

Chapter 2

Quadratic Robust Filter Design

Filtering is one of the fundamental issues in the areas of control, circuits and systems, and signal processing. Different filtering methods rely on different assumptions of the noises (including the process noise and the measurement noise) and the corresponding suitable performance index of the filtering error. For H_2 filtering, the noise is assumed to be Gaussian white noise processes with identity power spectrum density matrix, and the objective is to minimize the H_2 norm of the transfer function from the noise to the filtering error. For H_∞ filtering, the noise is assumed to have bounded energy and the objective is to minimize the H_∞ norm of the transfer function from the noise to the filtering error. For energy-to-peak filtering, it is self-explanatory that the noise is still assumed to be energy-bounded but the objective turns to minimizing the energy-to-peak gain of transfer function from the noise to the filtering error.

In this chapter, we present quadratic approaches to robust filtering for polytopic uncertain systems, and some common filtering schemes as mentioned, including H_2 filtering, H_∞ filtering, and energy-to-peak filtering, will be taken into consideration. The so-called *quadratic approach* means that for an uncertain system, design a filter such that the stability and the prescribed disturbance attenuation level of the filtering error system are guaranteed by a *single* quadratic Lyapunov function, the underlying idea of which is the generalization of the notion of quadratic stability.

Filtering for nominal systems will be addressed first and the main attention will be paid to linearizing the NLMI conditions for filtering performance analysis into LMI for the state-space realization of filters. Then, based on the notion of quadratic stability, the filter design methods for nominal systems will be extended to the uncertain case, that is, quadratic approaches to robust filter design. All the filter design methods, for both uncertain continuous- and discrete-time systems, are to be presented in terms of solving the corresponding optimization problem subject to LMI constraints. The effectiveness of the presented filter design methods will be demonstrated by several illustrative examples.

2.1 Quadratic Robust H_2 Filter Design

In this section, we consider the robust H_2 filtering problem in the framework of quadratic approaches. For the nominal systems, the results to be presented are classical and standard. Moreover, it is quite natural to extend the filter design methods in terms of LMI from the nominal case to the polytopic uncertainty by the quadratic stability notion.

2.1.1 Problem Formulation

Consider a stable, uncertain linear time-variant (LTI) system \mathcal{S} described by the following state-space model:

$$\begin{aligned}\mathcal{S} : \delta[x(t)] &= A(\lambda)x(t) + B(\lambda)w(t), \\ y(t) &= C(\lambda)x(t) + D(\lambda)w(t), \\ z(t) &= L(\lambda)x(t),\end{aligned}\tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^m$ is the measured output, $z(t) \in \mathbb{R}^p$ is the signal to be estimated, $w(t) \in \mathbb{R}^l$ is a zero-mean white noise with identity power spectrum density matrix. $\delta[\cdot]$ denotes the shift operator for discrete-time systems and the derivative operator for continuous-time systems, respectively. $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$, and $L(\lambda)$ are appropriately dimensioned real matrices. It is assumed that

$$M(\lambda) \triangleq (A(\lambda), B(\lambda), C(\lambda), D(\lambda), L(\lambda)) \in \mathcal{M} \tag{2.2}$$

where \mathcal{M} is a given convex polyhedral domain bounded by s vertices

$$\mathcal{M} \triangleq \left\{ M(\lambda) : M(\lambda) = \sum_{i=1}^s \lambda_i M_i; \lambda \in \Gamma \right\}$$

with $M_i \triangleq (A_i, B_i, C_i, D_i, L_i)$ denoting the vertices of the polytope, and Γ denoting the unit simplex

$$\Gamma \triangleq \left\{ (\lambda_1, \lambda_2, \dots, \lambda_s) : \sum_{i=1}^s \lambda_i = 1, \lambda_i \geq 0 \right\}. \tag{2.3}$$

Equation (2.2) indicates that system matrices can be expressed as the convex combination of several vertex matrices.

For system \mathcal{S} in (2.1), the filtering problem can be roughly thought as using the measurable signal $y(t)$ to construct an estimate $z_F(t)$ for the unmeasurable signal $z(t)$. Hence, the filter, denoted by \mathcal{F} , actually is an operator from $y(t)$ to $z_F(t)$, that

is, $z_F = \mathcal{F}[y]$. In this book, consider that the filter \mathcal{F} is a full-order LTI operator given by the following minimum state-space realization:

$$\begin{aligned}\mathcal{F} : \delta[x_F(t)] &= A_F x_F(t) + B_F y(t), \\ z_F(t) &= C_F x_F(t),\end{aligned}\tag{2.4}$$

where $x_F(t) \in \mathbb{R}^n$ is the filter state vector, and A_F , B_F , and C_F are appropriately dimensioned filter matrices to be determined.

Define the filtering error signal $e(t) \triangleq z(t) - z_F(t)$ and the augmented state vector $\xi(t) = [x(t)^T, x_F(t)^T]^T$. By connecting the filter \mathcal{F} to the system \mathcal{S} , the filtering error system \mathcal{E} is given by the following augmented state space model:

$$\begin{aligned}\mathcal{E} : \delta[\xi(t)] &= \bar{A}(\lambda)\xi(t) + \bar{B}(\lambda)w(t), \\ e(t) &= \bar{C}(\lambda)\xi(t)\end{aligned}\tag{2.5}$$

with

$$\bar{A}(\lambda) \triangleq \begin{bmatrix} A(\lambda) & 0 \\ B_F C(\lambda) & A_F \end{bmatrix}, \quad \bar{B}(\lambda) \triangleq \begin{bmatrix} B(\lambda) \\ B_F D(\lambda) \end{bmatrix}, \quad \bar{C}(\lambda) \triangleq [L(\lambda) - C_F].$$

For any fixed $\lambda \in \Gamma$, the transfer function of the filtering error system from the noise input $w(t)$ to the filtering error $e(t)$ is defined by

$$T(\delta, \lambda) \triangleq \bar{C}(\lambda) [\delta I - \bar{A}(\lambda)]^{-1} \bar{B}(\lambda).\tag{2.6}$$

With slight abuse of notations, keep in mind that, in the frequency domain, the operator δ can be seen as, respectively, the Laplacian operator s for the continuous-time case and the shift operator z for the discrete-time case.

It is assumed that the initial condition $x(0)$ of system \mathcal{S} in (2.1) is a zero-mean random signal uncorrelated with the input noise $w(t)$ for all $t \geq 0$, while the filter \mathcal{F} in (2.4) has zero initial condition $x_F(0) = 0$. The robust H_2 filtering problem to be addressed for system \mathcal{S} is formulated as follows.

Robust H_2 Filtering Problem: Given system \mathcal{S} in (2.1), design a filter \mathcal{F} of the form in (2.4) such that the filtering error system \mathcal{E} in (2.5), for all $\lambda \in \Gamma$, is robustly asymptotically stable, and for all zero-mean white noise $w(t)$ with identity power spectrum density matrix, satisfies

$$\sup_{\lambda \in \Gamma} \mathbf{E} [e(t)^T e(t)] < \gamma \left(\text{i.e., } \sup_{\lambda \in \Gamma} \|T(\delta, \lambda)\|_2^2 < \gamma \right),\tag{2.7}$$

where $\mathbf{E}[\cdot]$ denotes mathematical expectation of the random signal and γ is a given positive constant. Moreover, a filter satisfying the above conditions is referred to as a filter with a guaranteed robust H_2 performance bound $\sqrt{\gamma}$.

2.1.2 H_2 Filtering for Nominal Systems: Continuous-Time

First, we consider the H_2 filtering problem for nominal systems, i.e., the uncertain parameter λ is arbitrary but fixed in the unit simplex Γ .

For nominal systems, necessary and sufficient conditions in terms of LMI have been well-established for the H_2 performance, see [1–3]. In the continuous-time context, the H_2 performance criterion is given in the following lemma, which will be used for H_2 filter design.

Lemma 2.1 ([1, 3]) *Consider the continuous-time system \mathcal{S} in (2.1) and assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given filter \mathcal{F} in (2.4) and a scalar $\gamma > 0$, the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies $\mathbf{E}[e(t)^T e(t)] < \gamma$ if and only if the following matrix inequalities*

$$\text{Tr}[Z_c] < \gamma, \quad (2.8)$$

$$\begin{bmatrix} P_c & \bar{C}^T \\ \bar{C} & Z_c \end{bmatrix} > 0, \quad (2.9)$$

$$\begin{bmatrix} \bar{A}^T P_c + P_c \bar{A} & P_c \bar{B} \\ \bar{B}^T P_c & -I \end{bmatrix} < 0 \quad (2.10)$$

are feasible in the real matrix variables P_c and Z_c .

It is well-known that as $t \rightarrow +\infty$, the filtering error variance

$$\mathbf{E}[e(t)^T e(t)] = \text{Tr}[\bar{C} P \bar{C}^T],$$

where P is the solution to the Lyapunov equation

$$\bar{A} P + P \bar{A}^T + \bar{B} \bar{B}^T = 0.$$

Note that if system \mathcal{M} is stable, a unique solution exists for this equation. To briefly justify Lemma 2.1, pre- and postmultiplying the both hand sides of (2.10) by

$$\begin{bmatrix} P_c^{-1} & 0 \\ 0 & I \end{bmatrix} \text{ result in}$$

$$\begin{bmatrix} P_c^{-1} \bar{A}^T + \bar{A} P_c^{-1} & \bar{B} \\ \bar{B}^T & -I \end{bmatrix} < 0,$$

which, by the Schur complement, is equivalent to

$$\bar{A} P_c^{-1} + P_c^{-1} \bar{A}^T + \bar{B} \bar{B}^T < 0.$$

Due to the monotonicity property of the solution to the Lyapunov equation, it follows that $P < P_c^{-1}$, which together with (2.8) and (2.9) gives

$$\mathbf{E}[e(t)^T e(t)] = \text{Tr}[\bar{C} P \bar{C}^T] < \text{Tr}[\bar{C} P_c^{-1} \bar{C}^T] < \text{Tr}[Z_c] < \gamma.$$

Remark 2.1 In [4, 5], the LMI constraints corresponding to (2.9) and (2.10) are given, respectively, by

$$\begin{bmatrix} \bar{P} & \bar{P} \bar{C}^T \\ \bar{C} \bar{P} & Z \end{bmatrix} > 0, \begin{bmatrix} \bar{P} \bar{A}^T + \bar{A} \bar{P} & \bar{B} \\ \bar{B}^T & -I \end{bmatrix} < 0.$$

Simply matrix manipulations can verify that these two inequalities are equivalent to (2.9) and (2.10), respectively. Indeed, if letting $P_c \triangleq \bar{P}^{-1}$, it is readily obtained that

$$\begin{bmatrix} P_c & \bar{C}^T \\ \bar{C} & Z \end{bmatrix} = \begin{bmatrix} \bar{P}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{P} & \bar{P} \bar{C}^T \\ \bar{C} \bar{P} & Z \end{bmatrix} \begin{bmatrix} \bar{P}^{-1} & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} \bar{A}^T P_c + P_c \bar{A} & P_c \bar{B} \\ \bar{B}^T P_c & -I \end{bmatrix} = \begin{bmatrix} \bar{P}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{P} \bar{A}^T + \bar{A} \bar{P} & \bar{B} \\ \bar{B}^T & -I \end{bmatrix} \begin{bmatrix} \bar{P}^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

From these two formulations, it is found that the product terms between Lyapunov matrix P_c and system matrices in the latter formulation appear in both of the LMIs while similar terms in Lemma 2.1 exist only in (2.10). This feature is important for robustness analysis of the filtering error system, which will be further elucidated in the next chapter regarding the parameter-dependent approach to robust H_2 filter design. However, for nominal systems, it does not matter which formulation is employed for H_2 performance analysis. For consistency, this book adopts the one in Lemma 2.1.

When a filter is known, the above matrix inequality constraints are LMIs in the variables P_c , Z_c , and γ . Thus, the optimal H_2 filtering performance level of the given filter can be obtained by minimizing γ subject to (2.8)–(2.10).

When the state-space realization of a filter is unknown, (2.10) is not an LMI with respect to the filter matrices A_F , B_F , and C_F . Hence, Lemma 2.1 does not give an explicit solution to the realization of an H_2 filter. The possibility to handle H_2 filter design with Lemma 2.1 in light of the LMI technique relies on a fact that the inequality in (2.10) can be converted into an LMI by a reversible matrix transformation. In what follows, the main purpose of this subsection is to convert the inequality in (2.10) into an LMI such that the problem of H_2 filter design is cast into a convex optimization problem that can be efficiently solved by mature numerical algorithms.

To this end, partition matrix P_c into four blocks as the following form

$$P_c = \begin{bmatrix} P_{c1} & P_{c2} \\ P_{c2}^T & P_{c3} \end{bmatrix}$$

with each block being an $n \times n$ matrix. Due to the fact that P_c is nonsingular, it can be assumed that P_{c2} is nonsingular by invoking a small perturbation if necessary. Thus, this assumption does not lose generality.

Define multiplier matrix

$$J_{c0} \triangleq \begin{bmatrix} I & 0 \\ 0 & P_{c3}^{-1} P_{c2}^T \end{bmatrix}.$$

Introduce the following matrix transformations

$$\tilde{A}_F \triangleq P_{c2} A_F P_{c3}^{-1} P_{c2}^T, \quad (2.11)$$

$$\tilde{B}_F \triangleq P_{c2} B_F, \quad (2.12)$$

$$\tilde{C}_F \triangleq C_F P_{c3}^{-1} P_{c2}^T, \quad (2.13)$$

$$\begin{bmatrix} \tilde{P}_{c1} & \tilde{P}_{c2} \\ \tilde{P}_{c2} & \tilde{P}_{c2} \end{bmatrix} \triangleq J_{c0}^T P_c J_{c0} = \begin{bmatrix} P_{c1} & P_{c2} P_{c3}^{-1} P_{c2}^T \\ P_{c2} P_{c3}^{-1} P_{c2}^T & P_{c2} P_{c3}^{-1} P_{c2}^T \end{bmatrix}, \quad (2.14)$$

from which, the following relations can be readily obtained:

$$\begin{aligned} J_{c0}^T P_c \bar{A} J_{c0} &= \begin{bmatrix} \tilde{P}_{c1} A + \tilde{B}_F C & \tilde{A}_F \\ \tilde{P}_{c2} A + \tilde{B}_F C & \tilde{A}_F \end{bmatrix} \triangleq \begin{bmatrix} X_{c1} & \tilde{A}_F \\ X_{c2} & \tilde{A}_F \end{bmatrix}, \\ J_{c0}^T P_c \bar{B} &= \begin{bmatrix} \tilde{P}_{c1} B + \tilde{B}_F D \\ \tilde{P}_{c2} B + \tilde{B}_F D \end{bmatrix}, \\ \bar{C} J_{c0} &= [L \quad -\tilde{C}_F]. \end{aligned}$$

Now, by virtue of the transformations defined in (2.11)–(2.14) and further noting the following transformations

$$\begin{aligned} J_{c1}^T \begin{bmatrix} P_c & \bar{C}^T \\ \bar{C} & Z_c \end{bmatrix} J_{c1} &= \begin{bmatrix} \tilde{P}_{c1} & \tilde{P}_{c2} & L^T \\ \tilde{P}_{c2} & \tilde{P}_{c2} & -\tilde{C}_F^T \\ L & -\tilde{C}_F & Z_c \end{bmatrix}, \\ J_{c2}^T \begin{bmatrix} \bar{A}^T P_c + P_c \bar{A} & P_c \bar{B} \\ \bar{B}^T P_c & -I \end{bmatrix} J_{c2} &= \begin{bmatrix} X_{c1} + X_{c1}^T & \tilde{A}_F + X_{c2}^T & \tilde{P}_{c1} B + \tilde{B}_F D \\ * & \tilde{A}_F + \tilde{A}_F^T & \tilde{P}_{c2} B + \tilde{B}_F D \\ * & * & -I \end{bmatrix} \end{aligned}$$

with $J_{c1} \triangleq \text{diag}\{J_{c0}, I_n\}$ and $J_{c2} \triangleq \text{diag}\{J_{c0}, I_l\}$, it can be shown that the NLMIs in (2.8)–(2.10) with respect to variables

$$P_{c1}, P_{c2}, P_{c3}, Z_c, A_F, B_F, \text{ and } C_F$$

are converted into LMIs with respect to new variables

$$\tilde{P}_{c1}, \tilde{P}_{c2}, Z_c, \tilde{A}_F, \tilde{B}_F, \text{ and } \tilde{C}_F.$$

More importantly, since matrix J_{c0} is invertible, the above transformations are reversible; that is, the filter matrices can be obtained from the feasible solution to the transformed LMI constraints.

Now, it is in a position to present the first method for H_2 filter design for nominal systems in the continuous-time case.

Theorem 2.1 *Consider the continuous-time system \mathcal{S} in (2.1) and assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.4) exists such that the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies $\mathbf{E}[e(t)^T e(t)] < \gamma$ if and only if the inequalities in (2.8) and in the following*

$$\begin{bmatrix} \tilde{P}_{c1} & \tilde{P}_{c2} & L^T \\ \tilde{P}_{c2} & \tilde{P}_{c2} & -\tilde{C}_F^T \\ L & -\tilde{C}_F & Z_c \end{bmatrix} > 0, \quad (2.15)$$

$$\begin{bmatrix} X_{c1} + X_{c1}^T & \tilde{A}_F + X_{c2}^T \tilde{P}_{c1} B + \tilde{B}_F D \\ * & \tilde{A}_F + \tilde{A}_F^T \tilde{P}_{c2} B + \tilde{B}_F D \\ * & * & -I \end{bmatrix} < 0 \quad (2.16)$$

are feasible in the real matrix variables \tilde{P}_{c1} , \tilde{P}_{c2} , Z_c , \tilde{A}_F , \tilde{B}_F , and \tilde{C}_F , where

$$X_{c1} \triangleq \tilde{P}_{c1} A + \tilde{B}_F C, \quad X_{c2} \triangleq \tilde{P}_{c2} A + \tilde{B}_F C.$$

Moreover, if these conditions are feasible, an admissible state-space realization of the filter \mathcal{F} in (2.4) is given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} \tilde{P}_{c2}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & 0 \end{bmatrix} \quad (2.17)$$

or

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & 0 \end{bmatrix} \begin{bmatrix} \tilde{P}_{c2}^{-1} & 0 \\ 0 & I \end{bmatrix}. \quad (2.18)$$

Proof By combining the above discussion and Lemma 2.1, the necessity of the conditions in (2.15) and (2.16) can be obtained.

The sufficiency of the conditions in (2.15) and (2.16) follows from the fact that the matrix transformations embedded in the relation between Lemma 2.1 and Theorem 2.1 are reversible. To elucidate this more clearly, note that from (2.15), \tilde{P}_{c2} is nonsingular. Thus, it is always possible to find two matrices P_{c2} and P_{c3} such that $\tilde{P}_{c2} = P_{c2} P_{c3}^{-1} P_{c2}^T$ holds with both P_{c2} and P_{c3} invertible. Then, the following definitions are meaningful

$$\begin{aligned} \begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} &\triangleq \begin{bmatrix} P_{c2}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & 0 \end{bmatrix} \begin{bmatrix} P_{c2}^{-T} P_{c3} & 0 \\ 0 & I \end{bmatrix}, \\ P_c &\triangleq J_{c0}^{-T} \begin{bmatrix} \tilde{P}_{c1} & \tilde{P}_{c2} \\ \tilde{P}_{c2} & \tilde{P}_{c2} \end{bmatrix} J_{c0}^{-1}, \end{aligned} \quad (2.19)$$

which are just the reversed ones in (2.11)–(2.14). With these definitions, the conditions in (2.15) and (2.16) can be rewritten as

$$\begin{aligned} J_{c1}^T \begin{bmatrix} P_c & \bar{C}^T \\ \bar{C} & Z_c \end{bmatrix} J_{c1} &< 0, \\ J_{c2}^T \begin{bmatrix} \bar{A}^T P_c + P_c \bar{A} & P_c \bar{B} \\ \bar{B}^T P_c & -I \end{bmatrix} J_{c2} &< 0. \end{aligned}$$

Due to the nonsingularity of J_{c1} and J_{c2} , the above two conditions are equivalent to (2.9) and (2.10), respectively. Hence, (2.15) and (2.16) are sufficient conditions for (2.9) and (2.10), respectively. Consequently, (2.8), (2.15), and (2.16) are also necessary and sufficient conditions for the existence of a filter with a guaranteed H_2 performance level bound $\sqrt{\gamma}$.

