Chapter 2
Barodesy: The Next Generation of Hypoplastic Constitutive Models for Soils

D. Kolymbas

Abstract Barodesy is, like hypoplasticity, a frame for an evolution equation where the stress rate is expressed as tensorial function of stress, stretching and other parameters like void ratio. This equation being non-linear and non-integrable allows to express the path-dependent evolution of stress with deformation. The specific feature of barodesy is that it is based on two very simple theorems on asymptotic behavior of sand. The first theorem states that proportional strain paths starting from the stress-free state lead to proportional stress paths. Barodesy shows that this can be easily modeled with an exponential mapping. The second theorem refers to proportional strain paths starting form a non-vanishing stress state. They lead asymptotically to proportional stress paths that would have been obtained starting at the stress free state. Barodesy models this by adding a simple term in the constitutive relation, and this is now the complete new constitutive relation. The so obtained mathematical relation allows to embed in a simple and elegant way many known principles of soil mechanics, allowing additionally for some asymptotic effects due to cyclic loading. The striking simplicity of the new model not only facilitates its application in numerical applications but also offers a frame for understanding the behavior of soil and granular matter, in general. Moreover, it offers a good starting point for further investigations towards open problems such as rate sensitivity and behavior at small strains.

2.1 Introduction

Barodesy is a completely new frame of constitutive models for soils. In this article the structure of the new theory is outlined; the presentation of special applications and the results of simulations are left for a forthcoming paper. The present article

D. Kolymbas (✉)
Unit of Geotechnical and Tunnel Engineering, University of Innsbruck, Technikerstr. 13, A6020 Innsbruck, Austria
e-mail: Dimitrios.Kolymbas@uibk.ac.at

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refers to sand, barodesy, however, holds also for clay, as shown in the PhD thesis of Gertraud Medicus (in preparation). Clay, being also a particulate material consisting of minute particles has a behaviour very similar to sand. However, there are some differences that mainly arise from the fact that the stiffness of sand in monotonic compression is much higher than that of virgin consolidated clay.

2.2 Empirical Basis of Barodesy

Of basic importance for the following is the notion of a proportional path. Proportional stress and strain paths are characterized by constant ratios of the principal values $\sigma_1 : \sigma_2 : \sigma_3$ and $\varepsilon_1 : \varepsilon_2 : \varepsilon_3$, respectively.

There are two basic experimental findings for sand:

1. Starting from the stress-free state, proportional strain paths lead to proportional stress paths.
2. Starting from a non-vanishing stress state and applying a proportional strain path leads asymptotically to the proportional stress path that would be obtained starting from the stress-free state.

The two rules stated above are inferred by Goldscheider from his test results obtained with rectilinear extensions of sand [5]. These tests have been carried out in a so-called true triaxial apparatus. This apparatus allows to apply rectilinear extensions (i.e. motions without rotation of the principal axes of deformation) independently in all three directions of space.

Besides these rules, the generally observed lack of an elastic regime in soils contradicts a basic ingredient of the theory of plasticity.

2.3 Early Quests for Alternatives to Plasticity Theory

The theory of plasticity, based on the notions of yield surface, flow rule, consistency, decomposition of strain into elastic and plastic parts, etc. was for a long time the only mathematical tool to describe irreversible deformation. Thus, also soil mechanics has been developed along the principles of this theory. The first consistent model for soil, the Cam-Clay model is a particular plasticity theory adapted to clay. However, several researchers have called to depart from plasticity. In 1973 Palmer and Pearce published a paper titled "Plasticity theory without yield surfaces" [18]. Some sentences of this paper deserve being quoted here:

> It was quite natural that the idea of a yield surface should assume such importance in a theory built on experience with metals, since in most metals yield occurs at a fairly well-defined stress level. ...
> In soil mechanics the status of the yield surface concept is quite different, both in theory and experiment...

