

Emptiness and Discharge in Sequent Calculus and Natural Deduction

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Abstract We investigate the correlation between empty antecedent and succedent of the intuitionistic (respectively dual-intuitionistic) sequent calculus and discharge of assumptions and the constants absurdity (resp. discharge of conclusions and triviality) in natural deduction. In order to be able to express and manipulate the sequent calculus phenomena, we add two units to sequent calculus. Depending on the sequent calculus considered, the units can serve as discharge markers or as absurdity and triviality.

Keywords Sequent calculus · Natural deduction · Dual intuitionistic logic · Pseudo-constants

1 Introduction

In this article, we are interested in the correspondence between the structural features of the sequent calculus that are often neglected in comparisons with natural deduction, namely the features of empty antecedent or empty succedent of sequents. While these features are completely symmetric in the classical sequent calculus LK, this is not the case for the intuitionistic calculus LI. This becomes apparent when considering

M. Arndt and L. Tranchini—Supported by the French-German ANR-DFG project “Hypothetical Reasoning—Its Proof-Theoretic Analysis” (HYPOTHESES) (DFG Schr 275/16-2).
L. Tranchini—Supported by the German Research Agency (DFG grant Tr1112/1) as part of the project “Logical consequence. Proof-theoretic and epistemological perspectives”

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the correspondence between intuitionistic sequents and derivations in intuitionistic natural deduction NI.

In general, the correspondence of natural deduction and sequent calculus is based on the idea that formulae in the antecedent (resp. succedent) correspond to assumptions (resp. conclusions) in natural deduction derivations. Natural deduction introduction rules, those in which the logical constant figures in the conclusion, correspond to sequent calculus right rules, in which the logical constant figures on the right hand side of the conclusive sequent. Natural deduction elimination rules, in which the logical constant figures in one of the premises, correspond to sequent calculus left rules, in which the logical constant figures on the left hand side of the conclusive sequent.

In the intuitionistic case, sequents are restricted to at most one formula in the succedent (Fig. 1). We thus have:

Sequent $\Sigma \Rightarrow A$ is derivable in LI if and only if $\Sigma \vdash A$ holds in NI.

However, this does not exhaustively express the correspondence, because there is the critical case of deriving absurdity in natural deduction.

Sequent $\Sigma \Rightarrow \perp$ is derivable in LI if and only if $\Sigma \vdash \perp$ holds in NI.

To have a perfect correspondence, the sequent calculus is usually modified by introducing the constant $\perp$ as governed by particular axioms. Although this suffices to establish a correspondence between NI and LI we will argue that a more fine-grained correspondence can be attained.

The introduction of $\perp$ usually goes together with the definition of negation as $A \rightarrow \perp$. In this paper, we will rather take negation as primitive (not only in sequent calculus, but in natural deduction as well).

In contradistinction to the empty succedent, the empty antecedent expresses the fact that all the assumptions of the corresponding natural deduction derivation have been discharged. The empty succedent corresponds to the obtaining of a specific conclusion (i.e. $\perp$), whereas the empty antecedent corresponds to an agglomerate of discharges.

In order to obtain the complete picture, we consider also the sequent calculus and natural deduction systems for dual-intuitionistic logic LDI and NDI. Dually to what happens in LI, the antecedent of sequents of LDI are restricted to at most one formula (cf. [1, 8]). A natural deduction formulation of dual-intuitionistic logic can be obtained by considering a single-premise multiple-conclusion setting in which derivation trees branch downward, as described in Tranchini [5, 6]. The correspondence between LDI and NDI can be roughly stated as follows:

Sequent $A \Rightarrow \Gamma$ is derivable in LDI if and only if $A \vdash \Gamma$ holds in NDI.

In this dual case, it is the empty succedent that corresponds to an agglomerate of discharges, specifically, conclusion discharges, side effects of the application of some

$$\begin{array}{c}
 \frac{}{A \Rightarrow A} \text{ (ax)} \\
 \frac{\Sigma \Rightarrow \Gamma}{\Sigma, A \Rightarrow \Gamma} \text{ (WL)} \\
 \frac{\Sigma, A, A \Rightarrow \Gamma}{\Sigma, A \Rightarrow \Gamma} \text{ (CL)} \\
 \frac{\Sigma, A, B \Rightarrow \Gamma}{\Sigma, A \wedge B \Rightarrow \Gamma} \text{ (\wedge L)} \\
 \frac{\Sigma, A \Rightarrow \Gamma \quad \Sigma, B \Rightarrow \Gamma}{\Sigma, A \vee B \Rightarrow \Gamma} \text{ (\vee L)} \\
 \frac{\Sigma \Rightarrow A \quad \Theta, B \Rightarrow \Gamma}{\Sigma, \Theta, A \rightarrow B \Rightarrow \Gamma} \text{ (\rightarrow L)} \\
 \frac{\Sigma \Rightarrow A}{\Sigma, \neg A \Rightarrow} \text{ (\neg L)}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\Sigma \Rightarrow A \quad \Theta, A \Rightarrow \Gamma}{\Sigma, \Theta \Rightarrow \Gamma} \text{ (cut)} \\
 \frac{\Sigma \Rightarrow}{\Sigma, \Rightarrow A} \text{ (WR)} \\
 \frac{\Sigma \Rightarrow A \quad \Theta \Rightarrow B}{\Sigma, \Theta \Rightarrow A \wedge B} \text{ (\wedge R)} \\
 \frac{\Sigma \Rightarrow A}{\Sigma \Rightarrow A \vee B} \text{ (\vee R}_1\text{)} \quad \frac{\Sigma \Rightarrow B}{\Sigma \Rightarrow A \vee B} \text{ (\vee R}_2\text{)} \\
 \frac{\Sigma, A \Rightarrow B}{\Sigma \Rightarrow A \rightarrow B} \text{ (\rightarrow R)} \\
 \frac{\Sigma, A \Rightarrow}{\Sigma \Rightarrow \neg A} \text{ (\neg R)}
 \end{array}$$