From the expressions of A_F , B_F , and C_F in (2.19), the transfer function of the filter satisfies

$$\begin{aligned} T_F(\delta) &= C_F (\delta I - A_F)^{-1} B_F \\ &= \left(\tilde{C}_F P_{c2}^{-T} P_{c3} \right) \left(\delta I - P_{c2}^{-1} \tilde{A}_F P_{c2}^{-T} P_{c3} \right)^{-1} \left(P_{c2}^{-1} \tilde{B}_F \right) \\ &= \tilde{C}_F \left(P_{c2}^{-T} P_{c3} \right) \left(\delta I - P_{c2}^{-1} \tilde{A}_F P_{c2}^{-T} P_{c3} \right)^{-1} \left(P_{c2}^{-T} P_{c3} \right)^{-1} \left(\tilde{P}_{c2}^{-1} \tilde{B}_F \right) \\ &= \tilde{C}_F \left(\delta I - \tilde{P}_{c2}^{-1} \tilde{A}_F \right)^{-1} \left(\tilde{P}_{c2}^{-1} \tilde{B}_F \right) \end{aligned}$$

and

$$\begin{aligned} T_F(\delta) &= \left(\tilde{C}_F P_{c2}^{-T} P_{c3} \right) \left(\delta I - P_{c2}^{-1} \tilde{A}_F P_{c2}^{-T} P_{c3} \right)^{-1} \left(P_{c2}^{-1} \tilde{B}_F \right) \\ &= \left(\tilde{C}_F P_{c2}^{-T} P_{c3} \right) \left(P_{c2}^{-1} P_{c2} \right) \left(\delta I - P_{c2}^{-1} \tilde{A}_F P_{c2}^{-T} P_{c3} \right)^{-1} P_{c2}^{-1} \tilde{B}_F \\ &= \left(\tilde{C}_F \tilde{P}_{c2}^{-1} \right) \left(\delta I - \tilde{A}_F \tilde{P}_{c2}^{-1} \right)^{-1} \tilde{B}_F, \end{aligned}$$

which show that the transfer function is independent of the choice of P_{c2} and P_{c3} and the matrices in Eqs. (2.17) and (2.18) both are admissible realizations for the filter \mathcal{F} . The entire proof is completed. \square

Remark 2.2 Given system \mathcal{S} and filter \mathcal{F} , Theorem 2.1 can also be applied to filter performance analysis. An interesting fact is that the conditions in Theorem 2.1 involve fewer variables than those in Lemma 2.1. However, as is proven, Theorem 2.1 and Lemma 2.1 actually are equivalent to each other and their equivalence can be established by some reversible matrix transformations in (2.11)–(2.14). This point implies that some variables in Lemma 2.1 are redundant. The procedures in the proof of sufficiency show that the additional variables in Lemma 2.1 are introduced when \tilde{P}_{c2} is decomposed as $\tilde{P}_{c2} = P_{c2} P_{c3}^{-1} P_{c2}^T$. Indeed, the proof of sufficiency can be

completed by a simpler setting as $\tilde{P}_{c2} = P_{c2} = P_{c3}$. This observation reveals that matrix P_c in Lemma 2.1 can be set as

$$P_c = \begin{bmatrix} P_{c1} & P_{c2} \\ P_{c2} & P_{c2} \end{bmatrix}$$

without any conservatism introduced. Moreover, based on this interesting observation, the linearization procedure in the proof of Theorem 2.1 can be simplified.

By Theorem 2.1, the H_2 filter design problem for continuous-time nominal systems has been cast into a feasibility problem with LMI constraints. Thus, the H_2 filter design method based on Theorem 2.1 can be readily implemented. Moreover, γ is also a linear variable in (2.8), which can be directly minimized by virtue of the LMI tool. The optimal H_2 filter design based on Theorem 2.1 is summarized in Algorithm 1.

Algorithm 1 H_2 Filter Design I (Continuous-Time)

- Solve the minimization problem

$$\min_{\tilde{P}_{c1}, \tilde{P}_{c2}, Z_c, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \gamma} \gamma \quad \text{s.t. (2.8), (2.15), and (2.16)}$$

- Compute filter \mathcal{F} in (2.4) by (2.17) or (2.18).
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Revisiting the conditions in Theorem 2.1, it is found that the matrix variable \tilde{C}_F appears only in the LMI (2.15); thus, by utilizing Lemma 1.3, this variable can be eliminated from the condition such that the total number of variables involved in Algorithm 1 can be further reduced.

To verify this point, by using the Schur complement, the LMI in (2.15) can be rewritten in the equivalent form

$$\begin{bmatrix} \tilde{P}_{c2} - \tilde{P}_{c2}\tilde{P}_{c1}^{-1}\tilde{P}_{c2} & -\tilde{C}_F^T - \tilde{P}_{c2}\tilde{P}_{c1}^{-1}L^T \\ -\tilde{C}_F - L\tilde{P}_{c1}^{-1}\tilde{P}_{c2} & Z_c - L\tilde{P}_{c1}^{-1}L^T \end{bmatrix} > 0,$$

which, by applying Lemma 1.3, is further equivalent to the following pair of inequalities

$$\tilde{P}_{c2} - \tilde{P}_{c2}\tilde{P}_{c1}^{-1}\tilde{P}_{c2} > 0, \quad Z_c - L\tilde{P}_{c1}^{-1}L^T > 0$$

together with $\tilde{C}_F = -L\tilde{P}_{c1}^{-1}\tilde{P}_{c2}$. Again using the Schur complement, the pair of inequalities can be written in the LMI form

$$\begin{bmatrix} \tilde{P}_{c1} & \tilde{P}_{c2} \\ \tilde{P}_{c2} & \tilde{P}_{c2} \end{bmatrix} > 0, \quad \begin{bmatrix} \tilde{P}_{c1} & L^T \\ L & Z_c \end{bmatrix} > 0. \quad (2.20)$$

Furthermore, for any $n \times n$ invertible matrix P_{c2} , \tilde{P}_{c2} can always be decomposed as $\tilde{P}_{c2} = P_{c2}P_{c3}^{-1}P_{c2}^T$ with P_{c3} also being an invertible matrix. Based on the matrix transformations in (2.13) and (2.14), the above variable elimination procedure yields

$$\begin{aligned} C_F &= \tilde{C}_F P_{c2}^{-T} P_{c3} = -L \tilde{P}_{c1}^{-1} \tilde{P}_{c2} P_{c2}^{-T} P_{c3} \\ &= -L \tilde{P}_{c1}^{-1} P_{c2} P_{c3}^{-1} P_{c2}^T P_{c2}^{-T} P_{c3} \\ &= -L \tilde{P}_{c1}^{-1} P_{c2}. \end{aligned}$$

Due to the arbitrariness of P_{c2} , it can be chosen as $P_{c2} = -\tilde{P}_{c1}$ with no conservatism introduced, which results in $C_F = L$. This imposition obviously further reduces the number of free variables in the H_2 filter design method.

According to the above discussion, we present another method for H_2 filter design for continuous-time nominal systems.

Theorem 2.2 *Consider the continuous-time system \mathcal{S} in (2.1) and assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.4) exists such that the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies $\mathbf{E}[e(t)^T e(t)] < \gamma$ if and only if the inequalities in (2.8), (2.16), and (2.20) are feasible in the real matrix variables \tilde{P}_{c1} , \tilde{P}_{c2} , Z_c , \tilde{A}_F , and \tilde{B}_F . Moreover, if these conditions are feasible, an admissible state-space realization of the filter \mathcal{F} in (2.4) is given by*

$$A_F = \tilde{P}_{c1}^{-1} \tilde{A}_F \tilde{P}_{c2}^{-1} \tilde{P}_{c1}, \quad B_F = -\tilde{P}_{c1}^{-1} \tilde{B}_F, \quad C_F = L. \quad (2.21)$$

Proof The sufficiency and necessity of the conditions can be proven according to the above discussion. Now we demonstrate the admissible realization of A_F and B_F in (2.21). From $P_{c2} = -\tilde{P}_{c1}$, $\tilde{P}_{c2} = P_{c2}P_{c3}^{-1}P_{c2}^T$ and the expression of A_F and B_F in (2.19), it follows that

$$\begin{aligned} A_F &= P_{c2}^{-1} \tilde{A}_F (P_{c3}^{-1} P_{c2}^T)^{-1} \\ &= -\tilde{P}_{c1}^{-1} \tilde{A}_F (P_{c2}^{-1} P_{c2} P_{c3}^{-1} P_{c2}^T)^{-1} \\ &= \tilde{P}_{c1}^{-1} \tilde{A}_F \tilde{P}_{c2}^{-1} \tilde{P}_{c1}, \\ B_F &= P_{c2}^{-1} \tilde{B}_F = -\tilde{P}_{c1}^{-1} \tilde{B}_F. \end{aligned} \quad \square$$

Based on Theorem 2.2, the second method for the optimal H_2 filter design is summarized in Algorithm 2.

Remark 2.3 As shown in Theorem 2.2, to design H_2 filters, the filter matrix C_F can be prescribed as $C_F = L$, which does not bring about conservatism. The basis of this point lies in the relationship between H_2 filtering and Kalman filtering. For nominal systems, it is well known that the optimal H_2 filter and the optimal Kalman filter have the same filtering error variance level bound [6]. Note that $C_F = L$

Algorithm 2 H_2 Filter Design II (Continuous-Time)

- Solve the minimization problem

$$\min_{\tilde{P}_{c1}, \tilde{P}_{c2}, Z_c, \tilde{A}_F, \tilde{B}_F, \gamma} \gamma \quad \text{s.t. (2.8), (2.16), and (2.20)}$$

- Compute filter \mathcal{F} in (2.4) by (2.21).

is also the filter gain for the Kalman filter. Thus, the fact that C_F takes L for the optimal H_2 filter is quite natural. Indeed, the result of $C_F = L$ has already been pointed out in [4, 6]. Of course, it should be emphasized that the main advantage of Theorems 2.1 and 2.2 is not to handle precisely known systems or nominal systems, because the optimal H_2 filtering for nominal systems can be designed by the Kalman filtering method, which is much simpler than Theorem 2.1 and 2.2 (see Sect. 2.1.3). The main advantage of Theorem 2.1, by virtue of the quadratic stability notion, is its extendability to polytopic uncertain systems bounded in a convex polyhedral domain \mathcal{M} (see Sect. 2.1.6).

2.1.3 Connection to the Kalman filtering: Continuous-Time

As mentioned in Remark 2.3, the optimal H_2 filter and the Kalman filter achieve the same variance level of the estimation error for nominal systems. In this section, we provide some theoretical analysis to relate the H_2 filter design method in Sect. 2.1.2 and the continuous-time Kalman filtering method.

To design the Kalman filter for the system \mathcal{S} in (2.1) as in [6], we assume

$$BD^T = 0, \quad DD^T = I.$$

This assumption implicates that the noise $w(t)$ has two parts of remarkably different meanings: the first one acts as the real process noise through matrix B , the second one as the measurement noise through matrix D , and the two parts are independent of each other. The Kalman filter for the system \mathcal{S} with nominal parameters in the continuous-time case has the form of filter \mathcal{F} in (2.4) with state space matrices given by

$$A_F = A - PC^T C, \quad B_F = PC^T, \quad C_F = L \quad (2.22)$$

where matrix P is the solution to the following Riccati equation:

$$AP + PA^T - PC^T C P + BB^T = 0. \quad (2.23)$$

To relate the Kalman filter realization in (2.22) with the H_2 filter realization in (2.21), rewrite the LMI in (2.16) into

$$\Phi + X^T \tilde{A}_F Z + Z^T \tilde{A}_F^T X < 0 \quad (2.24)$$

where

$$\Phi = \begin{bmatrix} X_{c1} + X_{c1}^T & X_{c2}^T \tilde{P}_{c1} B + \tilde{B}_F D \\ * & 0 \\ * & \tilde{P}_{c2} B + \tilde{B}_F D \\ * & * & -I \end{bmatrix},$$

$$X = [I \ I \ 0], Z = [0 \ I \ 0].$$

Choose \mathcal{N}_X and \mathcal{N}_Z for the nullspaces of X and Z , respectively, as

$$\mathcal{N}_X = \begin{bmatrix} I & 0 \\ -I & 0 \\ 0 & I \end{bmatrix}, \mathcal{N}_Z = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}.$$

Then by applying the Projection Lemma, the inequality in (2.24) is equivalent to

$$\mathcal{N}_X^T \Phi \mathcal{N}_X = \begin{bmatrix} A^T(\tilde{P}_{c1} - \tilde{P}_{c2}) + (\tilde{P}_{c1} - \tilde{P}_{c2})A & (\tilde{P}_{c1} - \tilde{P}_{c2})B \\ * & -I \end{bmatrix} < 0, \quad (2.25)$$

$$\mathcal{N}_Z^T \Phi \mathcal{N}_Z = \begin{bmatrix} X_{c1} + X_{c1}^T & \tilde{P}_{c1} B + \tilde{B}_F D \\ * & -I \end{bmatrix} < 0. \quad (2.26)$$

Since system \mathcal{S} is assumed to be stable, there always exists a symmetric matrix $\hat{P} > 0$ such that

$$A^T \hat{P} + \hat{P} A < 0.$$

Hence, when we choose $\tilde{P}_{c1} - \tilde{P}_{c2} = \epsilon \hat{P}$ with $\epsilon > 0$ being a sufficiently small scalar, the inequality in (2.25) reduces to

$$\mathcal{N}_X^T \Phi \mathcal{N}_X = \begin{bmatrix} \epsilon A^T \hat{P} + \epsilon \hat{P} A & \epsilon \hat{P} B \\ * & -I \end{bmatrix} < 0$$

which automatically holds. What's more, under this choice, the first inequality of (2.20) is also automatically satisfied, because, according to the Schur complement, it is equivalent to

$$\begin{cases} \tilde{P}_{c2} > 0 \\ \tilde{P}_{c1} - \tilde{P}_{c2} = \epsilon \hat{P} > 0 \end{cases}.$$

Note that from the second inequality of (2.20) and the one in (2.8), the H_2 performance bound $\sqrt{\gamma}$ is not directly relevant to \tilde{P}_{c2} , which implies that the choice of \tilde{P}_{c2} satisfying $\tilde{P}_{c1} - \tilde{P}_{c2} = \epsilon \hat{P}$ does not introduce conservatism for computing γ . Consequently, γ_{H_2} is larger than but can get arbitrarily close to γ_1 , where γ_{H_2} is the minimum of γ in Algorithm 2 and γ_1 is defined as

$$\gamma_1 \triangleq \left(\min_{\tilde{P}_{c1}, \tilde{B}_F, Z_c, \gamma} \text{Tr}[Z_c] \text{ s.t. (2.26) and } \begin{bmatrix} \tilde{P}_{c1} & L^T \\ L & Z_c \end{bmatrix} > 0 \right).$$

By using the Schur complement and the assumption $BD^T = 0$ and $DD^T = I$, the inequality in (2.26) reduces to

$$\begin{aligned} & X_{c1} + X_{c1}^T + \left(\tilde{P}_{c1}B + \tilde{B}_FD \right) \left(\tilde{P}_{c1}B + \tilde{B}_FD \right)^T \\ &= \tilde{P}_{c1}A + A^T\tilde{P}_{c1} + \tilde{P}_{c1}BB^T\tilde{P}_{c1} + \tilde{B}_FC + C^T\tilde{B}_F^T + \tilde{B}_F\tilde{B}_F^T \\ &< 0. \end{aligned} \quad (2.27)$$

Noting the fact that

$$\tilde{B}_FC + C^T\tilde{B}_F^T + \tilde{B}_F\tilde{B}_F^T \geq -C^TC$$

always holds, where the equality holds only for $\tilde{B}_F = -C^T$, one sees that the inequality in (2.27) must hold for $\tilde{B}_F = -C^T$. Hence, the inequality in (2.26) can be replaced by

$$\tilde{P}_{c1}A + A^T\tilde{P}_{c1} + \tilde{P}_{c1}BB^T\tilde{P}_{c1} - C^TC < 0$$

or by performing a congruence transformation with \tilde{P}_{c1}^{-1} and defining $\tilde{P} \triangleq \tilde{P}_{c1}^{-1}$,

$$A\tilde{P} + \tilde{P}A^T + BB^T - \tilde{P}C^TC\tilde{P} < 0. \quad (2.28)$$

Consequently, by the Schur complement, we have

$$\begin{aligned} \gamma_1 &= \left(\min_{\tilde{P}, Z_c} \text{Tr}[Z_c] \text{ s.t. (2.28) and } \begin{bmatrix} \tilde{P}^{-1} & L^T \\ L & Z_c \end{bmatrix} > 0 \right) \\ &> \left(\min_{\tilde{P}} \text{Tr}[L\tilde{P}L^T] \text{ s.t. (2.28)} \right) \triangleq \gamma_2 \end{aligned}$$

Also, γ_1 can get arbitrarily close to γ_2 .

In view of (2.28), and according to the monotonicity of the solution to the Riccati equation in (2.23), we have $\tilde{P} > P$, and

$$\gamma_2 = \left(\min_{\tilde{P}} \text{tr}[L \tilde{P} L^T] \text{ s.t. (2.28)} \right) > \text{tr}[L P L^T] \triangleq \gamma_{\text{Kalman}},$$

where γ_{Kalman} is the H_2 performance of the Kalman filter. Moreover, $\gamma_2 \rightarrow \gamma_{\text{Kalman}}$ when $\tilde{P} \rightarrow P$, so that γ_2 can get arbitrarily close to γ_{Kalman} . Consequently, we have established that γ_{H_2} is larger than but can get arbitrarily close to γ_{Kalman} , that is, γ_{Kalman} is the optimal H_2 filtering performance that can be achieved by general LTI filters. This interprets the optimality of the Kalman filter in the H_2 sense.

