r_t = \Pr\{T > t\}$ $t = 0, 1, 2, \ldots$ and its generating function $r(z) = \sum_{t=0}^{\infty} r_t z^t$, $|z| \leq 1$. The following representation holds true

$$r(z) = \frac{1 - f(z)}{1 - z}.$$

Now note that $r(z) \neq 0$ for $|z| \leq 1$ and $z \neq 1$. For $z = 1$, we have $r(1) = f'(1) = \mu \neq 0$. From the result of Wiener we get that

$$\Lambda(z) = \frac{1}{r(z)} = \sum_{t=0}^{\infty} \Lambda_t z^t \quad \text{and} \quad \sum_{t=0}^{\infty} |\Lambda_t| < \infty.$$

It follows that $(1 - z)u(z) = \Lambda(z)$ and hence also that

$$u_0 = \Lambda_0, \quad \text{and} \quad (u_t - u_{t-1}) = \Lambda_t, \quad t \geq 1.$$

Taking sums, we get that

$$u_0 + \sum_{k=1}^{t}(u_k - u_{k-1}) = \sum_{k=0}^{t} \Lambda_k$$

or equivalently $u_t = \sum_{k=0}^{t} \Lambda_k$. Since $\sum_{k=0}^{\infty} \Lambda_k = \Lambda(1) = 1/r(1) = 1/\mu$, we get that

$$u_t \to 1/\mu, \quad \text{as} \quad t \to \infty,$$

which completes the proof of the theorem. △

Proof II of Theorem 2.1. (Barbu and Limnios [1], Chap. 2). As before, let $\{S_n, n \geq 0\}$ denote the renewal sequence with $S_0 = 0$ and $S_n = \sum_{i=1}^{n} T_i, n \geq 1$. We assume that $\Pr\{T < \infty\} = 1$ and that $\mu = \mathbf{E}[T] < \infty$. Let $\left\{S_n^*,\ n \geq 0\right\}$ denote the stationary renewal sequence associated with $\{S_n, n \geq 0\}$. More precisely, we have

$$S_n^* = T_0^* + \sum_{i=1}^{n} T_i^*,\ n \geq 0,$$

where the $T_i^*,\ i \geq 1$ are i.i.d. with $T_i^* \stackrel{d}{=} T$ and independent of the T_i^*,

$$\Pr\left\{T_0^* = n\right\} = \Pr\{T > n\}/\mu.$$

Let u_t and u_t^* denote the corresponding renewal sequences. Because of the definition, we have

$$u_t^* = \Pr\left\{S_k^* = t \ \text{ for some } \ k \in \mathbf{N}\right\} = 1/\mu.$$

Now, define $U_0 = T_0 - T_0^*$, $U_n = U_{n-1} + (T_n - T_n^*) = S_n - S_n^*, n \geq 0$ and denote by N the first time that we have the same number of renewals in the two processes, i.e.

$$N = \min\{n : U_n = 0\}.$$

Since $\mathbf{E}[T_i] = \mathbf{E}[T_i^*] = \mu$, then $\mathbf{E}[T_i - T_i^*] = 0$. Hence $U_n, n = 0, 1, 2 \ldots$ is a recurrent Markov chain. Consequently, $\Pr\{N < \infty\} = 1$.

Thus, for $n \geq N$ we have that $S_n \stackrel{d}{=} S_n^*$. Now we find

$$\begin{aligned}
u_t &= \Pr\{S_k = t \text{ for some } k \in \mathbf{N}\} \\
&= \Pr\{S_k = t \ \text{ for some } \ k \geq N\} + \Pr\{S_k = t \ \text{ for some } \ k < N\} \\
&= \Pr\left\{S_k^* = t \ \text{ for some } \ k \geq N\right\} + \Pr\{S_k = t \ \text{ for some } \ k < N\} \\
&= 1/\mu - \Pr\left\{S_k^* = t \ \text{ for some } \ k < N\right\} + \Pr\{S_k = t \ \text{ for some } \ k < N\}.
\end{aligned}$$

Now note that $\{S_k = t \text{ for some } k < N\} \subseteq \{S_N \geq t\}$. We have

$$\lim_{t\to\infty} \Pr\{S_k = t \text{ for some } k < N\} \leq \Pr\{S_N = \infty\} = 0.$$

In a similar way, we have

$$\lim_{t\to\infty} \Pr\left\{S_k^* = t \text{ for some } k < N\right\} \leq \Pr\left\{S_N^* = \infty\right\} = 0.$$

It follows that $u_t \to 1/\mu$. △

Remark 2.2 The general proof of the theorem given in the paper Erdös et al.[3] does not depend on the fact that $\mu < \infty$. It can be found also in Feller [4].