Fig. 1 The intuitionistic calculus LI

$$\begin{array}{c}
 \frac{}{A \Rightarrow A} \text{ (ax)} \\
 \frac{\Rightarrow \Gamma}{A \Rightarrow \Gamma} \text{ (WL)} \\
 \frac{A \Rightarrow \Gamma}{A \wedge B \Rightarrow \Gamma} \text{ (\wedge L}_1\text{)} \quad \frac{B \Rightarrow \Gamma}{A \wedge B \Rightarrow \Gamma} \text{ (\wedge L}_2\text{)} \\
 \frac{A \Rightarrow \Gamma \quad B \Rightarrow \Xi}{A \vee B \Rightarrow \Gamma, \Xi} \text{ (\vee L)} \\
 \frac{B \Rightarrow A, \Gamma}{A \neg B \Rightarrow \Gamma} \text{ (\neg L)} \\
 \frac{\Rightarrow A, \Gamma}{\neg A \Rightarrow \Gamma} \text{ (\neg L)}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\Sigma \Rightarrow A, \Gamma \quad A \Rightarrow \Xi}{\Sigma \Rightarrow \Gamma, \Xi} \text{ (cut)} \\
 \frac{\Sigma \Rightarrow \Gamma}{\Sigma, \Rightarrow A, \Gamma} \text{ (WR)} \\
 \frac{\Sigma \Rightarrow A, A, \Gamma}{\Sigma \Rightarrow A, \Gamma} \text{ (CR)} \\
 \frac{\Sigma \Rightarrow A, \Gamma \quad \Sigma \Rightarrow B, \Gamma}{\Sigma \Rightarrow A \wedge B, \Gamma} \text{ (\wedge R)} \\
 \frac{\Sigma \Rightarrow A, B, \Gamma}{\Sigma \Rightarrow A \vee B, \Gamma} \text{ (\vee R)} \\
 \frac{\Sigma \Rightarrow B, \Gamma \quad A \Rightarrow \Xi}{\Sigma \Rightarrow A \neg B, \Gamma, \Xi} \text{ (\neg R)} \\
 \frac{A \Rightarrow \Gamma}{\Rightarrow \neg A, \Gamma} \text{ (\neg R)}
 \end{array}$$

Fig. 2 The dual intuitionistic calculus LDI

rule. The empty antecedent corresponds to the obtaining of a specific assumption, namely *triviality* $\top$:

Sequent $\Rightarrow \Gamma$ is derivable in LDI if and only if $\top \vdash \Gamma$ holds in NDI.

Apart from $\top$ playing the dual role of $\perp$, in NDI and LDI implication and intuitionistic negation are replaced by co-implication ‘ $\neg$ ’ and dual-intuitionistic negation ‘ $\neg$ ’ (Fig. 2).

$$\begin{array}{c}
\frac{\Sigma, A, B \Rightarrow \Gamma}{\Sigma, A \wedge B \Rightarrow \Gamma} (\wedge L) \\
\\
\frac{\Sigma, A \Rightarrow \Gamma \quad \Theta, B \Rightarrow \Xi}{\Sigma, \Theta, A \vee B \Rightarrow \Gamma, \Xi} (\vee L) \\
\\
\frac{\Sigma \Rightarrow A, \Gamma}{\Sigma, \sim A \Rightarrow \Gamma} (\sim L) \\
\\
\frac{\Sigma \Rightarrow A, \Gamma \quad \Theta \Rightarrow B, \Xi}{\Sigma, \Theta \Rightarrow A \wedge B, \Gamma, \Xi} (\wedge) \\
\\
\frac{\Sigma \Rightarrow A, B, \Gamma}{\Sigma \Rightarrow A \vee B, \Gamma} (\vee R) \\
\\
\frac{\Sigma, A \Rightarrow \Gamma}{\Sigma \Rightarrow \sim A, \Gamma} (\sim R)
\end{array}$$

Fig. 3 The logical rules of LK

2 Emptiness in the Sequent Calculus

The rules of the sequent calculus which actually bring forth the features of empty antecedent or empty succedent have the following characteristic: they are single premise rules and: (i) in either the antecedent or succedent of their premise some formula A occurs in some context and (ii) on that side of the sequent in their conclusion the context figures alone.

If we restrict ourselves to the (classically representative) $\{\wedge, \vee, \sim\}$ -fragment of the calculus LK (Fig. 3), there are two rules that satisfy this characterization, namely $(\sim R)$ and $(\sim L)$:

$$\frac{\Sigma \Rightarrow A, \Gamma}{\Sigma, \sim A \Rightarrow \Gamma} (\sim L) \qquad \frac{\Sigma, A \Rightarrow \Gamma}{\Sigma \Rightarrow \sim A, \Gamma} (\sim R)$$

The result of applying the right rule is that the formula A is removed from the antecedent and—suitably amended by negation—placed in the succedent. Clearly, an application of this rule in the case of $\Sigma = \emptyset$ yields a sequent which has an empty antecedent.

Apparently, emptiness of the antecedent is simply a contextual residue. Formula A is shifted into the succedent, and the resulting emptiness of the antecedent is due to the fact that context Σ in which A occurred was already empty. In this formulation of the sequent calculus it is impossible to retain the nature of the local void that is due to the removal of A , call it an ‘occurrence’ of emptiness. This is the reason why emptiness of the antecedent is generally an agglomerate phenomenon in the classical calculus.

The same phenomenon can be observed also in the implication right rule as soon as the language is opportunely enriched. In the classical case, this is however not particularly significant, due to the definability of implication as $\sim A \vee B$.

Conversely, an application of $(\sim L)$ results in a formula A being removed from its context in the succedent. Thus, the phenomenon of emptiness can also occur in the succedent of a sequent as described for the antecedent case.

