

# Chapter 2

## Dynamical Systems

**Abstract** Dynamical systems are formally defined—may they be classical or quantum. We introduce important concepts for the analysis of classical non-linear systems. In order to focus on the essential questions, this chapter restricts to one-dimensional discrete maps as relatively simple examples of dynamical systems.

### 2.1 Evolution Law

A dynamical system is given by a set of states  $\Omega$  and an evolution law telling us how to propagate these states in (discrete or continuous) time. Let us assume that the propagation law is homogeneous in time. That means it depends on the initial state but not on the initial time. Mathematically speaking a dynamical system is given by a one-parameter flow or map

$$\begin{aligned} T : G \times \Omega &\rightarrow \Omega, \\ (g \in G, \omega \in \Omega) &\mapsto T^g(\omega) \in \Omega, \end{aligned} \tag{2.1.1}$$

such that the composition is given by

$$T^g \circ T^h = T^{g+h}. \tag{2.1.2}$$

For a *discrete dynamical system* we have  $G = \mathbb{N}$  or  $G = \mathbb{Z}$  (discrete time steps) and for a *continuous dynamical system*  $G = \mathbb{R}^+$  or  $G = \mathbb{R}$  (continuous time). The set  $\Omega$  contains all possible states. It describes the physical reality one wants to model and is called the phase space. From an algebraic point of view the above defined maps form a semi-group that operates on  $\Omega$ . If the map  $T$  is invertible for all  $t \in G$  this structure extends to a group<sup>1</sup> and we say that the dynamical system is invertible.

---

<sup>1</sup> Translational invariance of the flow with respect to the time parameter  $g$  is only given if the generator of the dynamics is itself time-independent. Any classical problem can formally be made

The sets  $\{T^t(\omega)\}_{t \in G}$  define *orbits* or *trajectories* of the map. They can be discrete or continuous, finite or infinite.

The above definitions are quite abstract and therefore we give a few examples.

1. The simplest example is a discrete dynamical system defined by an iterated map. Let  $f$  be a map of the interval  $\Omega$  onto itself. We define

$$T^n = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}, \quad G = \mathbb{N}. \quad (2.1.3)$$

If the map  $f$  is invertible so is the dynamical system, and we can extend time to  $G = \mathbb{Z}$ . Two concrete examples of such discrete maps will be given in Sect. 2.2.

2. An example from classical physics is the motion of  $N$  particles in three space dimensions. The dynamics are governed by Newton's equations of motion for the vector of positions  $\mathbf{x}(t)$

$$m\ddot{\mathbf{x}}(t) = F(\mathbf{x}(t)). \quad (2.1.4)$$

Defining the composite vector  $\mathbf{y}(t) = (\mathbf{y}_1(t) \equiv \mathbf{x}(t), \dot{\mathbf{x}}(t))$  and  $f(\mathbf{y}(t)) = (\mathbf{y}_1(t), F(\mathbf{y}_1(t))/m)$ , we obtain

$$\dot{\mathbf{y}}(t) = f(\mathbf{y}(t)). \quad (2.1.5)$$

To make the connection between this formulation and the definition of dynamical systems we write the solution as  $T^t(\mathbf{y}(0)) = \mathbf{y}(t)$ . We obtain the solution in one step but also in two steps if we insert the end of the first trajectory as an initial condition into the second trajectory,<sup>2</sup> i.e.  $T^{s+t}(\mathbf{y}(0)) = T^t(T^s(\mathbf{y}(0))) = (T^t \circ T^s)(\mathbf{y}(0))$ . Note that  $\mathbf{x}(t)$  is an element of the configuration space  $\mathbb{R}^{3N}$  while  $\mathbf{y}(t)$  is an element of the phase space  $\Omega = \mathbb{R}^{6N}$ . Equation (2.1.5) is equivalent to the Hamiltonian formulation of the problem. Motivated by our general definition (2.1.1) and (2.1.2) we see that this is actually the natural way to treat classical dynamics. We will therefore use the formalism of Hamiltonian mechanics (Sect. 3.2), operating in phase space rather than configuration space, throughout the Chap. 3.

