Preface

This book evolved from a series of lectures at the University of Sussex and is designed to provide an integrated course in real and complex analysis for undergraduates who have taken first steps in real analysis; the intention is to exhibit something of the interplay between these and other areas of mathematical study. The prerequisites are modest: it would be completely sufficient to have followed preliminary courses in real analysis (involving ε, δ ideas) and algebraic structures. There are many exercises, ranging from the elementary to the quite demanding. To establish notation and terminology, some prerequisites are reviewed in the appendices.

A persistent theme in the text is the search for a primitive. In the case of real analysis, the Riemann integral offers one route in this quest and, with an eye to complex analysis, the improper Riemann integral is an extension consonant with the demands of contour integrals.

Chapter 1 deals with the Riemann theory of integration on the real line using the simple and elegant approach due to Darboux that quickly leads to the basic properties of the integral together with means of evaluation and estimation. It also enables direct, elementary proofs to be given of the results that if f is Riemann-integrable, then (i) the set of its points of continuity is dense in the domain of f, and (ii) $g \circ f$ is Riemann-integrable if g is continuous. A characterization of the class of Riemann-integrable functions, from which these last two assertions follow, is postponed to the next chapter as it is technically more challenging. The Riemann integral is confined to bounded functions defined on closed bounded intervals and requires extension to cope with the demands of later chapters. To allow for some relaxation of these constraints, the improper Riemann integral is introduced. We indicate the limitations of the Riemann integral which led to the development of Lebesgue's integral (which itself would require slight extension for use in the later chapters), of which the former is a special case.

Metric spaces form the theme of Chap. 2; the earlier one provides a wealth of examples of such objects. Detailed coverage is given of the core properties of completeness, compactness, connectedness and simple connectedness: this last property is highlighted. While it has become more common in recent times to present such matters in the context of normed linear spaces, we believe it is important for the student to realize that linear structure is irrelevant to many of the results. Regarding completeness, Cantor's characterization is established as are

Banach's contraction mapping theorem and the Baire category theorem, the last leading to a proof of existence of a continuous, nowhere-differentiable function and also to the fact that the pointwise limit of a sequence of continuous real-valued functions on a complete metric space is continuous on a dense subset of that space. Compactness and connectedness are motivated in a variety of ways, the definitions chosen being intrinsic and applicable in more general contexts. Among the applications of compactness are differentiation under the integral sign, Peano's theorem on the existence of solutions of initial-value problems for certain nonlinear ordinary differential equations, and the characterization of Riemann-integrable functions as functions that are bounded and continuous almost everywhere. With the next chapter in mind, we conclude with the consideration of simply connected spaces. Various forms of homotopy are given especially detailed coverage, strenuous efforts being made to give complete proofs. We show that a metric space is simply connected if and only if it is path-connected and its fundamental group at any (and hence every) point of the space has only one element.

In Chap. 3, we reach our main goal, the theory of complex analysis, surely one of the most wonderful and fertile parts of mathematics. After some basic definitions and results, we deal with power series, branches of the argument and logarithm, continuous logarithms of continuous zero-free functions, the winding number for arbitrary paths in the plane and its invariance under free homotopy, and integrals over contours. Ample justification for the introduction of the winding number is provided by the demands of the proof of the Jordan curve theorem given later (for which the winding number is essential and the index is inadequate as it is undefined for general paths having no smoothness), but in addition we believe that there is a computational and pedagogical advantage in having this concept available. The homology version of Cauchy's theorem is derived by means of the elegant approach of Dixon [6]. Rudin [15] was one of the first to draw attention to the importance of Dixon's contribution and the organisation of complex analysis consequent upon it. Rather than appeal to an interchange of the order of integration, as Rudin does, we follow Dixon's original treatment and use differentiation under the integral sign. This leads to the residue theorem, from which flow such major theoretical results as Rouché's theorem and the open mapping and inverse function theorems; further, at a practical and technical level it is valuable in the evaluation of definite integrals. The penultimate section contains a result of exceptional aesthetic appeal which establishes, for connected open sets $G \subset \mathbf{C}$ (the space of all complex numbers) the equivalence of various statements of an analytic, algebraic and topological character. In particular, it shows that every function analytic on G has a primitive if and only if G is simply connected. In the course of the proof, such famous results as Montel's theorem and the Riemann mapping theorem are obtained. The final section reinforces the links between analysis and topology. Further study of topics introduced earlier, namely continuous logarithms of continuous zero-free functions and the winding number of a path, leads in a very natural way to a proof of the celebrated Jordan curve theorem. For this development of the theory, we acknowledge a major debt to the book [3] by Burckel. A beautiful result due to Borsuk concerning any compact set

$K \subset \mathbf{C}$ emerges in the course of the proof of the Jordan curve theorem: $\mathbf{C} \backslash K$ is connected if and only if every continuous function $f : K \to \mathbf{C} \backslash \{0\}$ has a continuous logarithm.

Our exposition covers aspects of classical analysis due to the efforts of generations of mathematicians. There is no claim to originality save for the selection and presentation of material. We have been greatly influenced by the scholarly and inspirational books by Burckel [3], Remmert [13] and Rudin [15], and hope readers of the present book will go on to consult these more advanced and wider-ranging works.

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