From the above derivations, we have recovered the Kalman filter gains

$$B_F = -\tilde{P}_{c1}^{-1} \tilde{B}_F = \tilde{P} C^T, C_F = L.$$

In the following, we show how to obtain the gain A_F for the Kalman filter from the solution of \tilde{P} . To this end, performing a congruence transformation to the inequality in (2.16), we have

$$\begin{aligned} & \begin{bmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} X_{c1} + X_{c1}^T & \tilde{A}_F + X_{c2}^T & \tilde{P}_{c1} B + \tilde{B}_F D \\ * & \tilde{A}_F + \tilde{A}_F^T & \tilde{P}_{c2} B + \tilde{B}_F D \\ * & * & -I \end{bmatrix} \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} X_{c1} + X_{c1}^T & \tilde{A}_F - X_{c1} + A^T(\tilde{P}_{c2} - \tilde{P}_{c1})^T & \tilde{P}_{c1} B + \tilde{B}_F D \\ * & (\tilde{P}_{c1} - \tilde{P}_{c2})A + A^T(\tilde{P}_{c2} - \tilde{P}_{c1})^T & (\tilde{P}_{c2} - \tilde{P}_{c1})B \\ * & * & -I \end{bmatrix} \\ &< 0. \end{aligned}$$

Substituting $\tilde{P}_{c1} - \tilde{P}_{c2} = \epsilon \hat{P}$ into the above inequality, we get

$$\begin{bmatrix} X_{c1} + X_{c1}^T & \tilde{A}_F - X_{c1} - \epsilon A^T \hat{P} & \tilde{P}_{c1} B + \tilde{B}_F D \\ * & \epsilon \hat{P} A + \epsilon A^T \hat{P} & -\epsilon \hat{P} B \\ * & * & -I \end{bmatrix} < 0. \quad (2.29)$$

The previous derivations show that the inequality in (2.16) holds for ϵ sufficiently small, so does the one in (2.29). As $\epsilon \rightarrow 0^+$, we need to impose $\tilde{A}_F = X_{c1}$, and the inequality in (2.29) finally reduces to the one in (2.26). Moreover, in view of the expression of A_F in (2.21), we have

$$\begin{aligned} A_F &= \tilde{P}_{c1}^{-1} \tilde{A}_F \tilde{P}_{c2}^{-1} \tilde{P}_{c1} \\ &= \tilde{P}_{c1}^{-1} (\tilde{P}_{c1} A + \tilde{B}_F C) \tilde{P}_{c2}^{-1} \tilde{P}_{c1} && (\text{since } \tilde{A}_F = X_{c1} = \tilde{P}_{c1} A + \tilde{B}_F C) \\ &= (A - \tilde{P} C^T C) \tilde{P}_{c2}^{-1} \tilde{P}^{-1} && (\text{since } \tilde{B}_F = -C^T \text{ and } \tilde{P}_{c1}^{-1} = \tilde{P}) \\ &= (A - \tilde{P} C^T C) (\tilde{P}^{-1} - \epsilon \hat{P})^{-1} \tilde{P}^{-1} && (\text{since } \tilde{P}_{c1} - \tilde{P}_{c2} = \epsilon \hat{P}) \\ &= (A - \tilde{P} C^T C) && (\text{as } \epsilon \rightarrow 0^+) \end{aligned}$$

Table 2.1 Step-by-step procedures relating the optimal H_2 design and the Kalman filter design

Filter	Procedure	Purpose	Matrix transfer
H_2			
\Downarrow	Theorem 2.1 \leftrightarrow Theorem 2.2	Eliminate C_F	$C_F = L$
H_2			
\Downarrow	$\gamma_{H_2} > \gamma_1$ and $\gamma_{H_2} \rightarrow \gamma_1$	Eliminate A_F and \tilde{P}_{c2}	$\tilde{P}_{c2} \rightarrow \tilde{P}_{c1}$
H_2			
\Downarrow	$\gamma_1 > \gamma_2$ and $\gamma_1 \rightarrow \gamma_2$	Eliminate B_F	$B_F = -C^T$
H_2			
\Downarrow	$\gamma_2 > \gamma_{\text{Kalman}}$ and $\gamma_2 \rightarrow \gamma_{\text{Kalman}}$	Recover Kalman filter	$A_F = A - \tilde{P}C^TC$
Kalman			

which is expressed in a form consistent with the Kalman gain for A_F in (2.22) but with P replaced by \tilde{P} . We thus have completed the recovery of all the gain matrices of the Kalman filter.

The above analysis has uncovered the relationship between the H_2 filter design method in Sect. 2.1.2 and the Kalman filtering method. For helping reader's understanding, we summarize these procedures and the main points in Table 2.1.

Remark 2.4 It is shown that the optimal H_2 filters can be designed by the Kalman filtering method. When recovering the Kalman filter gains from Theorem 2.1 or 2.2, we need to specifically choose some matrices that depend on the plant parameters, so as to keep the optimality of each procedure. Once the plant has uncertainty, the optimality of each procedure would be broken, resulting in the loss of optimality for the Kalman filter. Hence, an inevitable drawback of the Kalman filter is that its optimality depends strongly on the plant parameters, such that the Kalman filter inherently is not robust against parametric uncertainty.

2.1.4 H_2 Filtering for Nominal Systems: Discrete-Time

In Sect. 2.1.2, we have considered the analysis and synthesis problems of H_2 filters for continuous-time systems. In this subsection, we deal with the H_2 filtering problem for discrete-time systems. Sufficient and necessary conditions will be presented in terms of LMI for analysis and synthesis of H_2 filters. At first, an H_2 performance criterion for discrete-time nominal systems is given in terms of LMI as follows.

Lemma 2.2 ([1, 3]) *Consider the discrete-time system \mathcal{S} in (2.1) and assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given filter \mathcal{F} in (2.4) and a scalar $\gamma > 0$, the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies $\mathbf{E}[e(t)^T e(t)] < \gamma$ if and only if the following matrix inequalities*

$$\text{Tr}[Z_d] < \gamma, \quad (2.30)$$

$$\begin{bmatrix} P_d & \bar{C}^T \\ \bar{C} & Z_d \end{bmatrix} > 0, \quad (2.31)$$

$$\begin{bmatrix} P_d & P_d \bar{A} & P_d \bar{B} \\ \bar{A}^T P_d & P_d & 0 \\ \bar{B}^T P_d & 0 & I \end{bmatrix} > 0 \quad (2.32)$$

are feasible in the real matrix variables P_d and Z_d .

In the context of discrete-time systems, the above result can be briefly justified as follows. First note that $\mathbf{E}[e(t)^T e(t)] = \text{Tr}[\bar{C} P \bar{C}^T]$, where P is the solution to the discrete-time Lyapunov equation

$$\bar{A} P \bar{A}^T - P + \bar{B} \bar{B}^T = 0.$$

Pre- and post-multiplying the both hand sides of (2.32) by $\text{diag}\{P_d^{-1}, P_d^{-1}, I\}$ and using the Schur complement, we have

$$\bar{A} P_d^{-1} \bar{A}^T - P_d^{-1} + \bar{B} \bar{B}^T < 0.$$

The monotonicity property of the solution to the Lyapunov equation ensures $P < P_d^{-1}$, which together with (2.30) and (2.31) further results in

$$\mathbf{E}[e(t)^T e(t)] = \text{Tr}[\bar{C} P \bar{C}^T] < \text{Tr}[\bar{C} P_d^{-1} \bar{C}^T] < \text{Tr}[Z_d] < \gamma.$$

Remark 2.5 Note that in [7, 8], the LMI constraints corresponding to (2.31) and (2.32) are given, respectively, by

$$\begin{bmatrix} \bar{P} & \bar{P} \bar{C}^T \\ \bar{C} \bar{P} & Z \end{bmatrix} > 0, \quad \begin{bmatrix} \bar{P} & \bar{A} \bar{P} & \bar{B} \\ \bar{P} \bar{A}^T & \bar{P} & 0 \\ \bar{B}^T & 0 & I \end{bmatrix} > 0.$$

By performing some simple matrix manipulations, it can be verified that these two inequalities are equivalent to (2.31) and (2.32), respectively. The difference of these two formulations is very similar to that for the continuous-time case discussed before. For the case of nominal systems, it does not matter which formulation is employed for H_2 performance analysis. Nevertheless, the feature that product terms between Lyapunov matrix P_d and system matrices in Lemma 2.2 exist only in (2.32) is useful for robustness analysis of the filtering error system.

Given a filter, the above lemma gives a tractable LMI-based condition for computing the optimal H_2 filtering performance level. After performing some matrix

transformations to the conditions in Lemma 2.2, the first result on H_2 filter design for discrete-time nominal systems is obtained as follows.

Theorem 2.3 *Consider the discrete-time system \mathcal{S} in (2.1) and assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.4) exists such that the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies $\mathbf{E}[e(t)^T e(t)] < \gamma$ if and only if the inequalities in (2.30) and in the following*

$$\begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} & L^T \\ \tilde{P}_{d2} & \tilde{P}_{d2} & -\tilde{C}_F^T \\ L & -\tilde{C}_F & Z_d \end{bmatrix} > 0, \quad (2.33)$$

$$\begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} & X_{d1} & \tilde{A}_F & \tilde{P}_{d1}B + \tilde{B}_FD \\ * & \tilde{P}_{d2} & X_{d2} & \tilde{A}_F & \tilde{P}_{d2}B + \tilde{B}_FD \\ * & * & \tilde{P}_{d1} & \tilde{P}_{d2} & 0 \\ * & * & * & \tilde{P}_{d2} & 0 \\ * & * & * & * & I \end{bmatrix} > 0 \quad (2.34)$$

are feasible in the real matrix variables \tilde{P}_{d1} , \tilde{P}_{d2} , Z_d , \tilde{A}_F , \tilde{B}_F , and \tilde{C}_F , where

$$X_{d1} \triangleq \tilde{P}_{d1}A + \tilde{B}_FC, \quad X_{d2} \triangleq \tilde{P}_{d2}A + \tilde{B}_FC.$$

Moreover, if these conditions are feasible, an admissible state-space realization of the filter \mathcal{F} in (2.4) is given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} \tilde{P}_{d2}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & 0 \end{bmatrix} \quad (2.35)$$

or

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & 0 \end{bmatrix} \begin{bmatrix} \tilde{P}_{d2}^{-1} & 0 \\ 0 & I \end{bmatrix}. \quad (2.36)$$

Proof First, we prove that the feasibility of (2.30), (2.33), and (2.34) is a necessary and sufficient condition for a guaranteed H_2 performance of the filtering error system. The proof methodology and the employed techniques are similar to that of the proof of Theorem 2.1 for continuous-time systems.

Necessity: Suppose that there exist P_d and Z_d such that (2.30)–(2.32) are satisfied. Partition matrix P_d into four blocks as the following form

$$P_d = \begin{bmatrix} P_{d1} & P_{d2} \\ P_{d2}^T & P_{d3} \end{bmatrix}$$

with each block being an $n \times n$ matrix. The nonsingularity of P_d enables us to assume that, by invoking a small perturbation if necessary, P_{d2} is nonsingular. Define

multiplier matrix

$$J_{d0} \triangleq \begin{bmatrix} I & 0 \\ 0 & P_{d3}^{-1} P_{d2}^T \end{bmatrix}$$

and introduce the following matrix transformations

$$\tilde{A}_F \triangleq P_{d2} A_F P_{d3}^{-1} P_{d2}^T, \quad (2.37)$$

$$\tilde{B}_F \triangleq P_{d2} B_F, \quad (2.38)$$

$$\tilde{C}_F \triangleq C_F P_{d3}^{-1} P_{d2}^T, \quad (2.39)$$

$$\begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} \\ \tilde{P}_{d2} & \tilde{P}_{d2} \end{bmatrix} \triangleq J_{d0}^T P_d J_{d0} = \begin{bmatrix} P_{d1} & P_{d2} P_{d3}^{-1} P_{d2}^T \\ P_{d2} P_{d3}^{-1} P_{d2}^T & P_{d2} P_{d3}^{-1} P_{d2}^T \end{bmatrix}. \quad (2.40)$$

Then, the following equations can be obtained

$$\begin{aligned} J_{d0}^T P_d \bar{A} J_{d0} &= \begin{bmatrix} X_{d1} & \tilde{A}_F \\ X_{d2} & \tilde{A}_F \end{bmatrix}, \\ J_0^T P_d \bar{B} &= \begin{bmatrix} \tilde{P}_{d1} B + \tilde{B}_F D \\ \tilde{P}_{d2} B + \tilde{B}_F D \end{bmatrix}, \\ \bar{C} J_{d0} &= [L \quad -\tilde{C}_F]. \end{aligned}$$

Furthermore, by defining $J_{d1} \triangleq \text{diag} \{J_{d0}, I_n\}$ and $J_{d1} \triangleq \text{diag} \{J_{d0}, J_{d0}, I_l\}$ and noting these given equations, it readily follows that

$$\begin{aligned} J_{d1}^T \begin{bmatrix} P_d & \bar{C}^T \\ \bar{C} & Z_d \end{bmatrix} J_{d1} &= \begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} & L^T \\ \tilde{P}_{d2} & \tilde{P}_{d2} & -\tilde{C}_F^T \\ L & -\tilde{C}_F & Z_d \end{bmatrix}, \\ J_{d2}^T \begin{bmatrix} P_d & P_d \bar{A} & P_d \bar{B} \\ \bar{A}^T P_d & P_d & 0 \\ \bar{B}^T P_d & 0 & I \end{bmatrix} J_{d2} &= \begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} & X_{d1} & \tilde{A}_F & \tilde{P}_{d1} B + \tilde{B}_F D \\ * & \tilde{P}_{d2} & X_{d2} & \tilde{A}_F & \tilde{P}_{d2} B + \tilde{B}_F D \\ * & * & \tilde{P}_{d1} & \tilde{P}_{d2} & 0 \\ * & * & * & \tilde{P}_{d2} & 0 \\ * & * & * & * & I \end{bmatrix}, \end{aligned}$$

which together with (2.30) imply that if there exist variables P_d and Z_d such that (2.30)–(2.32) hold, there always exist variables \tilde{P}_{d1} , \tilde{P}_{d2} , \tilde{A}_F , \tilde{B}_F , and \tilde{C}_F defined in (2.37)–(2.40) and variable Z_d such that (2.30), (2.33), and (2.34) hold. Then by virtue of Lemma 2.2, the necessity has been established.

Sufficiency: Suppose that there exist \tilde{P}_{d1} , \tilde{P}_{d2} , $Z_d \tilde{A}_F$, \tilde{B}_F , and \tilde{C}_F such that (2.30), (2.33), and (2.34) are satisfied. Due to (2.33), \tilde{P}_{d2} is nonsingular, which means that it is always possible to find two invertible matrices P_{d2} and P_{d3} such that $\tilde{P}_{d2} = P_{d2} P_{d3}^{-1} P_{d2}^T$. Introduce the following definitions

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} \triangleq \begin{bmatrix} P_{d2}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & 0 \end{bmatrix} \begin{bmatrix} P_{d2}^{-T} P_{d3} & 0 \\ 0 & I \end{bmatrix}, \quad (2.41)$$

$$P_d \triangleq J_{d0}^{-T} \begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} \\ \tilde{P}_{d2} & \tilde{P}_{d2} \end{bmatrix} J_{d0}^{-1}. \quad (2.42)$$

By these definitions, the conditions in (2.33) and (2.34) can be rewritten as

$$J_{d1}^T \begin{bmatrix} P_d & \tilde{C}^T \\ \tilde{C} & Z_d \end{bmatrix} J_{d1} < 0,$$

$$J_{d2}^T \begin{bmatrix} P_d & P_d \tilde{A} & P_d \tilde{B} \\ \tilde{A}^T P_d & P_d & 0 \\ \tilde{B}^T P_d & 0 & I \end{bmatrix} J_{d2} < 0.$$

Since J_{d1} and J_{d2} both are nonsingular, the above two conditions imply that there exist variables P_d in (2.42), Z_d together with filter realization (A_F, B_F, C_F) in (2.41) such that (2.31) and (2.32) hold. Consequently, by Lemma 2.2, (2.30), (2.33), and (2.34) sufficiently ensure the existence of a filter with a guaranteed H_2 performance level bound $\sqrt{\gamma}$.

Substituting the expression of A_F , B_F , and C_F in (2.41) into the transfer function of the filter and performing some matrix manipulations similar to that in the proof of Theorem 2.1, one sees that (2.35) and (2.36) both are admissible realizations for the filter \mathcal{F} . The entire proof is completed. \square

Remark 2.6 Similar to continuous-time case, it is also valid that some variables in Lemma 2.2 are redundant, and these additional variables are also introduced when \tilde{P}_{d2} is decomposed as $\tilde{P}_{d2} = P_{d2} P_{d3}^{-1} P_{d2}^T$. Hence, with no conservatism introduced, it can be set as $\tilde{P}_{d2} = P_{d2} = P_{d3}$, and P_d in Lemma 2.1 reduces to the following form

$$P_d = \begin{bmatrix} P_{d1} & P_{d2} \\ P_{d2} & P_{d2} \end{bmatrix}.$$

By Theorem 2.3, the first method for designing the optimal H_2 filters for discrete-time nominal systems is formulated in Algorithm 3.

Algorithm 3 H_2 Filter Design I (Discrete-Time)

- Solve the minimization problem

$$\min_{\tilde{P}_{d1}, \tilde{P}_{d2}, Z_d, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \gamma} \gamma \quad \text{s.t. (2.30), (2.33) and (2.34)}$$

- Compute filter \mathcal{F} in (2.4) by (2.35) or (2.36).
-

Note that the LMI in (2.33) has the same form as the one in (2.15). Therefore, Lemma 1.3 can also be applied to (2.33) such that the number of the free variables

in Theorem 2.3 can be reduced. Similar to the derivation of Theorem 2.2, the choice for P_{d2} could be $P_{d2} = -\tilde{P}_{d1}$ and the elimination procedure finally results in a fixed value for \tilde{C}_F as $\tilde{C}_F = L$. Accordingly, we obtain another method for H_2 filter design for discrete-time nominal systems.

Theorem 2.4 *Consider the discrete-time system \mathcal{S} in (2.1) and assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.4) exists such that the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies $\mathbf{E}[e(t)^T e(t)] < \gamma$ if and only if the LMIs in (2.30), (2.34) and in the following*

$$\begin{bmatrix} \tilde{P}_{d1} & L^T \\ L & Z_d \end{bmatrix} > 0 \quad (2.43)$$

are feasible in the real matrix variables \tilde{P}_{d1} , \tilde{P}_{d2} , Z_d , \tilde{A}_F , and \tilde{B}_F . Moreover, if these conditions are feasible, an admissible state-space realization of the filter \mathcal{F} in (2.4) is given by

$$A_F = \tilde{P}_{d1}^{-1} \tilde{A}_F \tilde{P}_{d2}^{-1} \tilde{P}_{d1}, \quad B_F = -\tilde{P}_{d1}^{-1} \tilde{B}_F, \quad C_F = L. \quad (2.44)$$

Proof The proof is omitted due to its similarity to that of Theorem 2.2. A special point should be pointed out as follows. When Lemma 1.3 is applied to eliminate the variable \tilde{C}_F in (2.33), it reduces to two LMIs:

$$\begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} \\ \tilde{P}_{d2}^T & \tilde{P}_{d2} \end{bmatrix} > 0, \quad \begin{bmatrix} \tilde{P}_{d1} & L^T \\ L & Z_d \end{bmatrix} > 0.$$

Note that the first one has been implied by (2.34). So reserving the second one, i.e., (2.43), suffices to ensure the validity of the results. \square

Based on Theorem 2.4, another method for the optimal H_2 filter design for discrete-time nominal systems is given in Algorithm 4.

Algorithm 4 H_2 Filter Design II (Discrete-Time)

- Solve the minimization problem

$$\min_{\tilde{P}_{d1}, \tilde{P}_{d2}, Z_d, \tilde{A}_F, \tilde{B}_F, \gamma} \gamma \quad \text{s.t. (2.30), (2.34), and (2.43)}$$

- Compute filter \mathcal{F} in (2.4) by (2.44).
-

2.1.5 Connection to the Kalman Filtering: Discrete-Time

Following similar discussions as in Sect. 2.1.3, we can also establish the connection between the H_2 filter design methods in Sect. 2.1.4 and the Kalman filtering theory in the discrete-time case. Here, we only provide the main points for illustration.