3. The time evolution of a quantum mechanical spinless particle in three space dimensions with time-independent hermitian Hamiltonian. Here  $\Psi_0 \in \mathcal{L}^2(\mathbb{R}^3) = \Omega$  and

$$T^t(\Psi_0) = \hat{U}(t)\Psi_0 = e^{-i\hat{H}t/\hbar} \Psi_0. \quad (2.1.6)$$

The so-defined dynamical system is obviously invertible, with  $T^{-t} = \hat{U}^\dagger(t)$ .

---

(Footnote 1 continued)

time-independent, see Sect. 3.3.1; hence the property of a translationally invariant group is always obeyed in this generalized sense. The quantum evolution for periodically time-dependent systems can also be cast in a similar way using a theorem of Floquet [1]. For generally time-dependent quantum systems, time ordering [2] must be used to formally write down the evolution law.

<sup>2</sup> A formal proof is found in [3] based on the fact that every point in phase space has a unique time evolution.

4. Non-invertible quantum dynamics: A reduced quantum mechanical system (a subsystem of a larger one) can—under certain conditions—be described by a master equation for the density operator of the system

$$\dot{\hat{\rho}}(t) = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)] + \hat{\mathcal{L}}(\hat{\rho}(t)). \quad (2.1.7)$$

In this case the time evolution has only the property of a semi-group. With the above equation it is possible to model dissipation and decoherence by the Lindblad operator  $\hat{\mathcal{L}}$ , whilst the coherent evolution is induced by the first term on the right hand side of the equation. For more information see, e.g., [4, 5]. Replacing the density operator by a classical density distribution in phase space one may model the corresponding quantum evolution to some extent. On the classical level one then has to deal with a Fokker-Planck equation for phase space densities describing irreversible motion [6].

## 2.2 One-Dimensional Maps

In the following we discuss two seemingly simple discrete dynamical systems. Those are not Hamiltonian systems but one-dimensional mappings, i.e., the phase space is just one-dimensional. Yet, it will turn out that some general concepts can be easily introduced with the help of such maps, e.g. fixed points and their stability or periodic orbits. They have also the great advantage that much can be shown rigorously for them [7, 8]. Therefore, one-dimensional discrete maps form one of the bases for the mathematical theory of dynamical systems, see for instance [9].

### 2.2.1 The Logistic Map

The logistic map is a versatile and well understood example of a discrete dynamical map which was introduced in 1838 by Pierre Francois Verhulst as a mathematical model for demographic evolution [10]. Its nonlinear iteration equation is given by the formula

$$y_{n+1} = Ry_n(M - y_n), \quad (2.2.1)$$

where  $y_n$  is a population at time  $n$ ,  $R \geq 0$  is a growth rate and  $M$  is an upper bound for the population. The population at the next time step is proportional to the growth rate times the population (which alone would lead to exponential growth for  $R > 1/M$ ) and to the available resources assumed to be given by  $M - y_n$ . The system's evolution is relatively simple for  $R < 1/M$  having the asymptotic solution  $\lim_{n \rightarrow \infty} y_n = 0$  (extinct population) for all initial values. For general values of the growth rate, the system shows a surprisingly complicated dynamical behavior. Most

interestingly, in some parameter regimes the motion becomes chaotic, which means that the population  $y_n$  strongly depends on the initial condition  $y_0$ . Additionally, the system might not converge to an asymptotic value or show non-periodic behavior.