3 The Corresponding Phenomena in Natural Deduction

In sequent calculus, at each node of a derivation all assumptions and conclusions are written anew in the sequent labelling the node. In contrast to this, each node of natural deduction derivations is labelled with just one formula. Thus in natural deduction it is not possible to move or remove formulae from assumption or conclusion position apart by incrementally developing the derivation tree by adding nodes. Consequently, there is no notion of ‘no assumptions’ or of ‘no conclusions’ that directly corresponds to the features of emptiness in the sequent calculus. Instead, natural deduction mimics these phenomena of the sequent calculus by means of two devices: discharge of assumptions and a special propositional constant.

3.1 Assumption Discharge

In the intuitionistic case, a derivation of a sequent with empty antecedent corresponds to a *closed* derivation, that is a derivation in which the conclusion does not depend on any assumptions. One of the features of natural deduction is the fact that derivations originate in one or more assumptions. Thus, the closure of a derivation has to be obtained by successively discharging those assumptions. This is obtained by means of what Prawitz [2] calls ‘improper’ inference rules, that is inference rules that not only yield some new conclusion, but that additionally allow for the discharge of one or more of the assumptions from which the derivation started. This side effect of assumption discharge is displayed most prominently in the rule that governs the introduction of implicative formulae, ($\rightarrow$ I), whose applications look like:

$$\frac{\begin{array}{c} [A] \\ \mathcal{D} \\ B \end{array}}{A \rightarrow B} (\rightarrow I)$$

The rule expresses the fact that a derivation of B that may be based on zero or more assumptions of A can be turned into a derivation of $A \rightarrow B$ by cancelling any number of those assumptions. Prawitz uses the notation $\langle\langle \Sigma, B \rangle / \langle \Theta, A \rightarrow B \rangle\rangle$, where $\Theta = \Sigma - \{A\}$, for this deduction rule.

This rule is very similar to the sequent calculus’ ($\rightarrow$ R), although the latter removes exactly one occurrence of A from the antecedent. Of course, the sequent calculus’ structural rules of weakening and contraction allow the introduction of a formula A or the successive contraction of multiple occurrences of A into a single one before the application of the linear ($\rightarrow$ R).

However, while the application of ($\rightarrow$ R) in the sequent calculus makes a formula disappear from the antecedent, the application of ($\rightarrow$ I) in natural deduction (properly speaking) does not make any assumption disappear, but rather changes the status of the assumptions by marking it as discharged .

For example, compare the following derivation in NI and its corresponding LI derivation.

$$\frac{\frac{[A]^1}{B \rightarrow A} (\rightarrow I)}{A \rightarrow (B \rightarrow A)} (\rightarrow I)^1 \qquad \frac{\frac{\frac{A \Rightarrow A}{A, B \Rightarrow A} (WL)}{A \Rightarrow B \rightarrow A} (\rightarrow R)}{\Rightarrow A \rightarrow (B \rightarrow A)} (\rightarrow R) \quad (\dagger)$$

In natural deduction, negation is usually defined by means of absurdity and implication. In order to establish clear correspondences, we instead use the following rule of ($\neg I$) as primitive:

$$\frac{[A]}{\frac{\perp}{\neg A}} (\neg I)$$

This rule obviously matches the structure of ($\rightarrow I$) for the special case of B being the absurdity $\perp$, i.e. in Prawitz's notation the deduction rule is simply $\langle \langle \Sigma, \perp \rangle / \langle \Theta, \neg A \rangle \rangle$, where $\Theta = \Sigma - \{A\}$.

With the same caveat as was given for ($\rightarrow I$), this corresponds to the intuitionistic sequent calculus' ($\neg R$).

For the correspondence to properly work, we have to assume that a sequent with an empty succedent corresponds to a natural deduction derivation of conclusion $\perp$. We now turn to this.

3.2 Absurdity

It is perhaps too strong to say that natural deduction's absurdity outright mimics the empty succedent. For, contrary to the somewhat subtle mechanic of discharge which marks a formula occurrence as 'actually being absent', absurdity is a much more forthright signal. It occurs as a propositional constant that is actually inferred at some point in the construction of a derivation. However, absurdity has been treated with justified suspicion since the rule governing it, the so-called *ex falso quodlibet* rule, breaks the harmony that is exhibited between introduction and elimination rules for the other logical signs. Indeed the *ex falso* looks like an elimination rule for which there is no corresponding introduction:

$$\frac{\perp}{C} (efq)$$

Thus, there is some reason to not regard absurdity as a proper proposition. Tennant [4] proposed to treat $\perp$ as 'a sort of punctuation mark', registering that 'the derivation has reached a dead end'. Tennant's suggestion is indeed very natural if we look at $\perp$ from the sequent calculus perspective, where the empty succedent indeed marks

an extremal point in a derivation. Rules requiring the presence of some formula in the succedent can no longer be applied. Therefore, in a sense, the attention must be shifted to the formulae of the antecedent.

As mentioned previously, in sequent calculus one may end up with an empty succedent by applying negation left rule. If in the intuitionistic case this has to correspond to a derivation having the constant $\perp$ as its conclusion, it must be due to an application of the natural deduction rule that corresponds to $(\neg L)$, namely $(\neg E)$ ¹:

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\frac{\neg A \quad A}{\perp} (\neg E)}$$

However, only an application of cut yields the derivation corresponding to an application of $(\neg E)$.

$$\frac{\Sigma_1 \Rightarrow \neg A \quad \frac{\Sigma_2 \Rightarrow A}{\neg A, \Sigma_2 \Rightarrow} (\neg L)}{\Sigma_1, \Sigma_2 \Rightarrow} (\text{cut})$$

An application of the *ex falso quodlibet* rule following immediately $(\neg E)$ would correspond to extending the sequent calculus derivation with an application of right weakening.