In order to design the Kalman filter to the system \mathcal{S} in (2.1), similar to the continuous-time case, we need to assume

$$BD^T = 0 \text{ and } DD^T = I.$$

For the system \mathcal{S} in (2.1) with nominal parameters, the discrete-time Kalman filter in the form of the general filter \mathcal{F} in (2.4) is with matrices given by [9]

$$\begin{aligned} A_F &= A - APC^T (CPC^T + I)^{-1} C, \\ B_F &= APC^T (CPC^T + I)^{-1}, \\ C_F &= L, \end{aligned}$$

where P is a positive definite matrix that solves the following discrete-time Riccati equation:

$$APA^T - P - APC^T (CPC^T + I)^{-1} CPA^T + BB^T = 0. \quad (2.45)$$

To relate the Kalman filter gains with the H_2 filter realization obtained by Theorem 2.4, performing a congruence transformation to the inequality in (2.34), we have

$$\begin{aligned} & \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ -I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & -I & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} & X_{d1} & \tilde{A}_F & \tilde{P}_{d1}B + \tilde{B}_FD \\ * & \tilde{P}_{d2} & X_{d2} & \tilde{A}_F & \tilde{P}_{d2}B + \tilde{B}_FD \\ * & * & \tilde{P}_{d1} & \tilde{P}_{d2} & 0 \\ * & * & * & \tilde{P}_{d2} & 0 \\ * & * & * & * & I \end{bmatrix} \begin{bmatrix} I & -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & -I & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \tilde{P}_{d1} & \tilde{P}_{d2} - \tilde{P}_{d1} & X_{d1} & \tilde{A}_F - X_{d1} & \tilde{P}_{d1}B + \tilde{B}_FD \\ * & \tilde{P}_{d1} - \tilde{P}_{d2} & (\tilde{P}_{d2} - \tilde{P}_{d1})A & (\tilde{P}_{d1} - \tilde{P}_{d2})A & (\tilde{P}_{d2} - \tilde{P}_{d1})B \\ * & * & \tilde{P}_{d1} & \tilde{P}_{d2} - \tilde{P}_{d1} & 0 \\ * & * & * & \tilde{P}_{d1} - \tilde{P}_{d2} & 0 \\ ** & * & * & * & I \end{bmatrix} \\ &> 0. \end{aligned} \quad (2.46)$$

By analogy to the continuous-time case (details are omitted), we can take $\tilde{P}_{d1} - \tilde{P}_{d2} = \epsilon \tilde{P}$, where $\epsilon > 0$ is a sufficient small scalar and $\tilde{P} > 0$ satisfies $\tilde{P} - A^T \tilde{P} A > 0$. Under this choice, the inequality in (2.46) turns to be

$$\begin{bmatrix} \tilde{P}_{d1} - \epsilon \tilde{P} & X_{d1} & \tilde{A}_F - X_{d1} & \tilde{P}_{d1} B + \tilde{B}_F D \\ * & \epsilon \tilde{P} & -\epsilon \tilde{P} A & \epsilon \tilde{P} A & -\epsilon \tilde{P} B \\ * & * & \tilde{P}_{d1} & -\epsilon \tilde{P} & 0 \\ * & * & * & \epsilon \tilde{P} & 0 \\ * & * & * & * & I \end{bmatrix} > 0. \quad (2.47)$$

To keep the feasibility of this inequality for $\epsilon \rightarrow 0^+$, we need to prescribe $\tilde{A}_F - X_{d1} = 0$, under which, the inequality in (2.47), when $\epsilon \rightarrow 0^+$, reduces to

$$\begin{bmatrix} \tilde{P}_{d1} & X_{d1} & \tilde{P}_{d1} B + \tilde{B}_F D \\ * & \tilde{P}_{d1} & 0 \\ * & * & I \end{bmatrix} > 0.$$

Using the Schur complement and noting the assumption $BD^T = 0$ and $DD^T = I$, we further get

$$\begin{aligned} & \tilde{P}_{d1} - [X_{d1} \ \tilde{P}_{d1} B + \tilde{B}_F D] \begin{bmatrix} \tilde{P}_{d1} & 0 \\ 0 & I \end{bmatrix} [X_{d1} \ \tilde{P}_{d1} B + \tilde{B}_F D]^T \\ &= \tilde{P}_{d1} - X_{d1} \tilde{P}_{d1}^{-1} X_{d1}^T - \tilde{P}_{d1} B B \tilde{P}_{d1}^T - \tilde{B}_F \tilde{B}_F^T \\ &= \tilde{P}_{d1} - \tilde{P}_{d1} B B \tilde{P}_{d1}^T - (\tilde{P}_{d1} A + \tilde{B}_F C) \tilde{P}_{d1}^{-1} (\tilde{P}_{d1} A + \tilde{B}_F C)^T - \tilde{B}_F \tilde{B}_F^T \\ &= \tilde{P}_{d1} - \tilde{P}_{d1} B B \tilde{P}_{d1}^T - \tilde{P}_{d1} A \tilde{P}_{d1}^{-1} (\tilde{P}_{d1} A)^T - \tilde{P}_{d1} A \tilde{P}_{d1}^{-1} (\tilde{B}_F C)^T \\ &\quad - \tilde{B}_F C \tilde{P}_{d1}^{-1} (\tilde{P}_{d1} A)^T - \tilde{B}_F (C \tilde{P}_{d1}^{-1} C + I) \tilde{B}_F^T \end{aligned} \quad (2.48)$$

$$\begin{aligned} &= \tilde{P}_{d1} - \tilde{P}_{d1} B B \tilde{P}_{d1}^T - \tilde{P}_{d1} A \tilde{P}_{d1}^{-1} (\tilde{P}_{d1} A)^T \\ &\quad - \left(\tilde{B}_F + \tilde{P}_{d1} A \tilde{P}_{d1}^{-1} C^T (C \tilde{P}_{d1}^{-1} C^T + I)^{-1} \right) (C \tilde{P}_{d1}^{-1} C + I) \\ &\quad \times \left(\tilde{B}_F + \tilde{P}_{d1} A \tilde{P}_{d1}^{-1} C^T (C \tilde{P}_{d1}^{-1} C^T + I)^{-1} \right)^T \\ &\quad + \tilde{P}_{d1} A \tilde{P}_{d1}^{-1} C^T (C \tilde{P}_{d1}^{-1} C^T + I)^{-1} (\tilde{P}_{d1} A \tilde{P}_{d1}^{-1} C^T)^T \\ &> 0. \end{aligned} \quad (2.49)$$

Due to the fact that

$$\begin{aligned} & \left(\tilde{B}_F + \tilde{P}_{d1} A \tilde{P}_{d1}^{-1} C^T (C \tilde{P}_{d1}^{-1} C^T + I)^{-1} \right) (C \tilde{P}_{d1}^{-1} C + I) \\ & \times \left(\tilde{B}_F + \tilde{P}_{d1} A \tilde{P}_{d1}^{-1} C^T (C \tilde{P}_{d1}^{-1} C^T + I)^{-1} \right)^T \geq 0 \end{aligned}$$

and the equality holds only for $\tilde{B}_F = -\tilde{P}_{d1}A\tilde{P}_{d1}^{-1}C^T \left(C\tilde{P}_{d1}^{-1}C^T + I\right)^{-1}$, the inequality in (2.47) can be replaced by

$$\begin{aligned} & \tilde{P}_{d1} - \tilde{P}_{d1}BB\tilde{P}_{d1} - \tilde{P}_{d1}A\tilde{P}_{d1}^{-1}(\tilde{P}_{d1}A)^T + \tilde{P}_{d1}A\tilde{P}_{d1}^{-1}C^T \\ & \times \left(C\tilde{P}_{d1}^{-1}C^T + I\right)^{-1} \left(\tilde{P}_{d1}A\tilde{P}_{d1}^{-1}C^T\right)^T > 0 \end{aligned}$$

or equivalently, via performing a congruence transformation, changing the sign, and defining $\tilde{P} \triangleq \tilde{P}_{d1}^{-1}$,

$$A\tilde{P}A^T - \tilde{P} - A\tilde{P}C^T \left(C\tilde{P}C^T + I\right)^{-1} C\tilde{P}A^T + BB < 0. \quad (2.50)$$

Comparing this inequality with the discrete-time Riccati equation in (2.45) and noting the monotonicity of the solution to the Riccati equation, one sees that the optimal H_2 performance $\sqrt{\gamma}$ computed by Theorem 2.4 can get arbitrarily close to $\text{Tr}[LPL^T]$ for the H_2 performance of the Kalman filter. What's more, the filter realization computed from (2.44) is given by

$$\begin{aligned} A_F &= \tilde{P}_{d1}^{-1}\tilde{A}_F\tilde{P}_{d2}^{-1}\tilde{P}_{d1} = \tilde{P}_{d1}^{-1}X_{d1} \\ &= \tilde{P}_{d1}^{-1}(\tilde{P}_{d1}A + \tilde{B}_FC) \\ &= A - \tilde{P}_{d1}^{-1}\tilde{P}_{d1}A\tilde{P}_{d1}^{-1}C^T \left(C\tilde{P}_{d1}^{-1}C^T + I\right)^{-1} C \\ &= A - A\tilde{P}C^T \left(C\tilde{P}C^T + I\right)^{-1} C, \\ B_F &= -\tilde{P}_{d1}^{-1}\tilde{B}_F \\ &= \tilde{P}_{d1}^{-1}\tilde{P}_{d1}A\tilde{P}_{d1}^{-1}C^T \left(C\tilde{P}_{d1}^{-1}C^T + I\right)^{-1} \\ &= A\tilde{P}C^T \left(C\tilde{P}C^T + I\right)^{-1}, \\ C_F &= L, \end{aligned}$$

which are in a form consistent with the discrete-time Kalman filter.

2.1.6 Quadratic Robust H_2 Filtering

In Sects. 2.1.2 and 2.1.4, several sufficient and necessary conditions in terms of LMI have been obtained for designing optimal H_2 filters for nominal continuous- or discrete-time systems. The crucial linearization procedures have been established such that the problems of H_2 filter design are cast into the feasibility of

the corresponding convex optimization problems. In this subsection, these obtained results will be further extended and applied to design robust H_2 filters.

To handle the robustness of uncertain dynamical systems, a basic notion in control theory is quadratic stability [10–12]. In the framework of quadratic stability, many results regarding robust H_2 or Kalman filtering for systems with norm-bounded uncertainties have been proposed [6, 9, 13–26], where the filtering scheme is often termed as *quadratic guaranteed cost filtering or estimation* [19, 20, 22, 23]. This subsection will employ the notion of quadratic stability to investigate the robust H_2 filtering problem for systems with convex polyhedral uncertainties bounded in \mathcal{M} .

When uncertain but fixed parameter λ exists, the filtering error variance of system \mathcal{E} as $t \rightarrow +\infty$ satisfies

$$\sup_{\lambda \in \Gamma} \mathbf{E} \left[e(t)^T e(t) \right] = \sup_{\lambda \in \Gamma} \text{Tr}[C(\lambda)P(\lambda)C(\lambda)^T],$$

where parameter-dependent matrix $P(\lambda)$ is calculated from the following parameter-dependent Lyapunov equation

$$\bar{A}(\lambda)P(\lambda) + P(\lambda)\bar{A}(\lambda)^T + \bar{B}(\lambda)\bar{B}(\lambda)^T = 0 \quad (2.51)$$

for the continuous-time case, or

$$\bar{A}(\lambda)P(\lambda)\bar{A}(\lambda)^T - P(\lambda) + \bar{B}(\lambda)\bar{B}(\lambda)^T = 0 \quad (2.52)$$

for the discrete-time case.

For a given filter \mathcal{F} , suppose there exists a matrix $\bar{P} > 0$ satisfying the parameter-dependent LMI, for the continuous-time case,

$$\bar{A}(\lambda)\bar{P} + \bar{P}\bar{A}(\lambda)^T + \bar{B}(\lambda)\bar{B}(\lambda)^T < 0, \quad (2.53)$$

or for the discrete-time case,

$$\bar{A}(\lambda)\bar{P}\bar{A}(\lambda)^T - \bar{P} + \bar{B}(\lambda)\bar{B}(\lambda)^T < 0 \quad (2.54)$$

for all $M(\lambda) \in \mathcal{M}$. From the monotonicity of the solution of matrix $P(\lambda)$ to Lyapunov equations, there holds $P(\lambda) \leq \bar{P}$ for all $\lambda \in \Gamma$, which results in

$$\begin{aligned} \sup_{\lambda \in \Gamma} \mathbf{E} \left[e(t)^T e(t) \right] &= \sup_{\lambda \in \Gamma} \text{Tr}[C(\lambda)P(\lambda)C(\lambda)^T] \\ &\leq \sup_{\lambda \in \Gamma} \text{Tr}[\bar{C}(\lambda)\bar{P}\bar{C}(\lambda)^T]. \end{aligned}$$

Therefore for a prescribed robust H_2 performance bound $\sqrt{\gamma}$, a natural idea for solving the robust H_2 filtering problem is to look for a filter satisfying (2.53) or (2.54) together with $\sup_{\lambda \in \Gamma} \text{Tr}[C(\lambda)\bar{P}C(\lambda)^T] \leq \gamma$. Let $(\bar{A}_i, \bar{B}_i, \bar{C}_i)$ be the matrix

$(\bar{A}(\lambda), \bar{B}(\lambda), \bar{C}(\lambda))$ at the vertices of the polyhedral domain \mathcal{M} . Consequently, a quadratic approach to robust H_2 performance analysis for the filtering error system under a given filter can be obtained in the following lemma.

Lemma 2.3 *Given system \mathcal{S} in (2.1), filter \mathcal{F} in (2.4), and a scalar $\gamma > 0$, the filtering error system \mathcal{E} in (2.5) is robustly asymptotically stable and satisfies $\mathbf{E}[e(t)^T e(t)] < \gamma$ for all $\lambda \in \Gamma$ if the matrix inequalities, for the continuous-time case,*

$$\text{Tr}[Z] < \gamma, \begin{bmatrix} P & \bar{C}_i^T \\ \bar{C}_i & Z \end{bmatrix} > 0, \begin{bmatrix} \bar{A}_i^T P + P \bar{A}_i & P \bar{B}_i \\ \bar{B}_i^T P & -I \end{bmatrix} < 0, \quad (2.55)$$

or for the discrete-time case,

$$\text{Tr}[Z] < \gamma, \begin{bmatrix} P & \bar{C}_i^T \\ \bar{C}_i & Z \end{bmatrix} > 0, \begin{bmatrix} P & P \bar{A}_i & P \bar{B}_i \\ \bar{A}_i^T P & P & 0 \\ \bar{B}_i^T P & 0 & I \end{bmatrix} > 0 \quad (2.56)$$

are feasible in the real matrix variables P and Z for all $i = 1, 2, \dots, s$.

Proof Suppose that the matrix inequalities in (2.55) and (2.56) hold for some P and Z . Letting $\bar{P} = P^{-1}$, $T_1 = \text{diag}\{\bar{P}, I_p\}$, $T_2 = \text{diag}\{\bar{P}, I_l\}$, and $T_3 = \text{diag}\{\bar{P}, \bar{P}, I_l\}$ and noting that $\sum_{i=1}^s \lambda_i = 1$ and $\lambda_i \geq 0$, we have

$$\begin{aligned} \begin{bmatrix} \bar{P} & \bar{P} \bar{C}(\lambda)^T \\ \bar{C}(\lambda) \bar{P} & Z \end{bmatrix} &= T_1^T \begin{bmatrix} P & \bar{C}(\lambda)^T \\ \bar{C}(\lambda) & Z \end{bmatrix} T_1 \\ &= T_1^T \left(\sum_{i=1}^s \lambda_i \begin{bmatrix} P & \bar{C}_i^T \\ \bar{C}_i & Z \end{bmatrix} \right) T_1 \\ &> 0, \\ \begin{bmatrix} \bar{P} \bar{A}(\lambda)^T + \bar{A}(\lambda) \bar{P} & \bar{B}(\lambda) \\ \bar{B}(\lambda)^T & -I \end{bmatrix} &= T_2^T \begin{bmatrix} \bar{A}(\lambda)^T P + P \bar{A}(\lambda) & P \bar{B}(\lambda) \\ \bar{B}(\lambda)^T P & -I \end{bmatrix} T_2 \\ &= T_2^T \left(\sum_{i=1}^s \lambda_i \begin{bmatrix} \bar{A}_i^T P + P \bar{A}_i & P \bar{B}_i \\ \bar{B}_i^T P & -I \end{bmatrix} \right) T_2 \\ &< 0, \\ \begin{bmatrix} \bar{P} & \bar{A}(\lambda) \bar{P} & \bar{B}(\lambda) \\ \bar{P} \bar{A}(\lambda)^T & \bar{P} & 0 \\ \bar{B}(\lambda)^T & 0 & I \end{bmatrix} &= T_3^T \begin{bmatrix} P & P \bar{A}(\lambda) & P \bar{B}(\lambda) \\ \bar{A}(\lambda)^T P & P & 0 \\ \bar{B}(\lambda)^T P & 0 & I \end{bmatrix} T_3 \\ &= T_3^T \left(\sum_{i=1}^s \lambda_i \begin{bmatrix} P & P \bar{A}_i & P \bar{B}_i \\ \bar{A}_i^T P & P & 0 \\ \bar{B}_i^T P & 0 & I \end{bmatrix} \right) T_3 \\ &> 0. \end{aligned}$$

which, by the Schur complement, imply $\bar{C}(\lambda)\bar{P}\bar{C}(\lambda)^T < Z$ and the inequalities in (2.53) and (2.54). Therefore, we have

$$\sup_{\lambda \in \Gamma} \mathbf{E} \left[e(t)^T e(t) \right] \leq \sup_{\lambda \in \Gamma} \text{Tr}[\bar{C}(\lambda)\bar{P}\bar{C}(\lambda)^T] < \text{Tr}[Z] < \gamma. \quad \square$$

As the starting point of Lemma 2.2, the inequalities in (2.53) and (2.54) mean that an upper bound of the robust H_2 filtering performance level is evaluated by adopting a single Lyapunov matrix \bar{P} independent of the uncertain parameters, which is essentially the same as the notion of quadratic stability (then the meaning of *quadratic approaches*, as termed in book, is self-explanatory). Compared with Lemmas 2.1 and 2.2, it is easily found that Lemma 2.3 is a direct extension of Lemmas 2.1 and 2.2 to polytopic uncertain systems. Indeed, the last two inequalities in (2.55) and (2.56) include as many LMIs as the number of the vertices of the polytope \mathcal{M} with each LMI being the corresponding one in Lemmas 2.1 and 2.2 at the vertex of \mathcal{M} . Since \mathcal{M} is a convex combination of the matrices at the vertices, the conditions in Lemmas 2.1 and 2.2 for arbitrary matrix $M \in \mathcal{M}$ have been automatically guaranteed by those in Lemma 2.3.

The conditions in Lemma 2.3 can also be directly linearized with respect to the unknown filter matrices by the same procedures as in the derivation of H_2 filter design methods for nominal systems. This yields the following quadratic approach to robust H_2 filter design.