Let us now analyse the logistic map in more detail. In order to get the standard form of the logistic map we rescale the variable describing the population  $x_n = y_n/M$  and write the time step as the application of the evolution law  $T$  leading to

$$T(x) = rx(1 - x), \text{ for } x \in \Omega = [0, 1]. \quad (2.2.2)$$

Here  $r = MR$ . If we want  $T$  to map the interval  $\Omega$  onto itself we have to choose  $r \in [0, 4]$ . First, we look for fixed points of the map  $T$ , that means points for which

$$T(x^*) = x^* \Leftrightarrow x^* = rx^*(1 - x^*). \quad (2.2.3)$$

holds. The above equation has two solutions:

$$x_{1,1}^* = 0 \text{ is } \begin{cases} \text{an attractive fixed point for} & r < 1 \\ \text{an indifferent fixed point for} & r = 1 \\ \text{a repulsive fixed point for} & r > 1, \end{cases} \quad (2.2.4)$$

and

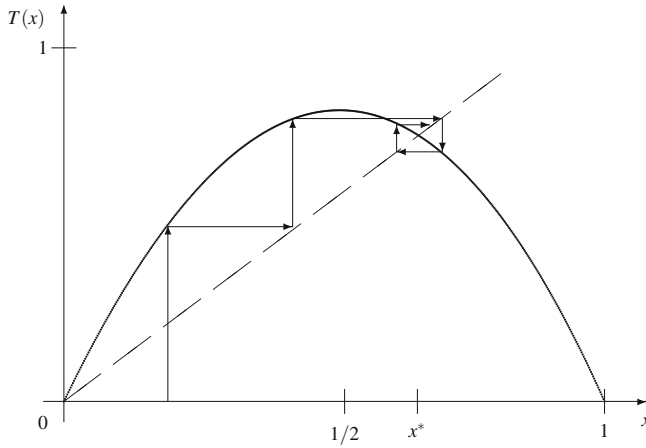
$$x_{1,2}^* = 1 - \frac{1}{r} \text{ is } \begin{cases} \text{an attractive fixed point for} & 1 < r < 3 \\ \text{an indifferent fixed point for} & r \in \{1, 3\} \\ \text{a repulsive fixed point for} & |r - 2| > 1. \end{cases} \quad (2.2.5)$$

Thereby we have used the notion that a fixed point  $x^*$  is called *attractive/repulsive* if the derivative with respect to  $x$   $|T'(x^*)| \leq 1$ , and *indifferent* if  $|T'(x^*)| = 1$ . Attractive fixed points are important quantities because they lead to asymptotically converging dynamics. For all initial conditions which lie in the subset of  $\Omega$  around the fixed point where the map  $T$  is contracting (modulus of the derivative smaller than one), the evolution converges towards the fixed point for large times. How the system reaches the fixed point  $x_{1,2}^*$  by iteration is shown schematically in Fig. 2.1.

Besides the two fixed points  $x_{1,1}^*$  and  $x_{1,2}^*$ , the logistic map has fixed points of higher order, too. That means fixed points of the  $p$ -times iterated map  $T^p$ . A fixed point of order  $p$  is defined by

$$T^p(x^*) = x^*. \quad (2.2.6)$$

This includes the possibility that for all  $n \leq p$  one has  $T^n(x^*) = x^*$ , as, for example, a fixed point of first order is also a fixed point of all higher orders. Usually, one assumes that  $p$  is the prime period of  $x^*$ , that is, we have  $T^n(x^*) \neq x^*$  for all  $n < p$ . Fixed points lead to *periodic orbits* (PO), that means orbits  $\{T^n(x^*)\}_{n \in \mathbb{N}}$  which consist only of finitely many points. If one starts the dynamics with a fixed



**Fig. 2.1** Schematic iteration shown by the *arrows* which converges to the fixed point of the logistic map for  $r = 2.5$

point of order  $p$ , the initial value is recovered after  $p$  time steps. The logistic map has, for example, two fixed points of order two

$$x_{2,(1,2)}^* = \frac{1}{2r} \left( r + 1 \pm \sqrt{(r+1)(r-3)} \right), \quad (2.2.7)$$