> . . . strain measurements in clay depend on direct observation of boundary displacements, so that only quite large strain increments are reliably measurable, creep and pore-pressure diffusion confuse results. . .
> . . . yield surface motions during strain-hardening are often too complex for the results to be helpful in constructing usable stress–strain relations.
> Might it be possible to resolve this (dilemma) by constructing a different kind of plasticity model, in which the yield surface concept had been dropped or relegated to a minor role?
> . . . it might be useful to idealise clay as a material in which the yield surface has shrunk to a point, so that all deformations are plastic and *any* changes of stress from the current state will produce plastic strain increments.

Palmer and Pearce present in their paper a concept for a plasticity theory without yield surfaces. This concept is based on two postulates by Ilyushin which are, in a sense, precursors of Goldscheider's theorems:

Isotropy postulate: *If the strain path is rotated in strain space, then the corresponding stress path is rotated by the same amount.* This postulate has nothing to do with isotropy, since it considers rotations in the strain and stress spaces, not in the natural space. It is controversial and certainly not valid in the full stress and strain spaces. It is only approximately valid in the deviatoric subspace: This postulate implies that the deviatoric directions of proportional strain and stress paths coincide. This is, however, not true, according to experimental results by Goldscheider [5].

Delay postulate: *The stress at some instant in a loading history does not depend on the whole previous history, but only on the last part of it.* This is a postulate of fading memory and is similar to Goldscheider's second theorem.

Based on Ilyushin's postulates, Palmer and Pearce present the following concept:

> The deviatoric stress has two components. The magnitude of the first component is a function of the octahedral shear strain, and its direction coincides with the principal strain vector (referring strain to an isotropically-consolidated initial state). The magnitude of the second component is constant, and its direction coincides with the current strain rate . . .
> Reversal of the strain path would reverse the second component but not the first . . .

The very last sentence strongly resembles to a basic concept of hypoplasticity and barodesy, to which presumably the authors would have concluded, had they used rate equations instead of finite ones.

2.4 Barodesy and Hypoplasticity

Constitutive models can't be *derived* from general principles, because they have to describe specific features of particular materials. Thus, besides intuition, trial and error is a basic tool in developing constitutive models. In hypoplasticity, trial and error has been guided by general principles of objectivity and representation theorems for tensor-valued functions. In barodesy, the amount of trial and error has been further reduced in favour of reasoning on asymptotic behaviour of granulates. Asymptotic states are attractive not only from conceptual reasons but also from the experimental viewpoint: If we consider long monotonic deformations, initial disturbances, related, for example, to sample preparation, fade out and do no more influence the measurements.

Barodesy can be seen as a hypoplastic implementation of the Critical States Concept. Previous attempts to incorporate Critical States into hypoplasticity have been published e.g. by Bauer [1], Gudehus [6], Herle and Kolymbas [7], Masin [16], Niemunis [17], and Wu et al. [21]. As in the original proposal by the author [9, 10], they are composed of two parts, the one being linear and the other non-linear in **D**. Barodesy is also composed of two parts, neither of which is linear in the stretching tensor **D**. The response envelopes of hypoplastic versions are ellipses, whereas in barodesy they have a similar form but are not ellipses. As these features are not essential, the author believes that the mathematical structure of barodesy is appealing and capable of useful extensions. Clearly, all mathematical models (including elastoplastic ones) succeeding to describe the same object, e.g. soil, must include a common mathematical kernel, which is however still hidden.