2.3 Rate of Convergence in Discrete Renewal Theorem

Following Rogozin [5], we define three classes of sequences.

$$\begin{aligned}
D &= \left\{\{x_n\} : x(z) = \sum_{n=0}^{\infty} x_n z^n \text{ is absolutely convergent for } |z| \leq 1\right\};\\
R_1(\alpha, L) &= \left\{\{x_n\} \in D : x_n = O(n^{-\alpha} L(n)\right\};\\
R_2(\alpha, L) &= \left\{\{x_n\} \in D : x_n = o(n^{-\alpha} L(n)\right\},
\end{aligned}$$

where $L(.)$ denotes a slowly varying function at infinity.

Rogozin proves the following Wiener type of results.

Lemma 2.2 (Rogozin [6], p. 664) (i) *Suppose that* $\{x_n\} \in R_1(\alpha, L)$ *and* $x(z) \neq 0$ *for* $|z| \leq 1$. *Then*

$$\frac{1}{x(z)} = y(z) = \sum_{n=0}^{\infty} y_n z^n$$

and $\{y_n\} \in R_1(\alpha, L)$.

(ii) *Suppose that* $\{x_n\} \in R_2(\alpha, L)$, $\alpha > 1$ *and* $x(z) \neq 0$ *for* $|z| \leq 1$. *Then*

$$\frac{1}{x(z)} = y(z) = \sum_{n=0}^{\infty} y_n z^n$$

and $\{y_n\} \in R_2(\alpha, L)$.

By replacing $\Phi(x) = 1/x$ by other suitable functions, we get the following general result.

Lemma 2.3 (Rogozin [5], Theorem 5) (i) *Suppose that* $\{x_n\} \in R_1(\alpha, L)$ *and suppose that* $\Phi(z)$ *is a complex function that is analytic in a region that contains the set* $\{x(z) : |z| \leq 1\}$. *Then*

$$q(z) = \Phi(x(z)) = \sum_{n=0}^{\infty} q_n z^n$$

and $\{q_n\} \in R_1(\alpha, L)$.

(ii) *A similar result holds for* $R_2(\alpha, L)$ *with* $\alpha > 1$.

The next lemma states the conditions under which one obtains an asymptotic equality.

Lemma 2.4 (Borovkov [2], p. 258, Theorem 1) *Suppose that* $\{x_n\} \in R_1(\alpha, L)$ *and that*

$$0 < An^{-\alpha}L(n) \leq |x_n| \leq Bn^{-\alpha}L(n), \quad n \geq 1.$$

With Φ *as in the previous lemma, we have*

$$q(z) = \Phi(x(z)) = \sum_{n=0}^{\infty} q_n z^n$$

and $q_n \sim x_n \Phi'(x(1))$ *as* $n \to \infty$.

Now we are ready to formulate the results for the rate of convergence in the renewal theorem. The results depend on the behavior of the tail sequence $r_n = \Pr\{T > n\}$. Using Lemma 2.3 and 2.4, we obtain the following result.

Theorem 2.2 *Suppose that* $\alpha > 1$, $L(x) \in RV(0)$ *and* $\mu < \infty$. *Then*

(i) $r_n = O(n^{-\alpha}L(n)) \Longleftrightarrow u_n - u_{n-1} = O(n^{-\alpha}L(n))$;
(ii) $r_n = o(n^{-\alpha}L(n)) \Longleftrightarrow u_n - u_{n-1} = o(n^{-\alpha}L(n))$;
(iii) $r_n \sim n^{-\alpha}L(n) \Longleftrightarrow \mu^2(u_n - u_{n-1}) \sim -n^{-\alpha}L(n)$.

