Chapter 2
Two Paths to Group Field Theories

Men's memories are uncertain and the past that was differs little from the past that was not.

Cormac McCarthy, *Blood Meridian*

In this chapter, we recall two historical and conceptual paths leading to the GFT formalism. The first follows the traditional quantization route, from canonical LQG, to spin foam models, and finally to GFTs, seen as generating functionals for the latter. The second approach relies from the start on the idea of discretization, and takes its root in the successes of matrix models. We will then argue in favor of taking these two strategies equally seriously to investigate the properties of GFTs, and the physical picture emerging from them.

2.1 Group Field Theories and Quantum General Relativity

2.1.1 Loop Quantum Gravity

Loop Quantum Gravity is a tentative approach to the canonical quantization of general relativity. It is conceptually very much in line with the old Wheeler-DeWitt (WdW) theory, at the notable exception that many previously formal expressions could be turned into rigorous equations thanks to LQG techniques. This breakthrough triggered many developments, from cosmology to black hole physics, all relying on the well-understood kinematical properties of LQG. Unfortunately, to this date the canonical quantization program could not be completed, a proper definition of the dynamics being still challenging, despite some recent progress [1, 2].

As in the WdW theory, one starts from a canonical formulation of general relativity, which requires hyperbolicity of the space-time manifold $\mathcal{M}$. This allows to introduce a global foliation, in terms of a time function $t \in \mathbb{R}$ indexing three-dimensional

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spacelike hypersurfaces Σ_t. One then recasts Einstein's equation in the form of a Hamiltonian system, dictating the evolution of spatial degrees of freedoms in this parameter t. In the WdW theory, the Hamiltonian formulation is the ADM one [3], for which the configuration space is the set of all spatial metrics g_{ab} $(1 \leq a, b \leq 3)$ on Σ_t. This is where LQG departs from the WdW theory, with the Ashtekar-Barbero formulation of classical GR [4, 5] as a starting point, hence shifting the emphasis to forms and connection variables.

Let us first introduce the four-dimensional formalism. Instead of using the space-time metric $g_{\mu\nu}$ $(0 \leq \mu, \nu \leq 3)$ as configuration variable, one instead introduces a tetrad field

$$e^I(x) = e^I_\mu(x)\mathrm{d}x^\mu, \tag{2.1}$$

which is a quadruple of 1-forms on $\mathcal{M}$, indexed by a Minkowski index $I \in [\![0, 3]\!]$. The metric is only a derived quantity, which simply writes as

$$g_{\mu\nu}(x) = e^I_\mu(x)e^J_\nu(x)\eta_{IJ} \tag{2.2}$$

where η is the Minkowski metric (with signature $[-, +, +, +]$). This redundant rewriting of the metric introduces a new local symmetry, corresponding to the action of $\mathrm{SO}(1, 3)$ on the Minkowski index labeling the tetrad:

$$e^I(x) \mapsto \Lambda^I{}_J(x)e^J(x), \qquad \Lambda(x) \in \mathrm{SO}(1, 3). \tag{2.3}$$

Einstein's equations can then be recovered from a first order variational principle, depending on an additional spin connection ω, which is a 1-form with values in the Lie algebra $\mathfrak{so}(1, 3)$. The particular action lying at the foundations of current LQG and spin foams is the Holst action (for vacuum gravity without cosmological constant):

$$S[e, \omega] = \frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{Tr}\left[\left(\star e \wedge e + \frac{1}{\gamma} e \wedge e\right) \wedge F(\omega)\right], \tag{2.4}$$

where κ is a constant containing Newton's G, γ is the Barbero-Immirzi parameter, and $F(\omega) = \mathrm{d}\omega + \omega \wedge \omega$ is the curvature of the spin connection. Varying S with respect to e and ω independently provides two sets of equations: the first forces ω to be the unique torsion-free connection compatible with e, and the second gives Einstein's equations. The term proportional to the Barbero-Immirzi parameter is topological, and therefore plays no role as far as classical equations are concerned. It is however a key ingredient of the loop quantization, and as we will see is a cornerstone of the current interpretation of quantum geometry in LQG.

Assuming hyperbolicity of $\mathcal{M}$, it is possible to perform a canonical analysis of (2.4) leading to the Ashtekar-Barbero formulation of GR. Details can for example be found in [6, 7], and at the end of the day a $3+1$ splitting is obtained for both the tetrad and the spin connection in terms of canonical pairs (E^a_i, A^j_b):

$$E^a_i \equiv \det(e) e^a_i, \qquad A^i_a \equiv \omega^i_a + \gamma \omega^{0i}_a = \frac{1}{2} e^i{}_{jk} \omega^{jk}_a + \gamma \omega^{0i}_a,$$
$$\{E^a_i(x), A^j_b(y)\} = \kappa \gamma \delta^a_b \delta^i_j \delta(x, y), \tag{2.5}$$

where now x and y refer to coordinates on the spatial hypersurface Σ. A is an $\mathrm{SU}(2)$ connection encoding parallel transport on Σ, and the densitized triad E can be used to define geometric quantities such as areas and volumes embedded in Σ. The most interesting aspect of such a formulation is that the dynamics of GR is encoded in three sets of constraints, bearing some similarities with Yang-Mills theory on a background space-time:

$$\mathcal{G}_i = \mathcal{D}_a E^a_i, \qquad \mathcal{C}_a = E^b_k F^k_{ba}, \qquad \mathcal{S} = e^{ijk} E^a_i E^b_j F_{abk} + 2 \frac{\gamma^2 - 1}{\gamma^2} E^a_{[i} E^b_{j]} (A^i_a - \omega^i_a)(A^j_b - \omega^j_b). \tag{2.6}$$

These constraints are first class, which means that the Poisson bracket of two constraints is itself a linear combination of constraints, and therefore weakly vanishes. In addition to defining a submanifold of admissible states in phase space, they therefore also generate gauge transformations, which encode all the symmetries of the triad formulation of GR. More precisely, the Gauss constraint $\mathcal{G}_i$ generates $\mathrm{SU}(2)$ local transformations induced by the $\mathrm{SO}(1, 3)$ symmetry (2.3), while the vector constraint generates spatial diffeomorphism on Σ. The third and so-called scalar constraint $\mathcal{S}$ is responsible for time reparametrization invariance, therefore extending the gauge symmetries to space-time diffeomorphisms.

In its canonical formulation, GR is a fully constrained system, which can be quantized following Dirac's program [8, 9]. The first step consists in finding a representation of the basic phase space variables as operators acting on a kinematical Hilbert space $\mathcal{H}_{kin}$, and such that Poisson brackets are turned into commutation relations in the standard way. Then, one has to promote the constraints themselves to operators on $\mathcal{H}_{kin}$. Third, one should perform the equivalent of finding the constraint surface in the classical theory, which means finding the states annihilated by the constraint operators. This step is crucial in the sense that it is only at this stage that the Hilbert space of physical sates $\mathcal{H}_{phys}$ is defined. Finally, one has to look for a complete set of physical observables, that is a maximal set of operators commuting with the constraints. The main achievements of LQG regarding this quantization program concerns the first three steps. Contrary to the WdW theory, a kinematical Hilbert space could explicitly be constructed. Moreover, a representation of the constraint algebra on this Hilbert space could be specified, without any anomaly, therefore completing the second step. Ambiguities remain, but they mainly concern the scalar constraint. The Gauss and vector constraints, which form a closed subalgebra, could moreover be solved explicitly. The resulting Hilbert space of gauge and diffeomorphism invariant states has also been proved unique under certain additional assumptions [10]. This provides a solid arena to study the dynamics of LQG further, as encoded by the scalar constraint. However, its quantum definition remains ambiguous, and a complete set of solutions out of reach, hence precluding the completion of the canonical program. It should also be noted that, even if a satisfactory space of solutions to the scalar constraint were to be found, completing Dirac's program would remain incredibly difficult: even at the classical level, finding a complete (and manageable) set of Dirac observables for GR remains an outstanding task, and possibly forever so.

The structures of the kinematical Hilbert space of LQG, and of the solutions to the Gauss and vector constraints, are one of the main inputs for the covariant approach, and therefore deserve to be described in some details here. A distinctive feature of the loop approach is the choice of basic variables to be quantized, which are holonomies of the connection rather than the connection itself. To any path ℓ in Σ is associated the unique holonomy

$$h_\ell(A) = \mathcal{P} \exp\left(-\int_\ell A\right) \in \mathrm{SU}(2), \tag{2.7}$$

where $\mathcal{P}$ stands for path ordering. This $\mathrm{SU}(2)$ element encodes parallel transport of the triad along ℓ, and collectively the set of all the holonomy functionals fully capture the information contained in the connection. There is an important subtlety however, lying in the fact that smooth connections are replaced by a generalized space of connections before the quantization. It can be given an inductive definition in terms of holonomy functionals associated to closed embedded graphs [11], which play a central role in the definition of the kinematical Hilbert space. As far as we know, alternative constructions might be proposed at this level, relying on slightly different generalizations of smooth connections. This concerns for example regularity properties of the graphs (usually assumed to be piecewise analytic), as was already explored in [12]. But more importantly so, it seems to us that the relevance of the combinatorial properties of the graphs supporting LQG states might have been underestimated. As we will see in the GFT context, combinatorial restrictions can dramatically improve analytic control over the theory, without affecting its conceptual aspects in any clear way. It seems to us that this freedom, so far overlooked in the canonical approach, would deserve to be investigated and taken advantage of. In any case, the standard kinematical Hilbert space of LQG is spanned by so-called *cylindrical* functionals $\Psi_{\mathcal{G},f}$, labeled by a closed oriented graph $\mathcal{G}$ with L links, and a function $f : \mathrm{SU}(2)^L \to \mathbb{C}$. $\Psi_{\mathcal{G},f}(A)$ only depends on the holonomies $h_\ell(A)$ along the links of $\mathcal{G}$, through the following formula:

$$\Psi_{\mathcal{G},f}(A) = f(h_{\ell_1}(A), \ldots, h_{\ell_L}(A)). \tag{2.8}$$

f is moreover assumed to be square-integrable with respect to the Haar measure on $\mathrm{SU}(2)^L$, which allows to define a scalar product between cylindrical functionals associated to a same graph:

$$\langle \Psi_{\mathcal{G},f_1} | \Psi_{\mathcal{G},f_2} \rangle \equiv \int [\mathrm{d}h_\ell]^L \overline{f_1(h_{\ell_1}, \ldots, h_{\ell_L})} f_2(h_{\ell_1}, \ldots, h_{\ell_L}). \tag{2.9}$$