Theorem 2.5 *Consider system \mathcal{S} in (2.1). Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.4) exists such that the filtering error system \mathcal{E} in (2.5) is robustly asymptotically stable and satisfies $\mathbf{E} [e(t)^T e(t)] < \gamma$ for all $\lambda \in \Gamma$ if the matrix inequalities*

$$\text{Tr}[Z] < \gamma, \quad (2.57)$$

$$\begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 & L_i^T \\ \tilde{P}_2 & \tilde{P}_2 & -\tilde{C}_F^T \\ L_i & -\tilde{C}_F & Z \end{bmatrix} > 0 \quad (i = 1, 2, \dots, s), \quad (2.58)$$

and

- for the continuous-time case,

$$\begin{bmatrix} X_{1,i} + X_{1,i}^T & \tilde{A}_F + X_{1,i}^T & \tilde{P}_1 B_i + \tilde{B}_F D_i \\ * & \tilde{A}_F + \tilde{A}_F^T & \tilde{P}_2 B_i + \tilde{B}_F D_i \\ * & * & -I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s), \quad (2.59)$$

- for the discrete-time case,

$$\begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 & X_{1,i} & \tilde{A}_F & \tilde{P}_1 B_i + \tilde{B}_F D_i \\ * & \tilde{P}_2 & X_{2,i} & \tilde{A}_F & \tilde{P}_2 B_i + \tilde{B}_F D_i \\ * & * & \tilde{P}_1 & \tilde{P}_2 & 0 \\ * & * & * & \tilde{P}_2 & 0 \\ * & * & * & * & I \end{bmatrix} > 0 \quad (i = 1, 2, \dots, s), \quad (2.60)$$

are feasible in the real matrix variables \tilde{P}_1 , \tilde{P}_2 , Z , \tilde{A}_F , \tilde{B}_F , and \tilde{C}_F , where

$$X_{1,i} \triangleq \tilde{P}_1 A_i + \tilde{B}_F C_i, \quad X_{2,i} \triangleq \tilde{P}_2 A_i + \tilde{B}_F C_i.$$

Moreover, if these conditions are feasible, an admissible state-space realization of the filter \mathcal{F} in (2.4) is given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} \tilde{P}_2^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & 0 \end{bmatrix} \quad (2.61)$$

or

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & 0 \end{bmatrix} \begin{bmatrix} \tilde{P}_2^{-1} & 0 \\ 0 & I \end{bmatrix}. \quad (2.62)$$

Proof For the continuous-time case and the discrete-time case, the proof is quite similar to that of Theorems 2.1 and 2.3, respectively. \square

Based on Theorem 2.5, the first quadratic approach to robust H_2 filter design for systems with polytopic uncertainty is summarized in Algorithm 5.

Algorithm 5 Quadratic Approach to Robust H_2 Filter Design

- Solve the minimization problem:

- Continuous-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, Z, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \gamma} \gamma \quad \text{s.t. (2.57), (2.58), (2.59)}$$

- Discrete-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, Z, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \gamma} \gamma \quad \text{s.t. (2.57), (2.58), (2.60)}$$

- Compute filter \mathcal{F} in (2.4) by (2.61) or (2.62).
-

According to Theorems 2.2 and 2.4, when matrix L is known, i.e., $L(\lambda) = L = L_1 = \dots = L_s$, matrix variable \tilde{C}_F in (2.58) can be eliminated and filter matrix C_F can take $C_F = L$ with no loss of generality. In this situation, (2.58) reduces to

$$\begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 \end{bmatrix} > 0, \begin{bmatrix} \tilde{P}_1 & L^T \\ L & Z \end{bmatrix} > 0$$

and an admissible state-space realization of the filter \mathcal{F} in (2.4) is given by

$$A_F = \tilde{P}_1^{-1} \tilde{A}_F \tilde{P}_2^{-1} \tilde{P}_1, \quad B_F = -\tilde{P}_1^{-1} \tilde{B}_F, \quad C_F = L. \quad (2.63)$$

Consequently, we have Algorithm 6, another quadratic approach to robust H_2 filter design based on the LMI technique.

Algorithm 6 Quadratic Approach to Robust H_2 Filter Design (Known L)

- Solve the minimization problem:

- Continuous-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, Z, \tilde{A}_F, \tilde{B}_F, \gamma} \gamma \quad \text{s.t.} \quad \begin{cases} (2.57), (2.59) \\ \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 \end{bmatrix} > 0, \begin{bmatrix} \tilde{P}_1 & L^T \\ L & Z \end{bmatrix} > 0 \end{cases}$$

- Discrete-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, Z, \tilde{A}_F, \tilde{B}_F, \gamma} \gamma \quad \text{s.t.} \quad \begin{cases} (2.57), (2.60) \\ \begin{bmatrix} \tilde{P}_1 & L^T \\ L & Z \end{bmatrix} > 0 \end{cases}$$

- Compute filter \mathcal{F} in (2.4) by (2.63).
-

2.2 Quadratic Robust H_∞ Filter Design

In Sect. 2.1.6, quadratic approaches have been proposed for the design of robust H_2 filters for polytopic uncertain systems. In the H_2 setting, the noise process or series is assumed to be Gaussian, which means that the performance of an H_2 filter relies much on the knowledge of the statistic property of the noise input. However, in real situations, this assumption is not always completely satisfied. To overcome this drawback of H_2 filters, other assumptions independent of the statistic information of the noise input are often made and new filtering schemes based on the new assumptions are taken into consideration. The most frequently used assumption of such types is the square integrable function or square summable series with respect to time, that is, the noise $w(t) \in L_2[0, +\infty)$ for the continuous-time case and $w(t) \in l_2[0, +\infty)$ for the discrete-time case. As for the physical meaning of this type of noises, $w(t) \in L_2[0, +\infty)$ or $w(t) \in l_2[0, +\infty)$ implies the energy-bounded property of noises. Obviously, this type of assumption differs from the Gaussian white noise type, and moreover, does not need any information on the statistics of the noise. H_∞ filtering to be considered in this section is one of the filtering schemes that are based on this assumption for noises.

2.2.1 Problem Formulation

Consider a stable uncertain system \mathcal{S} described by the following state-space model:

$$\begin{aligned}\mathcal{S} : \delta[x(t)] &= A(\lambda)x(t) + B(\lambda)w(t), \\ y(t) &= C(\lambda)x(t) + D(\lambda)w(t), \\ z(t) &= L(\lambda)x(t) + E(\lambda)w(t),\end{aligned}\tag{2.64}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^m$ is the measured output, $z(t) \in \mathbb{R}^p$ is the signal to be estimated, and $w(t) \in \mathbb{R}^l$ is the noise. The meaning of $\delta[\cdot]$ is the same as that in (2.1). $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$, $L(\lambda)$, and $E(\lambda)$ are appropriately dimensioned matrices, and are assumed to satisfy

$$R(\lambda) \triangleq (A(\lambda), B(\lambda), C(\lambda), D(\lambda), L(\lambda), E(\lambda)) \in \mathcal{R} \tag{2.65}$$

where \mathcal{R} is a given convex polyhedral domain bounded by s vertices

$$\mathcal{R} \triangleq \left\{ R(\lambda) : R(\lambda) = \sum_{i=1}^s \lambda_i R_i; \lambda \in \Gamma \right\}$$

with $R_i \triangleq (A_i, B_i, C_i, D_i, L_i, E_i)$ denoting the vertices of the polytope, and Γ denoting the unit simplex defined in (2.3).

In this section, we are interested in designing a nonproper LTI filter as:

$$\begin{aligned}\mathcal{F} : \delta[x_F(t)] &= A_F x_F(t) + B_F y(t), \\ z_F(t) &= C_F x_F(t) + D_F y(t),\end{aligned}\tag{2.66}$$

where $x_F(t) \in \mathbb{R}^n$ is the filter vector, and A_F , B_F , C_F , and D_F are appropriately dimensioned real-valued matrices to be determined.

Defining the filtering error signal $e(t) \triangleq z(t) - z_F(t)$ and augmenting the system \mathcal{S} to include the dynamics of the filter \mathcal{F} , we have the following state-space representation of the filtering error system:

$$\begin{aligned}\mathcal{E} : \delta[\xi(t)] &= \bar{A}(\lambda)\xi(t) + \bar{B}(\lambda)w(t), \\ e(t) &= \bar{C}(\lambda)\xi(t) + \bar{D}(\lambda)w(t)\end{aligned}\tag{2.67}$$

where $\xi(t)$, $\bar{A}(\lambda)$, and $\bar{B}(\lambda)$ are defined in (2.5), and

$$\bar{C}(\lambda) \triangleq [L(\lambda) - D_F C(\lambda) - C_F], \quad \bar{D}(\lambda) \triangleq E(\lambda) - D_F D(\lambda).$$

For any fixed $\lambda \in \Gamma$, the transfer function of the filtering error system from the noise input $w(t)$ to the filtering error $e(t)$ can be written as:

$$T(\delta, \lambda) \triangleq \bar{C}(\lambda) [\delta I - \bar{A}(\lambda)]^{-1} \bar{B}(\lambda) + \bar{D}(\lambda). \quad (2.68)$$

To design filter \mathcal{F} in the H_∞ setting, suppose that the noise $w(t) \in L_2[0, +\infty)$ for the continuous-time case and $w(t) \in l_2[0, +\infty)$ for the discrete-time case. It is still assumed that both system \mathcal{S} and filter \mathcal{F} have zero initial conditions, that is, $x(0) = 0$ and $x_F(0) = 0$. Then, the robust H_∞ filtering problem to be addressed for system \mathcal{S} in this section is formulated as follows.

Robust H_∞ Filtering Problem: Given system \mathcal{S} in (2.64), design a filter \mathcal{F} of the form in (2.66) such that the filtering error system \mathcal{E} in (2.67), for all $\lambda \in \Gamma$, is robustly asymptotically stable, and for all nonzero $w(t) \in L_2[0, +\infty)$ for the continuous-time case or all nonzero $w(t) \in l_2[0, +\infty)$ for the discrete-time case, satisfies

$$\sup_{\lambda \in \Gamma} \frac{\|e\|_2}{\|w\|_2} < \gamma, \quad \left(\text{i.e., } \sup_{\lambda \in \Gamma} \|T(\delta, \lambda)\|_\infty < \gamma \right) \quad (2.69)$$

where $\|e\|_2 = \|e\|_{L_2}$ for the continuous-time case and $\|e\|_2 = \|e\|_{l_2}$ for the discrete-time case, and γ is a given positive constant. Moreover, a filter satisfying the above conditions is referred to as *a filter with a guaranteed robust H_∞ performance bound γ* .

The specification in (2.69) indicates that γ is an upper bound of the worst energy-to-energy gain, i.e., L_2 -induced or l_2 -induced gain, of the filtering error system. For time-varying systems and nonlinear systems that do not have a transfer function representation, H_∞ filtering is usually termed as *energy-to-energy* filtering; as such, the physical meaning of this filtering scheme is apparent. When uncertainty exists, this bound should be guaranteed for all the uncertainties. If the specification in (2.69) is satisfied, the filtering error energy will be bounded by $\gamma \|w\|_2$ for all energy-bounded noise $w(t)$. Smaller γ means the better noise energy attenuation level of a filter. Hence, if possible, it is desirable to (globally or locally) minimize γ in the robust H_∞ filtering problem.

2.2.2 H_∞ Filtering for Nominal Systems

First, we consider the H_∞ filtering problem for nominal systems. A given upper bound of the H_∞ norm of the filtering error system can be characterized by the so-called bounded real lemma (BRL) in terms of LMI [1, 27–29].

Lemma 2.4 (Continuous-Time [1, 27]) *Consider system \mathcal{S} in (2.64) and assume that $R \in \mathcal{R}$ is fixed but arbitrary. Given filter \mathcal{F} in (2.66) and a scalar $\gamma > 0$, the filtering error system \mathcal{E} in (2.67) is asymptotically stable and satisfies (2.69) if and only if the following matrix inequalities*

$$P > 0, \begin{bmatrix} \bar{A}^T P + P \bar{A} & P \bar{B} & \bar{C}^T \\ \bar{B}^T P & -\gamma^2 I & \bar{D}^T \\ \bar{C} & \bar{D} & -I \end{bmatrix} < 0 \quad (2.70)$$

are feasible in the real matrix variable P .

Lemma 2.5 (Discrete-Time [27–29]) Consider system \mathcal{S} in (2.64) and assume that $R \in \mathcal{R}$ is fixed but arbitrary. Given filter \mathcal{F} in (2.66) and a scalar $\gamma > 0$, the filtering error system \mathcal{E} in (2.67) is asymptotically stable and satisfies (2.69) if and only if the following matrix inequality

$$\begin{bmatrix} -P & P \bar{A} & P \bar{B} & 0 \\ \bar{A}^T P & -P & 0 & \bar{C}^T \\ \bar{B}^T P & 0 & -\gamma^2 I & \bar{D}^T \\ 0 & \bar{C} & \bar{D} & -I \end{bmatrix} < 0 \quad (2.71)$$

is feasible in the matrix variable P .

Since the big inequalities in (2.70) and (2.71) are not linear simultaneously with respect to the filter matrices (A_F, B_F, C_F, D_F) and the Lyapunov matrix P , one cannot directly obtain the filter realization from Lemma 2.4 or 2.5 by solving LMI-based convex optimization problems. To convert the NLMIs in (2.70) and (2.71) into LMIs, we also resort to a linearization procedure similar to the one used in dealing with H_2 filtering problems in Sect. 2.1. Now, we give the following necessary and sufficient condition for the existence of H_∞ filters for both continuous- and discrete-time nominal systems.

Theorem 2.6 Consider system \mathcal{S} in (2.64) and assume that $R \in \mathcal{R}$ is fixed but arbitrary. Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.66) exists such that the filtering error system \mathcal{E} in (2.67) is asymptotically stable and satisfies (2.69) if and only if the following inequalities

- for the continuous-time case,

$$\begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 \end{bmatrix} > 0, \begin{bmatrix} X_1 + X_1^T & \tilde{A}_F + X_2^T & Y_1 & L^T - C^T \tilde{D}_F^T \\ * & \tilde{A}_F + \tilde{A}_F^T & Y_2 & -\tilde{C}_F^T \\ * & * & -\gamma^2 I & E^T - D^T \tilde{D}_F^T \\ * & * & * & -I \end{bmatrix} < 0 \quad (2.72)$$

- for the discrete-time case,

$$\begin{bmatrix} -\tilde{P}_1 & -\tilde{P}_2 & X_1 & \tilde{A}_F & Y_1 & 0 \\ * & -\tilde{P}_2 & X_2 & \tilde{A}_F & Y_2 & 0 \\ * & * & -\tilde{P}_1 & -\tilde{P}_2 & 0 & L^T - C^T \tilde{D}_F^T \\ * & * & * & -\tilde{P}_2 & 0 & -\tilde{C}_F^T \\ * & * & * & * & -\gamma^2 I & E^T - D^T \tilde{D}_F^T \\ * & * & * & * & * & -I \end{bmatrix} < 0 \quad (2.73)$$

are feasible in the real matrix variables \tilde{P}_1 , \tilde{P}_2 , \tilde{A}_F , \tilde{B}_F , \tilde{C}_F , and \tilde{D}_F , where

$$\begin{aligned} X_1 &\triangleq \tilde{P}_1 A + \tilde{B}_F C, \quad X_2 \triangleq \tilde{P}_2 A + \tilde{B}_F C, \\ Y_1 &\triangleq \tilde{P}_1 B + \tilde{B}_F D, \quad Y_2 \triangleq \tilde{P}_2 B + \tilde{B}_F D, \end{aligned}$$

Moreover, if these conditions are feasible, an admissible state-space realization of the filter \mathcal{F} in (2.66) is given by

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \begin{bmatrix} \tilde{P}_2^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & \tilde{D}_F \end{bmatrix} \quad (2.74)$$

or

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & \tilde{D}_F \end{bmatrix} \begin{bmatrix} \tilde{P}_2^{-1} & 0 \\ 0 & I \end{bmatrix}. \quad (2.75)$$

Proof In the first part, we prove that (2.72) \Leftrightarrow (2.70) while (2.73) \Leftrightarrow (2.71).

(2.70) \Rightarrow (2.72) and (2.71) \Rightarrow (2.73): Suppose that there exist P such that (2.70) and (2.71) are satisfied. Partition matrix P into four blocks as the following form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}$$

with each block being an $n \times n$ matrix. The nonsingularity of P enables us to assume that, by invoking a small perturbation if necessary, P_2 is nonsingular. Define multiplier matrix

$$J_0 \triangleq \begin{bmatrix} I & 0 \\ 0 & P_3^{-1} P_2^T \end{bmatrix} \quad (2.76)$$

and introduce the following matrix transformations

$$\begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & \tilde{D}_F \end{bmatrix} \triangleq \begin{bmatrix} P_2 A_F P_3^{-1} P_2^T & P_2 B_F \\ C_F P_3^{-1} P_2^T & D_F \end{bmatrix}, \quad (2.77)$$

$$\begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 \end{bmatrix} \triangleq J_0^T P J_0 = \begin{bmatrix} P_1 & P_2 P_3^{-1} P_2^T \\ P_2 P_3^{-1} P_2^T & P_2 P_3^{-1} P_2^T \end{bmatrix}. \quad (2.78)$$

Then, the following equations can be obtained

$$\begin{aligned} J_0^T P \bar{A} J_0 &= \begin{bmatrix} X_1 & \tilde{A}_F \\ X_2 & \tilde{A}_F \end{bmatrix}, \\ J_0^T P \bar{B} &= \begin{bmatrix} \tilde{P}_1 B + \tilde{B}_F D \\ \tilde{P}_2 B + \tilde{B}_F D \end{bmatrix}, \\ \bar{C} J_0 &= [L - \tilde{D}_F C - \tilde{C}_F]. \end{aligned}$$

Furthermore, by defining

$$J_c \triangleq \text{diag} \{J_0, I_l, I_p\}, \quad J_d \triangleq \text{diag} \{J_0, J_0, I_l, I_p\} \quad (2.79)$$

and noting the equations given above, it readily follows that

$$\begin{aligned} J_c^T \begin{bmatrix} \bar{A}^T P + P \bar{A} & P \bar{B} & \bar{C}^T \\ \bar{B}^T P & -\gamma^2 I & \bar{D}^T \\ \bar{C} & \bar{D} & -I \end{bmatrix} J_c &= \begin{bmatrix} X_1 + X_1^T & \tilde{A}_F + X_2^T & Y_1 & L^T - C^T \tilde{D}_F^T \\ * & \tilde{A}_F + \tilde{A}_F^T & Y_2 & -\tilde{C}_F^T \\ * & * & -\gamma^2 I & E^T - D^T \tilde{D}_F^T \\ * & * & 0 & -I \end{bmatrix}, \\ J_d^T \begin{bmatrix} -P & P \bar{A} & P \bar{B} & 0 \\ \bar{A}^T P - P & 0 & \bar{C}^T \\ \bar{B}^T P & 0 & -\gamma^2 I & \bar{D}^T \\ 0 & \bar{C} & \bar{D} & -I \end{bmatrix} J_d &= \begin{bmatrix} -\tilde{P}_1 - \tilde{P}_2 & X_1 & \tilde{A}_F & Y_1 & 0 \\ * & -\tilde{P}_2 & X_2 & \tilde{A}_F & Y_2 & 0 \\ * & * & -\tilde{P}_1 - \tilde{P}_2 & 0 & L^T - C^T \tilde{D}_F^T \\ * & * & * & -\tilde{P}_2 & 0 & -\tilde{C}_F^T \\ * & * & * & * & -\gamma^2 I & E^T - D^T \tilde{D}_F^T \\ * & * & * & * & * & -I \end{bmatrix}, \end{aligned}$$

which together with (2.78) imply that (2.70) \Rightarrow (2.72) and (2.71) \Rightarrow (2.73).

