

CHAPTER 2

Dante's 3-Sphere Universe

Dante Alighieri (1265–1321) AD wrote what is now considered one of the greatest works of world literature, the *Divine Comedy* — an allegorical vision of Christian afterlife.¹ The focus here is a description of a universe that includes the *Empyrean*, which, among Christian poets, is the abode of God or the firmament. Dante constructs the Empyrean as a mirror image of the classical Aristotle universe, and then “glues their 2-sphere boundaries” to form a 3-sphere. We essentially follow the article *Dante and the 3-sphere* by Mark Peterson, *American Journal of Physics* 47 (1979).

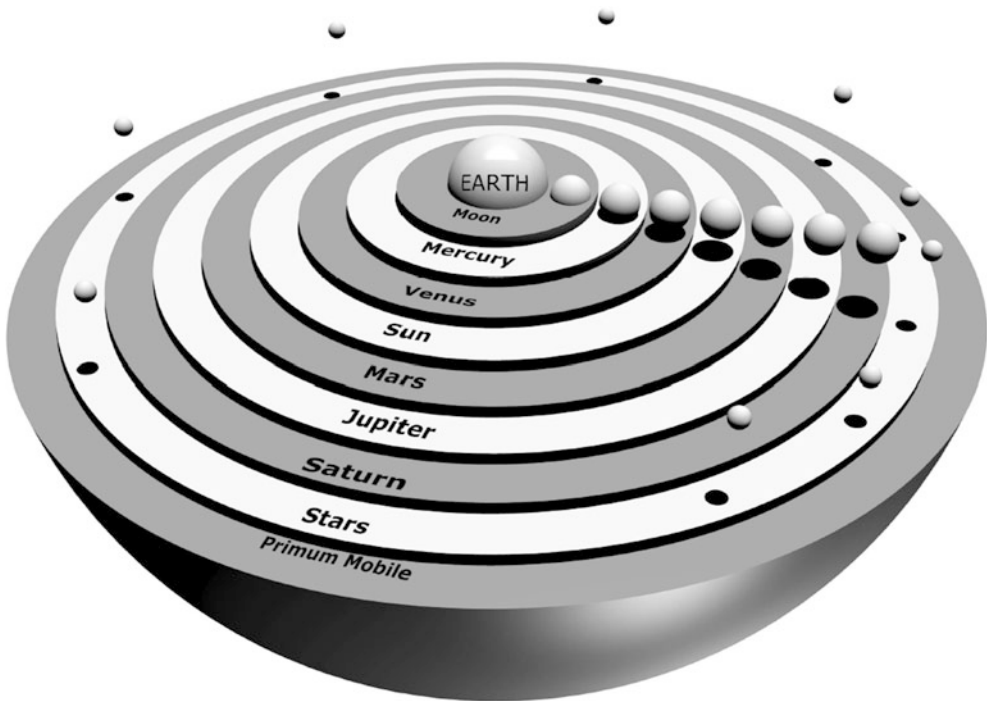


FIG. 2.1 *Heavenly sphere* — classical Aristotle/Greek model of the universe.

¹An *allegory* is a description of one thing under the image of another.

§7 ARISTOTLE UNIVERSE

Underlying Dante's *Divine Comedy* is the Aristotle (Greek) model of our universe. The illustration above enhances the following quote from John Stillwell's *Yearning for the Impossible* published by A. K. Peters, Ltd., 2006 (Figure 2.1).

[Page 102 of Stillwell's book; square brackets "[·]" indicate my comments.]

The Greeks believed the universe should reflect the geometric perfection of circles and spheres, and they imagined space structured by a system of spheres. ... The earth is the innermost sphere, surrounded by eight concentric "heavenly" spheres carrying the known celestial bodies, and an outermost sphere called the *Primum Mobile* [in the Ptolemaic system, the tenth and outermost concentric sphere, revolving from east to west around the earth and causing all celestial bodies to revolve with it]. (For example, "seventh heaven" is the sphere of Saturn.) The motion of the sun, moon, planets, and stars was attributed to the rotation of the spheres carrying them, with the *Primum Mobile* ("first mover") controlling them all. Somehow, the ancient universe stopped at the *Primum Mobile*, ...