Proof As in the first proof of Theorem 2.1, we use the equation

$$r(z) = \frac{1 - f(z)}{1 - z}, \text{ and } r(1) = \mu.$$

Then by Lemma 2.1 we get

$$\Lambda(z) = \frac{1}{r(z)} = \sum_{n=0}^{\infty} \Lambda_n z^n$$

where $\sum_0^\infty |\Lambda_n| < \infty$. Since $\Lambda(z) = (1-z)u(z)$, then $\Lambda_0 = u_0$ and $\Lambda_n = u_n - u_{n-1}, n \geq 1$. Now (i) and (ii) follow from these relations and Lemma 2.3.

(iii) First suppose that $r_n \sim n^{-\alpha} L(n)$. Using Lemma 2.4 with $\Phi(z) = 1/z$, it follows that

$$\Lambda_n \sim r_n \Phi'(r(1)) = -r_n/\mu^2.$$

Since $\Lambda_n = (u_n - u_{n-1})$, the result follows. The converse follows in a similar way, which completes the proof of the theorem. △

From Theorem 2.2 (iii), we have

$$\mu^2(u_n - u_{n-1}) \sim -r_n.$$

By taking sums, we have

$$-(1+\varepsilon)\sum_{n=k}^{K} r_n \leq \mu^2 \sum_{n=k}^{K}(u_n - u_{n-1}) \leq -(1-\varepsilon)\sum_{n=k}^{K} r_n$$

or equivalently

$$-(1+\varepsilon)\sum_{n=k}^{K} r_n \leq \mu^2(u_K - u_{k-1}) \leq -(1-\varepsilon)\sum_{n=k}^{K} r_n.$$

Taking limits with respect to K, we obtain that

$$-(1+\varepsilon)\sum_{n=k}^{\infty} r_n \leq \mu^2(1/\mu - u_{k-1}) \leq -(1-\varepsilon)\sum_{n=k}^{\infty} r_n.$$

Using $r_n \sim n^{-\alpha} L(n), \alpha > 1$, it follows that

$$\sum_{n=k}^{\infty} r_n \sim \frac{1}{\alpha - 1} k r_k$$

The final conclusion is that

$$u_n - \frac{1}{\mu} \sim \frac{1}{\mu^2(\alpha - 1)} n r_n.$$

Under the conditions of Theorem 2.2 (i), (ii) in a similar way we obtain $O(.)$ and $o(.)$ results.

Corollary 2.1 *Suppose that* $\alpha > 1$, $L(x) \in RV(0)$ *and* $\mu < \infty$. *Then*

(i) $r_n = O(n^{-\alpha}L(n)) \Longrightarrow u_n - 1/\mu = O(n^{1-\alpha}L(n))$;
(ii) $r_n = o(n^{-\alpha}L(n)) \Longrightarrow u_n - 1/\mu = o(n^{1-\alpha}L(n))$;
(iii) $r_n \sim n^{-\alpha}L(n) \Longrightarrow \mu(u_n - 1/\mu) \sim n^{1-\alpha}L(n)/(\alpha - 1)$.

2.4 Lifetime Processes

Let us have a look at lifetime processes $A(t) = t - S_{N(t)}$, $B(t) = S_{M(t)} - t$ and $C(t) = T_{M(t)}$. Now they are discrete time integer valued processes. We will derive the representation for their probability distributions. Then we will prove limit theorems.

For the distribution of $A(t)$, we get for $n \geq 0$, $t \geq n$,

$$\begin{aligned}
\Pr\{A(t) = n\} &= \Pr\{S_{N(t)} = t - n\} \\
&= \sum_{k=0}^{\infty} \Pr\{S_{N(t)} = t - n, N(t) = k\} \\
&= \sum_{k=0}^{\infty} \Pr\{S_k = t - n, S_k \leq t < S_{k+1}\} \\
&= \sum_{k=0}^{\infty} \Pr\{S_k = t - n, t - n \leq t < t - n + T_{k+1}\} \\
&= \sum_{k=0}^{\infty} \Pr\{S_k = t - n\} \Pr\{n < T_{k+1}\} \\
&= \Pr\{T > n\} \sum_{k=0}^{\infty} \Pr\{S_k = t - n\} \\
&= u_{t-n} \Pr\{T > n\}.
\end{aligned}$$