This scalar product naturally extends to arbitrary couples of cylindrical functionals, via embeddings of two distinct graphs into a common bigger one. The kinematical Hilbert space $\mathcal{H}_{kin}$ is the completion with respect to this scalar product of the vector space generated by the cylindrical functionals. It can be shown to be nothing but the space $L^2(\overline{\mathcal{A}}, \mathrm{d}\mu_{AL})$ of square-integrable functions on the space of generalized connections $\overline{\mathcal{A}}$, with respect to the Ashtekar-Lewandowski measure $\mathrm{d}\mu_{AL}$ [13]. The (matrix elements of the) holonomies are turned into operators, acting by multiplication on the kinematical states. As for the momenta E, there are also smeared out before quantization. For any two-dimensional surface S in Σ, and any $\mathfrak{su}(2)$-valued function α, one defines the *flux*:

$$E(S, \alpha) = \int_S \alpha^i E_i. \tag{2.10}$$

The action of an elementary flux on a kinematical state can be found by formally turning the densitized triad into a derivative operator with respect to the connection. This construction provides an anomaly-free representation of the classical *holonomy-flux algebra*, that is the Poisson algebra formed by the smeared variables just introduced, which is moreover unique under certain assumptions regarding the diffeomorphism symmetry [10]. Solving the Gauss constraint is relatively straightforward once this representation has been constructed, since its (quantum) action on cylindrical functionals can be simply inferred from its (classical) action on holonomies. A finite (generalized) gauge transformation is labeled by one SU(2) element $g(x)$ at each point $x \in \Sigma$, acting on holonomies and cylindrical functionals as:

$$h_\ell(A) \mapsto (g \triangleright h_\ell)(A) \equiv g_{s(\ell)} h_\ell(A) g_{t(\ell)}^{-1}, \tag{2.11}$$

$$\Psi_{\mathcal{G},f}(A) = f(h_\ell(A)) \mapsto (g \triangleright \Psi_{\mathcal{G},f})(A) \equiv f((g \triangleright h_\ell)(A)). \tag{2.12}$$

Thanks to group-averaging techniques, any cylindrical functional can be projected down to a gauge invariant one. Combined with the Peter-Weyl theorem, this allows to construct an explicit orthonormal basis for the gauge invariant Hilbert space. It is the subspace of $\mathcal{H}_{kin}$ spanned by the *spin network functionals*, which are special cylindrical functionals $\Psi_{\mathcal{G},\{j_\ell\},\{\iota_n\}}$. The half-integers $\{j_\ell\}$ label representations of SU(2) associated to the links $\{\ell\}$ of $\mathcal{G}$, while ι_n is an SU(2) intertwiner attached to the node n, compatible with the different representations meeting at n. $\Psi_{\mathcal{G},\{j_\ell\},\{\iota_n\}}$ is the contraction of representations matrices D^{j_ℓ} with the intertwiners, according to the pattern of the decorated graph $(\mathcal{G}, \{j_\ell\}, \{\iota_n\})$, which can schematically be denoted:

$$\Psi_{\mathcal{G},\{j_\ell\},\{\iota_n\}}(A) = \prod_n \iota_n \cdot \prod_\ell D^{j_\ell}(h_\ell(A)). \tag{2.13}$$

Solving the vector constraint is more intricate, because its solutions have to be found in the topological dual of an appropriate dense subspace of $\mathcal{H}_{kin}$. This aspect of the canonical theory is largely unrelated to the present thesis, therefore it is sufficient to recall the main conceptual idea, and skip all the technical details. When acted upon a spin network, a diffeomorphism simply moves and deforms the graph on which it is supported. One therefore needs to repackage spin network functionals into equivalent classes of decorated graphs which can be deformed into one another. All the information about the embeddings of the graphs is washed out, except for their knot structures, which provide additional quantum numbers characterizing the diffeomorphism invariant classes. As we will see, a different route has been developed in the covariant theory as regards the diffeomorphism symmetry. The embedding information is altogether dispensed with, and the spin networks states are instead labeled by abstract graphs. The topological structure is then recovered from the graph itself, and likewise smooth diffeomorphisms (be them spatial or four-dimensional) are only expected to be (approximately) recovered in a yet to be defined continuum limit.

The essential outcome of this partial completion of Dirac's quantization program is the notion of *quantum geometry*, which provides a physical interpretation for spin network states. From the quantum fluxes, it is possible to construct geometric operators quantizing the area of a 2-dimensional surface, or the volume of a 3-dimensional region. Because the spin network functionals are eigenstates of such operators, it was possible to determine their spectra, which turned out to be discrete. More precisely, it was shown that the quanta of area are carried by the links of a spin network: each link with spin j_ℓ, puncturing a surface S, contributes with a term $\pm 8\pi\gamma\ell_P^2$ to the (oriented) area of S, where ℓ_P is the Planck length. Similarly, the nodes of a spin network carry infinitesimal volumes, in a rather complicated but still discrete fashion. These results are at the chore of the applications of LQG techniques to cosmology and black hole physics, and also enter crucially into the semi-heuristic constructions on which spin foam models are based. The fact that the Immirzi parameter γ parametrizes these spectra is the reason why we claimed earlier it crucially enters the geometric interpretation of spin networks states.

2.1.2 Spin Foams

Spin Foam Models (SFMs) are a covariant approach to the quantization of GR, initially introduced to circumvent the difficulties with the scalar constraint encountered in canonical LQG. Like any known quantum theory, LQG is expected to have a second equivalent representation, in the form of a path-integral à? la Feynman. SFMs provide tentative formulations of this covariant theory, with the idea that LQG techniques can again make old formal theories well-defined, and that the dynamics of quantum GR might be more amenable in a four-dimensional setting. In the gravitational context, Feynman's formulation of quantum mechanics suggests to define transition amplitudes between three-dimensional boundary states of the gravitational field by integrating over histories (i.e. space time geometries interpolating between such boundary states), with a weight given by an action for GR. Schematically, call h_1 and h_2 two boundary 3-geometries, and denote by g a space time manifold having h_1 and h_2 as boundaries. Then one would like to define the transition amplitude between h_1 and h_2 by:

$$\langle h_1 | h_2 \rangle_{phys} = \int \mathcal{D}g \, \exp\left(\mathrm{i}\mathrm{S}_{\mathrm{GR}}(g)\right), \tag{2.14}$$

where $\mathcal{D}g$ is a suitable probability measure on the space of interpolating 4-geometries, and S_{GR} is an action for GR. This goal is mathematically very challenging, essentially because of the dubious existence of probability measures on spaces of continuous geometries. Indeed, if we were to use coordinates in order to define such quantities, in addition to having to handle the usual difficulties associated to measures on infinite dimensional spaces, one would also have to face the even harder question of diffeomorphism invariance. Even the left-hand side of (2.14) is problematic, as it assumes the availability of a well controlled space of 3-geometries. Finally, even at this very abstract level, what exactly should be summed over is not really clear (metric degrees of freedom only or topologies as well?).

In order to make (2.14) more concrete, one can follow the ideas behind LQG, but this time in a four-dimensional framework. This can be done in several ways, which all seem to point in a same direction. They can be classified according to the amount of inspiration and results which can be directly traced back to the canonical theory. The first, historical route, has been to reproduce Feynman's heuristic construction from canonical LQG. While this allows to deduce the general form of spin foam amplitudes, no definite model can be inferred, since the dynamics of canonical LQG is itself not well understood. Rather, one hopes in reverse to be able to pin-point the right model on the covariant side (right-hand side of Eq. (2.14)), and deduce the definition of the physical Hilbert space (scalar product on the left-hand side of Eq. (2.14)). In this approach, one takes as many features of the canonical theory as possible for granted, such as the boundary $SU(2)$ spin network states, and relies on covariant quantization techniques for the dynamics only. In the second approach, one starts the quantization from scratch in the covariant setting, but like in canonical LQG one hopes that shifting the emphasis from metric to tetrads and connection variables is profitable. Finally, the third strategy consists in taking seriously the type of degrees of freedom and discrete features of canonical LQG, but being critical towards strict quantization procedures. The latter will be discussed in details in the next section. Here, we only present and comment on the advantages and shortcomings of the first two strategies.