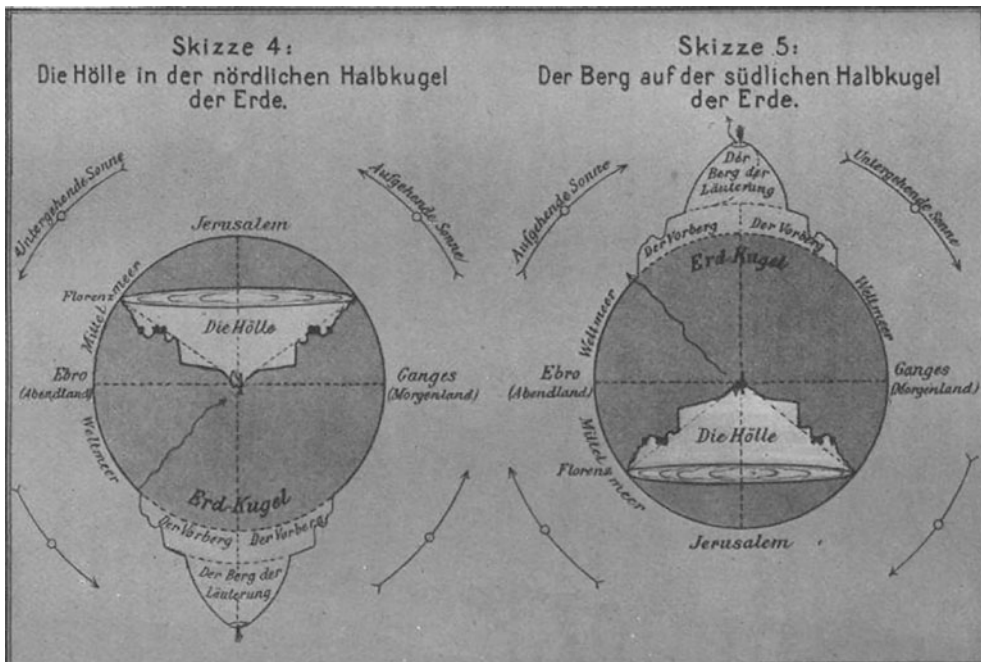


FIG. 2.2 Albert Ritter's sketches of *Comedy's* geography.

§8 DANTE'S JOURNEY

Centered around Dante's afterlife journey (the journey of his Christian soul) into Heaven, the poem is divided into three major parts, the *Inferno*, *Purgatorio*, and *Paradiso*. In *Inferno* we have the beginning of his journey where Dante descends into Hell. Satan's fall created Hell as a funnel-shaped gateway (rock displacement) under Jerusalem that leads to the center of the earth. The rock displacement in turn created *Mount Purgatory* on the surface of the earth diametrically opposite to Jerusalem (Figure 2.2).

The sphere on the right in Figure 2.2 is an inversion of the one on the left. Starting at Florence, located on the left side of the "funnel to Hell" in the left-side sphere in Figure 2.2, Dante, moving toward the center of the earth, descends into Hell. He then continues down, and thereby up in the right-side sphere, to Mount Purgatory's shores in the southern hemisphere. Then he ascends to the top of Mount Purgatory, the location of the *first sphere of Heaven* — the "ANGELS" part of the *Angelic sphere* (Figure 2.3).