2.4.1 About the Name "Barodesy"

One should not be fast in introducing new names, as too many neologisms create confusion. However, sometimes new names are needed to denote ideas that are really new. There is an abundance of elastic and plastic concepts equipped with prefixes such as hypo-, para-, hyper- etc. Therefore, the author suggests to avoid using the words elasticity and plasticity (to the extend the latter is associated with notions such as yield surface, elastic regime etc., originally created for metals), since they are not the only framework to describe granular materials such as soil. It should be admitted that soil behaviour can—in principle—also be described in the framework of the theory of elastoplasticity. The author believes, however, that yield surfaces and the other concepts of plasticity theory may prejudice our perception and sometimes obscure soil mechanics, which suffers from the long lasting fragmentation in constitutive modelling [11]. Hypoplasticities have been developed, independently of each other, in California [3], Karlsruhe and Grenoble [2]. The Grenoble and Karlsruhe branches are inherently related. The California and the Karlsruhe/Grenoble perceptions of hypoplasticity have nothing in common but the name, the first one being designed in the frame of elastoplasticiy. It should be stressed that there is no use in seeking rigorous definitions of what *is* hypoplasticity. Taking that a constitutive relation is a mathematical expression being continuously developed, any attempt to provide a strict definition and distinctive characteristics ends up in a sterile exercise of dogmatism. As for the Karlsruhe branch, many different versions have emanated since the publication of the first proposal by the author in 1977.[1] This proposal was motivated by the quest to describe the mechanical behaviour of soil on the basis of Rational Mechanics without any recourse to the formalism of elastoplasticity.

The new approach presented in this paper pays tribute to a basic idea of Gudehus, who guided the research team in Karlsruhe in the years 1973–2006: Asymptotic states, as represented by proportional strain paths, are attractors and play a paramount role in mechanics of granulates. In this paper is shown that *almost the entire constitutive relation for granulates can be derived from the consideration of proportional paths.* The here presented theory, which yields a variety of more or less realistic predictions, is based on a few reasonable assumptions. Therefore it claims generality and deserves a new name. The name barodesy has been coined motivated by the fact that granular materials gain their stiffness ($\delta\varepsilon\sigma\iota\varsigma$ = bond, hence stiffening, hardening) from externally applied pressure ($\beta\alpha\rho o\varsigma$). Thus, the names "barodesy" and "barodetic" are proposed for granular materials to distinguish them from what traditionally is denoted as "elastic" or "plastic".

2.5 Symbols and Notation

The notation in the "Non-Linear Field Theories of Mechanics" [19] is mainly followed in this article. Compared to the notation of tensors with indices, the symbolic notation facilitates insight into the prevailing relationships.

$\mathbf{T}$:	Cauchy-stress. Its principal components are denoted with $\sigma_1, \sigma_2, \sigma_3$.
$\mathbf{D}$:	Stretching tensor, i.e. the symmetric part of the velocity gradient $\nabla\mathbf{v}$. It can be set approximately equal to the strain rate, $D_{ij} \approx \dot{\varepsilon}_{ij}$.
e:	Void ratio, i.e. the ratio V_p/V_s, where V_p and V_s are the volumes of pores and solids (grains), respectively.
exponent 0:	Denotes normalization of a tensor $\mathbf{A}$, i.e. $\mathbf{A}^0 := \mathbf{A}/|\mathbf{A}|$, with $|\mathbf{A}| := \sqrt{\mathrm{tr}\mathbf{A}^2}$.
σ:	$|\mathbf{T}|$
$\dot{\varepsilon}$:	$|\mathbf{D}|$
ζ:	$\mathrm{tr}\mathbf{D}^0$
$\dot{\mathbf{T}}$:	Time rate of stress. In the general case, $\dot{\mathbf{T}}$ should be replaced by a co-rotational stress rate $\mathring{\mathbf{T}}$. For rectilinear extensions it is $\mathring{\mathbf{T}} \equiv \dot{\mathbf{T}}$.
c_1, c_2, c_3, c_4:	Material constants.

[1] The first versions were not yet named "hypoplastic".