The dynamics of canonical LQG is encoded into the vector and scalar constraints, which together must ensure space-time diffeomorphism invariance. The vector constraint can be imposed at the kinematical level, and with the additional combinatorial abstraction previously mentioned, provides an intermediate Hilbert space spanned by (combinatorial) spin networks. The physical Hilbert space should be deduced by projecting down to states annihilated by the quantum scalar constraint $\widehat{\mathcal{S}}$. The trick lies in the fact that formally

$$\widehat{\mathcal{S}}|s\rangle = 0 \;\Leftrightarrow\; \forall t \in \mathbb{R},\; \exp(it\widehat{\mathcal{S}})|s\rangle = |s\rangle, \tag{2.15}$$

and therefore the projector on physical states can be given the formal definition:

$$P \equiv \int \mathrm{d}t \, \exp(it\widehat{\mathcal{S}}). \tag{2.16}$$

The parameter t is not a time variable, but an abstract parameter of gauge transformations generated by $\widehat{\mathcal{S}}$. However, formally one can reproduce Feynman's original derivation of the path-integral, by cutting the integral on t into infinitesimal pieces and inserting resolutions of the identity in terms of the spin network basis. This heuristic argument [14, 15] leads to the spin foam general ansatz replacing (2.14):

$$\langle s_1 | s_2 \rangle_{phys} = \sum_{\mathcal{F}: s_1 \to s_2} A_{\mathcal{F}}, \tag{2.17}$$

where $s_1 = (\mathcal{G}_1, \{j_{\ell_1}\}, \{\iota_{v_1}\})$ and $s_2 = (\mathcal{G}_2, \{j_{\ell_2}\}, \{\iota_{v_2}\})$ are two spin network states and $\mathcal{F}$ is a spin foam interpolating between them. In addition to s_1 and s_2, $\mathcal{F}$ is labeled by a triplet $(\mathcal{C}, \{j_f\}, \{\iota_e\})$, where $\mathcal{C}$ is a 2-complex with boundary $\mathcal{G}_1 \cup \mathcal{G}_2$, $\{j_f\}$ are SU(2) representations associated to its faces $\{f\}$, and $\{\iota_e\}$ are intertwiners associated to its edges $\{e\}$. Compatibility between $\mathcal{F}$ and its boundary requires a face f touching a boundary link ℓ to be labeled by $j_f = j_\ell$, and an edge e touching a boundary vertex v to be associated to an intertwiner $\iota_e = \iota_v$. The main advantages of (2.17) over (2.14) is that boundary states have well-defined background-independent labels, and the sum-over-path is combinatorial in nature. Apart from that, the conceptual setting is identical: boundary states represent quantum spatial geometries, while the foams are one dimensional higher analogues which can be given the interpretation of quantum space-times. The main shortcoming is that no clear-cut derivation of this ansatz from canonical LQG is available, and therefore viewing SFMs as the covariant realization of the same thing might be misleading, at least in a literal sense. In any case, this general heuristic argument does not provide much clue as to how the amplitudes $\mathcal{A}_{\mathcal{F}}$ should be defined, nor as to which exact combinatorial structures should be summed over. We discard the question of the combinatorics for the moment, since it will be addressed at length in the core chapters of this thesis. At this stage we only point out that in most of the literature on spin foams, the 2-complexes are assumed to be dual to simplicial decompositions of some topological 4-manifold. Taking seriously the fact that the scalar constraint of LQG (as defined by Thiemann [16]) only acts at the nodes of LQG states, one can further assume that the amplitudes $\mathcal{A}_{\mathcal{F}}$ can be factorized over elementary contributions $\mathcal{A}_v$, only depending on the group-theoretic labels related to the vertex v. It is therefore on the definition of the so-called *vertex amplitude* that most of the efforts have been concentrated, and we refer to [17] for details or additional references. We would like to see this assumption as a 'locality principle', akin to the fact that S_{GR} is best understood as an integral over the 4-manifolds entering the formal definition (2.14). In this respect, it is worth-mentioning one particular derivation of spin foam dynamics, outlined in the review [17], which is tightly related to the canonical theory: LQG provides the degrees of freedom and their quantum geometric interpretation, the dynamics of the

Engle-Peirera-Rovelli-Livine (EPRL) [18] model is then argued for on the basis of Lorentz covariance and some locality principle. This approach is conceptually similar to the original parts of this thesis, and it seems to us could benefit from the new 'locality principle' we will introduce. For more insights into the relation between canonical LQG and SFMs, we refer to [19].

The other approach to SFMs, more independent from canonical LQG, originates from the Ponzano-Regge model for quantum gravity, which is a spin foam model for Euclidean quantum gravity in three dimensions. The basic idea is that, because the classical theory is topological, it can be discretized before quantization without loss of information. Applying a path-integral quantization to the discretized classical theory, which takes the form of an $\mathrm{SU}(2)$ BF theory, provides a SFM with the same structure as inferred from the heuristic reasoning based on LQG. This model has then been generalized, in several ways, to four-dimensional models with Euclidean or Lorentzian signatures (see for instance [18, 20–26]). The classical starting point is not the Holst action itself, but rather the Holst-Plebanski one, which recasts gravity as an $\mathrm{SO}(1,3)$ BF theory with additional constraints. The Holst-Plebanski action is

$$S[B,\omega,\lambda] = \frac{1}{\kappa}\int_{\mathcal{M}} \left[\left(\star B^{IJ} + \frac{1}{\gamma} B^{IJ} \right) \wedge F_{IJ}(\omega) + \lambda_{IJKL} B^{IJ} \wedge B^{KL} \right], \tag{2.18}$$

where λ^{IJKL} is a Lagrange multiplier symmetric under the exchange of pairs (IJ) and (KL), and such that $\varepsilon_{IJKL}\lambda^{IJKL} = 0$. B is an $\mathfrak{so}(1,3)$-valued 2-form, the bivector, and the $B \wedge B$ term ensures that on shell:

$$B = \pm \star (e \wedge e), \;\; \text{or B} = \pm \text{e} \wedge \text{e}, \tag{2.19}$$

where e is a triad. Solving for the equations of motion the variation of λ provides therefore gives back the Holst action (in one sector). The main interest of the Holst-Plebanski action principle is that BF theory can be rigorously quantized by spin foam techniques. This consists in two steps: (1) define a discretization of the B-field and the curvature on some adequate cell complex, which thanks to the topological nature of BF, captures all its dynamical features; (2) quantize the discrete theory through path-integral methods. In three dimensions gravity and BF theory coincide, hence the initial interest in such a strategy. In four dimensions, the idea is to start from the exact quantization of BF, and use a discrete version of the Plebanski constraints to reintroduce the tetrad degrees of freedom *at the quantum level*. Note that in such approaches, one works with the full $\mathrm{SO}(1,3)$ (or $\mathrm{SO}(4)$) group rather than $\mathrm{SU}(2)$, and therefore slightly generalizes the formalism. This strategy of quantizing first, and only then constraining the degrees of freedom, is at the same time one of the main appeals and the most cumbersome issue of the spin foam quantization. While this allowed to construct interesting quantum gravity models, some of them well-connected to the canonical theory, it is certainly problematic from the conceptual point of view, and to a large extent explains the variety of models one can construct:

ill-defined recipes necessary introduce a handful of ambiguities. We refer again to [19] for a review of quantization and discretization ambiguities in SFMs.

We finish this discussion of SFMs by a brief introduction to the Ponzano-Regge model. Classically, three-dimensional Euclidean gravity can be formulated through the first order action:

$$S_{3d}[e, \omega] = \int_{\mathcal{M}} \mathrm{Tr}\left(e \wedge F(\omega)\right), \tag{2.20}$$

where the triad e is an $\mathfrak{su}(2)$-valued 1-form (the B field), ω is an $\mathfrak{su}(2)$-connection with curvature $F(\omega)$, and Tr stands for the trace in $\mathfrak{su}(2)$. As in the four-dimensional context, variation with respect to ω imposes the torsion-free equation, and the variation of e provides Einstein's equation: $F(\omega) = 0$; hence space-time is flat. The only degrees of freedom of vacuum $3d$ Euclidean gravity are global, which is the sense in which the theory is said to be topological, and this flatness condition should therefore be preserved at the quantum level. Indeed, the triad can formally be integrated in the (ill-defined) continuous partition function

$$\mathcal{Z}_{3d} = \int \mathcal{D}\omega \int \mathcal{D}e \exp\left(\mathrm{i}S_{3d}[e, \omega]\right) = \int \mathcal{D}\omega\, \delta(F(\omega)), \tag{2.21}$$

suggesting that a rigorous definition should in a way measure the 'volume' of the set of flat connections on $\mathcal{M}$. This expectation can be made more rigorous in the discrete: let us introduce a cellular decomposition Δ of $\mathcal{M}$, and its dual 2-complex Δ^*. For definiteness, one can for example assume that Δ is a three-dimensional simplicial complex: elementary cells are tetrahedra, glued along their boundary triangles. Δ^* is a set of nodes n, lines ℓ and faces f such that: inside each tetrahedron of Δ one finds a unique node, two nodes are connected by a line whenever the two dual tetrahedra share a triangle, and faces are collections of lines closing around the edges of Δ. Similarly to lattice gauge theory, one then discretizes the connection by extracting its holonomy along each line of Δ^*, noted $h_\ell \in \mathrm{SU}(2)$. As for the triad, it can be integrated along the edges of Δ, and provides a Lie algebra element $X_e = X_f$ for each edge $e \in \Delta$ dual to the face $f \in \Delta^*$. This allow to discretize the action in the following way:

$$S_\Delta(X, h) = \sum_{f \in \Delta^*} \mathrm{Tr}\left(X_f H_f\right), \tag{2.22}$$

where

$$H_f = \overrightarrow{\prod_{\ell \in f}} h_\ell \tag{2.23}$$

is the oriented product of the holonomies around the face f. The formal partition function (2.21) is made concrete by summing holonomies with the Haar measure $\mathrm{d}h_\ell$ over $\mathrm{SU}(2)$, and triads with the Lebesgue measure $\mathrm{d}X_f$ on $\mathfrak{su}(2) \simeq \mathbb{R}^3$:

$$Z_{PR}(\Delta) = \int [\mathrm{d}X_f] \int [\mathrm{d}h_\ell] \exp\left(\mathrm{i} \sum_f \mathrm{tr}(X_f H_f) \right) = \int [\mathrm{d}h_\ell] \prod_{f\in\Delta^*} \delta\left(H_f\right). \tag{2.24}$$

This formulation of the Ponzano-Regge partition function, in lattice gauge theoretic language making flatness of the geometry apparent, will be primary in the GFT context. Two other formulations are however possible. One could instead integrate the connection degrees of freedom, and express Z_{PR} as an integral over Lie algebra variables [27]. The subtlety is that these are non-commutative variables, but we will see how it can be done and put to good use in later chapters. The other rewriting, which we describe now, makes the connection to spin foam models explicit. One simply relies on the Peter-Weyl theorem, and expands the δ-functions in representations:

$$\delta = \sum_{j\in\frac{\mathbb{N}}{2}} (2j+1)\chi_j, \tag{2.25}$$

where χ_j is the character of the jth representation of SU(2). Each character can be decomposed in products of Wigner matrices with individual line holonomies as variables. Each variable h_ℓ will appear exactly three times, one for each edge of the dual triangle of ℓ. The h_ℓ holonomies can therefore be integrated explicitly, yielding one 3-valent intertwiner per line, which is nothing but a $3j$ symbol. Finally, these can be contracted together, four by four, following the tetrahedral pattern associated to each vertex of Δ^*. The partition function then takes the original form proposed by Ponzano and Regge in their seminal work [28]:

$$Z_{PR}(\Delta) = \sum_{\{j_f\}} \prod_{f\in\Delta^*} (-1)^{2j_f}(2j_f+1) \prod_{v\in\Delta^*} \{6j\}(v), \tag{2.26}$$

where the sum runs over all possible spin attributions to the faces of Δ^*, and $\{6j\}(v)$ denotes the evaluation of the $6j$ symbol on the six spin labels of the faces of Δ^* running through v. This is the first SFM ever proposed, and we see that in the simplicial setting, the vertex amplitude is essentially captured by the $6j$ symbol.