(2.70) \Leftarrow (2.72) and (2.71) \Leftarrow (2.73): Suppose that there exist \tilde{P}_1 , \tilde{P}_2 , \tilde{A}_F , \tilde{B}_F , \tilde{C}_F , and \tilde{D}_F such that (2.72) and (2.73) are satisfied. Obviously, \tilde{P}_2 is nonsingular, which means that it is always possible to find two invertible matrices P_2 and P_3 such that $\tilde{P}_2 = P_2 P_3^{-1} P_2^T$. Introduce the following definitions

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \triangleq \begin{bmatrix} P_2^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_F & \tilde{B}_F \\ \tilde{C}_F & \tilde{D}_F \end{bmatrix} \begin{bmatrix} P_2^{-T} P_3 & 0 \\ 0 & I \end{bmatrix}, \quad (2.80)$$

$$P \triangleq J_0^{-T} \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 \end{bmatrix} J_0^{-1} \quad (2.81)$$

with J_0 defined in (2.76). By these definitions, the conditions in (2.72) and (2.73) can be rewritten as

$$J_c^T \begin{bmatrix} \bar{A}^T P + P \bar{A} & P \bar{B} & \bar{C}^T \\ \bar{B}^T P & -\gamma^2 I & \bar{D}^T \\ \bar{C} & \bar{D} & -I \end{bmatrix} J_c < 0,$$

$$J_d^T \begin{bmatrix} -P & P \bar{A} & P \bar{B} & 0 \\ \bar{A}^T P - P & 0 & \bar{C}^T \\ \bar{B}^T P & 0 & -\gamma^2 I & \bar{D}^T \\ 0 & \bar{C} & \bar{D} & -I \end{bmatrix} J_d < 0,$$

with J_c and J_d defined in (2.79). Since both J_c and J_d are nonsingular, the above two conditions imply that there exist a variable P in (2.81) together with filter realization (A_F, B_F, C_F, D_F) in (2.80) such that (2.70) and (2.71) hold. Hence, we have (2.70) \Leftrightarrow (2.72) and (2.71) \Leftrightarrow (2.73). Consequently, by Lemma 2.4 and 2.5, Theorem 2.6 gives a necessary and sufficient condition that can ensure the existence of a filter with a guaranteed H_∞ performance level bound γ .

From the expression of (A_F, B_F, C_F, D_F) in (2.80), admissible state-space realizations of the filter \mathcal{F} can be obtained from (2.74) or (2.75) by following the second part of the proof of Theorem 2.1. The entire proof is completed. \square

All the variables, including the square of the H_∞ performance bound γ , are in a linear form in (2.72) and (2.73). Hence, applying Theorem 2.6, we can design the optimal H_∞ filters for nominal systems by the LMI technique, which is summarized in Algorithm 7.

Algorithm 7 H_∞ Filter Design

- Solve the minimization problem:

- Continuous-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \tilde{D}_F, \mu} \gamma^2 = \mu \quad \text{s.t. (2.72)}$$

- Discrete-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \tilde{D}_F, \mu} \gamma^2 = \mu \quad \text{s.t. (2.73)}$$

- Compute filter \mathcal{F} in (2.66) by (2.74) or (2.75).
-

2.2.3 Quadratic Robust H_∞ Filtering

It is well-known that, from a time-domain point of view, the sufficiency of the BRLs can be easily verified through the Lyapunov function or functional method [1]. For instance, the conditions in Lemma 2.4 and 2.5 can be established by the following procedures. Suppose (2.70) and (2.71) hold for some $P > 0$, then by the Schur complement, the quadratic Lyapunov function

$$V(t) = \bar{x}(t)^T P x(t) \tag{2.82}$$

satisfies, for the continuous-time case,

$$\begin{aligned} \dot{V}(t) + e(t)^T e(t) - \gamma^2 w(t)^T w(t) &= \xi(t)^T \begin{bmatrix} \bar{A}^T P + P \bar{A} + \bar{C}^T \bar{C} & P \bar{B} \\ \bar{B}^T P & -\gamma^2 I \end{bmatrix} \xi(t) \\ &< 0, \end{aligned}$$

and, for the discrete-time case,

$$\begin{aligned} \Delta V(t) + e(t)^T e(t) - \gamma^2 w(t)^T w(t) &= \xi(t)^T \begin{bmatrix} \bar{A}^T P \bar{A} - P + \bar{C}^T \bar{C} & \bar{A}^T P \bar{B} \\ \bar{B}^T P \bar{A} & \bar{B}^T P \bar{B} - \gamma^2 I \end{bmatrix} \\ &\quad \times \xi(t) \\ &< 0, \end{aligned}$$

with $\xi(t) = \text{col}\{\bar{x}(t), w(t)\}$. Noting $V(+\infty) \geq 0$ and integrating or summing the inequalities from 0 to $+\infty$, we have, under the initial condition $\bar{x}(0) = 0$,

$$\begin{aligned} \|e\|_{L_2}^2 - \gamma^2 \|w\|_{L_2}^2 &= \int_0^{+\infty} \left(\dot{V}(t) + e(t)^T e(t) - \gamma^2 w(t)^T w(t) \right) dt - V(+\infty) \\ &\leq \int_0^{+\infty} \left(\dot{V}(t) + e(t)^T e(t) - \gamma^2 w(t)^T w(t) \right) dt < 0, \\ \|e\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 &= \sum_0^{+\infty} \left(\Delta V(t) + e(t)^T e(t) - \gamma^2 w(t)^T w(t) \right) - V(+\infty) \\ &\leq \sum_0^{+\infty} \left(\Delta V(t) + e(t)^T e(t) - \gamma^2 w(t)^T w(t) \right) < 0. \end{aligned}$$

Consequently, the specification in (2.69) is established.

The above derivation shows that the results in Lemma 2.4 and 2.5 are essentially based on the specific *single quadratic* Lyapunov function in (2.82). Moreover, when parameter uncertainties exist in system \mathcal{S} , the above derivation is still valid by using this quadratic Lyapunov function, corresponding to the application of the notion of quadratic stability [10–12]. Accordingly, for the case of parametric uncertainty, system matrices in (2.70) and (2.71) should be modified to be in the parameter-dependent form, that is,

$$\begin{aligned} &\begin{bmatrix} \bar{A}(\lambda)^T P + P \bar{A}(\lambda) & P \bar{B}(\lambda) & \bar{C}(\lambda)^T \\ \bar{B}^T P(\lambda) & -\gamma^2 I & \bar{D}(\lambda)^T \\ \bar{C}(\lambda) & 0 & -I \end{bmatrix} < 0, \\ &\begin{bmatrix} -P & P \bar{A}(\lambda) & P \bar{B}(\lambda) & 0 \\ \bar{A}(\lambda)^T P & -P & 0 & \bar{C}(\lambda)^T \\ \bar{B}(\lambda)^T P & 0 & -\gamma^2 I & \bar{D}(\lambda)^T \\ 0 & \bar{C}(\lambda) & \bar{D}(\lambda) & -I \end{bmatrix} < 0. \end{aligned}$$

Due to the convex polytopic assumption of $R(\lambda)$ in (2.65), the above infinite number of parameter-dependent LMIs hold if and only if the corresponding LMIs at the vertices are satisfied. Hence, the following BRL based on a quadratic approach is obtained for the robust H_∞ performance analysis of the filtering error system with polytopic uncertainty.

Lemma 2.6 *Given system \mathcal{S} in (2.64), filter \mathcal{F} in (2.66), and a scalar $\gamma > 0$, the filtering error system \mathcal{E} in (2.67) is robustly asymptotically stable and satisfies (2.69) for all $\lambda \in \Gamma$ if the following inequalities*

- *for the continuous-time case,*

$$P > 0, \begin{bmatrix} \bar{A}_i^T P + P \bar{A}_i & P \bar{B}_i & \bar{C}_i^T \\ \bar{B}_i^T P & -\gamma^2 I & \bar{D}_i^T \\ \bar{C}_i & \bar{D}_i & -I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s), \quad (2.83)$$

- *for the discrete-time case,*

$$\begin{bmatrix} -P & P \bar{A}_i & P \bar{B}_i & 0 \\ \bar{A}_i^T P & -P & 0 & \bar{C}_i^T \\ \bar{B}_i^T P & 0 & -\gamma^2 I & \bar{D}_i^T \\ 0 & \bar{C}_i & \bar{D}_i & -I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s), \quad (2.84)$$

are feasible in the real matrix variable P .

Proof Noting that $\sum_{i=1}^s \lambda_i = 1$ and $\lambda_i \geq 0$, we have

$$\begin{aligned} \begin{bmatrix} \bar{A}(\lambda)^T P + P \bar{A}(\lambda) & P \bar{B}(\lambda) & \bar{C}(\lambda)^T \\ \bar{B}^T P(\lambda) & -\gamma^2 I & 0 \\ \bar{C}(\lambda) & 0 & -I \end{bmatrix} &= \sum_{i=1}^s \lambda_i \begin{bmatrix} \bar{A}_i^T P + P \bar{A}_i & P \bar{B}_i & \bar{C}_i^T \\ \bar{B}_i^T P & -\gamma^2 I & 0 \\ \bar{C}_i & 0 & -I \end{bmatrix} < 0, \\ \begin{bmatrix} -P & P \bar{A}(\lambda) & P \bar{B}(\lambda) & 0 \\ \bar{A}(\lambda)^T P & -P & 0 & \bar{C}(\lambda)^T \\ \bar{B}(\lambda)^T P & 0 & -\gamma^2 I & 0 \\ 0 & \bar{C}(\lambda) & 0 & -I \end{bmatrix} &= \sum_{i=1}^s \lambda_i \begin{bmatrix} -P & P \bar{A}_i & P \bar{B}_i & 0 \\ \bar{A}_i^T P & -P & 0 & \bar{C}_i^T \\ \bar{B}_i^T P & 0 & -\gamma^2 I & 0 \\ 0 & \bar{C}_i & 0 & -I \end{bmatrix} < 0, \end{aligned}$$

which together with the above discussion can complete the proof. \square

By applying the linearization procedures in the derivation of Theorem 2.6 to the results in Lemma 2.6, we have the following theorem that provides a quadratic approach to robust H_∞ filter design.

Theorem 2.7 *Consider system \mathcal{S} in (2.64). Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.66) exists such that the filtering error system \mathcal{E} in (2.67) is robustly asymptotically stable and satisfies (2.69) if the following inequalities*

- for the continuous-time case,

$$\begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 \end{bmatrix} > 0, \quad (2.85)$$

$$\begin{bmatrix} X_{1,i} + X_{1,i}^T & \tilde{A}_F + X_{2,i}^T & Y_{1,i} & L_i^T - C_i^T \tilde{D}_F^T \\ * & \tilde{A}_F + \tilde{A}_F^T & Y_{2,i} & -\tilde{C}_F^T \\ * & * & -\gamma^2 I & E_i^T - D_i^T \tilde{D}_F^T \\ * & * & * & -I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s), \quad (2.86)$$

- for the discrete-time case,

$$\begin{bmatrix} -\tilde{P}_1 - \tilde{P}_2 & X_{1,i} & \tilde{A}_F & Y_{1,i} & 0 \\ * & -\tilde{P}_2 & X_{2,i} & \tilde{A}_F & Y_{2,i} & 0 \\ * & * & -\tilde{P}_1 - \tilde{P}_2 & 0 & L_i^T - C_i^T \tilde{D}_F^T \\ * & * & * & -\tilde{P}_2 & 0 & -\tilde{C}_F^T \\ * & * & * & * & -\gamma^2 I & E_i^T - D_i^T \tilde{D}_F^T \\ * & * & * & * & * & -I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s), \quad (2.87)$$

are feasible in the real matrix variables \tilde{P}_1 , \tilde{P}_2 , \tilde{A}_F , \tilde{B}_F , \tilde{C}_F , and \tilde{D}_F , where

$$\begin{aligned} X_{1,i} &\triangleq \tilde{P}_1 A_i + \tilde{B}_F C_i, \quad X_{2,i} \triangleq \tilde{P}_2 A_i + \tilde{B}_F C_i, \\ Y_{1,i} &\triangleq \tilde{P}_1 B_i + \tilde{B}_F D_i, \quad Y_{2,i} \triangleq \tilde{P}_2 B_i + \tilde{B}_F D_i. \end{aligned}$$

Moreover, if these conditions are feasible, an admissible state space realization of the filter \mathcal{F} in (2.66) is given by (2.74) or (2.75).

Based on Theorem 2.7, Algorithm 8 gives a quadratic approach to the design of robust filters with a suboptimal robust H_∞ performance bound γ .

2.3 Quadratic Robust Energy-to-Peak Filter Design

In Sect. 2.2.3, under the energy-bounded assumption for noises, a robust H_∞ filter design method has been proposed for designing filters that guarantee a prescribed noise energy attenuation level γ for all the uncertainties.

In this section, also under the energy-bounded assumption for noises, we consider another filtering scheme, i.e., energy-to-peak filtering (in contrast to the energy-to-energy gain meaning of H_∞ filtering). For this filtering strategy, it is self-explanatory that the worst possible peak value, instead of the energy value, is applied to evaluate the size of the filtering error.

Algorithm 8 Quadratic Approach to Robust H_∞ Filter Design

- Solve the minimization problem:

- Continuous-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \tilde{D}_F, \mu} \gamma^2 = \mu \quad \text{s.t. (2.85) and (2.86)}$$

- Discrete-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \tilde{D}_F, \mu} \gamma^2 = \mu \quad \text{s.t. (2.87)}$$

- Compute filter \mathcal{F} in (2.66) by (2.74) or (2.81).
-

2.3.1 Problem Formulation

Also consider the uncertain stable system \mathcal{S} in (2.1), the full-order proper filter \mathcal{F} in (2.4), and the resulting filtering error system \mathcal{E} in (2.5). To design a filter \mathcal{F} in the energy-to-peak setting, suppose that the noise $w(t) \in L_2[0, +\infty)$ for the continuous-time case and $w(t) \in l_2[0, +\infty)$ for the discrete-time case, and that both system \mathcal{S} and filter \mathcal{F} have zero initial conditions, $x(0) = 0$ and $x_F(0) = 0$. The robust energy-to-peak filtering problem to be addressed for system \mathcal{S} in this section is formulated as follows.

Robust Energy-to-Peak Filtering Problem: Given system \mathcal{S} in (2.1), design a filter \mathcal{F} of the form in (2.4) such that the filtering error system \mathcal{E} in (2.5), for all $\lambda \in \Gamma$, is robustly asymptotically stable, and for all nonzero $w(t) \in L_2[0, +\infty)$ for the continuous-time case or for all nonzero $w(t) \in l_2[0, +\infty)$ for the discrete-time case, satisfies

$$\sup_{\lambda \in \Gamma} \frac{\|e\|_\infty}{\|w\|_2} < \gamma \quad (2.88)$$

where $\|e\|_\infty = \|e\|_{L_\infty}$ and $\|w\|_2 = \|w\|_{L_2}$ for the continuous-time case, $\|e\|_\infty = \|e\|_{l_\infty}$ and $\|w\|_2 = \|w\|_{l_2}$ for the discrete-time case and γ is a given positive constant. Moreover, a filter satisfying the above conditions is referred to as a *filter with a guaranteed robust energy-to-peak performance bound* γ .

The physical meaning of the specification in (2.69) is that the worst energy-to-peak ($L_2 - L_\infty$ or $l_2 - l_\infty$) gain of the filtering error system is bounded by a given level γ for all the uncertainties. If the specification in (2.69) is satisfied, the peak value of the filtering error will not exceed $\gamma \|w\|_2$ for all energy-bounded noise $w(t)$. The strictly proper form of the filter \mathcal{F} in (2.4) guarantees that the energy-to-peak gain of the filtering error system is finite, that is, there always exists some sufficiently large γ such that (2.88) can be satisfied. Smaller γ means the better filtering performance of a filter in the energy-to-peak setting. Hence, it is often expected to (globally or locally) minimize γ in the above energy-to-peak filtering problem.

2.3.2 Energy-to-Peak Filtering for Nominal Systems

For nominal systems, the following result provides an energy-to-peak performance criterion in terms of LMI for filtering performance analysis.

Lemma 2.7 ([28, 29]) *Consider system \mathcal{S} in (2.1) and assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given filter \mathcal{F} in (2.4) and a scalar $\gamma > 0$, the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies (2.88) if and only if the following matrix inequalities*

- *for the continuous-time case*

$$\begin{bmatrix} -\gamma^2 I & \bar{C} \\ \bar{C}^T & -P \end{bmatrix} < 0, \begin{bmatrix} \bar{A}^T P + P \bar{A} & P \bar{B} \\ \bar{B}^T P & -I \end{bmatrix} < 0 \quad (2.89)$$

- *for the discrete-time case*

$$\begin{bmatrix} -\gamma^2 I & \bar{C} \\ \bar{C}^T & -P \end{bmatrix} < 0, \begin{bmatrix} -P & P \bar{A} & P \bar{B} \\ \bar{A}^T P & -P & 0 \\ \bar{B}^T P & 0 & -I \end{bmatrix} < 0 \quad (2.90)$$

are feasible in the real matrix variable P .

Although the result in Lemma 2.7 does not give the filter realization explicitly, the NLMI in (2.89) and (2.90) can be converted into LMIs by noting the similarity between the inequalities in (2.89) and (2.90) and those in Lemmas 2.1 and 2.2 for H_2 filtering. Hence, by performing a series of similar matrix transformations, the following necessary and sufficient condition can be obtained for energy-to-peak filter design, leading to an LMI approach to solve the energy-to-peak filtering problem for nominal systems.

Theorem 2.8 *Assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.4) exists such that the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies (2.88) if and only if the following inequality*

$$\begin{bmatrix} -\gamma^2 I & L & -\tilde{C}_F \\ L^T & -\tilde{P}_1 & -\tilde{P}_2 \\ -\tilde{C}_F^T & -\tilde{P}_2 & -\tilde{P}_2 \end{bmatrix} < 0 \quad (2.91)$$

and

- *for the continuous-time case*

$$\begin{bmatrix} X_1 + X_1^T & \tilde{A}_F + X_1^T \tilde{P}_1 B + \tilde{B}_F D \\ * & \tilde{A}_F + \tilde{A}_F^T \tilde{P}_2 B + \tilde{B}_F D \\ * & * & -I \end{bmatrix} < 0 \quad (2.92)$$

- for the discrete-time case

$$\begin{bmatrix} -\tilde{P}_1 & -\tilde{P}_2 & X_1 & \tilde{A}_F & \tilde{P}_1 B + \tilde{B}_F D \\ * & -\tilde{P}_2 & X_2 & \tilde{A}_F & \tilde{P}_2 B + \tilde{B}_F D \\ * & * & -\tilde{P}_1 & -\tilde{P}_2 & 0 \\ * & * & * & -\tilde{P}_2 & 0 \\ * & * & * & * & I \end{bmatrix} < 0 \quad (2.93)$$

are feasible in the real matrix variables \tilde{P}_1 , \tilde{P}_2 , \tilde{A}_F , \tilde{B}_F , and \tilde{C}_F , where

$$X_1 \triangleq \tilde{P}_1 A + \tilde{B}_F C, \quad X_2 \triangleq \tilde{P}_2 A + \tilde{B}_F C.$$

Moreover, if these conditions are feasible, an admissible state-space realization of the filter \mathcal{F} in (2.4) is given by (2.61) or (2.62).

Based on Theorem 2.8, we have Algorithm 9 for designing the optimal energy-to-peak filters for nominal systems by the LMI technique.

Algorithm 9 Energy-to-Peak Filter Design I

- Solve the minimization problem:

- Continuous-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \mu} \gamma^2 = \mu \quad \text{s.t. (2.91), (2.92)}$$

- Discrete-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \mu} \gamma^2 = \mu \quad \text{s.t. (2.91), (2.93)}$$

- Compute filter \mathcal{F} in (2.4) by (2.61) or (2.62).
-

One may note that matrix variable \tilde{C}_F appears only in the inequality in (2.91). Hence, by Lemma 1.3, it is also possible to eliminate \tilde{C}_F from the result in Theorem 2.8, so that the number of free variables can be reduced. The derivation and the result will be similar to the case of Theorems 2.2 and 2.4 dealing with H_2 filtering. With the proof omitted, a modified algorithm (Algorithm 10) is presented for the optimal energy-to-peak filter design for nominal systems, where \tilde{C}_F can be prescribed as $\tilde{C}_F = L$ in advance and no conservatism is introduced.