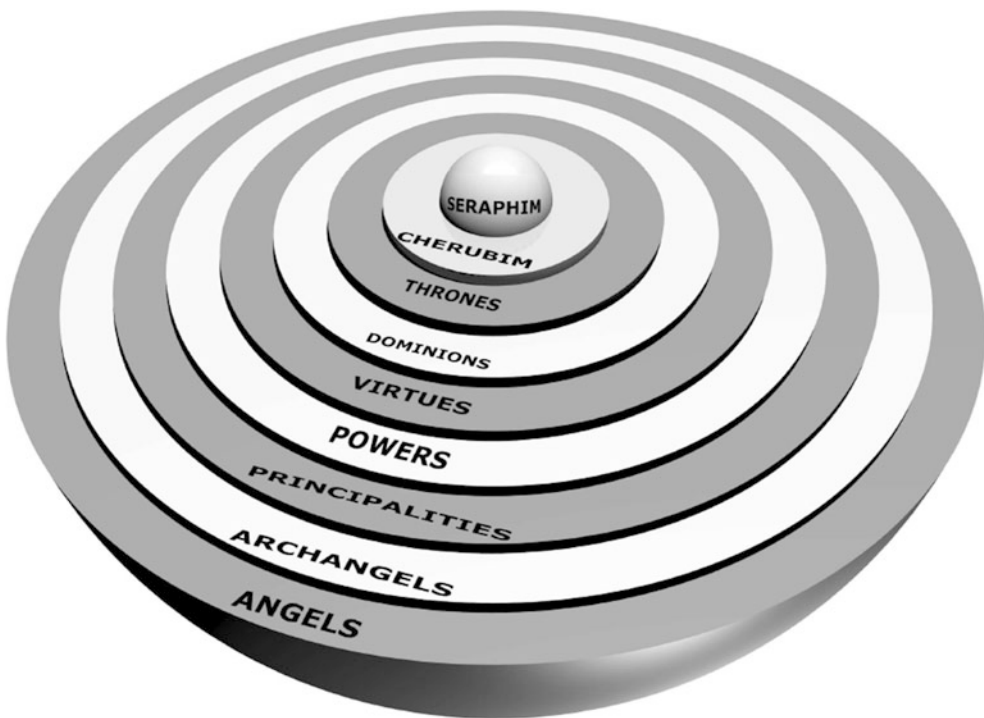


FIG. 2.3 *Angelic sphere* (Empyrean: home of God and Angels).

As Dante descends into Hell, his soul sees sin for what it really is — self-indulgent sins, violent sins, and malicious sins. But escorted by an Angel, Dante's soul survives Hell, and then Dante and the Angel move from the center of the Earth to Mount Purgatory. *Purgatorio* allegorically represents the Christian life — angels bring Christian souls for conversion from sin to the state of grace.

In Paradiso, we find Dante after he leaves the Aristotle universe and enters "Heaven" which is the *Empyrean* (among Christian poets, the abode of God, the firmament).

§9 ANGELIC SPHERE

The *Angelic Sphere* is a mirror image of the Aristotle Universe. The "spheres of angels" or *Angelic Choirs* illustrate an ordering relative to "closeness to God," who resides at the center of the center sphere.

[Page 104 of Stillwell's book; square brackets "[·]" indicate my comments.]

In Canto [main division of poem, analogous to "chapter" division of a book of prose] XXVIII Dante views the Empyrean as not only the *complement* but also the *reflection* of the heavens visible from earth. He makes a smooth transition from the heavens to the Empyrean by using the Primum Mobile as a half-way stage between two worlds, the "model" and the "copy." From this vantage point, he sees the heavenly spheres on one side as an image of the angelic spheres on the other.

*as one who in a mirror catches sight
of candlelight aglow behind his back
before he sees it or expects it,
and, turning from the looking-glass to test
the truth of it, he sees that glass and flame
are in accord as notes to music's beat²*



FIG. 2.4 Dante and Beatrice gaze upon the Empyrean.²

We continue with our quote from Stillwell's book:

With this sophisticated model of a finite universe, the Church was able to hold out against infinite space for a few centuries. But eventually infinite flat space came to be generally accepted for its greater simplicity, despite some uneasiness about infinity ...

In the twentieth century, cosmology returned to the idea of a finite universe, and physicists now look back in admiration to Dante's *Paradiso*, seeing in it a good description of the simplest finite universe, which we now call the 3-*sphere*

²Translation by Mark Musa of line 4-9, Canto XXVIII, of Dante's *Paradiso*. Art by Gustave Doré (1832-1883).

§10 DANTE'S 3-SPHERE

To reinforce Dante's construction of his universe let us consider the Aristotle Universe, abstractly a 3-disc (solid ball whose boundary is a 2-sphere), and its mirror image the Empyrean, another 3-disc (Figure 2.5).