Consider now the residual lifetime $B(t) = S_{M(t)} - t$. By conditioning on $M(t)$, for $n \geq 1$, $t \geq 0$, we have

$$\begin{aligned}
\Pr\{B(t) = n\} &= \sum_{i=1}^{\infty} \Pr\{S_i = t + n, M(t) = i\} = \sum_{i=1}^{\infty} \Pr\{S_i = t + n, S_{i-1} \leq t < S_i\} \\
&= \Pr\{T = t + n\} + \sum_{i=2}^{\infty} \sum_{j=1}^{t} \Pr\{T_i = t + n - j\} \Pr\{S_{i-1} = j\} \\
&= \Pr\{T = t + n\} + \sum_{j=1}^{t} \Pr\{T = t + n - j\} \sum_{i=2}^{\infty} \Pr\{S_{i-1} = j\}
\end{aligned}$$

$$= \Pr\{T = t + n\} + \sum_{j=1}^{t} \Pr\{T = t + n - j\} u_j$$

$$= \sum_{j=0}^{t} \Pr\{T = t + n - j\} u_j.$$

For the distribution of $C(t)$, we get

$$\begin{aligned}
\Pr\{C(t) = n\} &= \Pr\{T_{N(t)+1} = n\} \\
&= \sum_{j=0}^{\infty} \Pr\{T_{j+1} = n, N(t) = j\} = \sum_{j=0}^{\infty} \Pr\{T_{j+1} = n, S_j \le t < S_j + T_{j+1}\} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Pr\{T_{j+1} = n, S_j \le t < S_j + T_{j+1}, S_j = k\} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Pr\{T_{j+1} = n, k \le t < k + n, S_j = k\} \\
&= \sum_{j=0}^{\infty} \sum_{k=t-n+1}^{t} \Pr\{T = n\} \Pr\{S_j = k\} \\
&= \Pr\{T = n\} \sum_{k=t-n+1}^{t} u_k.
\end{aligned}$$

Now we are ready to prove the following limit theorem.

Theorem 2.3 *Assume that $\mu < \infty$. Then*

$$\lim_{t \to \infty} \Pr\{A(t) = n\} = \frac{1}{\mu} \Pr\{T > n\}.$$

$$\lim_{t \to \infty} \Pr\{B(t) = n\} = \frac{1}{\mu} \Pr\{T \ge n\}.$$

$$\lim_{t \to \infty} \Pr\{C(t) = n\} = \frac{n}{\mu} \Pr\{T = n\}.$$

Proof Using renewal theorem, we easily obtain the first and the third limits from $\Pr\{A(t) = n\} = u_{t-n} \Pr\{T > n\}$ and $\Pr\{C(t) = n\} = \Pr\{T = n\} \sum_{k=t-n+1}^{t} u_k$, respectively

The proof of the second relation is not so evident. For $t > t^\circ$ and t° sufficiently large, we have

$$\Pr\{B(t) = n\} = \left(\sum_{j=0}^{t^\circ} + \sum_{j=t^\circ+1}^{t}\right) \Pr\{T = t + n - j\} u_j$$
$$= J_1(t) + J_2(t).$$

In $J_2(t)$, we have

$$\left(\frac{1}{\mu} - \varepsilon\right) \Pr\{n \le T \le t + n - t^\circ - 1\} \le J_2(t) \le \left(\frac{1}{\mu} + \varepsilon\right) \Pr\{n \le T \le t + n - t^\circ - 1\},$$

and it follows that

$$\left(\frac{1}{\mu} + \varepsilon\right) \Pr\{n \le T\} \le \liminf_{n\to\infty} J_2(t) \le \limsup_{n\to\infty} J_2(t) \le \left(\frac{1}{\mu} + \varepsilon\right) \Pr\{n \le T\}.$$

For $J_1(t)$, we have

$$J_1(t) \le \left(\sup_{j \le t^\circ} u_j\right) \Pr\{t + n - k^\circ \le T \le t + n\} \to 0, \text{ as } t \to \infty.$$

This completes the proof of the theorem. △

Remark 2.3 This theorem is the discrete time analog of Theorem 1.18.

Let us note that we derive the equations for the distributions of the lifetime processes conditioning on the last renewal event before time t. As in the continuous time case these distributions are solutions of three renewal equations. We will derive one of them using the standard renewal argument.