2.3.3 Quadratic Robust Energy-to-Peak Filtering

When there exists uncertain but fixed parameter λ , the energy-to-peak gain of the filtering system satisfies [28–30]

Algorithm 10 Energy-to-Peak Filter Design II

- Solve the minimization problem:

- Continuous-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \mu} \gamma^2 = \mu \quad \text{s.t.} \quad \begin{cases} (2.92) \\ \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 \end{bmatrix} > 0, \begin{bmatrix} \gamma^2 I & L \\ L^T & \tilde{P}_1 \end{bmatrix} > 0 \end{cases}$$

- Discrete-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \mu} \gamma^2 = \mu \quad \text{s.t.} \quad \begin{cases} (2.93) \\ \begin{bmatrix} \gamma^2 I & L \\ L^T & \tilde{P}_1 \end{bmatrix} > 0 \end{cases}$$

- Compute filter \mathcal{F} in (2.4) by

$$A_F = \tilde{P}_1^{-1} \tilde{A}_F \tilde{P}_2^{-1} \tilde{P}_1, \quad B_F = -\tilde{P}_1^{-1} \tilde{B}_F, \quad C_F = L.$$

$$\sup_{\lambda \in \Gamma} \frac{\|e\|_\infty}{\|w\|_2} = \sup_{\lambda \in \Gamma} \rho_{\max}^{1/2}[C(\lambda)P(\lambda)C(\lambda)^T],$$

where $\rho_{\max}[\cdot]$ represents the spectral radius of a square matrix, and Lyapunov matrix $P(\lambda)$ is the solution to parameter-dependent Lyapunov equation

$$\bar{A}(\lambda)P(\lambda) + P(\lambda)\bar{A}(\lambda)^T + \bar{B}(\lambda)\bar{B}(\lambda)^T = 0$$

for the continuous-time case, or

$$\bar{A}(\lambda)P(\lambda)\bar{A}(\lambda)^T - P(\lambda) + \bar{B}(\lambda)\bar{B}(\lambda)^T = 0$$

for the discrete-time case. Following the notion of quadratic stability, suppose that (2.89) and (2.90) with matrices $(\bar{A}, \bar{B}, \bar{C})$ replaced by its parameter-dependent counterpart $(\bar{A}(\lambda), \bar{B}(\lambda), \bar{C}(\lambda))$ are still valid for all $M(\lambda) \in \mathcal{M}$. By the Schur complement, the second inequalities in (2.89) and (2.90) imply

$$\begin{aligned} \bar{A}(\lambda)P^{-1} + P^{-1}\bar{A}(\lambda)^T + \bar{B}(\lambda)\bar{B}(\lambda)^T &< 0, \\ \bar{A}(\lambda)P^{-1}\bar{A}(\lambda)^T - P^{-1} + \bar{B}(\lambda)\bar{B}(\lambda)^T &< 0. \end{aligned}$$

From the monotonic property of the solution to Lyapunov equations, there holds $P(\lambda) < P^{-1}$ for all $\lambda \in \Gamma$, which together with the first inequalities in (2.89) and (2.90), respectively, gives rise to

$$\begin{aligned}
\sup_{\lambda \in \Gamma} \frac{\|e\|_\infty}{\|w\|_2} &= \sup_{\lambda \in \Gamma} \rho_{\max}^{1/2}[C(\lambda)P(\lambda)C(\lambda)^T] \\
&< \sup_{\lambda \in \Gamma} \rho_{\max}^{1/2}[C(\lambda)P^{-1}C(\lambda)^T] \\
&< \sup_{\lambda \in \Gamma} \rho_{\max}^{1/2}[\gamma^2 I] = \gamma.
\end{aligned}$$

Consequently, along the quadratic stability notion, the following lemma provides a quadratic approach to robust energy-to-peak performance analysis for the filtering error system under a given filter.

Lemma 2.8 *Consider system \mathcal{S} in (2.1) and assume that $M \in \mathcal{M}$ is fixed but arbitrary. Given filter \mathcal{F} in (2.4) and a scalar $\gamma > 0$, the filtering error system \mathcal{E} in (2.5) is robustly asymptotically stable and satisfies (2.88) if the following matrix inequalities*

- *for the continuous-time case*

$$\begin{bmatrix} -\gamma^2 I & \bar{C}_i \\ \bar{C}_i^T & -P \end{bmatrix} < 0, \begin{bmatrix} \bar{A}_i^T P + P \bar{A}_i & P \bar{B}_i \\ \bar{B}_i^T P & -I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s)$$

- *for the discrete-time case*

$$\begin{bmatrix} -\gamma^2 I & \bar{C}_i \\ \bar{C}_i^T & -P \end{bmatrix} < 0, \begin{bmatrix} -P & P \bar{A}_i & P \bar{B}_i \\ \bar{A}_i^T P & -P & 0 \\ \bar{B}_i^T P & 0 & -I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s)$$

are feasible in the real matrix variable P .

Furthermore, linearizing the NLMI conditions in this lemma results in the following quadratic approach to robust energy-to-peak filter design.

Theorem 2.9 *Consider system \mathcal{S} in (2.1). Given a scalar $\gamma > 0$, a filter \mathcal{F} in (2.4) exists such that the filtering error system \mathcal{E} in (2.5) is asymptotically stable and satisfies (2.88) if and only if the following inequality*

$$\begin{bmatrix} -\gamma^2 I & L_i & -\tilde{C}_F \\ L_i^T & -\tilde{P}_1 & -\tilde{P}_2 \\ -\tilde{C}_F^T & -\tilde{P}_2 & -\tilde{P}_2 \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s) \quad (2.94)$$

and

- *for the continuous-time case*

$$\begin{bmatrix} X_{1,i} + X_{1,i}^T & \tilde{A}_F + X_{2,i}^T & \tilde{P}_1 B_i + \tilde{B}_F D_i \\ * & \tilde{A}_F + \tilde{A}_F^T & \tilde{P}_2 B_i + \tilde{B}_F D_i \\ * & * & -I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s) \quad (2.95)$$

- for the discrete-time case

$$\begin{bmatrix} -\tilde{P}_1 & -\tilde{P}_2 & X_{1,i} & \tilde{A}_F & \tilde{P}_1 B_i + \tilde{B}_F D_i \\ * & -\tilde{P}_2 & X_{2,i} & \tilde{A}_F & \tilde{P}_2 B_i + \tilde{B}_F D_i \\ * & * & -\tilde{P}_1 & -\tilde{P}_2 & 0 \\ * & * & * & -\tilde{P}_2 & 0 \\ * & * & * & * & I \end{bmatrix} < 0 \quad (i = 1, 2, \dots, s) \quad (2.96)$$

are feasible in the real matrix variables \tilde{P}_1 , \tilde{P}_2 , \tilde{A}_F , \tilde{B}_F , and \tilde{C}_F , where

$$X_{1,i} \triangleq \tilde{P}_1 A_i + \tilde{B}_F C_i, \quad X_{2,i} \triangleq \tilde{P}_2 A_i + \tilde{B}_F C_i.$$

Moreover, if these conditions are feasible, an admissible state-space realization of the filter \mathcal{F} in (2.4) is given by (2.61) or (2.62).

Based on Theorem 2.9, two quadratic approaches (Algorithms 11 and 12) to design suboptimal robust energy-to-peak filters can be obtained for unknown and known L , respectively.

Algorithm 11 Quadratic Approach to Robust Energy-to-Peak Filter Design

- Solve the minimization problem:

- Continuous-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \mu} \gamma^2 = \mu \quad \text{s.t. (2.94), (2.95)}$$

- Discrete-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \mu} \gamma^2 = \mu \quad \text{s.t. (2.94), (2.96)}$$

- Compute filter \mathcal{F} in (2.4) by (2.61) or (2.62).
-

Finally, it deserves pointing out that, for single-input-single-output (SISO) systems, the energy-to-peak performance of a transfer function coincides with its H_2 norm. According to [31], the energy-to-peak performance of a continuous-time transfer function can be defined in the frequency-domain as

$$\|T(j\omega)\|_{L_2-L_\infty} = \sqrt{\frac{1}{2\pi} \lambda_{\max} \left[\int_{-\infty}^{\infty} T(j\omega) T(j\omega)^* d\omega \right]}$$

where $\lambda_{\max}[\cdot]$ denotes the maximum eigenvalue of a matrix, while in the frequency-domain, the H_2 norm of a transfer function is given by [1].

Algorithm 12 Quadratic Approach to Robust Energy-to-Peak Filter Design (Known L)

- Solve the minimization problem:

- Continuous-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \mu} \gamma^2 = \mu \quad \text{s.t.} \quad \begin{cases} (2.95) \\ \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 \end{bmatrix} > 0, \begin{bmatrix} \gamma^2 I & L \\ L^T & \tilde{P}_1 \end{bmatrix} > 0 \end{cases}$$

- Discrete-time case

$$\min_{\tilde{P}_1, \tilde{P}_2, \tilde{A}_F, \tilde{B}_F, \mu} \gamma^2 = \mu \quad \text{s.t.} \quad \begin{cases} (2.96) \\ \begin{bmatrix} \gamma^2 I & L \\ L^T & \tilde{P}_1 \end{bmatrix} > 0 \end{cases}$$

- Compute filter \mathcal{F} in (2.4) by

$$A_F = \tilde{P}_1^{-1} \tilde{A}_F \tilde{P}_2^{-1} \tilde{P}_1, \quad B_F = -\tilde{P}_1^{-1} \tilde{B}_F, \quad C_F = L.$$

$$\|T(j\omega)\|_2 = \sqrt{\frac{1}{2\pi} \text{Tr} \left[\int_{-\infty}^{\infty} T(j\omega) T(j\omega)^* d\omega \right]}$$

Obviously, for SISO systems, there holds $\|T(j\omega)\|_{L_2-L_\infty} = \|T(j\omega)\|_2$. On the other hand, this fact can also be obtained from Lemma 2.1 and the continuous-time part of Lemma 2.7, since they are obviously equivalent to each other for SISO systems. Hence, for SISO systems, the quadratic approach to robust energy-to-peak filter design in this section is actually that for robust H_2 filter design in Sect. 2.1.

2.4 Examples

In this section, several numerical examples are provided to demonstrate the effectiveness of the presented filter design methods.

Example 2.1 (H_2 filter design, continuous-time [4, 6, 32–35]) Consider an uncertain continuous-time system given by

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & -1 + 0.3\alpha \\ 1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} w(t), \\ y(t) &= \begin{bmatrix} -100 + 10\beta & 100 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t), \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \end{aligned} \tag{2.97}$$

where α and β are unknown but bounded constant parameters. It is assumed that $\alpha = \beta = 0$ corresponds to the nominal system of (2.97). The goal of this example

is to design H_2 filters in the form in (2.5) by Algorithms 1 and 2 for the nominal system of (2.97) and by Algorithms 5 and 6 (continuous-time case) for the uncertain system of (2.97).

(1) *Nominal system with $\alpha = \beta = 0$*

Solving the minimization problems in Algorithms 1 and 2 yields the same minimum upper bound of the error variance as

$$\gamma^* = 0.0266.$$

The decision matrix variables in Algorithm 1 are also obtained as

$$\begin{aligned}\tilde{P}_{c1} &= \begin{bmatrix} 41.7362 & -16.6945 \\ -16.6945 & 66.7779 \end{bmatrix}, \\ \tilde{P}_{c2} &= \begin{bmatrix} 41.6376 & -16.6605 \\ -16.6605 & 66.6418 \end{bmatrix}, \\ \tilde{A}_F &= \begin{bmatrix} -9,994 & 9,946 \\ 10,044 & -9,996 \end{bmatrix}, \\ \tilde{B}_F &= \begin{bmatrix} 99.9999 \\ -99.9999 \end{bmatrix}, \\ \tilde{C}_F &= \begin{bmatrix} -0.9976 & 0 \end{bmatrix},\end{aligned}$$

which, from (2.17), generate a filter \mathcal{F} with the state-space realization given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} -199.6846 & 198.7382 & 2.0015 \\ 100.7881 & -100.3155 & -1.0002 \\ \hline -0.9976 & 0 & 0 \end{array} \right], \quad (2.98)$$

and, from (2.18), generate another filter \mathcal{F} given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} -199.6846 & 99.1667 & 99.9999 \\ 201.3334 & -99.6667 & -99.9999 \\ \hline -0.0266 & -0.0067 & 0 \end{array} \right]. \quad (2.99)$$

The matrix variables in Algorithm 2 are obtained as

$$\begin{aligned}\tilde{P}_{c1} &= \begin{bmatrix} 41.7362 & -16.6945 \\ -16.6945 & 66.7779 \end{bmatrix}, \\ \tilde{P}_{c2} &= \begin{bmatrix} 41.6361 & -16.6595 \\ -16.6595 & 66.6389 \end{bmatrix}, \\ \tilde{A}_F &= \begin{bmatrix} -9,993 & 9,946 \\ 10,043 & -9,996 \end{bmatrix},\end{aligned}$$

$$\tilde{B}_F = \begin{bmatrix} 100.002 \\ -100.002 \end{bmatrix}.$$

Accordingly, the filter \mathcal{F} generated from (2.21) is given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} -199.6673 & 198.6673 & -1.9967 \\ 100.8336 & -100.3336 & 0.9983 \\ \hline 1 & 0 & 0 \end{array} \right]. \quad (2.100)$$

Note that $BD^T = 0$ and $DD^T = I$. Thus, for the nominal system of this example, the well-known Kalman filter (see Sect. 2.1.3) can be utilized to estimate $z(t)$. To design the Kalman filter for (2.97), we solve the following ARE:

$$AP + PA^T - PC^T C P + BB^T = 0, \quad P \geq 0$$

and obtain the unique solution P as

$$P = \begin{bmatrix} 0.0266 & 0.0067 \\ 0.0067 & 0.0166 \end{bmatrix}.$$

Then, the Kalman filter gain matrices in (2.23) are obtained from

$$\begin{aligned} A_F &= A - PC^T C = \begin{bmatrix} -199.6669 & 198.6669 \\ 100.8335 & -100.3335 \end{bmatrix}, \\ B_F &= PC^T = \begin{bmatrix} -1.9967 \\ 0.9983 \end{bmatrix}, \\ C_F &= L = \begin{bmatrix} 1 & 0 \end{bmatrix}, \end{aligned} \quad (2.101)$$

with the optimal filtering error variance bound given by

$$\mathbf{E} \left[e(t)^T e(t) \right] \leq \text{Tr} \left[L P L^T \right] = 0.0266.$$

It is shown that the filtering error variance bound of the filters in (2.98), (2.99), and (2.100) designed by the optimal H_2 approaches presented in this chapter is equal to that of the Kalman filter in (2.101). Furthermore, comparing the filter parameters presented above, it is realized that, if omitting the numerically computational error, the filter state-space realization in (2.100) is “equivalent” to the Kalman filter in (2.101). Indeed, according to the discussions in Sect. 2.1.3, these results are not surprising due to the fact that the objective of the optimal H_2 filter for the nominal system is the same as that of the Kalman filter.

(2) *Polytopic system with $|\alpha| \leq 1$ and $|\beta| \leq 1$*

For the uncertain case, system (2.97) can be represented by the form of polytopic system \mathcal{S} in (2.1) with four vertices. Applying the quadratic approaches, Algorithms

5 and 6 (continuous-time case), we obtain the same minimum filtering error variance upper bound

$$\gamma^* = 5.7281.$$

Accordingly, the filter state-space realizations obtained from (2.61), (2.62), and (2.63) are, respectively, given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} -0.8748 & 0.0946 & 0.0129 \\ 1.8493 & -1.3812 & -0.0095 \\ \hline -0.7698 & 0.0746 & 0 \end{array} \right], \quad (2.102)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} -2.2372 & -0.3924 & 0.0036 \\ 2.5257 & -0.0189 & -0.0035 \\ \hline -5.7281 & -2.8533 & 0 \end{array} \right], \quad (2.103)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} -1.0428 & 0.1279 & -0.0107 \\ 1.8127 & -1.2135 & 0.0082 \\ \hline 1 & 0 & 0 \end{array} \right]. \quad (2.104)$$

To illustrate the effectiveness of these filters in (2.102) and (2.104) (we omit (2.103) because its transfer function is the same as the one of the filter in (2.102)), the actual values of the H_2 norm square of the filtering error systems under different parameters α and β are depicted in Fig. 2.1, where $T(\delta, \alpha, \beta)$ denotes the transfer function of the filtering error system. It is shown that within the parameter uncertainty domain, the actual values of the H_2 norm square are obviously smaller than the obtained minimum error variance upper bound $\gamma^* = 5.7281$.

In addition, to illustrate the robustness difference of the designed filters against parameter uncertainties, Table 2.2 involves the actual values of the H_2 norm square of the filtering error systems generated from the Kalman filter in (2.101) and the robust H_2 filters in (2.102) and (2.104). Although the Kalman filter can guarantee the optimal filtering error variance level $\gamma^* = 0.0266$ for the nominal system, its performance deteriorates greatly when encountering parameter uncertainties. For instance, the actual filtering error variance level of the Kalman filter for $(\alpha, \beta) = (1, 1)$ is 31.4293, 1181 times larger than the value for the nominal $(\alpha, \beta) = (0, 0)$. The results in Table 2.2 display that the robustness of the Kalman filter against parameter uncertainties is the weakest among the listed filters. Thus, it is necessary to take uncertainties into consideration in the process of designing filters, so that the designed filters are robust enough in practical applications.

Example 2.2 (H_2 filter design, discrete-time, [7, 8, 33, 34, 36, 37]). Consider the following uncertain discrete-time system:

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0.9 & 0.1 + 0.06\alpha \\ 0.01 + 0.05\beta & 0.9 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} w(t), \\ y(t) &= [1 \ 0] x(t) + [0 \ 0 \ \sqrt{2}] w(t), \end{aligned}$$

Fig. 2.1 Actual performance $\|T(\delta, \alpha, \beta)\|_2^2$ with different α and β in Example 2.1 [**a** the filter in Eq. (2.102); **b** the filter in Eq. (2.104)]

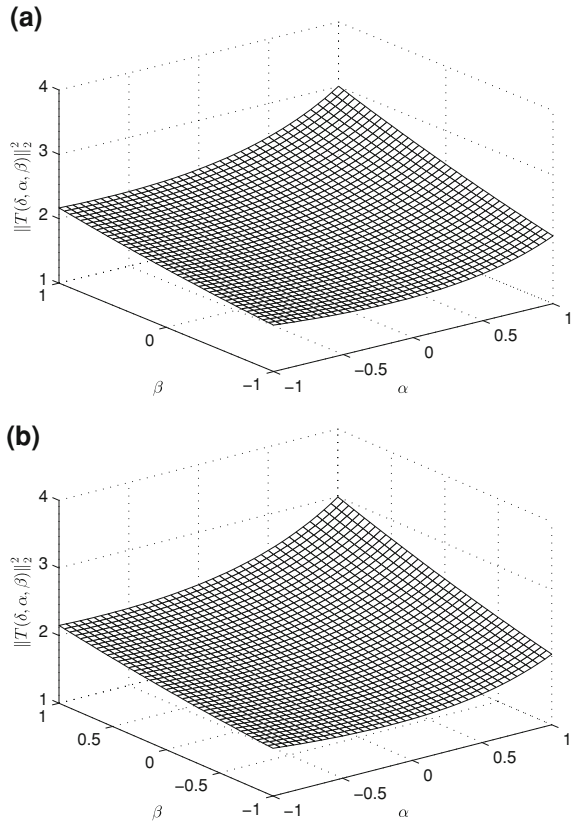


Table 2.2 Robustness comparison in Example 2.1 ((α, β) are in the bracket)

Filter	Bound γ^*	$\ T(\delta, \alpha, \beta)\ _2^2$				
		(0, 0)	(-1, -1)	(-1, 1)	(1, -1)	(1, 1)
Filter in (2.101)	—	0.0266	10.7343	0.2963	2.8822	31.4293
Filter in (2.102)	5.7281	1.8922	1.7217	2.1682	2.0528	3.0028
Filter in (2.104)	5.7281	1.8771	1.7062	2.1492	2.0452	2.9954

$$z(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t), \quad (2.105)$$

where α and β are uncertain parameters satisfying $|\alpha| \leq 1$ and $|\beta| \leq 1$. For the nominal system, it is assumed that $\alpha = \beta = 0$. We will design H_2 filters in the form (2.5) for the nominal system of (2.105) by Algorithms 3 and 4 and for the uncertain system of (2.105) by Algorithms 5 and 6 (discrete-time case), respectively.