2.1.3 Summing Over Spin Foams

From the point of view of the heuristic path towards spin foams from canonical LQG (formula (2.17)), it is clear that when computing transition amplitudes, one should not only sum over the group-theoretic data, but also on the combinatorial structure of the foam. The first question to come to mind is then: with which measure? Since the precise form of the amplitudes is dictated by a quantization of the discretized theory, where the foam is chosen once and for all at the classical level, this is a difficult question to address. On the other hand, if we do not give too much credit to the heuristic construction à la Feynman, and focus instead on the Ponzano-Regge and related spin foam models for three-dimensional gravity, it is possible to argue for a different strategy. Indeed, an interesting property of the Ponzano-Regge model is that it is formally triangulation independent: that is, performing any possible local move which do not change the topology (Pachner moves) on the triangulation only affects the amplitude through a generically divergent overall factor. The unpleasant formal character of this invariance can even be cured at the price of trading the $\mathrm{SU}(2)$ group for its quantum-deformed version. Is obtained in this way the so-called Turaev-Viro model, which is usually interpreted as 3d quantum gravity with a non-vanishing positive cosmological constant [29]. Since 3d quantum gravity turns out to be equivalent in this sense to the definition of a topological invariant for 3-manifolds, it is tempting to assume that 4d quantum gravity will be likewise equivalent to the definition of a diffeomorphism invariant for 4-manifolds. Again, in the realm of triangulated manifolds, diffeomorphism invariance can be understood as triangulation invariance, and more precisely invariance under local Pachner moves. This key property is what is generally understood as entailing the definition of a well-behaved *state-sum model*, applying tools from topological quantum field theory to quantum gravity [30].

In this thesis we tend to favor the heuristic argument leading to the idea of SFMs over the very formal idea of a quantum invariant. The outstanding difficulties paving the way to a satisfactory diffeomorphism invariant in 4d, and the rigidity of such a strategy are discouraging to us. As explained in the introduction, we would rather favor an approach in which the notions of scale and renormalization have a role to play, thus allowing for more flexible models, where approximate rather than exact invariance matters. This route is currently explored in details by Bianca Dittrich and collaborators [31–35]. Rather than looking for an exact state-sum model, they are instead developing coarse-graining and renormalization methods allowing to consistently improve the dynamics, and hopefully reach a diffeomorphism-invariant phase in some *infinite refinement limit* of the foams. In addition to providing a constructive method towards diffeomorphism invariance, this framework aims at developing the necessary tools to perform approximate calculations. From the point of view of applications to realistic physical situations, this seems to us a good alternative to the more abstract incarnations of state-sum models.

More prosaically, most of the SFMs currently under investigation not only fail to implement topological or diffeomorphism invariance, but even fail to verify the axioms which are at the basis of the state-sum approach [20]. For instance, the 'projector' defining the intertwiner space of the Lorentzian EPRL model does actually not square to itself. As a result, spin foam amplitudes do not compose well: if $\mathcal{F}_{12}$ is a spin foam mapping a spin network state s_1 to s_2, and $\mathcal{F}_{23}$ maps s_2 to s_3, one has in general

$$A_{\mathcal{F}_{23}\circ\mathcal{F}_{12}} \neq A_{\mathcal{F}_{23}} A_{\mathcal{F}_{12}}, \tag{2.27}$$

contrary to the formalism advocated in [20]. Even in the Ponzano-Regge model, such requirements cannot be implemented in full generalities. Because of the formal of nature equation (2.24), plagued with divergences, one needs to introduce a regulator cutting-off large spin labels. Such a cut-off spoils both the triangulation invariance and the morphism properties of the Ponzano-Regge amplitudes. It is true that in this case one could resort to the well-behaved Turaev-Viro model. However, from a more physical point of view it is not clear why the cosmological constant should be necessary to the very definition of spin foam models. These pathologies seem on the contrary to suggest that the strict notion of state-sum model, with rigid composition rules, is too narrow to accommodate interesting and physically sound proposals such as the EPRL model.

Let us summarize shortcomings of SFMs such as the Ponzano-Regge model or the EPRL one. First, they do not provide any canonical measure on the space of foams to be summed over, and even this space is not clearly constrained. Second, the presence of divergences (at least when the cosmological constant vanishes) calls for regularization and renormalization procedures, which as in usual quantum field theories are expected to map different combinatorial structures to one another (via coarse-graining or renormalization steps). Again, the quantization procedure leading to specific models does not seem to provide any hint as to how this should be done. Third, and this is certainly the main drawback, the first issue together with the absence of triangulation independence precludes any complete definition of the transition amplitudes (2.17), even at the formal level. We would like to add a fourth trouble, which we will address in more details later one: the question of the *continuum limit* of SFMs. If spin networks boundary states are thought of as encoding elementary excitations of the gravitational field, or equivalently atoms of space, one is naturally inclined to address the question of the dynamics of *very many* such states, collectively representing macroscopic and approximately smooth geometries. Indeed, if LQG and SFMs can ever be related to continuous GR in full generality, and hence established as proper quantum theories of gravity, it seems unlikely to be at the level of spin networks with a few links and nodes. For if such states were to describe macroscopic spaces, these would be highly discrete ones (i.e. sampled with very few points), hardly comparable to the smooth structure we experience in everyday life and physics. It is true that they might be appropriate to describe specific physical situations where the number of relevant degrees of freedom are effectively small, such as for instance in cosmology, but they cannot themselves explain the emergence of classical space-time. On the contrary, handling combinatorially large and complicated spin networks can only be made possible if new approximation tools are developed. Individually, and even more altogether, these four challenges seem to point towards essentially two alternatives, both relying on renormalization. The first, inspired by lattice gauge theories, is to find a consistent way of *refining* SFMs, as already explained. The second is to instead look for a way of consistently *summing* spin foam amplitudes.

2.1.4 Towards Well-Defined Quantum Field Theories of Spin Networks

The application of Group Field Theory to spin foam models originates from seminal work by De Pietri, Freidel, Krasnov, Reisenberger and Rovelli [36, 37]. The central idea is to construct a generating functional for spin foam amplitudes, which allows to sum them with quantum field theory techniques. Let us give a concrete illustration with the first GFT ever proposed, the Boulatov model [38], which generates Ponzano-Regge amplitudes. Following the general QFT procedure, we want to encode the boundary states of the model into functionals of one or several fields, and for definiteness let us say a single scalar field φ. Let us moreover aim at the lattice gauge formulation of the Ponzano-Regge amplitudes given in Eq. (2.24). It is then natural to assume the field φ to have support on several copies of $\mathrm{SU}(2)$. In its simplicial version, the boundary states of the Ponzano-Regge model are labeled by closed graphs with three-valent vertices, whose analogue in the field theory formalism are convolutions of the fields φ. It is thus necessary to work with three copies of $\mathrm{SU}(2)$, and hence natural to assume $\varphi \in L^2(\mathrm{SU}(2)^3)$ (with respect to the Haar measure). Incidentally, one immediately recognizes how to recover the spin network boundary states of the Ponzano-Regge model: through the harmonic expansion of φ in terms of Fourier modes labeled by triplets of spins. The partition function of the looked for field theory will have the generic structure:

$$\mathcal{Z}_{GFT} = \int \mathrm{d}\mu_C(\varphi) \exp(-S_{int}(\varphi)), \tag{2.28}$$

where $\mathrm{d}\mu_C$ is a Gaussian measure defining the notion of propagation of boundary data, and S_{int} is the interaction part of an action, encoding the non-Gaussian part of the dynamics. Note that, for reasons which will become clear shortly, the kinetic part of the action is encoded in the Gaussian measure $\mathrm{d}\mu_C$, together with the (ill-defined) Lebesgue measure on $L^2(\mathrm{SU}(2)^3)$. Let us focus on the combinatorics first. A boundary field is associated to a node of a spin network, while its variables label the three links connected to this node. Therefore in the bulk the GFT fields must label spinfoam edges, and a field variable be associated to a couple $w = (e, f)$ where f is a face incident on the edge e, called a wedge (see Fig. 2.1b). We want moreover to recover the 4-valent interaction vertices of the Ponzano-Regge model when constructing the perturbative theory, through an expansion of the exponential term in (2.28). Therefore S_{int} must consist in a single term, which precisely convolutes four GFT fields according to a tetrahedral pattern. Since in any QFT we are free to encode the non-combinatorial aspects of the dynamics in the propagator rather than the interaction, we can, without loss of generality fix:

$$S_{int}(\varphi) = \lambda \int [\mathrm{d}g_i]^6 \, \varphi(g_1, g_2, g_3)\varphi(g_3, g_5, g_4)\varphi(g_5, g_2, g_6)\varphi(g_4, g_6, g_1), \tag{2.29}$$

where λ is a new coupling constant. This ensures that, whatever the precise form of the propagator C, the perturbative expansion of (2.28) will be labeled by arbitrary 2-complexes verifying the condition that at each vertex meet exactly 4 edges and 6 faces, with a 'tetrahedral' pattern. See Fig. 2.1.