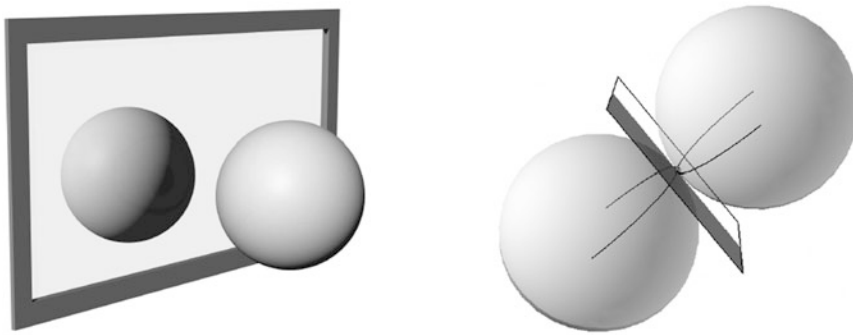


FIG. 2.5 Aristotle Universe just touching its mirror image the Empyrean.³

On the right side of Figure 2.5 we see how Dante views the gluing of *a single point* on the boundary of the Aristotle Universe with *a single point* on the boundary of the Empyrean. The single point (where the two spheres meet) may also be viewed — within Gustave Doré's art in Figure 2.4 — as the location of Dante and (the angel) Beatrice with the Empyrean in the background.

Dante's construct implies that each point on the boundary of the Aristotle Universe corresponds to one and only one point on the boundary of the Empyrean, and conversely, each point on the boundary of the Empyrean corresponds to one and only one point on the boundary of the Aristotle Universe. That is, using "Dante's mirror," we see that as the sphere in the left-side graphic in Figure 2.5 rotates, its mirror image would likewise rotate, yielding a *faithful matching* between the boundaries of the Aristotle Universe and the Empyrean.

It follows that Dante *faithfully glues the S^2 -boundary of the Aristotle Universe to the S^2 -boundary of the Empyrean*. Dante is constructing a 3-sphere.

One benefit of Dante's construction is that of removing the "edge" of the Aristotle Universe.

[Mark Peterson - *Dante and the 3-sphere*] The belief that the earth must be round goes back at least to Aristotle, whose doctrine of "natural place" required a round earth at the center of the universe. This same model became central to Christian theology with the work of Thomas Aquinas, and it forms the cosmological framework for Dante's *Divine Comedy*.

³The "mirror graphic" on the left was provided by Mary Fugier, who used Rhinoceros 3-D Software (www.rhino3d.com).

The belief that the universe as a whole might be round (or more generally, curved) is a much more recent one. It seems to require mathematics of the 19th century (non-Euclidean geometry) even to formulate the notion.

It is therefore a considerable surprise to find, on closer reading, that Dante's cosmology is not as simple geometrically as it at first appears, but actually seems to be a so-called "closed" universe, the 3-sphere, a universe which also emerges as a cosmological solution of Einstein's equations in general relativity theory.

I came upon this suggestion about Dante and the 3-sphere in wondering how Dante would treat an evidently unsatisfactory feature of the Aristotelian cosmology when he, as narrator in the *Paradiso*, got to the "edge" or "top" of the universe. How would he describe the edge? It is the same problem every child has wondered about: unless the universe is infinite, it must (the argument goes) have an edge — but then what is beyond? Dante faces this very problem at the end of the *Divine Comedy* where he must describe the Empyrean not in terms of principles or abstractions, as the standard cosmology did, but as someone actually there.

Within the last four words of Peterson's quote, namely, *as someone actually there*, the last word *there* means *at the boundary of the Aristotle Universe* where the gluing takes place. The "there" may be graphically viewed as the point in the right-side illustration of Figure 2.5 where the two spheres are just touching.

But how can we picture the "locally glued area" near the "there" point?

§11 LOCALLY GLUED AREAS

Referring again to Gustave Doré's art in Figure 2.4, we see Dante and Beatrice standing at the *there* point where we also see "land" or a "local area" on Mount Purgatory that surrounds Dante and Beatrice.

Expressed differently, if as illustrated on the right side of Figure 2.5 we select a point p on the 2-sphere boundary of the Aristotle Universe that we shall glue to a point p' on the 2-sphere boundary of the Empyrean, then can we view an *area of glued points* that are "close" to a single glued point $p = p'$?