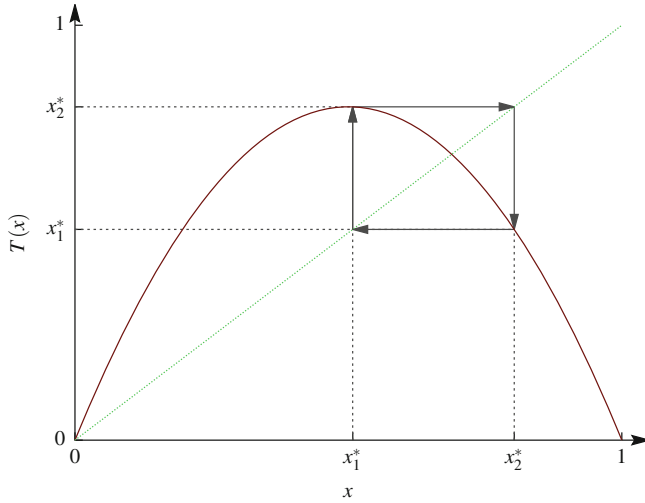
which are attractive for  $3 < r < 1 + \sqrt{6} \approx 3.45$ . As  $r$  is increased, attractive fixed points of higher order (4, 8, 16, ...) emerge. The fixed points are, however, determined by the solutions of high-order polynomials and analytical values are difficult to obtain, see, for example, [11] for more details.

We note that there is a natural relation between all fixed points of a given order  $p > 1$ . Assume, for instance, the two second-order fixed points  $x_{2,1}^*$  and  $x_{2,2}^*$ . When we apply  $T^2$  on  $x_{2,1}^*$  we find  $T(T(x_{2,1}^*)) = T(x_{2,2}^*) = x_{2,1}^*$ . This behavior is found for all fixed points of order  $p > 1$ . Let  $x_{p,i}^*$ ,  $i = 1, \dots, p$ , be the  $p$  fixed points of order  $p$  (the number of fixed points always equals the order). One finds

$$T^p(x_{p,1}^*) = \underbrace{T(T(\dots(x_{p,1}^*)\dots))}_{p \text{ times}} = \underbrace{T(T(\dots(x_{p,2}^*)\dots))}_{p-1 \text{ times}} = \dots = T(x_{p,p}^*) = x_{p,1}^*. \quad (2.2.8)$$

Hence, if one fixed point of order  $p$  is given, the  $p - 1$  other fixed points of that order can be computed by applying  $T$  repeatedly. We illustrate the described phenomenon in (2.2.8) for  $p = 2$ . The two fixed-points of second order are mapped onto each other by  $T$  (Fig. 2.2).

The dynamical behavior of the logistic map is summarized in its bifurcation diagram, see Fig. 2.3. For each value of  $r$  on the abscissa, the value of  $x$  is plotted after 400 iterations for random initial data. It turns out that outside the chaotic regime



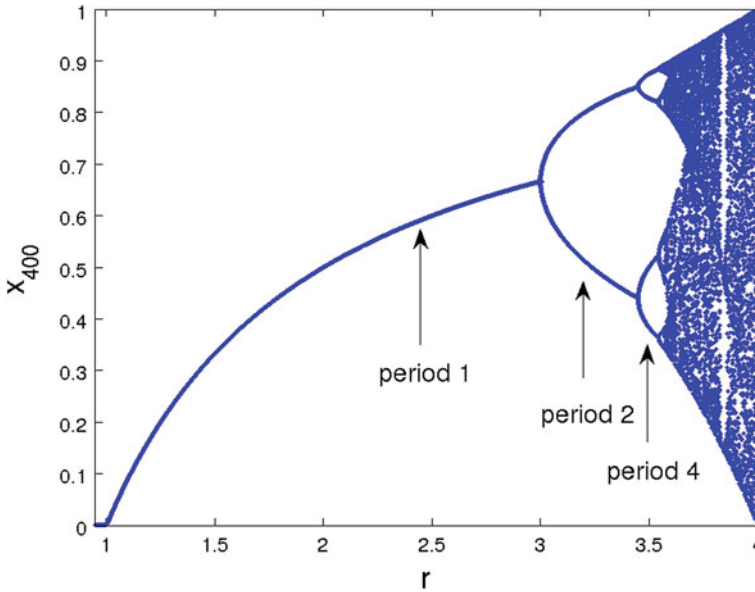
**Fig. 2.2**  $T$  maps  $x_1^*$  onto  $x_2^*$  and vice versa. Accordingly, both  $x_1^*$  and  $x_2^*$  are fixed points of  $T^2 = T \circ T$ . The plot was made for the parameter  $r = 3.214$ , giving  $x_1^* \approx 0.5078$  and  $x_2^* \approx 0.8033$