We now need to find the correct covariance, so that the amplitudes contain the relevant δ-functions ensuring triviality of the holonomies around any face. Recall that the covariance defines a kernel $C(g_1, g_2, g_3; g'_1, g'_2, g'_3)$ (the propagator), through:

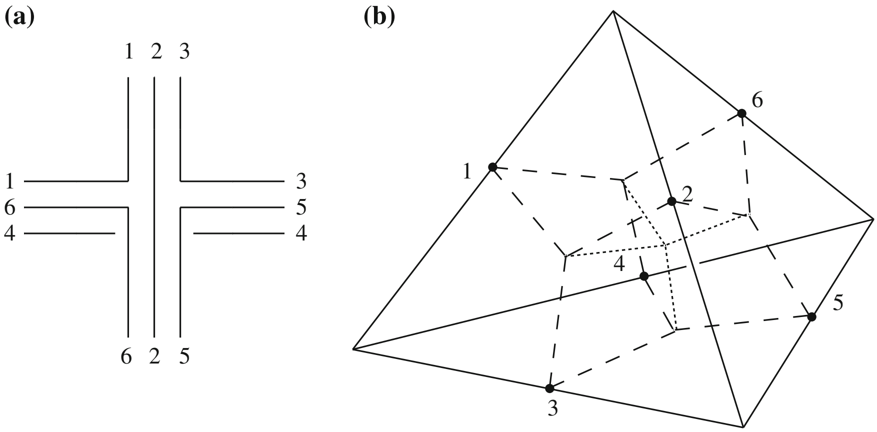

Fig. 2.1 **a** Interaction vertex of the Boulatov model: each half-line, dual to a triangle, is composed of three strands, themselves dual to the strands of the triangle; **b** elementary tetrahedron, dual to the vertex, with its six wedges represented in dashed lines. **a** $\omega = 1$, **b** $\omega = 0$

$$\int \mathrm{d}\mu_C(\varphi)\, \varphi(g_1, g_2, g_3)\varphi(g'_3, g'_2, g'_1) = C(g_1, g_2, g_3; g'_1, g'_2, g'_3) \tag{2.30}$$

The first data which should be introduced in this kernel are the holonomies h_ℓ associated to the lines of the foam. They cannot be the field variables g_i or g'_i themselves, which as we pointed out rather label the wedges of the foam. Therefore C should be expressed as an integral over a single SU(2) parameter. It should also contain three δ functions depending on h, such that when constructing the amplitudes and integrating out the g's one recovers formula (2.24). It is then not difficult to convince oneself and check that the solution is:

$$C(g_1, g_2, g_3; g'_1, g'_2, g'_3) = \int \mathrm{d}h\, \delta(g_1 h g'^{-1}_1)\delta(g_2 h g'^{-1}_2)\delta(g_3 h g'^{-1}_3). \tag{2.31}$$

In this case, one can indeed show that:

$$\mathcal{Z}_{GFT} = \sum_{\Delta*} \frac{(-\lambda)^{V(\Delta*)}}{s(\Delta^*)} Z_{PR}(\Delta), \tag{2.32}$$

where Δ runs over all the 2-complexes (without boundaries) verifying the aforementioned combinatorial constraints, $V(\Delta^*)$ is the number of vertices in Δ^*, and $s(\Delta^*)$ is a symmetry factor. It is now clear why we did not want to write the kinetic term of this field theory in an exponentiated form: because of the integration over the holonomy h, C does not define an invertible operator, and therefore this representation is strictly speaking not available. A different but equivalent way of understanding this point is sometimes adopted (see for instance the presentation adopted in the classical references [15, 39–41]). Instead of integrating over the whole Hilbert space $L^2(\mathrm{SU}(2)^3)$ with respect to a degenerate Gaussian measure, one can restrict beforehand the space of fields to the *gauge invariant* ones:

$$\forall h \in \mathrm{SU}(2)\,, \qquad \varphi(g_1, g_2, g_3) = \varphi(hg_1, hg_2, hg_3), \tag{2.33}$$

and use an invertible propagator (the trivial one with no h variable). The constraint (2.33) is nothing but the Gauss one in the GFT setting. It implies that boundary states are really gauge invariant (abstract) spin networks. Using an unconstrained space of fields with the degenerate covariance (2.31) is morally equivalent, in the sense that only the gauge invariant part of the field is propagated in this framework. The reasons why we prefer to adopt the degenerate covariance formalism are: (a) it is mathematically rigorous; (b) it allows to use simple notions of locality for the interactions [42]; (c) it seems convenient to encode as much as the dynamics as possible in the propagator, this way combinatorial issues and the problem of how gravitational constraints should be imposed get clearly separated. In particular, this thesis explores ways of defining a locality principle, and covariances implementing the Gauss constraint, such that a well-defined theory of renormalization becomes available.

The correspondence between GFTs and SFMs we have just illustrated is very general, as the example of the Boulatov model hopefully makes it clear. GFTs can from this perspective be characterized as quantum field theories for spin networks, generalizing the correspondence between relativistic QFTs and scattering states one finds at the root of particle physics. The only constraints are on the valencies of the spin networks nodes on the boundary, and the structures of the spin foam vertices in the bulk. The former are determined by the field content, while the latter depend on the choice of interactions. More precisely, if we want to have n-valent spin networks nodes in the boundary, we need to bring in one new GFT field with n variables; and to any type of vertex in the spin foam amplitudes must be associated a certain convolution of fields in S_{int}. Therefore the formalism is in principle general enough to accommodate any finite numbers of nodes in the boundary, and any finite number of vertices in the bulk. Likewise, the type of boundary data and dynamics can be specified at will by changing the group and the covariance. In particular, any constraint arising from the spin foam quantization, such as simplicity constraints, can be included in the covariance. From this point of view, the Gauss constraint is a defining feature of GFTs, since it is responsible for the presence of holonomy degrees of freedom. This explains why we will concentrate on this aspect in the remainder of this thesis.

The appeals of the GFT formalism from the point of view of SFMs and LQG are numerous. First and foremost, the QFT formalism allows to fix canonical weights for the sum over foams. While a full justification of these weights from first principles is not available, they have the merit of being well-defined, therefore completing the definition of SFMs. They can moreover be partially justified: with an appropriate definition of the GFT coupling constant, the combinatorial weight of a complex is the order of its automorphism group, which can be argued to be a discrete, purely combinatorial analogue of the diffeomorphism group. Second, the divergences of SFMs are now understood as divergences of the amplitudes of a field theory. Such features have to be expected in any QFT, and tools to control them are well understood both conceptually and mathematically, thanks to renormalization theory. Through the embedding of SFMs into GFTs, the question of the value of the cosmological constant and that of the occurrence of divergences become clearly separated, as it seems to us they should be. A third feature which can be seen as an advantage too, is that in GFT the topology of space-time is itself dynamical. For instance in the version of the Boulatov model introduced above, the foams contributing to the partition function are dual to arbitrary gluings of tetrahedra, which include in particular all types of triangulated topological manifolds, but also more pathological structures. Therefore GFT can potentially explain the local topology of macroscopic space-time, in addition to its metric properties. This feature is sometimes called third quantization, and we refer to [43] for a more detailed exposition in the context of GFTs. Finally, as quantum field theories on local symmetry groups rather than space-time, GFTs make possible to incorporate all the tools which are so crucial to quantum field theories in a background independent context. Especially, perturbative methods in the QFT sense are available, without having to resort to perturbation in the space-time sense. This is very different from the usual perturbative approach to quantum gravity, which is so to speak doubly perturbative: perturbative QFT methods are used to analyze quantum perturbations over a background metric. This aspect of GFTs has already been taken advantage of, in the limited context of 'graviton propagator' calculations, where the existence and physical relevance of the perturbative expansion in the coupling constants of GFTs is assumed from the start [44–46].

In order to determine whether this list of merits is truly realized in GFTs, a lot of work is needed, both at the conceptual and mathematical level. The key result to achieve in this respect is to establish GFTs as well-defined perturbative quantum field theories. To this effect, one needs to generalize renormalization theory to this new context. It is only equipped with such a new toolbox that we will be able to determine whether a specific GFT model can be taken seriously as a field theory or not, and in which sense. It seems that the key physical questions are to be settled down only then. The most pressing one is to determine in which sector of a given theory (if any) the classical effective dynamics of GR lies. In particular, can we relate the first few orders of the perturbative expansion to continuum macroscopic physics? Or, on the contrary, does it emerge from the interaction of very many degrees of freedom, and can as a result only be captured by non-perturbative techniques, or by perturbation in a different phase of the theory? The somehow mysterious interpretation of the coupling constants of a GFT from the gravity perspective is certainly related. Whatever the answers to these questions, for which in our view renormalization methods applied to phase transitions could play a discriminant role, it must be admitted that if GFTs are used to complete the definition of SFMs, the problem of their renormalizability have to be faced head on and resolved.

This concludes our summary of motivations for studying GFTs, from the point of view of loop quantum gravity and spin foam models. As is certainly clear to the reader, several points of the reasoning are semi-heuristic and formal. Moreover, several key combinatorial aspects of GFTs seem poorly motivated by the quantization procedures, and it seems to us essentially independent of them. It is therefore of paramount importance to take some distance with the quantization, and present the more combinatorial motivations behind GFTs, which we do in the next section.