3.3 The Dual Perspective

We restrict ourselves to a brief exhibition of the natural deduction system for dual-intuitionistic logic NDI; for further details we recommend Schroeder-Heister [3], Wansing [9], Tranchini [5, 6]. The system NDI is a single assumption and multiple conclusions calculus in which derivations are built bottom-up.² Implication is replaced by *co-implication* ' $\multimap$ ' which, instead of discharging assumptions, is a operator that can discharge conclusions.

Co-implication is governed by the following elimination and introduction rules:

$$\frac{A \multimap B}{B} (\multimap E) \qquad \frac{B}{A \multimap B \quad A} (\multimap I)$$

[A]

¹ As already mentioned, negation could also be understood as a defined notion, in which case the rule is simply an instance of $(\rightarrow E)$.

² The terminology of 'assumption' and 'conclusion' is employed for the purpose of retaining the correspondences to antecedent and succedent of sequents. The terminology of 'introduction' and 'elimination' rules follows this perspective and is thereby counter-intuitive to the direction in which derivations are constructed.

As intuitionistic negation is usually viewed as implication of $\perp$, dual-intuitionistic negation can be defined as co-implication of $\top$, where $\top$ denotes the universal trivial statement, governed by the following rule:

$$\frac{C}{\top}$$

As intuitionistic negation, we will however take dual-intuitionistic negation as primitive and governed by the following rules:

$$\frac{\frac{\neg A}{\top} \text{ (-E)}}{[A]} \qquad \frac{\top}{\frac{\neg A}{A} \text{ (-I)}}$$

Note that in dual-intuitionistic logic the picture we described concerning the relation between emptiness in the sequent calculus and natural deduction is reversed. In LDI, empty succedents correspond to NDI derivations in which all conclusions are discharged, and empty antecedents correspond to NDI derivations in which the only assumption is the propositional constant $\top$. This means that the passage from sequent calculus to natural deduction introduces again an asymmetry in how emptiness on the left and on the right side of a sequent are rendered.

4 Units in the Sequent Calculus

We have referred to the phenomena of empty antecedent and empty succedent as structural features of the sequent calculus. We have also mentioned that assumption discharge in NI can be viewed as closely mimicking this structural feature of the emptiness of sequents, in the sense that it can be seen as a structural feature of natural deduction, at least as far as it is only a side effect of an improper inference rule. On the other hand, the absurdity constant $\perp$ provides an entirely propositional correlation to the structural feature of empty succedent.

In NDI, the situation is reversed with discharge of conclusions being a structural feature and triviality $\top$ being a propositional constant.

If we were to follow Tennant's suggestion of giving $\perp$ (resp. $\top$ in the dual case) a structural meaning (and if that were feasible without incurring other problems), both calculi would address these extremal phenomena in a purely structural manner. For the remainder of this article, we shall rather go in the opposite direction. We shall present the empty spaces as having a propositional nature already in sequent calculus.

To do this we define in each sequent calculus an infinite set of *pseudo-constants*, nullary constants labelled with formulae. We will refer to pseudo-constants as *units*.

In LK, LI and LDI, the units are uniformly defined as follows (we use ∇ , Δ and $*$ as meta-symbols for the units and for the negation of each system):

$$\nabla A =_{\text{def}} A \vee *A \qquad \Delta A =_{\text{def}} A \wedge *A$$

In LK, where $*$ is $\sim$, ∇ and Δ are respectively $\forall$ and $\exists$. In LI, where $*$ is $\neg$, ∇ and Δ are respectively $\vee$ and $\perp$. In LDI, where $*$ is $\neg$, ∇ and Δ are respectively $\top$ and $\wedge$. The choice of symbols will become clear in the next sections.

The main reason for labelling pseudo-constants with formulae is that it will permit to interpret them as representing the discharge of some formula A in the corresponding natural deduction derivations.³

While the set of formulae is supplemented by the infinitely many units, one for each formula, the units must remain distinct from proper formulae. We enforce the following restriction on the use of the units in the sequent calculi we will consider: *Units cannot be connected to other formulae through applications of left and right rules for the connectives*. Therefore, formula variables A and B that occur in the premises of rules of the calculi and are connected by a connective in the conclusion must never be instantiated by units. Note that this does not apply to the cut rule, since the cut formula does not occur in the conclusion. That is why cuts on units are permitted.

We now face the task of adding the two units to the sequent calculus. We will not add the units to the intuitionistic and the dual-intuitionistic calculi right away, but instead consider their effects on the classical calculus LK. Only afterwards will we move on to LI and LDI.

4.1 Units and Negation Rules in LK

We consider LK with conjunction and disjunction rules formulated in the multiplicative fashion.

Given the definition of Δ , instances of $(\wedge R)$ with principal formulae A and $\sim A$ read as follows:

$$\frac{\Sigma \Rightarrow A, \Gamma \quad \Theta \Rightarrow \sim A, \Xi}{\Sigma, \Theta \Rightarrow \Delta A, \Gamma, \Xi} (\wedge R)$$

³ In this we follow Gentzen's example in the second part of the *Untersuchungen*, where he suggests the formula $p \wedge \neg p$ (for some arbitrary propositional variable p) as propositional representation of $\perp$ in the sequent calculus as part of his translation of derivations in NI into derivations in LI. Gentzen's choice of some arbitrary p is however a merely utilitarian *ad hoc* choice. If one wished to abstract over the choice of p , one could define units as second-order formulae.

$$\Delta =_{\text{def}} \exists \alpha. \alpha \wedge \neg \alpha \qquad \nabla =_{\text{def}} \forall \alpha. \alpha \vee \neg \alpha$$

However, we will actually profit from labels and thereby we chose to stay within the propositional setting.