Consider the residual lifetime $B(t) = S_{M(t)} - t$. Note that $B(0) = T_1$. By conditioning on T_1, we have

$$\Pr\{B(t) > n\} = \sum_{i=1}^{\infty} \Pr\{B(t) > n \mid T_1 = i\} p_i.$$

Now note that the process restarts at T_1 and we get

$$\begin{aligned}
&\Pr\{B(t) > n \mid T_1 = i\} = 1, \quad \text{if } i > t + n\\
&\Pr\{B(t) > n \mid T_1 = i\} = 0, \quad \text{if } t < i \le t + n\\
&\Pr\{B(t) > n \mid T_1 = i\} = \Pr\{B(t - i) > n\} \quad \text{if } i \le t.
\end{aligned}$$

It follows that

$$\Pr\{B(t) > n\} = \Pr\{T > t + n\} + \sum_{i=0}^{t} \Pr\{B(t - i) > n\} p_i, \ t \ge 0. \tag{2.4}$$

This renewal equation can be solved by using generating functions. Let

$$p_n(z) = \sum_{t=0}^{\infty} \Pr\{B(t) > n\} z^t$$
$$g(z) = \sum_{t=0}^{\infty} \Pr\{T > t + n\} z^t.$$

Multiplying Eq. (2.4) by z^t and summing on $t = 0, 1, 2, \ldots, \infty$, we get that

$$p_n(z) = g(z) + f(z) p_n(z).$$

Using also (2.3) we get

$$p_n(z) = \frac{g(z)}{1 - f(z)} = g(z)u(z).$$

Therefore we get the following representation for the tail of the distribution of $B(t)$,

$$\Pr\{B(t) > n\} = \sum_{i=0}^{t} \Pr\{T > t + n - i\} u_i.$$

In a similar way one can derive the renewal equations for the distributions of the processes $A(t)$ and $C(t)$.

We conclude this chapter with two examples.

Example 2.2 Let T be a Bernoulli random variable with $\Pr\{T = 1\} = p$ and $\Pr\{T = 0\} = q = 1 - p$. Then $f(z) = \mathbf{E}[z^T] = f(z) = q + pz$ and $\mu = p$. The renewal measure has generating function

$$u(z) = \frac{1}{1 - f(z)} = \frac{1}{p(1 - z)}$$

and it follows that $u_t = 1/p = 1/\mu$ for every $t = 1, 2, \ldots$.

Example 2.3 Let T have a geometric distribution $\Pr\{T = k\} = pq^{k-1}, k \geq 1$. We have $f(z) = \mathbf{E}[z^T] = pz/(1 - qz)$ and $\mu = \mathbf{E}[T] = 1/p$. The renewal measure has generating function

$$u(z) = \frac{1}{1 - f(z)} = \frac{1 - qz}{1 - qz - pz} = \frac{1 - qz}{1 - z},$$

which can be rewritten as

$$u(z) = p\frac{1}{1-z} + q.$$

Hence, $u_0 = 1$ and $u_t = p = 1/\mu$ for every $t = 1, 2, \ldots$.

References

1. Barbu, V.S., Limnios, N.: Semi-Markov Chains and Hidden Semi-Markov Models Toward Applications. Springer, New York (2008)
2. Borovkov, A.A.: Stochastic Processes in Queueing Theory. Springer, New York (1976)
3. Erdös, P., Feller, W., Pollard, H.: A property of power series with positive coefficients. Bull. Am. Math. Soc. **55**(2), 201–204 (1949)
4. Feller, W.: An Introduction to Probability Theory and its Applications, vol. I. Wiley, New York (1968)
5. Rogozin, B.A.: Asymptotics of the coefficients in the Lévy-Wiener theorems on absolutely convergent trigonometric series. Sib. Math. J. **14**, 1304–1312 (1973)
6. Rogozin, B.A.: An estimate of the remainder term in limit theorems of renewal theory. Theory Prob. Appl. **18**, 662–677 (1973)
7. Rudin, W.: Real and Complex Analysis. McGraw-Hill, New York (1970)

Springer

http://www.springer.com/978-3-319-05854-2

Renewal Processes
Mitov, K.V.; Omey, E.
2014, VIII, 122 p. 1 illus., Softcover
ISBN: 978-3-319-05854-2