2.2 Group Field Theories and Random Discrete Geometries

2.2.1 Matrix Models and Random Surfaces

Since GFTs are in a sense higher dimensional incarnations of matrix models, we start with a brief presentation of basic aspects of the latter [47–50]. Matrix models are statistical models for matrix-like degrees of freedom, in the sense that 'locality' is based on a matrix rather than point-wise product. For instance, let M be an hermitian matrix of size $N \times N$. We can construct an action $S(M)$ for M by requiring it to be invariant under conjugation of M. This plays the role of 'locality' principle in the same way as Lorentz and gauge invariance are at the root of local quantum field theories, i.e. by providing a set of allowed interactions. It is then possible to show that $S(M)$ has to be a sum of products of invariants of the form: $\mathrm{tr} M^k$, with $k \in \mathbb{N}^*$. If we further restrict to *connected* invariants, then $S(M)$ is simply a sum of such terms. The simplest non-trivial connected invariant action retains the first two terms only:

$$S(M) = \frac{1}{2}\mathrm{tr} M^2 - \lambda \mathrm{tr} M^3 , \tag{2.34}$$

where λ is a coupling constant. The partition function of the matrix model is then defined by:

$$\exp \mathcal{Z} = \int \mathrm{d}M \, \exp\left(-S(M)\right) = \int \mathrm{d}M \, \exp\left(-\frac{1}{2}\mathrm{tr} M^2 + \lambda \mathrm{tr} M^3\right) , \tag{2.35}$$

where $\mathrm{d}M$ is the invariant measure on the set of $N \times N$ hermitian matrices. This theory can be expanded in perturbations in λ and shown to generate Feynman amplitudes labeled by *ribbon graphs*. The propagator, deduced from the quadratic Gaussian term in $S(M)$, can be pictured as a line with two strands, each strand carrying one index of the matrix M. More precisely, the covariance of this theory identifies indices as follows:

$$C_{ij;kl} = \int [\mathrm{d}M] M_{ij} M_{kl} \exp\left(-\frac{1}{2}\mathrm{tr} M^2\right) = \delta_{j,k}\delta_{l,i} . \tag{2.36}$$

The interaction part of the action introduces a single 3-valent vertex, which identifies 6 strands pairwise. The propagator and interaction vertex are represented in Fig. 2.2. The free energy $\mathcal{Z}$ is then indexed by closed and connected ribbon graphs $\mathcal{G}$, an example of an open ribbon graph being given in Fig. 2.3. The ribbon structure of the graphs allows to define the notion of face: a face of $\mathcal{G}$ is a set of strands forming a loop. We note $n(\mathcal{G})$ the number of (3-valent) vertices of $\mathcal{G}$, and $F(\mathcal{G})$ its number of faces. It is then not difficult to see that:

$$\mathcal{Z} = \sum_{\mathcal{G} \text{ connected}} \frac{1}{s(\mathcal{G})} \lambda^{n(\mathcal{G})} \mathcal{A}_{\mathcal{G}}, \tag{2.37}$$

$$\mathcal{A}_{\mathcal{G}} = N^{F(\mathcal{G})}, \tag{2.38}$$

where $s(\mathcal{G})$ is a symmetry factor. The fact that the amplitude $\mathcal{A}_{\mathcal{G}}$ is weighted by the number of faces is easy to understand: since in $\mathcal{G}$, the indices of the strands are identified by δ-functions across propagators and vertices, one can trivially sum all of them but one per face; we are therefore left with one free index per face, which sums to N. The main interest of the occurrence of such ribbon graphs in the perturbative expansion of the matrix models, is that their duals are triangulated surfaces. To see this, it suffices to associate a transverse line to each propagator, and a triangle to each vertex, as represented in Figs. 2.2 and 2.3. In this dual picture, the role of the propagator is to identify pairwise the edges of the $n(\mathcal{G})$ triangles generated by $\mathcal{G}$, yielding a closed triangulated surface. Note that, very importantly, the vertices of the triangulation are dual to the faces of $\mathcal{G}$, so that a complete and unambiguous correspondence is established between a ribbon graph and its dual triangulation. This interpretation of matrix models as statistical models for discretized surfaces is very general. The statistical ensemble one is spanning depends on which matrix ensemble and which interactions are being used. For instance, in the model we just presented, the hermiticity of M restricts the sum to orientable triangulations, while both orientable and non-orientable triangulations would be included had we used symmetric matrices instead. Also, one could work with quadrangulations upon replacing the third order interaction by a fourth order one, or include arbitrarily wild kinds of discretizations thanks to all the other matrix invariants.

The first technical aspect which makes the link between matrix models and random triangulations so interesting is the existence of a $1/N$ expansion, bringing powerful analytical control over the formal sum (2.37). One can use the arbitrary size of the matrix N to unravel universal properties of the statistical model, in the limit of infinitely large matrices. In practice, one looks for a rescaling of the coupling constant

$$\lambda \mapsto \frac{\lambda}{N^{\alpha}}, \tag{2.39}$$

such that, when $N \to +\infty$: (a) the most divergent configurations have a uniform divergence degree at each order in λ; (b) these configurations are infinitely many. In such a situation, N can be used as a new perturbative parameter, in such a way that the leading order term in N captures infinitely many orders in λ. This is in this sense that a $1/N$ expansion captures non-perturbative effects. In our context, large orders in the coupling constants correspond to discrete surfaces with many building blocks, hence the $1/N$ expansion is particularly relevant to the question of the continuum

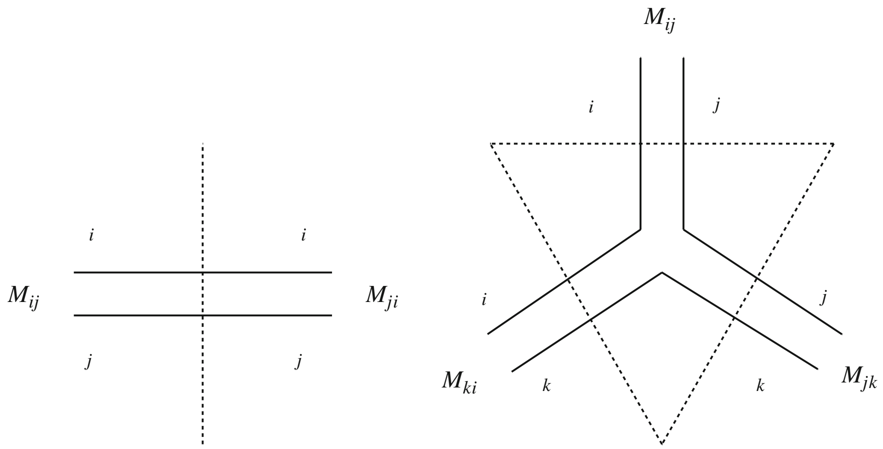

Fig. 2.2 Propagator (*Left*) and interaction vertex (*Right*) interaction of the matrix model (2.35). In *dashed lines*: the dual edge of the propagator and the dual triangle of the vertex

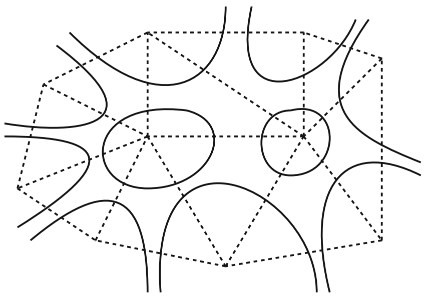

Fig. 2.3 An open ribbon graph, with two closed faces, and its dual triangulation

limit of such discrete models. Let us consider again the model (2.35). The unique value of α verifying the two previous conditions is $1/2$, and in this case the amplitude of a graph $\mathcal{G}$ becomes:

$$\mathcal{A}_{\mathcal{G}} = N^{F(\mathcal{G})-\frac{1}{2}n(\mathcal{G})}. \tag{2.40}$$

Let us call $V(\Delta_{\mathcal{G}})$, $E(\Delta_{\mathcal{G}})$ and $T(\Delta_{\mathcal{G}})$ the numbers of vertices, edges, and triangles in the triangulation $\Delta_{\mathcal{G}}$ dual to $\mathcal{G}$. Each edge of $\Delta_{\mathcal{G}}$ being shared by exactly two triangles, we have $2E(\Delta_{\mathcal{G}}) = 3T(\Delta_{\mathcal{G}})$ and therefore:

$$F(\mathcal{G})-\frac{1}{2}n(\mathcal{G}) = V(\Delta_{\mathcal{G}})-\frac{1}{2}T(\Delta_{\mathcal{G}}) = V(\Delta_{\mathcal{G}})-E(\Delta_{\mathcal{G}})+T(\Delta_{\mathcal{G}}) = \chi(\Delta_{\mathcal{G}}) = 2-2g(\Delta_{\mathcal{G}}), \tag{2.41}$$

where $\chi(\Delta_{\mathcal{G}})$ is the Euler characteristic of $\Delta_{\mathcal{G}}$, and $g(\Delta_{\mathcal{G}})$ its genus. Recall that closed two-dimensional topological manifolds are fully characterized by two invariants, the Euler characteristic and the orientability, so that in the orientable case we are considering the genus fully determines the topology of the triangulation. Therefore the $1/N$ expansion is actually a topological expansion:

$$\mathcal{Z} = \sum_{\mathcal{G} \text{ connected}} \frac{1}{s(\mathcal{G})} \lambda^{n(\mathcal{G})} N^{2-2g(\Delta_{\mathcal{G}})} \tag{2.42}$$

$$= \sum_{g \in \mathbb{N}^*} N^{2-2g} \mathcal{Z}_g(\lambda), \tag{2.43}$$

where

$$\mathcal{Z}_g(\lambda) \equiv \sum_{\mathcal{G} \text{ connected} | g(\Delta_{\mathcal{G}})=g} \frac{1}{s(\mathcal{G})} \lambda^{n(\mathcal{G})} \tag{2.44}$$

sums all the triangulations with genus g. As a result, in the limit of infinitely large matrices, spherical topologies ($g = 0$) dominate; and the more holes a non-spherical surface has, the more it is suppressed.