According to the definition of the pseudo-constant $\wedge_A$, the consequence of this inference is actually the sequent $\Sigma, \Theta \Rightarrow A \wedge \sim A, \Gamma, \Xi$. Note the special case of a right premise that is an instance of the axiom:

$$\frac{\Sigma \Rightarrow A, \Gamma \quad \overline{\sim A \Rightarrow \sim A}^{(ax)}}{\Sigma, \sim A \Rightarrow \wedge_A, \Gamma}^{(\wedge R)}$$

Accordingly, we can *define* negation left rule through simultaneous introduction of a unit into the succedent:

$$\frac{\Sigma \Rightarrow A, \Gamma}{\Sigma, \sim A \Rightarrow \wedge_A, \Gamma}^{(\sim L_\wedge)}$$

This definition is simply a special case of the more general unit introduction into the succedent.

Dually, negation right rule can be defined by introducing the dual unit into the antecedent. This time the introduction of the unit is the result of a disjunction left rule.

$$\frac{\Sigma, A \Rightarrow \Gamma}{\Sigma, \vee_A \Rightarrow \sim A, \Gamma}^{(\sim R^\vee)} \quad =_{\text{def}} \quad \frac{\Sigma, A \Rightarrow \Gamma \quad \overline{\sim A \Rightarrow \sim A}^{(ax)}}{\Sigma, \vee_A \Rightarrow \sim A, \Gamma}^{(\vee L)}$$

Applications of $(\sim L^\wedge)$ and $(\sim R^\vee)$ in which Γ and Σ are empty yield sequents in which emptiness no longer occurs. In a sense, the rules give a propositional content to the empty succedent and antecedent. (In this sense we do the opposite of what Tennant suggests.)

More precisely, call $\text{LK}_\Delta^\vee$ the result of replacing $(\sim L)$ and $(\sim R)$ with $(\sim L^\wedge)$ and $(\sim R^\vee)$ and let $\wedge_\Delta =_{\text{def}} \{\wedge_A : A \in \Delta\}$ and $\vee_\Delta =_{\text{def}} \{\vee_A : A \in \Delta\}$ for any set of formulae Δ . The following holds:

Theorem 1 (Units as emptiness)

1. If $\Sigma \Rightarrow$ is derivable in LK then $\vee_\Theta, \Sigma \Rightarrow \wedge_\Xi$ for some Θ, Ξ ($\Xi \neq \emptyset$) is derivable in $\text{LK}_\Delta^\vee$.
2. If $\Rightarrow \Gamma$ is derivable in LK then $\vee_\Theta \Rightarrow \Gamma, \wedge_\Xi$ for some Θ, Ξ ($\Theta \neq \emptyset$) is derivable in $\text{LK}_\Delta^\vee$.

Proof By induction on the structure of the derivation. The critical cases are those in which negation rules are applied. Details are left to the reader. $\square$

4.2 Units and (Co-)Implication Rules in LK

As already mentioned, in classical logic implication is defined as $\sim A \vee B$. In standard LK the implication right rule is derived as follows:

$$\frac{\Sigma, A \Rightarrow B, \Gamma}{\Sigma \Rightarrow A \rightarrow B, \Gamma} (\rightarrow R) \quad =_{\text{def}} \quad \frac{\Sigma, A \Rightarrow B, \Gamma}{\Sigma \Rightarrow \sim A, B, \Gamma} (\sim R) \quad \frac{\Sigma \Rightarrow \sim A, B, \Gamma}{\Sigma \Rightarrow \sim A \vee B, \Gamma} (\vee R)$$

By replacing the application of $(\sim R)$ with one of $(\sim R^\forall)$, we can derive an alternative right implication rule in which the movement of A from one side to the other leaves a unit as its trace:

$$\frac{\Sigma, A \Rightarrow B, \Gamma}{\Sigma, \forall A \Rightarrow A \rightarrow B, \Gamma} (\rightarrow R^\forall) \quad =_{\text{def}} \quad \frac{\Sigma, A \Rightarrow B, \Gamma \quad \overline{\sim A \Rightarrow \sim A}^{(\text{ax})}}{\Sigma, \forall A \Rightarrow \sim A, B, \Gamma} (\vee L) \quad \frac{\Sigma, \forall A \Rightarrow \sim A, B, \Gamma}{\Sigma, \forall A \Rightarrow \sim A \vee B, \Gamma} (\vee R)$$

It should be observed that also in this case, the “real” shift of side taking place in $(\rightarrow R)$ is replaced by an only “apparent” shift of side in $(\rightarrow R^\forall)$. The occurrence of the formula A in the premise of the rule gets locked into the unit. The occurrence of A as sub-formula of $A \rightarrow B$ in the conclusion comes from the axiom which acts as premise of $(\vee L)$.

It is also worth noticing that the definition of $(\rightarrow R^\forall)$ is essentially classical, in the sense that putting $\Gamma = \emptyset$ would not make the rule intuitionistically derivable: One needs multiplicity of formulae in the succedent in order to apply the multiplicative disjunction right rule. However, this does not forbid to take an intuitionistic version of $(\rightarrow R^\forall)$ as primitive. Given the correspondence of **NI** and **LI**, this would enable to view the occurrence of the unit in the rule as corresponding to the discharge of the assumption A in applications of the natural deduction $(\rightarrow I)$ rule. In the next section, we will develop this suggestion in a fully-fledged manner.