The gluing process is that of gluing *points near p to points near p'* . But as illustrated on the right side of Figure 2.5, we see how the 2-sphere containing p curves away from p and how the 2-sphere that contains p' curves away from p' . So it is not immediately obvious how we could picture such a "local gluing."

With Figures 2.6 and 2.7 we nevertheless provide a way to view a *local gluing*. Beginning with the left-side of Figure 2.6, we locally-flatten two

2-spheres. Then we *faithfully match* the points within the small white area with their counterpart points on the adjacent flattened sphere — we glue (right-side illustration) the locally-matched areas.

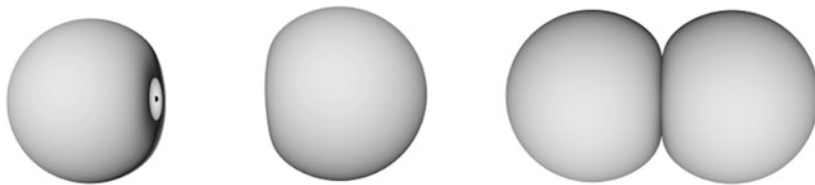


FIG. 2.6 We locally flatten two 2-spheres, and then glue locally-matched areas.

By making one of the flattened spheres transparent, we see the *locally glued area* illustrated as the relatively-white circular area in Figure 2.7.

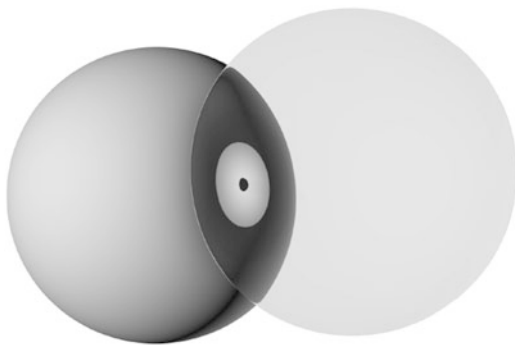


FIG. 2.7 The white area near the black dot is a “locally glued area.”

Using Figure 2.7 we may make another observation. If the black dot is viewed as the location of Dante and Beatrice in Figure 2.4, then in reverse, the area of land surrounding Dante and Beatrice corresponds to the white circular area surrounding the black dot.

Because every point on the 2-sphere boundary of the Aristotle Universe is surrounded by a “locally glued area,” we may reason that one may *cross over* from the Aristotle Universe to the Empyrean at every “glued point.”

This observation shows that Dante’s Universe has *no edge*, as Peterson implies in the last paragraph of his quote (page 17).

Keep in mind that the 3-sphere structure is not unlike its lower-dimensional analogues — the 1-sphere has two *semicircles* that meet in a 0-sphere (Figure 1.5); the 2-sphere has two *hemispheres* that meet in a 1-sphere (Figure 1.6); and, likewise, the 3-sphere has two *hyperhemispheres* — the Aristotle universe and the Empyrean — that meet in a 2-sphere. The *equator* of the 3-sphere is a 2-sphere, and the 3-sphere has no edge.

§12 DANTE ORGANIZES 3-SPHERE SLICES

In Figure 2.8 we slice a 1-sphere S^1 (circle) with vertical lines to obtain 0-spheres, and in Figure 2.9 we slice a 2-sphere S^2 with vertical planes to obtain 1-spheres:

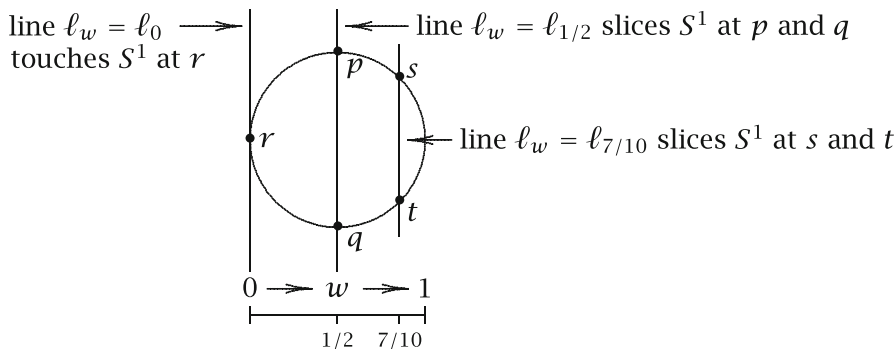


FIG. 2.8 As the values of w increase, the line ℓ_w moves across S^1 .