2.6 Proportional Paths A

Let us first consider proportional strain paths starting from the stress-free state. Such paths can be volume-decreasing (we will call them "consolidations"), characterized by tr$\mathbf{D} < 0$, or volume preserving ("isochoric" or "undrained"), characterized by tr$\mathbf{D} = 0$, or volumes increasing, characterized by tr$\mathbf{D} > 0$. Clearly, the latter are not feasible with cohesionless sand. Let us denote with $\mathbf{R}$ a tensor that has the direction of a proportional stress path. The question arises, how $\mathbf{R}$ depends on the direction of the corresponding proportional strain path. The latter is characterized by the direction of stretching $\mathbf{D}$, i.e. by the normalized stretching $\mathbf{D}^0$. How can we determine the relation $\mathbf{R}(\mathbf{D}^0)$? This question can be easily answered if we observe that all consolidations are mapped into a specific part of the principal stress space formed by the stress components σ_1, σ_2 and σ_3. This part is the octant, where all principal stresses are compressive, i.e. negative. Hence, the product $\sigma_1\sigma_2\sigma_3$ must also be negative. Now, for a proportional stress path we have $\sigma_i = \mu R(D_i)$, $\mu > 0, i = 1, 2, 3$.[2] Thus, the following condition must hold:

$$R_1(D_1)R_2(D_2)R_3(D_3) < 0 \qquad \text{for} \quad \mathrm{tr}\mathbf{D} = D_1 + D_2 + D_3 < 0\,. \tag{2.1}$$

This implies that $R_1(D_1)R_2(D_2)R_3(D_3)$ must be a function of $D_1 + D_2 + D_3$, a requirement which is fulfilled by the exponential mapping

$$\mathbf{R}(\mathbf{D}) = \exp(c_1\mathbf{D}^0)\,. \tag{2.2}$$

Equation (2.2) maps all volume-reducing (tr$\mathbf{D} < 0$) proportional strain paths into a cone in the stress space with appex at $\mathbf{T} = \mathbf{0}$, which can be called the $\mathbf{R}$-cone. Its boundary is the critical state surface and corresponds to paths with tr$\mathbf{D} = 0$. Consider the intersection of the $\mathbf{R}$-cone with a plane tr$\mathbf{T} =$ const, as shown in Fig. 2.1. This curve expresses the critical limit state in a so-called deviatoric plane in the stress space. The mathematical representation of this curve can be easily derived from Eq. (2.2): For isochoric deformations (tr$\mathbf{D}^0 = 0$) we can eliminate $\mathbf{D}^0$ from (2.2) and obtain:

$$\mathbf{D}^0 = \frac{1}{c_1}\ln(-\mathbf{R})\,. \tag{2.3}$$

The requirement tr$\mathbf{D}^0 = 0$ results in $\ln(-R_1R_2R_3) = 0$ or $R_1R_2R_3 = -1$. From the additional requirement $|\mathbf{D}^0| = 1$ we obtain:

$$(\ln R_1)^2 + (\ln R_2)^2 + (\ln R_3)^2 = c_1^2. \tag{2.4}$$

[2]Herein, D_i are the principal values of $\mathbf{D}$, and $R_j(D_i)$ are the principal values of $\mathbf{R(D)}$.

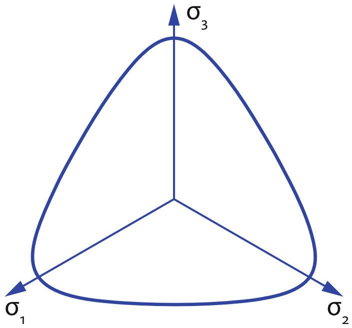

Fig. 2.1 Cross section of the **R**-cone with a deviatoric plane. Numerically obtained with Eq. (2.2) in the following way: the shown curve collects stress states that correspond to isochoric stretchings $\mathbf{D}^0$, $\mathrm{tr}\mathbf{D}^0 = 0$. For any such stretching, Eq. (2.2) yields a stress ray $\mathbf{T} = \lambda\mathbf{R}$, $\lambda > 0$. Its intersection with a π-plane ($\mathrm{tr}\mathbf{T} = \mathrm{const}$) is a point of the shown curve