Dual considerations apply to co-implication. Given the classical definition of $A \multimap B$ as $\sim A \wedge B$, the co-implication left rule can be derived by means of $(\sim L)$ and $(\wedge L)$:

$$\frac{\Sigma, B \Rightarrow A, \Gamma}{\Sigma, A \multimap B \Rightarrow \Gamma} (\multimap L) \quad =_{\text{def}} \quad \frac{\Sigma, B \Rightarrow A, \Gamma}{\Sigma, \sim A, B \Rightarrow \Gamma} (\sim L) \quad \frac{\Sigma, \sim A, B \Rightarrow \Gamma}{\Sigma, \sim A \wedge B \Rightarrow \Gamma} (\wedge L)$$

By replacing the application of $(\sim L)$ with one of $(\sim L^\wedge)$, an alternative co-implication left rule $(\multimap L^\wedge)$ can be derived in which the (now only apparent) shift of A from the succedent to the antecedent leaves a unit as trace.

$$\frac{\Sigma, B \Rightarrow A, \Gamma}{\Sigma, A \multimap B \Rightarrow \wedge_A \Gamma} (\multimap L^\wedge) \quad =_{\text{def}} \quad \frac{\Sigma, B \Rightarrow A, \Gamma \quad \overline{\sim A \Rightarrow \sim A}^{(\text{ax})}}{\Sigma, \sim A, B \Rightarrow \wedge_A \Gamma} (\sim L^\wedge) \quad \frac{\Sigma, \sim A, B \Rightarrow \wedge_A \Gamma}{\Sigma, \sim A \wedge B \Rightarrow \wedge_A \Gamma} (\wedge L)$$

Again, in a dual-intuitionistic version of the rule, the unit may be viewed as standing for the discharge of a conclusion of form A resulting by applications of $(\rightarrow E)$ in **NDI**. This will be properly spelled out in the next sections.

5 Introducing Units into LI and LDI

The sequent calculi LI and LDI for intuitionistic logic and its dual are obtained by imposing an asymmetric restriction on sequent contexts. These substructural restrictions can be viewed as modifying the meaning of the empty spaces on the side of the sequent arrow to which the restriction applies.

5.1 Units in LI

In Sect. 2 the rules ($\sim$ R) and ($\sim$ L) have been identified as the causes for emptiness into the antecedent or succedent in the classical system. In the previous section, we showed how these rules, as well as the defined rule ($\rightarrow$ R) can be derived using the units.

We now do the same in LI. As already remarked, both ($\rightarrow$ R) and its modified version displaying the unit cannot be derived in the intuitionistic system due to the restriction on the succedent of sequents. To overcome this, we will simply take implication rules as primitive.

In LI negation is denoted by $\neg$. We also use different symbols for the units, namely, $\top$ and $\perp$. The choice of $\perp$ is motivated by the fact that applying ($\wedge$ I) to the same premises of ($\neg$ E) yields the formula defining the unit:

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\frac{A \quad \neg A}{\perp}} (\neg E) \qquad \frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\frac{A \quad \neg A}{A \wedge \neg A}} (\wedge I)$$

A direct correspondence of the inference on the right is obtained through the replacement of ($\neg$ L) by the following rule that both shifts a negated A into the antecedent and at the same time introduces an instance of the pseudo-absurdity $\perp_A$:

$$\frac{\Theta \Rightarrow A}{\Theta, \neg A \Rightarrow \perp_A} (\neg L_{\perp})$$

In serving as a marker for the empty succedent, the unit $\perp_A$ strengthens the intuitionistic requirement of sequents having to contain no more than one succedent formula to that of sequents having to contain exactly one succedent formula or otherwise a unit.

This strengthened condition entails a complication, however. In LI, an instance of weakening of the succedent enables the transition from a sequent with empty succedent to a corresponding one that contains an arbitrary succedent formula C . With $\perp_A$ filling in a formerly empty succedent, a weakening of the succedent is no longer possible, rendering (WR) obsolete. A possible remedy of the situation would be to introduce an *ex falso quodlibet* rule:

$$\frac{\Theta \Rightarrow \perp_A}{\Theta \Rightarrow C} \text{ (efq)}$$

Following Troelstra and Schwichtenberg [7], we use an axiom instead that allows the derivation of the rule above by means of a cut on the unit:

$$\frac{}{\perp_A \Rightarrow C} \text{ (}\perp\text{L)}$$

In the previous section we suggested to view the units in $(\sim R^\forall) (\rightarrow R^\forall)$ as discharge markers. This is fully substantiated in their intuitionistic versions, given the correspondence between LI and NI:

$$\frac{\Theta, A \Rightarrow B}{\Theta, \forall_A \Rightarrow A \rightarrow B} (\rightarrow R^\forall) \qquad \frac{\Theta, A \Rightarrow \perp_C}{\Theta, \forall_A \Rightarrow \neg A} (\neg R^\forall)$$

In both rules, the unit $\forall_A$ is retained in place of an antecedent (assumptive) formula that is shifted (discharged) in the usual intuitionistic rule. In the case of negation introduction, a further modification of the premise guarantees that its succedent contains the unit $\perp_C$ for some formula C .

Let $LI_\perp^\forall$ be the calculus that is obtained from LI through the replacement of $(\rightarrow R)$, $(\neg R)$ and $(\neg L)$ by, respectively, $(\rightarrow R^\forall)$, $(\neg R^\forall)$ and $(\neg L_\perp)$ as well as the replacement of (WR) by the axiom $(\perp L)$.

Note that instances of the unit that serves as cancellation marker can simply accumulate in the antecedent of sequents over the course of a derivation. For example, compare the following $LI_\perp^\forall$ derivation to the LI derivation ($\dagger$):

$$\frac{\frac{\frac{A \Rightarrow A}{A, B \Rightarrow A} \text{ (WL)}}{A, \forall_B \Rightarrow B \rightarrow A} (\rightarrow R^\forall)}{\forall_A, \forall_B \Rightarrow A \rightarrow (B \rightarrow A)} (\rightarrow R^\forall)$$

The formula $A \rightarrow (B \rightarrow A)$ is no longer derivable in the usual sense (i.e. as the succedent of a sequent with an empty antecedent), since the antecedent contains discharge markers $\forall_A$ and $\forall_B$.

An interesting correspondence is obtained, however, when we modify the derivability relation of NI in such a way that discharged formulae are not simply dropped from the multiset of assumptions. Instead, a single instance of the discharged assumptions (regardless of whether it is a vacuous or a multiple discharge) is collected into a second multiset that keeps track of discharged assumptions. As the derivability relation for NI is merely a notational tool, this does not change the calculus. Thus, $[\Theta]\Sigma \vdash^* A$ expresses that A is derivable from Σ in NI with discharged assumptions collected in Θ in the manner just described. A closed derivation of A in NI is then a derivation $[\Theta] \vdash^* A$ for some Θ . Hence the fact that closed derivations are seen as relative to a set of discharged assumptions makes ' $\vdash^*$ ' more fine grained than ' $\vdash$ '.