Assuming that one can first take the large N limit to restrict oneself to the spherical sector, one can then look for a continuum limit in this sector. Because $\mathcal{Z}_0(\lambda)$ contains all the spherical triangulations, with arbitrary number of triangles, it is meaningful to ask whether it is dominated by large triangulations in some regime of λ. Such large order terms will start dominating the behavior of $\mathcal{Z}_0$ close to its convergence radius. It can be shown that $\mathcal{Z}_0$ has a non-zero and finite convergence radius, providing a critical value λ_c for the coupling constant, and such that:

$$\mathcal{Z}_0(\lambda) \underset{\lambda \to \lambda_c}{\sim} |\lambda - \lambda_c|^{2-\gamma}. \tag{2.45}$$

γ is the critical exponent, and is equal to $-1/2$ in this model, so that the free energy associated to the spherical sector diverges close to the critical point. At least at a formal level, this supports the idea that a continuum phase of the theory is reached by tuning λ to its critical value. We can illustrate this in the following way. Suppose that each triangle of the model is assumed equilateral, and attributed some elementary area a. Then the mean area of a triangulation in the statistical ensemble is given by

$$\langle A \rangle = a \langle n(\mathcal{G}) \rangle = a \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln(\mathcal{Z}_0(\lambda)) \underset{\lambda \to \lambda_c}{\sim} \frac{a}{|\lambda - \lambda_c|}. \tag{2.46}$$

So that if we send the fiducial area to 0 when approaching criticality, in such a way that the mean value of the area is kept fixed, one obtains a statistical model for infinitely refined spherical triangulations with a given area. This has certainly the flavor of the continuum!

One can go further and construct statistical ensembles containing all possible (oriented) manifolds. This is achieved in the so-called *double scaling* limit, which consists in taking the large N and critical limits in a correlated manner. It is possible at all thanks to the fact that $\mathcal{Z}_g$ can be shown to have the same critical point as $\mathcal{Z}_0$, for any $g \in \mathbb{N}$. More precisely, one has

$$\mathcal{Z}_g(\lambda) \underset{\lambda \to \lambda_c}{\sim} |\lambda - \lambda_c|^{\frac{(2-\gamma)(2-2g)}{2}}, \tag{2.47}$$

which suggests to take the joint limit $N \to +\infty$ and $\lambda \to \lambda_c$, keeping the ratio

$$\kappa^{-1} \equiv N|\lambda - \lambda_c|^{\frac{2-\gamma}{2}} \tag{2.48}$$

fixed. In such a limit, the damping effect due to the large N limit and the enhancement of higher genera close to the critical point compensate, in such a way that all topologies contribute to the free energy:

$$\mathcal{Z} \sim \sum_{g \in \mathbb{N}} \kappa^{2g-2} f_g, \tag{2.49}$$

where f_g are some computable constants.

The way these results are usually proven is through a rewriting of the free energy in terms of $\mathrm{U}(N)$ invariant quantities, namely the eigenvalues of M. Such a tool is not (yet?) available in higher dimensions, so we do not want to provide any details along this line here. The interested reader may refer to [47–50]. It should also be noted that these results are only a first step, and that the precise correspondence between the discrete theory and its continuum limit is studied by different means, for instance thanks to Schwinger-Dyson equations for the so-called *loop observables*, which in the large N limit reproduce the Witt algebra [48–50], and can be related to the Wheeler-DeWitt equation for 2d quantum gravity [51]. Finally, mathematically rigorous characterizations of the continuum limit of matrix models have been obtained thanks to random planar maps, see [52] and references therein.

As far as quantum gravity is concerned, what makes matrix models interesting is their connection to quantum gravity in two dimensions. We only recall the correspondence at the discrete level, which can be shown to extend to the continuum phase thanks to the Schwinger-Dyson equations. The action for gravity in two dimensions with cosmological constants is

$$S_{2d} = \frac{1}{G}\int_S \mathrm{d}x^2 \sqrt{-g}\,(-R(g) + \Lambda) = -\frac{4\pi}{G}\chi(S) + \frac{\Lambda}{G}A_S, \tag{2.50}$$

where the second equality is a consequence of the Gauss-Bonnet theorem. The Einstein term being topological in two dimensions, only the Euler characteristic and the area A_S can be dynamical in the vacuum. Intuitively, this suggests that a fine enough discretization of the theory is enough to encode the dynamics of these two global degrees of freedom. We can for instance introduce an equilateral triangulation Δ of S, each individual triangle having a same area a. The Einstein term of (2.50) can then be replaced by $\chi(\Delta)$, and the area approximated by $aT(\Delta)$. This provides an action principle for a discrete theory, from which we can deduce a path-integral quantization. Remarkably, the resulting partition function is:

$$\mathcal{Z}_{2d} = \sum_{\Delta} \exp\left(\frac{4\pi}{G}\chi(\Delta) - \frac{\Lambda a}{G}T(\Delta)\right), \tag{2.51}$$

and can therefore be matched to the free energy (2.37) of the matrix model (with the large N rescaling). The sum over Δ can include arbitrary topologies, so that one really obtains a third quantization of $2d$ gravity. The correspondence imposes the following identifications:

$$\lambda \leftrightarrow \exp\left(-\frac{\Lambda a}{G}\right), \qquad N \leftrightarrow \exp\left(\frac{4\pi}{G}\right). \tag{2.52}$$

Therefore the large N limit of the matrix model corresponds to the weak coupling regime of $2d$ gravity. Thanks to the double scaling, one can moreover extend the sum over topologies to the continuum phase.

Let us summarize the situation. Matrix models are at first sight algebraic models, with no particular relation to gravity or geometry in general. However the relation to discrete geometry is readily seen at the perturbative level, since Feynman amplitudes are labeled by discretized surfaces. Still at the discrete level, the amplitudes can be related to discrete gravity path integrals, the two coupling constants of gravity being encoded in the coupling constant of the matrix model and the size parameter N. Finally, the matrix model formulation allows to reach a continuum phase when sending N to infinity, dominated by spherical topologies, and which can also be extended to all types of manifolds. In a sense, what matrix models achieve is a definition of a measure on continuous geometries thanks to discrete methods, which reminds the relationship between Riemann sums and the Riemann integral.

2.2.2 Higher Dimensional Generalizations

The success of matrix models motivated extensions to higher dimensions already in the nineties, called *tensor models* [53–55]. Viewing a matrix as a 2-tensor, it is natural to introduce d-tensors in d dimensions. The action $S(T)$ of a tensor is inspired by its matrix counter-part: the kinetic term convolutes indices of two tensors pairwise, and is interpreted as the identification of two $(d-1)$-simplices; and the interaction is built in such a way as to represent elementary d-cells. It is also natural to work with a single interaction, dual to a d-simplex, as it is the interaction with the smallest number of fields which can be given a d-dimensional interpretation, and also any discretization of a manifold can be subdivided in such a way that all cells are simplices. For example, in three dimensions, one can define:

$$S(T) = \frac{1}{2} \sum_{i_1, i_2, i_3} T_{i_1 i_2 i_3} T_{i_3 i_2 i_1} - \lambda \sum_{i_1, \ldots, i_6} T_{i_1 i_2 i_3} T_{i_3 i_5 i_4} T_{i_5 i_2 i_6} T_{i_4 i_6 i_1}, \tag{2.53}$$

where all the indices run from 1 to N. As we will see later on, the type of tensors one is working with has important implications. This part of the generalization from matrix models is also not obvious, and the strategy which was adopted in the early nineties was to symmetrize the tensor indices. Since one does not want to give them any physical meaning, it seems at first sight a reasonable assumption, but it is not the only way to fulfill the requirement, and is responsible for key difficulties of this approach. One major challenge is to control the overwhelmingly complicated sum over triangulations generated in perturbative expansion. Contrary to matrix models, not all of them are discretization of topological manifolds: quite differently, mild singular contributions such as pseudo-manifolds are included, but also highly degenerate triangulations with extended singularities (see for instance [56], in the context of GFTs). This comes from the fact that in higher than two dimensions, prescribing simple local gluing rules for d-simplices along their $(d-1)$-subsimplices is not restrictive enough to eliminate these pathological structures. In particular, and as we will explore in more details in the next section, the data encoded in these simple combinatorial models is not rich enough to capture the structure of the simplices of dimension strictly less than $(d-2)$. For instance, in the model defined by the action (2.53), no data is associated to the vertices of the triangulation, therefore the topological structure around the vertices of the triangulations is essentially arbitrary [56].

With hindsight, this lack of combinatorial structure is what prevented all the achievements of matrix models from being reproduced in higher dimensions. Without the necessary analytical control over the perturbative series, no $1/N$ expansion could be formulated for these early versions of tensor models, and therefore none of the other appeals of matrix models could be investigated either. The only part of the story which remained true was the interpretation of the amplitudes as discrete gravity path integrals, at least for triangulations of manifolds. Interpreting the elementary d-simplices as equilateral, a discrete metric can be assigned to each configuration, in the same way as in two dimensions. The amplitudes themselves can then be matched to the exponential of a discrete version of the Einstein-Hilbert action. Volume terms coming from the cosmological constant are again related to the coupling constant λ, and the curvature is captured by deficit angles around $(d-2)$-simplices. Such considerations gave birth to the Dynamical Triangulations program, and later on their Causal versions, where ensembles of discrete space-times are generated and summed over numerically rather than by analytical means. We refer to the lecture notes [57] for details and references.

This situation changed dramatically thanks to the pioneering work of Gurau, which upon slightly restricting the combinatorial structures of tensor models, could define a $1/N$ expansion. A wealth of results could be gathered after this breakthrough, giving new support in favor of analytical studies of random triangulations.

2.2.3 Bringing Discrete Geometry In

We can now explain how GFT comes about in this discrete approach to quantum gravity. Tensor models and dynamical triangulations are the minimalistic backbone of discrete gravity path integrals, in the sense that metric degrees of freedom are encoded in a purely combinatorial way. While this was fine enough in the case of two dimensional gravity, where only topology matters, the strategy can be questioned when it comes to three or four dimensions. In general relativity enter several important background structures. A topological manifold of the appropriate dimension is one such structure, and is arguably exhaustive in two dimensions. However, this topological manifold has to be endowed with a differential structure, which is not uniquely specified by the topology in four dimensions. Most importantly, local Lorentz invariance can also be argued to be a primary ingredient of any quantum theory of gravity. It is experimentally tested with an overwhelming precision, and seems to be rather hard to make emergent from a more fundamental theory. In this respect, getting the continuum of GR out of a fundamentally discrete model seems ambitious enough, so that it seems reasonable to save ourselves the burden of explaining Lorentz invariance as well.