For the here considered proportional paths holds: $\mathbf{T} = \mu\mathbf{R}$, $0 < \mu < \infty$, hence we can replace in this equation $\mathbf{R}$ by $\mathbf{T}/\mu$ and obtain finally the equation of critical states in the stress space:

$$\left(\ln \frac{T_1}{\sqrt[3]{T_1T_2T_3}}\right)^2 + \left(\ln \frac{T_2}{\sqrt[3]{T_1T_2T_3}}\right)^2 + \left(\ln \frac{T_3}{\sqrt[3]{T_1T_2T_3}}\right)^2 = c_1^2. \qquad (2.5)$$

Equation (2.5) is homogeneous of the zero-th degree in $\mathbf{T}$ and describes thus a conical surface in the stress space with apex at $\mathbf{T} = \mathbf{0}$. Its intersection with a plane $\mathrm{tr}\mathbf{T} = \mathrm{const}$ is shown in Fig. 2.1. Note that its shape *practically coincides* [4] with the curve obtained by the expression of Matsuoka and Nakai:

$$\frac{(T_1 + T_2 + T_3)(T_1T_2 + T_1T_3 + T_2T_3)}{T_1T_2T_3} = \mathrm{const}. \qquad (2.6)$$

Equation (2.2) also relates the critical friction angle with K_0, the so-called coefficient of earth pressure at rest. We consider a critical state and use the abbreviation $K_c := (1 - \sin\varphi_c)/(1 + \sin\varphi_c)$. The equation $R_2/R_1 = K_c$ yields:

$$c_1 = \sqrt{\frac{2}{3}}\,\ln K_c. \qquad (2.7)$$

Now we consider an oedometric proportional stress path. The corresponding stretching is

$$\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (2.8)$$

Herewith we obtain:

$$K_0 = \frac{T_2}{T_1} = \frac{R_2}{R_1} = \frac{\exp(0)}{\exp(-c_1)} = \exp(c_1) = \exp(\ln(K_c^{\sqrt{2/3}})) = K_c^{\sqrt{2/3}}. \tag{2.9}$$

2.7 Proportional Paths B

Now we start from a stress state $\mathbf{T} \neq \mathbf{0}$ and apply the stretching $\mathbf{D}$. In order to asymptotically approach the corresponding proportional stress path $\mathbf{T} = \mu\mathbf{R}(\mathbf{D})$, the stress rate $\dot{\mathbf{T}}$ must point to the point $\mu_1\mathbf{R}(\mathbf{D})$, i.e.

$$\mathbf{T} + \lambda\dot{\mathbf{T}} = \mu_1\mathbf{R}(\mathbf{D}), \tag{2.10}$$

where the positive constants μ, μ_1 and λ need not be further specified here. If we eliminate $\dot{\mathbf{T}}$ we obtain an evolution equation for the stress:

$$\dot{\mathbf{T}} = \nu_1\mathbf{R}(\mathbf{D}) + \nu_2\mathbf{T} \tag{2.11}$$

with appropriately defined scalar quantities ν_1 and ν_2. Equation (2.11) is the final general form of the barodetic constitutive relation. To comply with barotropy, pyknotropy and rate independence of sand, ν_1 and ν_2 are further specified, such that the barodetic constitutive equation for sand obtains the following specific form:

$$\dot{\mathbf{T}} = h(\sigma) \cdot (f\mathbf{R}^0 + g\mathbf{T}^0) \cdot \dot{\varepsilon}, \tag{2.12}$$

where f and g are functions of the stress and the void ratio. In the sequel it will be shown how all known concepts of soil mechanics can be cast in the frame given by Eq. (2.12).

2.8 Limit States

A limit state is obtained when the stiffness vanishes:

$$\dot{\mathbf{T}} = \mathbf{0}. \tag{2.13}$$

Considering stress–strain curves, the limit states are manifested either as peak or residual limit states, where the curve obtains a horizontal slope. In barodesy (see Eq. (2.12)), yield is modeled by the equation

$$f\mathbf{R}^0 + g\mathbf{T}^0 = \mathbf{0}. \tag{2.14}$$

This tensorial equation implies two equations:

1. The flow rule

$$\mathbf{R}^0 = \mathbf{T}^0. \tag{2.15}$$

Note that $\mathbf{R}$ depends on $\mathbf{D}$. Thus, the flow rule gives (via an implicit equation) the direction of strain that pertains to a limit state $\mathbf{T}$.
2. The scalar equation

$$f + g = 0, \tag{2.16}$$

which takes into account the actual void ratio e and the stress magnitude σ. This scalar equation somehow corresponds to the yield surface of plasticity theory.