Theorem 2 *Let the formulae in Σ , Θ and formula A be formulae of **NI**, i.e. unit-free, and let $\forall_{\Theta} = \{\forall_A : A \in \Theta\}$. Then the following holds:*

1. $[\Theta]\Sigma \vdash^* A$ holds in **NI** if and only if $\forall_{\Theta}, \Sigma \Rightarrow A$ is derivable in $\text{LI}_{\perp}^{\forall}$.
2. $[\Theta]\Sigma \vdash^* \perp$ holds in **NI** if and only if $\forall_{\Theta}, \Sigma \Rightarrow \perp_A$ is derivable in $\text{LI}_{\perp}^{\forall}$.

Rather than establishing this result directly, we first relate derivations in $\text{LI}_{\perp}^{\forall}$ to derivations in **LI** and thus, via the well-known correspondence, to derivations in **NI**. We will describe the correspondence stated by the theorem in some more detail further below.

To establish such a relation we have to get rid of residual discharge markers occurring in sequents. This can be done by adding the following axiom to the rules and the axioms that comprise $\text{LI}_{\perp}^{\forall}$:

$$\frac{}{\Rightarrow \forall_A} (\forall R)$$

Through applications of cut, this axiom allows for the removal of all units $\forall_A$ that were introduced by means of $(\rightarrow R^{\forall})$ or $(\neg R^{\forall})$. This allows us to state the following correspondence between the intuitionistic sequent calculi with and without units.

Lemma 1 *Let the formulae in Σ and formula A be formulae of **LI**, i.e. unit-free. Then the following holds:*

1. $\Sigma \Rightarrow A$ is derivable in **LI** if and only if $\Sigma \Rightarrow A$ is derivable in $\text{LI}_{\perp}^{\forall} + (\forall R)$.
2. $\Sigma \Rightarrow \perp$ is derivable in **LI** if and only if $\Sigma \Rightarrow \perp_A$ is derivable in $\text{LI}_{\perp}^{\forall} + (\forall R)$.

Proof The direction from left to right is rather straightforward. In order to translate a **LI** derivation into a $\text{LI}_{\perp}^{\forall}$ derivation, translate and compose the subderivations by following this procedure:

1. Replace every application of a critical rule $(\neg R)$ or $(\rightarrow R)$ by, respectively, $(\neg R^{\forall})$ and $(\rightarrow R^{\forall})$, followed by a cut with the axiom $(\forall R)$.
2. Replace every application of $(\neg L)$ by an application of $(\neg L_{\perp})$, and, while tracing the derivation downwards from that point on, replacing empty succedents by the corresponding unit. Replace any application of (RW) by a cut with the corresponding instance of $(\perp L)$.
3. If any application of a rule in the **LI** derivation has two premises with empty succedent, and their translations into $\text{LI}_{\perp}^{\forall}$ have succedents $\perp_A$ and $\perp_B$, let the conclusion have the succedent $\perp_{A \vee B}$. This applies to the rule $(\vee L)$ the only one that can merge empty succedents.

The reverse direction is somewhat more involved, because formulae C introduced via the axiom $(\perp L)$ can be side formulae of further logical rules that are applied before $\perp_A$ is eventually cut. For this reason, all cuts on units $\perp_A$ have to be pushed upwards in the derivation by following the usual cut elimination procedure until a cut that has an instance of $(\perp L)$ as its premise is obtained. Only then can subderivations of the resulting $\text{LI}_{\perp}^{\forall}$ derivation be translated and composed. In doing this, the following points have to be observed:

1. Whenever one of the new rules $(\neg R^\vee)$, $(\rightarrow R^\vee)$ or $(\neg L_\perp)$ is applied in the $LI_\perp^\vee$ derivation, simply employ the original LI rules.
2. Any application of cut that has one of the new axioms as one of its premises is simply dropped. $\square$

Corollary 1 *Let the formulae in Σ and formula A be formulae of NI, i.e. unit-free. Then the following holds:*

1. $\Sigma \vdash A$ holds in NI if and only if $\Sigma \Rightarrow A$ is derivable in $LI_\perp^\vee + (\vee R)$.
2. $\Sigma \vdash \perp$ holds in NI if and only if $\Sigma \Rightarrow \perp_A$ is derivable in $LI_\perp^\vee + (\vee R)$.

Proof This follows from the preceding lemma by means of the well-known correspondence result between derivability in LI and NI. $\square$

Proof of Theorem 2 The correspondence between the modified derivability in NI and derivability in $LI_\perp^\vee$ without axiom $(\vee R)$ is established analogously to Corollary 1.⁴ The difference is that in the direction from NI to $LI_\perp^\vee$, applications of $(\rightarrow R^\vee)$ and $(\neg R^\vee)$ are *not* followed by the cuts removing the units. $\square$

5.2 Units in LDI

In LDI negation is denoted by $\neg$. We also use different symbols for the units, namely, $\wedge$ and $\top$. The choice of $\top$ is motivated by the fact that applying $(\vee E)$ to the same conclusions of $(\neg I)$ yields the formula defining the unit:

$$\frac{\top}{A} \quad \frac{\neg A}{D_2} \quad (\neg I) \qquad \frac{A \vee \neg A}{A} \quad \frac{\neg A}{D_2} \quad (\vee E)$$

As in LI $\vee$ is properly interpreted as an assumption discharge marker, in LDI $\wedge$ is a proper conclusion discharge marker.