In GFTs, a notion of local Lorentz invariance (or Euclidean invariance in Euclidean models) is assumed from scratch. To this effect, the purely combinatorial indices of tensor models become instead elements of some subgroup of the Lorentz or rotation group. For example in a 3d Euclidean context, one goes from the interaction part of (2.53) to (2.29) by turning the indices into $\mathrm{SU}(2)$ group elements. It is in this combinatorial sense that a GFT field can be considered a tensor. However, that is not all: after this new type of data has been introduce, one needs to provide them with a discrete geometric meaning. We simply follow the matrix/tensor models reasoning: if the GFT field φ is assumed to represent an elementary building block of geometry, then the geometric data should refer to this building block. Let us again use the Boulatov model as an example, in which case $\varphi(g_1, g_2, g_3)$ is to be interpreted as a flat triangle, and the variables g_i label its edges. It is the role of the constraint (2.33) to introduce an $\mathrm{SU}(2)$ flat discrete connection at the level of the amplitudes, encoded in the elementary line holonomies h_ℓ. The natural interpretation of the variable g_i is as the holonomy from a reference point inside the triangle, to the center of the edge i. Thanks to the flatness assumption, this holonomy is independent of the path one chooses to compute g_i. The meaning of the constraint (2.33) is also clear: it simply encodes the freedom in the choice of reference point. From the discrete geometric perspective, the Boulatov model can therefore naturally be called a *second quantization of a flat triangle*: the GFT field φ is the wave-function of a quantized flat triangle, and the path integral provides an interacting theory for such quantum geometric degrees of freedom. This point of view was already present in the early stages of SFMs [58, 59], and has guided the development of this research field ever since. It was more recently advocated in [27, 60], providing a new look at the construction of four-dimensional models [25, 26].

The natural question which can arise at this point is: which type of geometric data one should introduce? Or more specifically, why should we work with holonomy variables rather than simply edge vectors, as is done for example in Regge calculus [61]? As it turns out, there is a general correspondence between these two alternatives, which gives us the opportunity to introduce the Lie algebra formalism for GFTs, initially introduced in [27]. To avoid unnecessary complications, we illustrate this dual representation on the Boulatov model restricted to the SO(3) group. The technical tool allowing to use (non-commuting) Lie algebra variables $x_i \in \mathfrak{su}(2) \sim \mathbb{R}^3$, is the group Fourier transform [62–64]. In our case, it maps $L^2(\mathrm{SO}(3)^3)$ to a space $L^2_\star(\mathfrak{so}(3)^3)$, endowed with a non-commutative $\star$-product. The Fourier transform of $\varphi \in L^2(\mathrm{SO}(3)^3)$ is defined as:

$$\widehat{\varphi}(x_1, x_2, x_3) := \int [\mathrm{d}g_i]^3 \, \varphi(g_1, g_2, g_3) \, e_{g_1}(x_1) e_{g_2}(x_2) e_{g_3}(x_3), \tag{2.54}$$

where $e_g \colon \mathfrak{su}(2) \sim \mathbb{R}^3 \to \mathrm{U}(1)$ are non-commutative plane-waves, and functions on SO(3) are now identified with functions on SU(2) invariant under $g \to -g$. The definition of the plane-waves involves a choice of coordinates on the group. Following [27], we adopt:

$$\forall g \in \mathrm{SU}(2), \qquad e_g \colon \, x \mapsto e^{\mathrm{i Tr}(\mathrm{x}|\mathrm{g}|)} \tag{2.55}$$

where for $g \in \mathrm{SU}(2)$ we denote $|g| \equiv \mathrm{sign}(\mathrm{Tr}\, \mathrm{g})\mathrm{g}$, and Tr is the trace in the fundamental representation of SU(2). Note that other choices are possible, and some may be more convenient than others [65]. The Lie algebra variables can be given a simple metric interpretation, as vectors associated to the edges of the triangles [27]. One can therefore start from the Lie algebra representation, and provide an independent construction of the theory, following a similar procedure as in group space. The same combinatorial structure of the action can be assumed, entailing the same simplicial interpretation, except that the pointwise product for functions on SU(2) is replaced by the non-commutative and non-local $\star$-product. The latter is induced by the group structure of SU(2), as dual to the convolution product for functions on the group. Defined first on plane-waves:

$$(e_g \star e_{g'})(x) := e_{gg'}(x), \tag{2.56}$$

it is then extended to the image of the non-commutative Fourier transform, i.e. $L^2_\star(\mathfrak{so}(3)^3)$, by linearity. This formalism becomes particularly interesting when it comes to the geometric constraints. Indeed, if x_1, x_2, x_3 have to be interpreted as the edge vectors of a flat triangle, they should close, as for i

$$x_1 + x_2 + x_3 = 0. \tag{2.57}$$

This condition needs to be imposed at the operator level on the GFT field $\widehat{\varphi}$. A possible version of this projector is constructed out of a non-commutative notion of

δ-function. Defining

$$\delta_x(y) \equiv \int \mathrm{d}h\, e_{g^{-1}}(x) e_g(y), \tag{2.58}$$

it is easy to verify that

$$\int \mathrm{d}y\, (\delta_x \star f)(y) = \int \mathrm{d}y\, (f \star \delta_x)(y) = f(x), \tag{2.59}$$

and therefore δ_x plays the role of Dirac distribution at point x in $L^2_\star(\mathfrak{so}(3))$. We can therefore impose the following closure constraint on the GFT field:

$$\widehat{\varphi} = \widehat{C} \star \widehat{\varphi}, \tag{2.60}$$

with:

$$\widehat{C}(x_1, x_2, x_3) \equiv \delta_0(x_1 + x_2 + x_3). \tag{2.61}$$

It is then easy to check that transforming back (2.60) to $L^2(\mathrm{SO}(3)^3)$ gives back the gauge invariance condition (2.33). This confirms that the Boulatov model can be understood as a second quantization of a flat triangle, and we refer to [27] for more details.

In four dimensions, the same correspondence between group and Lie algebra representation has been put to profit [25, 26]. There, the GFT field represents a quantum tetrahedron, and Lie algebra elements correspond to bivectors associated to its boundary triangles. In addition to the closure constraint (again equivalent to the Gauss constraint in group space), additional geometricity conditions have to be imposed to guarantee that the bivectors are built from edge vectors of a geometric tetrahedron. These additional constraints are nothing by the simplicity constraints, and non-commutative δ-functions can again be used to implement them.

2.3 A Research Direction

Let us summarize the two possible takes on GFTs we have been presenting so far. We first focused on quantization procedures, either in canonical or covariant form, applied to Einstein's field equations. The key steps entering this line of thoughts were: (a) adopt connection and flux rather than metric variables at the classical level; (b) thanks to canonical LQG techniques, show that spin network functionals are good kinematical states for quantum GR; (c) resort to a semi-heuristic covariant formulation, spin foams, to identify the dynamics of these states; (d) introduce GFTs as generating functionals for spin foam amplitudes. Already from this point of view, we could see non-trivial combinatorial assumptions entering GFT models, as well as unorthodox quantization rules motivating spin foam models for four-dimensional gravity. Actually, discrete geometric considerations rather than strict quantization procedures are arguably at the root of most of the modern spin foam models. As a result, the combinatorics should play an important role, but seems essentially unconstrained by these procedures: one usually works with simplicial complexes because it seems like a natural starting point. However, there is no strong case for it, even less from the GFT point of view: as in any quantum field theory, it would be preferable to have a large set of interactions at our disposal, determined by a symmetry principle. The second set of works we focused on, matrix and tensor models, put on the contrary most of their emphasis on the combinatorics. GFTs in this perspective are simply enriched tensor models, where tensor indices carry geometrical information. The conceptual gap between 2d and 4d gravity supports the idea that such additional data might be needed in 4d, contrary to 2d where purely combinatorial models are able to capture the very limited metric aspects of gravity.

As for example put forward in [60], we would like to use these two sets of incomplete motivations for GFTs as a way to find new research directions. If we allow ourself to make a simple synthesis between the two, we could say that GFTs are quantum field theories of discrete geometries, in which boundary states are LQG like, namely a subset of spin network functionals (for the appropriate group). The key questions to ask can then be split into two classes. The first concerns model building, in the sense of finding the appropriate notion of quantum discrete geometry, both in the boundary and in the bulk. The second concerns more generic features of the formalism, such as the combinatorics, symmetry principles, regularization and renormalization. It should be noted that these two sets of questions are rather independent and should therefore better be explored in parallel. In this thesis, we are only concerned with the generic aspects of GFTs, and therefore we will mostly work in situations where discrete geometric aspects are either irrelevant, or well-controlled and unambiguous. More precisely, we would first like to understand to which extent GFTs can be defined rigorously as perturbative quantum field theories. And second, one would like to develop tools to explore the large triangulations/foams regime of GFTs. Whatever the position one adopts as regards discreteness in GFTs and SFMs, such regimes exist and therefore deserve to be studied. This is especially true in the current situation, in which the connection between the most refined models and continuum GR remain rather elusive. Whether this connection is to show up at the perturbative level already seems rather unlikely in the discrete gravity interpretation of these models, and if any such connection exists one would rather expect an emergence scenario [66]. In such a mindset, the large triangulations regime is the physically relevant one. These two open problems will be addressed by two complementary means: large N methods similar to the $1/N$ expansion of matrix models; and renormalizability studies of enriched GFT models, called Tensorial Group Field Theories (TGFTs). The latter will be established as the first well-behaved perturbative quantum field theories related to spin foam models, which is in our opinion the strongest result of this thesis. Key to these two series of results are new combinatorial tools which recently revived tensor models. In the next section, we therefore give some motivations for introducing them, and a detailed account of the combinatorial backbone provided by these new improved tensor models.

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