2.9 Incremental Non-Linearity

Incremental non-linearity (or "non-linearity in the small") means different stiffnesses at loading and unloading and, in general, irreversible or hysteretic mechanical behaviour. Both, elastoplastic and hypoplastic relations comprise incremental non-linearity. The elastoplastic approach consists in introducing two different stiffnesses, one for loading and one for unloading. A criterion has to be added to distinguish when we have loading and when unloading. In the frame of hypoplasticity a unique expression for the stress rate (or stiffness) is used, and the distinction between loading and unloading is accomplished by the non-linearity of this equation. In barodesy, the difference of stiffness at loading and unloading is modelled by the fact that the second term $g\mathbf{T}^0$ in Eq. (2.12) is not changed if $\mathbf{D}$ is switched to $-\mathbf{D}$, whereas the first term (i.e. $f\mathbf{R}^0$) undergoes a change.

2.10 Consolidations and Critical States

Noting that $\mathbf{T}^0 = \mathbf{R}^0$ holds true for proportional paths, we obtain from Eq. (2.12)

$$\dot{\mathbf{T}} = h(\sigma)\, \mathbf{T}^0\, (f + g)\, \dot{\varepsilon}. \tag{2.17}$$

For proportional paths holds also $\dot{\mathbf{T}} = \dot{\sigma}\mathbf{T}^0$, hence Eq. (2.12) reduces to

$$\dot{\sigma} = h(\sigma)\, (f + g)\, \dot{\varepsilon}. \tag{2.18}$$

The quantities f, g and, hence, $f+g$ are functions (still to be defined) of the void ratio e, the stress magnitude σ and of ζ, introduced in Sect. 2.5, which is a measure of dilatancy.

Vanishing stiffness for critical states implies $f+g=0$ for $\zeta=0$. Hence, we can set

$$f+g=c_2\zeta. \tag{2.19}$$

We require Eq. (2.19) to be valid not only for critical states but also for all consolidations, i.e. for proportional paths with $\epsilon<0$. Introducing Eq. (2.19) into (2.18) we obtain:

$$\dot{\sigma}=h(\sigma)\,c_2\,\zeta\,\dot{\varepsilon}, \tag{2.20}$$

Using

$$\zeta\,\dot{\varepsilon}=\frac{\mathrm{tr}\mathbf{D}}{\dot{\varepsilon}}\,\dot{\varepsilon}=\mathrm{tr}\mathbf{D}=\frac{\dot{e}}{1+e}, \tag{2.21}$$

we obtain for consolidations:

$$\dot{\sigma}=h(\sigma)\,c_2\frac{\dot{e}}{1+e}. \tag{2.22}$$

It follows that the slope of the e vs. σ curves is the same for oedometric, hydrostatic and, in general, for all consolidations. Adapting $h(\sigma)$, and thus Eq. (2.22), to a compression curve from a laboratory test allows to determine the compression curve $e=\kappa(\sigma)$. If we choose

$$h=\sigma^{c_3}, \tag{2.23}$$

we obtain

$$e=\kappa(\sigma)=(1+e_0)\exp\frac{\sigma^{1-c_3}}{(1-c_3)c_2}-1 \tag{2.24}$$

with $c_2<0$.