(1) *Nominal system with $\alpha = \beta = 0$*

By Algorithms 1 and 2, the same minimum upper bound of the error variance can be obtained as

$$\gamma^* = 8.0759.$$

And the state-space realizations of the filters generated from (2.35), (2.36), and (2.44) are, respectively, given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0.4514 & 0.0616 & -0.5375 \\ -0.1996 & 0.8914 & -0.2509 \\ \hline -0.9705 & -0.4273 & 0 \end{array} \right], \quad (2.106)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0.4811 & 0.0894 & -0.2405 \\ -0.0010 & 0.8616 & 0.0005 \\ \hline -2.6250 & -5.4509 & 0 \end{array} \right], \quad (2.107)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0.4427 & 0.1000 & 0.4573 \\ -0.1615 & 0.9000 & 0.1715 \\ \hline 1 & 1 & 0 \end{array} \right]. \quad (2.108)$$

(2) *Uncertain system with $|\alpha| \leq 1$ and $|\beta| \leq 1$*

The uncertainty polytope of the system in (2.105) consists of four vertices. By Algorithms 5 and 6 (discrete-time case) with $s = 4$, we get the optimal filtering error variance upper bound as

$$\gamma^* = 100.0276,$$

and the state-space realizations of the filters computed from (2.61), (2.62), and (2.63) are, respectively, given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0.0365 & 0.0640 & -0.9239 \\ -0.5890 & 0.9004 & -0.6506 \\ \hline -1.1228 & -0.2914 & 0 \end{array} \right], \quad (2.109)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0.0826 & -0.0768 & -0.0413 \\ -0.0002 & 0.8543 & 0.0001 \\ \hline -29.8402 & -70.1874 & 0 \end{array} \right], \quad (2.110)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0.0383 & 0.0982 & 0.8490 \\ -0.3681 & 0.8986 & 0.3779 \\ \hline 1 & 1 & 0 \end{array} \right]. \quad (2.111)$$

For each filter, the actual values of the H_2 norm square of the filtering error system $T(\delta, \alpha, \beta)$ with respect to uncertain parameters α and β are shown in Fig. 2.2. Note that the actual values of the H_2 norm square are far beneath the obtained guaranteed filtering error variance bound $\gamma^* = 100.0276$. Thus, the designed robust H_2 filters in (2.109) and (2.111) are effective.

Fig. 2.2 Actual performance $\|T(\delta, \alpha, \beta)\|_2^2$ with different α and β in Example 2.2 [**a** the filter in Eq. (2.109); **b** the filter in Eq. (2.111)]

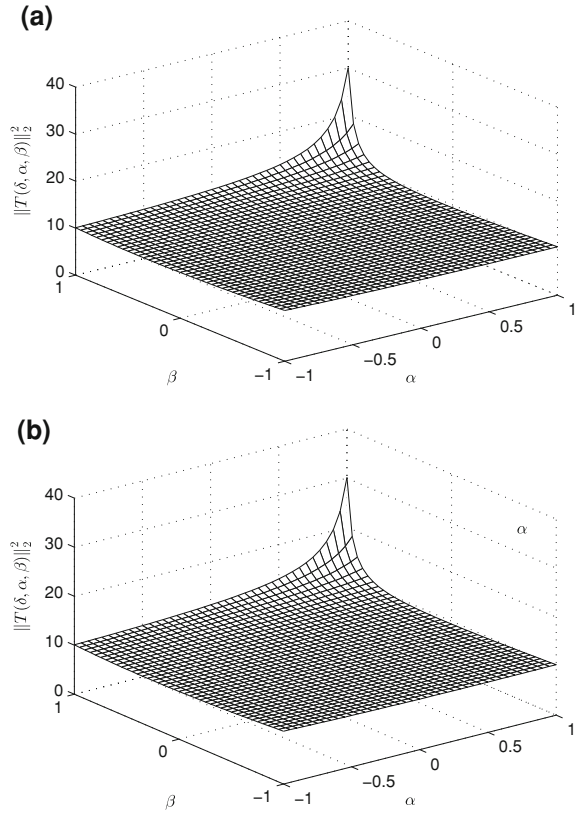


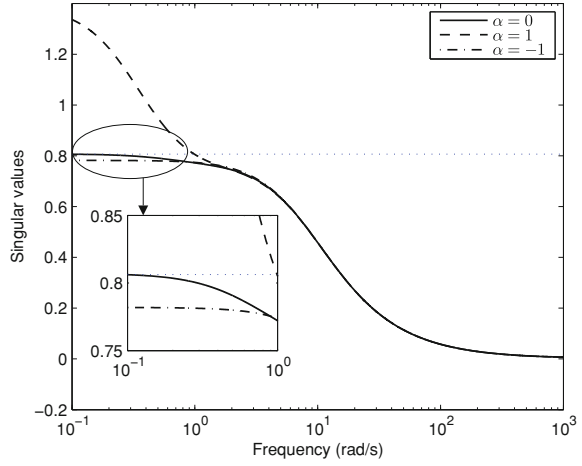
Table 2.3 Robustness comparison in Example 2.2

Filters	Eq. (2.106)	Eq. (2.108)	Eq. (2.109)	Eq. (2.111)
$\sup_{ \alpha \leq 1, \beta \leq 1} \ T(\delta, \alpha, \beta)\ _2^2$	49.4385	49.5109	30.0769	30.1377

Furthermore, Table 2.3 collects results on the robustness comparison between the filters in (2.106), (2.108), (2.109), and (2.111). Although the filters in (2.106) and (2.108) are the optimal with $\gamma^* = 8.0759$ for the nominal system, they are less robust against parameter uncertainties than the robust H_2 filters in (2.109) and (2.111).

Example 2.3 (H_∞ filter design, continuous-time). Consider an uncertain continuous-time system given by

Fig. 2.3 Singular value curves with the filter in Eq. (2.113) under $\alpha = 0, 1$ and -1 in Example 2.3



$$\begin{aligned}
 \dot{x}(t) &= \begin{bmatrix} -2 & 0 \\ -1 & -0.7 + 0.4\alpha \end{bmatrix} x(t) + \begin{bmatrix} -0.5 & 0 \\ 2 & 0 \end{bmatrix} w(t), \\
 y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t), \\
 z(t) &= \begin{bmatrix} 2 & 1 \end{bmatrix} x(t),
 \end{aligned} \tag{2.112}$$

where parameter α satisfies $|\alpha| \leq 1$ for the uncertain system and $\alpha = 0$ for the nominal system. Design H_∞ filters of the form in (2.5), respectively, for the nominal system of (2.112) by Algorithm 7 (continuous-time case) and for the uncertain system of (2.112) by Algorithm 8 (continuous-time case).

For the nominal system, Algorithm 7 provides the optimal H_∞ filtering performance upper bound

$$\gamma^* = 0.8063$$

and yields filters with the following state-space realizations:

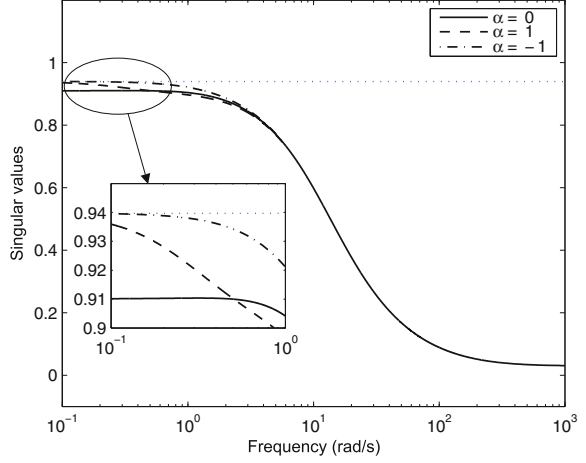
$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \left[\begin{array}{cc|c} -3.1789 & 0.6728 & 0.8479 \\ -2.4339 & -8.2371 & -7.3671 \\ -1.9077 & -0.9881 & 0.0039 \end{array} \right], \tag{2.113}$$

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \left[\begin{array}{cc|c} -5.1363 & 14.4562 & -0.0623 \\ 0.3066 & -6.2797 & -0.6517 \\ 0.8361 & -8.7680 & 0.0039 \end{array} \right], \tag{2.114}$$

which are obtained from (2.74) and (2.75), respectively.

Connecting the filter in (2.113) to the original system in (2.112), we plot the singular values of the filtering error system for different parameter α , shown in Fig. 2.3. On one hand, one can see that the designed optimal H_∞ filter in (2.113) is effective since the maximum singular value of the filtering error system with respect

Fig. 2.4 Singular value curves with the filter in Eq. (2.115) under $\alpha = 0, 1$ and -1 in Example 2.3



to frequency ω is tightly bounded by the obtained optimal H_∞ filtering performance upper bound $\gamma^* = 0.8063$ for the nominal system. On the other hand, Fig. 2.3 clearly shows that the filter cannot guarantee the optimal H_∞ performance level when parameter uncertainty exists. Hence, in the sequel, we focus on design robust H_∞ filters for the system in (2.112).

For the uncertain system, applying Algorithm 8 to design robust H_∞ filters, we obtain the minimum H_∞ filtering performance upper bound as $\gamma^* = 0.9397$ and the following two filters with following state-space realizations:

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \left[\begin{array}{cc|c} -4.1377 & 0.6820 & 0.8987 \\ 0.5488 & -10.1544 & -10.1829 \\ \hline -2.0160 & -0.9713 & 0.0296 \end{array} \right], \quad (2.115)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \left[\begin{array}{cc|c} -5.0838 & 11.4492 & 0.0930 \\ 0.4517 & -9.2084 & -0.8533 \\ \hline 0.4929 & -9.4139 & 0.0296 \end{array} \right]. \quad (2.116)$$

We still connect the filter in (2.115) to the original system in (2.112) and plot the singular values of the filtering error system for different parameter α , shown in Fig. 2.4. Figure 2.4 demonstrates that all the maximum singular values of the filtering error system at the two vertices ($\alpha = 1, -1$) and at the nominal parameter ($\alpha = 0$) are smaller than the guaranteed H_∞ filtering performance upper bound $\gamma^* = 0.9397$. Thus, the effectiveness of the filter in (2.115) (as well as the one in (2.116)) is validated.

Moreover, comparing the results in Figs. 2.3 and 2.4, one clearly sees that the maximum singular value under the filter in (2.113) over the uncertainty domain is larger than 1.2 (seeing the curve for $\alpha = -1$ in Fig. 2.3), which is beyond the guaranteed H_∞ performance upper bound $\gamma^* = 0.9397$ under the filter in (2.115). This

demonstrates that the filter in (2.115) designed by the quadratic approach Algorithm 8 for the uncertain system is more robust than the one in (2.113) designed by the optimal approach Algorithm 7 for the nominal system.

Example 2.4 (H_∞ filter design, discrete-time) Consider an uncertain discrete-time system given by

$$\begin{aligned} x(t+1) &= \begin{bmatrix} \alpha & -0.5 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -6 & 0 \\ 2 & 0 \end{bmatrix} w(t), \\ y(t) &= \begin{bmatrix} -100 & 10 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t), \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \end{aligned} \quad (2.117)$$

Uncertain parameter α is assumed as $\alpha = 0$ for the nominal system and $|\alpha| \leq 0.25$ for the uncertain system. We apply Algorithm 7 (discrete-time case) and Algorithm 8 (discrete-time case) to design H_∞ filters for the nominal system and the uncertain system of (2.117), respectively.

For the nominal system, the optimal H_∞ performance upper bound obtained by Algorithm 7 is $\gamma^* = 1.0000$, which can be guaranteed by a filter with the following two state-space realizations:

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \left[\begin{array}{cc|c} -0.8726 & -0.3981 & 9.2822 \times 10^{-3} \\ 1.9190 & 0.8756 & 9.9001 \times 10^{-3} \\ \hline -0.1680 & 7.8408 \times 10^{-2} & 8.2567 \times 10^{-3} \end{array} \right], \quad (2.118)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \left[\begin{array}{cc|c} 4.1549 \times 10^{-3} & 1.0249 \times 10^{-2} & 1.4551 \times 10^{-3} \\ 1.2184 \times 10^{-2} & -1.1731 \times 10^{-3} & 6.5640 \times 10^{-4} \\ \hline -0.8725 & -10.0037 & 8.2567 \times 10^{-3} \end{array} \right] \quad (2.119)$$

which are generated from (2.74) and (2.75), respectively.

Connect the obtained filter in (2.118) to the original system in (2.117) and depict the singular value curves of the resulting filtering error system with $\alpha = 0, 0.25, -0.25$. The results are displayed in Fig. 2.5, which clearly demonstrates the effectiveness of the filter in (2.118) in guaranteeing the optimal H_∞ performance upper bound $\gamma^* = 1.0000$ when $\alpha = 0$. Figure 2.5 also exposes that the filter cannot well endure the parameter uncertainty. For instance, the maximum singular value under $\alpha = 0.25$ is about 2, almost twice of $\gamma^* = 1.0000$ under $\alpha = 0$.

Next, robust H_∞ filters are designed by Algorithm 8 for the uncertain system of (2.117). The obtained minimum H_∞ filtering performance upper bound and the corresponding filter realizations are given as

$$\gamma^* = 1.2492$$

and

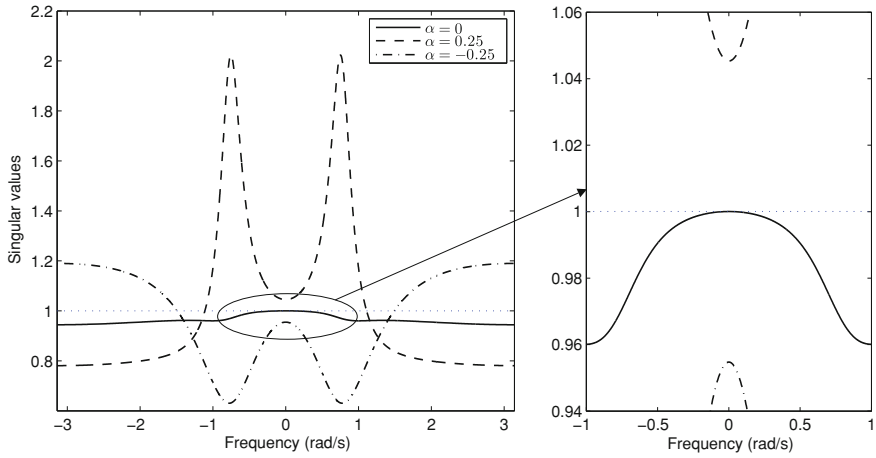
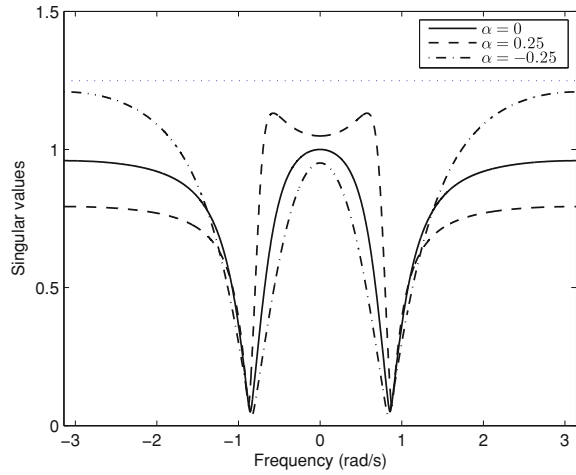


Fig. 2.5 Singular value curves with the filter in (2.118) under $\alpha = 0, 0.25$ and -0.25 in Example 2.4

Fig. 2.6 Singular value curves with the filter in Eq. (2.120) under $\alpha = 0, 0.25$ and -0.25 in Example 2.4



$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \begin{bmatrix} -1.0426 & -0.6039 \\ 1.6584 & 0.9605 \\ -0.1372 & -8.6277 \times 10^{-2} \end{bmatrix} \begin{array}{l} 1.2911 \times 10^{-2} \\ -7.0043 \times 10^{-3} \\ -8.6277 \times 10^{-3} \end{array}, \quad (2.120)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \begin{bmatrix} 3.4435 \times 10^{-2} & -5.4779 \times 10^{-2} \\ 7.3264 \times 10^{-2} & -0.1165 \\ -5.9589 \times 10^{-2} & -11.4952 \end{bmatrix} \begin{array}{l} 1.6021 \times 10^{-4} \\ 1.0072 \times 10^{-4} \\ -8.6277 \times 10^{-3} \end{array}. \quad (2.121)$$

Figure 2.6 depicts the singular value curves of the filtering error system resulting from the filter in (2.120) with $\alpha = 0, 0.25, -0.25$. The effectiveness of the filter is apparent. Moreover, for the uncertain system of this example, the minimum H_∞ filtering performance upper bound $\gamma^* = 1.2492$ guaranteed by the filter in (2.120) is obviously smaller than the maximum singular values for the case of $\alpha = 0.25$ in Fig. 2.5. Thus, the robustness of the filter in (2.120) is better than that of the filter in (2.118).

2.5 Summary and Notes

2.5.1 Summary

This chapter has addressed the quadratic approaches to robust filter design for polytopic uncertain systems. By a linearization procedure, filter design methods are firstly derived for nominal systems and then the notion of quadratic stability is employed to extend the design results to the polytopic uncertainty case. Systematic results on the commonly used H_2 , H_∞ , and energy-to-peak filtering schemes have been established in Sects. 2.1, 2.2, and 2.3, respectively. The relationship between the optimal H_2 filtering and the Kalman filtering has been revealed. All the design methods have been formulated in the form of solving a set of LMIs. Numerical examples have been presented to illustrate the effectiveness of the design methods.

For nominal systems, the derived filter design methods can design the optimal filters in the corresponding performance; for uncertain systems, suboptimal robust filters can be effectively designed by the quadratic approaches. However, it may have been noted from, e.g., Table 2.2 or 2.3, that the guaranteed performance upper bound of each filter designed by the quadratic approaches is still far from the actual optimal performance bound of this filter. This implies that the quadratic approaches are too conservative and still leave much room that deserves further investigation. In Chap. 3, the parameter-dependent idea will be employed to reduce this conservatism and improve the filter design methods in the quadratic framework.

2.5.2 Notes

For robust H_2 filtering subject to polytopic uncertainty, quadratic approaches in the LMI framework were investigated in [4, 5, 8, 38, 39], among which [4, 5, 38] are focused on continuous-time systems and [8, 38, 39] on discrete-time systems, respectively. For robust H_∞ filtering, related results are reported in [5, 8, 38–43], where [5, 38, 40, 41, 43] are concerned with polytopic uncertain continuous-time systems, and [8, 39, 41, 42] are about polytopic uncertain discrete-time systems, respectively. Quadratic approaches to robust energy-to-peak filtering can be found in [41, 44]. Among these mentioned results, except [39] where observer-type filters

are designed for state estimation, all other references design linear filters with the general state-space form.

For converting the conditions for filter design into the LMI form, the change-of-variable procedures in (2.11)–(2.14) are based on the results reported in [35]. Connections between the optimal H_2 filtering and the Kalman filtering can be found in [8, 38], where the relationship between the optimal H_∞ filter and the central H_∞ filter is also established.

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