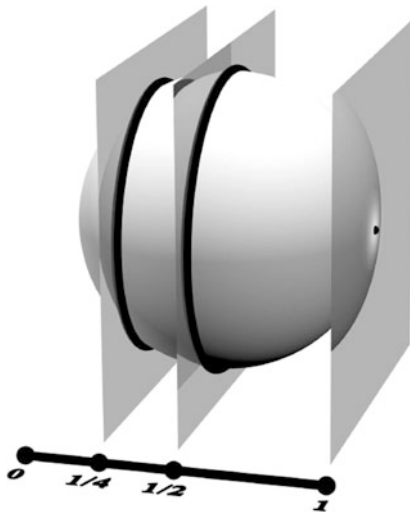


FIG. 2.9 As values of w increase, the plane π_w moves across S^2 .

We begin with Figure 2.8: For each number “ w ” between zero and one, there is a vertical line ℓ_w ; and as w moves left to right from zero to one, ℓ_w moves from left to right slicing the circle.

Letting the *size of a 0-sphere be the distance between its two points*, we start on the left with the ℓ_0 -slice as a point, then the sizes of the ℓ_w -slices increase to a maximum at $\ell_{1/2}$, and then decrease as ℓ_w moves further to the right until the final ℓ_1 -slice is simply a point.

Turning to Figure 2.9, we see the slicing of a 2-sphere with vertical planes that yield 1-spheres slices. Again, as the values of w increase from zero to

one, the corresponding planes π_w move across a 2-sphere S^2 , yielding slices that are one-spheres — the first π_0 -slice is a point, then the sizes (diameters) of the π_w -slices (one-spheres) increase to a maximum at $\pi_{1/2}$, and then the π_w -slices decrease in size as w moves from $1/2$ to 1 , where we find that the π_1 -slice is a point.

Slicing a 3-sphere is similar — using 3-dimensional *hyperplanes* (human vision spaces) we obtain slices that are 2-spheres, except for the $w = 0$ slice and the $w = 1$ slice which are points. And even though humans cannot picture a 3-sphere, there are mathematical proofs (see Appendix 1) that show the behavior is exactly as described.⁴

Against this backdrop there emerges another way to view these spheres. Consider the one-sphere S^1 (circle). Rather than starting with the S^1 , the *whole loaf of bread*, and then creating the *individual slices*, which are 0-spheres, suppose we reverse the process. That is, suppose we start with a *bunch of slices*, say a bunch of 0-spheres, and then consider *how to put these slices together to obtain the whole loaf* S^1 . The key is using the values of the *extra dimension* encoded as “ w ” in Figure 2.8. To start we could consider the largest $\ell_w = \ell_{1/2}$ slice, and go from there.

How can we make it clearer? Perhaps by saying that the slices must simply be lined up according to the values of w . The first slice is a point — the ℓ_0 -slice. The unique largest slice would be the $\ell_{1/2}$ slice, and so on. In summary, we could place the slices according to size until they were lined up as in Figure 2.8. And once together, we would see the entire loaf, i.e., we would see the 1-sphere, not the individual slices.

The key idea is the observation that the 0-spheres expand/contract in a *vertical direction*, while it is the *horizontal direction* induced by w that adds the *extra dimension* that is required if we desire to generate a 1-sphere from 0-spheres.

The idea of *building the whole loaf from its slices using an extra dimension induced by w* lies at the heart of Dante’s argument. Dante’s approach is discussed within the Peterson quote below, which is consistent with a careful reading and consideration of the examples above. In fact, Dante’s introduction of the *extra dimension w* allows him to construct his 3-sphere universe inside 4-dimensional space.

⁴The proof presented in Appendix 1 requires some background in either vector analysis or linear algebra. It is included for those readers who desire easy access to such a proof.