The incremental stiffness of compression tests, in particular of oedometric compression tests, denoted by $E_s:=d\sigma_1/d\varepsilon_1$, is known to be stress-dependent according to a relation attributed to Ohde and/or Janbu [8]:

$$E_s=E_{s0}\left(\frac{\sigma}{\sigma_0}\right)^w. \tag{2.25}$$

With $d\sigma_1 = -d\sigma/\sqrt{1+2K_0^2}$ and $d\varepsilon_1 = de/(1+e)$ we obtain from Eq. (2.22):

$$E_s = \frac{-c_2}{\sqrt{1+2K_0^2}}\sigma^{c_3} = \frac{-c_2\sigma_0^{c_3}}{\sqrt{1+2K_0^2}}\left(\frac{\sigma}{\sigma_0}\right)^{c_3} \tag{2.26}$$

in accordance with Eq. (2.25). A typical value for c_3 is ca. 0.5.

As said, the equation $f+g=0$ or $f+g=c_2\epsilon$ expresses for $\epsilon=0$ the critical state line (CSL) $e-e_c(\sigma)=0$. Thus, we can set

$$f+g = c_2\zeta + c_4(e_c(\sigma)-e). \tag{2.27}$$

For peak limit states we also have $f+g=0$. This can be fulfilled by Eq. (2.27) if $e<e_c$ for $c_2\epsilon<0$, i.e. for dilatant deformation with tr$\mathbf{D}>0$.

For Eq. (2.27) to be also valid for consolidations (i.e. to obtain $f+g=c_2\epsilon$, cf. Eq. (2.22)) we have to require that $e-e_c(\sigma)$ vanishes for consolidations, see Eq. (2.24). For this to hold, we have to require:

$$e_c(\sigma) = \kappa(\sigma). \tag{2.28}$$

In other words, the dependence of the critical void ratio e_c on stress σ is given by the same function that holds for consolidations (i.e. proportional compressions). Of course, the initial void ratio $\kappa(0)$ must be appropriately chosen in each case. Note that the general opinion in soil mechanics is not unique in that question. Many authors accept that the dependence $e_c(\sigma)$ is the same as in compression tests, other authors contradict this view. Barodesy leads to the acceptance of this view.

However, it has to be admitted, that the CSL is hard to determine by experiments. Wood [20] writes:

> The paths of tests on loose and dense samples head towards a somewhat diffuse, but clearly pressure dependent, zone of critical void ratios.

The final step in determining the barodetic constitutive equation consists in partitioning equation (2.27) into f and g by setting, e.g.

$$f = c_2\zeta - c_4 e, \tag{2.29}$$

$$g = c_4 e_c(\sigma). \tag{2.30}$$

2.11 Cyclic Loading, Limit Cycles and Shake-Down

Proportional paths are not the only attractors in the constitutive relation presented so far. Being ordinary differential equations, constitutive relations "of the rate type" [19] may exhibit also limit cycles or cyclic orbits as further attractors. In fact, Eq. (2.12) exhibits periodic orbits (limit cycles) at cyclic loading. In terms of mechanics, this effect is related to "shake-down" and means that stress cycles lead asymptotically to cyclic changes of void ratio. In case of, for example, oedometric deformation (but not for conventional triaxial tests), this implies also cyclic strain, i.e. strains due to cyclic stress are bounded, i.e. they do not increase to infinity. Generally, shake-down is one of the possible responses of sand to cyclic loading, the other one being "incremental collapse" (i.e. unlimited growth of strain with increasing number of cycles). It is yet unclear when exactly shake-down and when incremental collapse are to be expected. However, Eq. (2.12) exhibits shake-down (and periodic orbits) e.g. at cyclic oedometric loading: If the axial stress component σ_1 is periodically changed between a lower and an upper limit, then the corresponding radial stress component σ_2, which is bounded, will also become cyclic, i.e. a limit cycle will eventually be obtained in the stress space.[3]

Cyclic stress, asymptotically obtained with strain cycles of infinitesimally small amplitude, is related with the void ratio $\check{e}$, which can be called the *cyclic void ratio*. Little is known from experiments on the dependence of $\check{e}$ on actual stress $\mathbf{T}$ and on the direction $\mathbf{D}^0$ of strain cycles.