Let $LDI_\wedge^\top$ be the calculus that is obtained from LDI through the replacement of $(\neg L)$, $(\neg L)$ and $(\neg R)$ by, respectively, $(\neg L_\wedge)$, $(\neg L_\wedge)$ and $(\neg R^\top)$ as well as the replacement of (WL) by the axiom $(\top R)$. Furthermore, the additional axiom $(\wedge L)$ can be used to remove markers for discharged conclusions:

$$\frac{}{\wedge A \Rightarrow} \quad (\wedge L)$$

We can state the dual correspondence results.

⁴ If one wished a direct correspondence between discharged formulae in the $\ast$ -red derivability and formulae (and not just labelled units) in sequent calculus, one could introduce a sequent calculus $LI^\ast$ which instead of introducing units labelled with the discharged formulae, just put the latter ones in a special “discharge” context.

Fig. 4 The roles of the four units in the two calculi

	∇_A	Δ_A
$\text{LI}_\perp^Y$	$\frac{}{\Rightarrow \nabla_A} (\nabla R)$ discharge of assumptions	$\frac{\perp_A \Rightarrow C}{\perp_A \Rightarrow C} (\perp L)$ absurdity
$\text{LDI}_\wedge^T$	$\frac{C \Rightarrow \top_A}{C \Rightarrow \top_A} (\top R)$ triviality	$\frac{}{\wedge_A \Rightarrow} (\wedge L)$ discharge of conclusions

Theorem 3 Let the formulae in Γ , Ξ and formula A be formulae of NDI, i.e. unit-free, and let $\wedge_\Xi = \{\wedge_A : A \in \Xi\}$. Then the following holds:

1. $A \vdash^* \Gamma[\Xi]$ holds in NDI if and only if $A \Rightarrow \Gamma, \wedge_\Xi$ is derivable in $\text{LDI}_\wedge^T$.
2. $\top \vdash^* \Gamma[\Xi]$ holds in NDI if and only if $\top_A \Rightarrow \Gamma, \wedge_\Xi$ is derivable in $\text{LDI}_\wedge^T$.

Lemma 2 Let the formulae in Γ and formula A be formulae of LDI, i.e. unit-free. Then the following holds:

1. $A \Rightarrow \Gamma$ is derivable in LDI if and only if $A \Rightarrow \Gamma$ is derivable in $\text{LDI}_\wedge^T + (\wedge L)$.
2. $\Rightarrow \Gamma$ is derivable in LDI if and only if $\top_A \Rightarrow \Gamma$ is derivable in $\text{LDI}_\wedge^T + (\wedge L)$.

Corollary 2 Let the formulae in Γ and formula A be formulae of NDI, i.e. unit-free. Then the following holds:

1. $A \vdash \Gamma$ holds in NDI if and only if $A \Rightarrow \Gamma$ is derivable in $\text{LDI}_\wedge^T + (\wedge L)$.
2. $\top \vdash \Gamma$ holds in NDI if and only if $\top_A \Rightarrow \Gamma$ is derivable in $\text{LDI}_\wedge^T + (\wedge L)$.

6 Summary

Adding the two units to the intuitionistic and dual-intuitionistic sequent calculi has shed some light on the status of emptiness in antecedent and succedent of those calculi through the axioms that are required to guarantee interderivability of sequents between LI and $\text{LI}_\perp^Y$ and derivability statements for NI on one hand and between LDI and $\text{LDI}_\wedge^T$ and derivability statements for NDI on the other hand. Units as discharge markers are addressed by axioms which mention the units alone and may thus be viewed as purely structural. The axioms for units that represent constants relate the unit to some arbitrary formula and thus represent the *ex falso quodlibet* and its dual in the sequent calculus.

We employed four units to emphasize the difference in behaviour that depends on the impact of the structural restrictions imposed on the intuitionistic and dual-intuitionistic sequent calculi. This is summarized in Fig. 4. Note, however, that, strictly speaking, (∇R) is not an axiom of $\text{LI}_\perp^Y$, nor is $(\wedge L)$ an axiom of $\text{LDI}_\wedge^T$.

References

1. Czermak, J. (1977). A remark on Gentzen's calculus of sequents. *Notre Dame Journal of Formal Logic*, 18(3), 471–4.
2. Prawitz, D. (1965). *Natural deduction. A proof-theoretical study*. Stockholm: Almqvist & Wiksell.
3. Schroeder-Heister, P. (2009). Schluß und Umkehrschluß: ein Beitrag zur Definitionstheorie. In: Gethmann CF (ed) Akten des XXI Deutschen Kongresses für Philosophie (Essen, 15.-19.9.2008), Deutsches Jahrbuch Philosophie, Band 3, Felix Meiner Verlag, Hamburg.
4. Tennant, N. (1999). Negation, absurdity and contrariety. In D. M. Gabbay & H. Wansing (Eds.), *What is negation?*. The Netherlands: Kluwer Academic Publishers.
5. Tranchini, L. (2010). Refutation: A proof-theoretic account. In C. Marletti (Ed.), *First Pisa colloquium in logic*. Pisa: Language and Epistemology, ETS.
6. Tranchini, L. (2012). Natural deduction for dual-intuitionistic logic. *Studia Logica*, 100(3), 631–48.
7. Troelstra, A. S., & Schwichtenberg, H. (1996). *Basic proof theory*. Cambridge: Cambridge University Press.
8. Urbas, I. (1996). Dual-intuitionistic logic. *Notre Dame Journal of Formal Logic*, 37(3), 440–451.
9. Wansing, H. (2010). Proofs, disproofs, and their duals. In C. Areces & R. Goldblatt (Eds.), *Advances in Modal Logic 7, Papers from the 7th Conference on "Advances in Modal Logic"* (pp. 483–505). Nancy, France: College Publications (9–12 September 2008).

Recent Trends in Philosophical Logic

Ciuni, R.; Wansing, H.; Willkommen, C. (Eds.)

2014, VIII, 209 p. 12 illus., Hardcover

ISBN: 978-3-319-06079-8