[Mark Peterson - *Dante and the 3-sphere*] Dante himself believed he was expressing something entirely new at this juncture. He asserts this by describing the difficulty of the notion as being like a knot that has grown tight, because no one has ever before tried to untie it.⁵ There can be little doubt, however, that his new idea had no effect on cosmological thinking whatever — the 3-sphere in the *Paradiso* went unnoticed, or misunderstood. In recent times it has probably been dismissed by readers with less geometrical aptitude than Dante as mysticism.

To make the case, then, I first point out that Dante assumes from the outset that the nine angelic spheres and the nine heavenly spheres are analogous. In the notes to his translation, Ciardi makes this point by describing the angelic spheres as “a sort of counter-universe.”⁶ In fact, what interests Dante as narrator is a seeming *breakdown* in the analogy, about which Beatrice quickly reassures him.⁷ The problem is that the various heavenly spheres revolve faster in proportion as they are bigger, while just the reverse is true of the angelic spheres: the innermost and smallest of these are revolving the fastest, and the outer ones are slower. Beatrice replies that if he will shift his attention away from the spheres’ sizes to an intrinsic ranking they possess, he will see a marvelous consistency in the whole. The innermost angelic sphere turns faster than the other angelic spheres because it ranks higher, just as the Primum Mobile turns faster than the other heavenly spheres because it ranks higher. In other words, the spheres have a ranking, a “greatness,” which does not necessarily correspond to their size (although for the first nine it does), but is rather indicated to the eye by their speed. This explanation strongly suggests our construction of the 3-sphere as sliced up into 2-spheres which at first grow and then diminish in size, labeled by a fourth coordinate w , which simply increases. Indeed, Dante has actually introduced such a fourth coordinate to label the spheres as they grow and diminish, namely their speed. In all our visualizations of the 3-sphere it was the second hemisphere, composed of the diminishing sequence of 2-spheres, which was hardest to fit into the model — Dante embeds the model in four dimensions, which does, as we know, solve the problem. His fourth dimension is speed of revolution. Of course he would never have said it that way, but it amounts to the same thing. The overall organization of the 2-spheres is that of a 3-sphere.

⁵Dante, *Paradiso* (Harvard Univ., Cambridge, 1972), Canto 28, lines 58–60.

⁶John Ciardi in Dante, *The Paradiso* (Mentor, New York, 1970), notes to Canto 28, lines 21–36, p. 313.

⁷Dante, *Paradiso* (Harvard Univ., Cambridge, 1972), Cantor 28, lines 46–78.

Dante's elation with this idea — a feeling we may readily share — has traditionally left readers somewhat puzzled. That is just another way of saying that if this passage is not taken as a description of the organization of 2-spheres into a 3-sphere, then it is hard to see what the point of it is.

§13 COMMENTS

Students of 20th Century mathematics know of the construction of the 3-sphere as the object obtained by gluing the 2-sphere boundaries of two 3-discs (two solid balls).