Considering strain cycles with infinitesimal amplitude with the constitutive relation (2.12) and denoting with "+" and "−" loading and unloading, respectively, it is observed that at a limit cycle must hold: $\dot{\mathbf{T}}^+ = -\dot{\mathbf{T}}^-$. Hence, the condition for cyclic response reads

$$(f^+\mathbf{R}^{0+} + f^-\mathbf{R}^{0-}) + (g^+ + g^-)\mathbf{T}^0 = \mathbf{0}. \tag{2.31}$$

This equation constitutes a relation between the direction of the strain amplitude, $\mathbf{D}$, the cyclic void ratio $\check{e}$ and the stress $\sigma\mathbf{T}^0$, around which the stress oscillation occurs. Eliminating $\mathbf{T}^0$ from Eq. (2.31) yields:

$$\mathbf{T}^0 = \frac{-1}{g^+ + g^-}(f^+\mathbf{R}^{0+} + f^-\mathbf{R}^{0-}). \tag{2.32}$$

Using this equation and the additional condition $|\mathbf{T}^0| = 1$ makes it possible to determine for a given $\mathbf{D}^0$ the stress direction $\mathbf{T}^0$ of the corresponding cyclic state and also the pertaining cyclic void ratio $\check{e}(\sigma)$. A discussion of Eq. (2.32) is left for a future paper.

It should be added that the here presented model still exhibits ratcheting at cycles of small amplitude, e.g. in conventional triaxial tests.

[3]Integrity of grains (or permanence of the grain size distribution) has not been assumed for the derivation of the constitutive relation so far. In fact, a constitutive relation that does not contain any measure for the strength of grains presupposes that grain crushing does not occur. In reality, however, grain crushing is inevitable, especially at higher stresses. The corresponding changes of the grain size distribution curve are hard to measure.

2.12 Significance of Barodesy

Compared with the "classical" elastoplastic approaches, Eq. (2.12) constitutes a substantial change of paradigm and introduces not only new concepts but also a remarkable simplicity in a field of paramount complexity[4] dominated by a "morass of equations". Equation (2.12) is a convincing implementation of Noll's[5] program to formulate a constitutive equation as a rate equation of the type $\dot{\mathbf{T}} = \mathbf{h}(\mathbf{T}, \mathbf{D})$, and the importance of this achievement is enhanced by the fact that Eq. (2.12) is *derived* from general properties of sand. The implications of Eq. (2.12) are amazing. Despite its simplicity it captures almost every aspect of the behaviour of granular materials: stress dependent stiffness, hysteretic behaviour, dilatancy, contractancy, hardening up to the peak and subsequent softening to critical states, stress–strain curves and stress-paths for all types of tests, including drained and undrained triaxial tests. In a series of papers [12–15] are shown simulation results including drained and undrained triaxial tests with loose and dense sand, cyclic oedometric tests and cyclic simple shear tests with constant normal stress and constant volume. The range of applicability is huge, as it covers all particulate materials such as soils, granulates and powders. Such materials are addressed not only by geotechnical engineering but also by many other technological branches, such as offshore, mining, petroleum engineering, metallurgy, chemical and food industry.

2.13 Open Questions

Despite its simplicity and elegance, the present version of barodesy cannot cover all aspects of sand behaviour. The memory is still contained only in the actual stress $\mathbf{T}$ and the actual porosity e, and this is not sufficient to cover all aspects of re-loading, in particular the so-called aspects of "small strain stiffness". Though, it is interesting to note how many aspects of memory can be covered with $\mathbf{T}$ and e.

The barodetic equations are homogeneous of the first degree in the stretching $\mathbf{D}$ and, hence, rate-independent. To change this, the degree of homogeneity has to be modified.

[4]"Many properties of sand are equally puzzling to science as the big bang is", Neue Zuercher Zeitung, 13.2.2008.

[5]A prominent representative of a school of thought called Rational Mechanics. The main reference is the classical book "The Non-Linear Field Theories of Mechanics" [19].

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