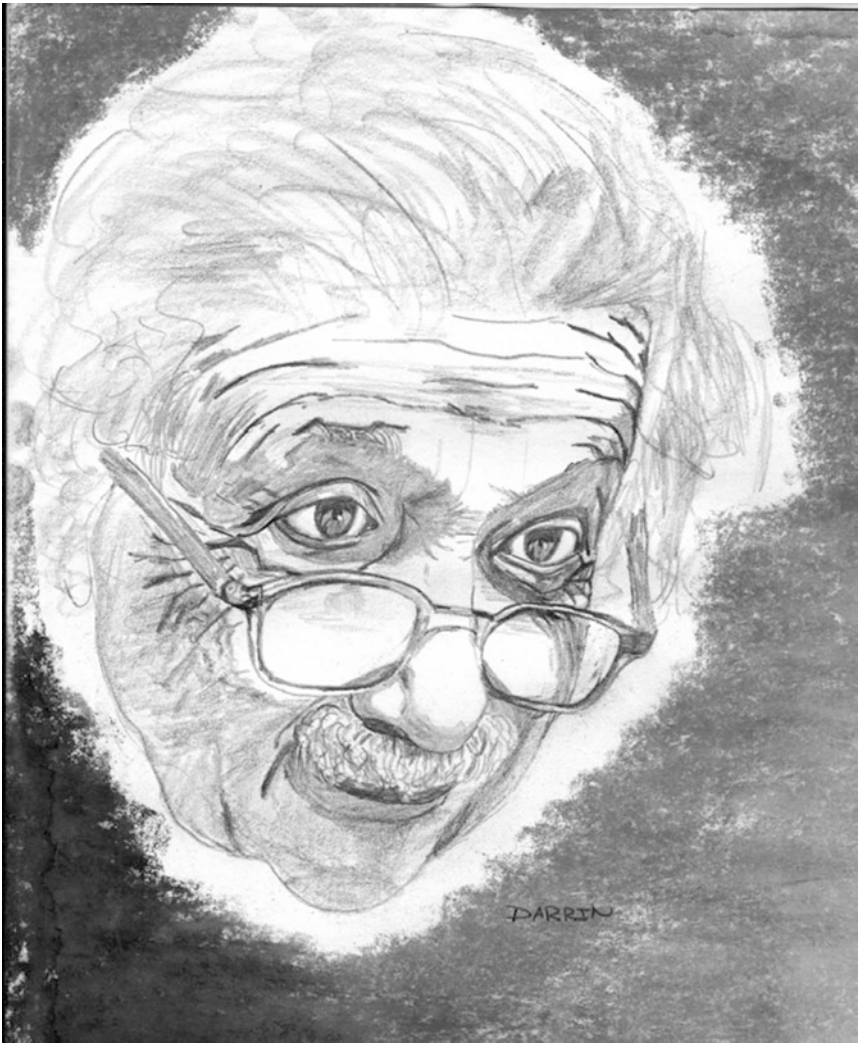
It is simply amazing that anyone living in the 14th Century could contemplate a universe as a 3-sphere, let alone its slicing into 2-sphere slices. Think about it — more than six centuries passed before Einstein proposed that our universe (at any instant in time) may be viewed globally as a 3-sphere.

To supplement this chapter, which obviously contains no literary summary, let us sample the summary on page 485 of Volume 5 of the *Macropædia* in the 15th Edition of *The New Encyclopædia Britannica*. (Statements within square brackets “[.]” are mine.)

“The Divine Comedy.” ... A poet above all, he [Dante] felt that only in poetry would he be able to express fully his dream of a spiritual and civilized renewal of the whole of humanity. The poem, though unique, ... is inspired by the poetry of the Bible and by the Christian wisdom of the Holy Scriptures. Divided into three books, or *cantiche* (treating of Hell, Purgatory, and Paradise — ...) ... the number 3, a symbol of the Trinity, is always present in every part of the work, with its multiples and in its unity. ... The literal subject of the work is the journey Dante makes through the world beyond the grave, ... At age 35, on the evening of Good Friday, 1300, the poet finds himself wandering astray in a dark wood [allegorically very depressed]. After a night of anguish, he sets out toward a hill illuminated by the sun, but three wild beasts — a female ounce (a species of leopard), a lion, and a wolf: symbols of lust, pride, and avarice [greed for riches] — bar his path and force him back toward the darkness of wood. Virgil, however, sent by the Virgin Mary, St. Lucy, and Beatrice, appears to help him. He guides Dante through the infernal realm and the mountain of Purgatory, at the summit of which the Roman poet [Virgil] is replaced by Beatrice, who then conducts Dante (raising him from heaven to heaven by the brilliant and loving power of her glance, which is that of a blessed soul contemplating God) as far as the Empyrean [the abode of God], where the poet enjoys for a brief moment the supreme vision of the divinity.

For details surrounding Albert Ritter's and Gustave Doré's art works, Figures 2.2 and 2.4 respectively, visit Wikipedia on the Internet. (The computer files that generated these figures were obtained from Wikimedia Commons, a freely licensed media file repository.)

For additional references concerning Dante's 3-sphere see §A7 *Dante's 3-sphere construct*.



Art Meets Mathematics in the Fourth Dimension

Lipscomb, S.

2014, XVII, 184 p. 147 illus., 28 illus. in color.,

Hardcover

ISBN: 978-3-319-06253-2