the asymptotic evolution does not depend on the initial conditions. For  $r < 1$ , the motion converges towards the fixed point  $x_{1,1}^* = 0$ . For  $1 < r < 3$ , it goes to  $x_{1,2}^* = 1 - \frac{1}{r}$  for almost every  $x_0$ . At  $r = 3$  is a *bifurcation* point. For larger values of the growth rate, the asymptotic dynamics converge towards the two fixed points of second order,  $x_{2,1}^*$  and  $x_{2,2}^*$  (defining a periodic orbit). This phenomenon is known as *period doubling*. In Fig. 2.3 we see both solutions, because  $x_{400}$  is plotted for many initial points. Note that, in the limit of many iterations, the dynamics do not converge to one of the fixed points, but the evolution jumps between them for all times.<sup>3</sup> The two fixed points of order two split again into two new ones at  $r = 1 + \sqrt{6}$ . This scheme repeats itself infinitely often while the distance between the bifurcation points decreases rapidly. It can be shown [7, 12] that the ratios of the lengths between two subsequent bifurcation points approach a limiting value

$$\lim_{k \rightarrow \infty} \delta_k = \frac{r_k - r_{k-1}}{r_{k+1} - r_k} = 4.669201 \dots \in \mathbb{R} \setminus \mathbb{Q}, \quad (2.2.9)$$

known as the Feigenbaum constant. For most  $r$  beyond the critical value  $r_\infty = 3.569945 \dots$  (known as accumulation point) the system becomes chaotic, which means

<sup>3</sup> When we say that the dynamics (for the initial condition  $x_0$ ) converge towards a periodic orbit with  $p$  elements (or towards the fixed point  $x^*$  of order  $p$ , which is an element of the periodic orbit) we mean that  $\lim_{n \rightarrow \infty} T^{np}(x_0) = x^*$ .



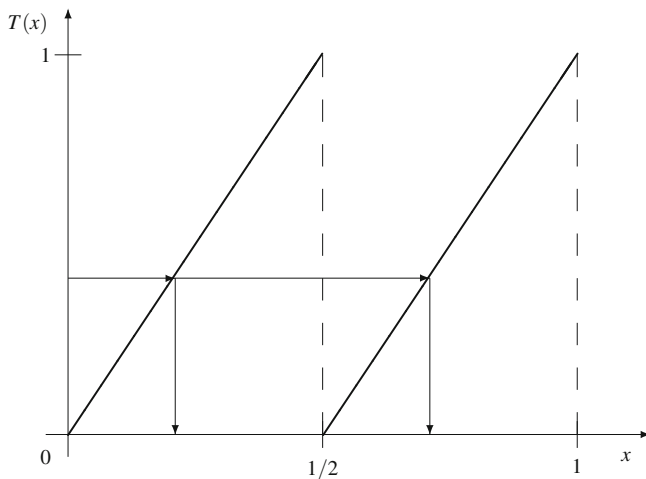
**Fig. 2.3** Bifurcation diagram of the logistic map. For each value on the abscissa, the value  $x_{n=400} \approx x_\infty$ , i.e. after 400 iterations of the map, for random initial data is represented on the ordinate. The path into chaos goes along a cascade of accumulating bifurcation points. Up to  $r < 3.5$  stable fixed points of order 1, 2 and 4 are shown. Above  $r_\infty \approx 3.57$  chaos develops as motivated in the text

that the asymptotic evolution will not converge towards periodic orbits any more.<sup>4</sup> Here the values  $x_{400}$  (for many random initial conditions) cover quasi-uniformly the whole phase space  $\Omega = [0, 1]$ . Note that in this regime the motion strongly depends on the initial value  $x_0$ . Nevertheless there exist values  $r > r_\infty$  for which new attractive fixed points (of order 3, 5, 6, 7 ...) appear, see [8]. The route into chaos via a cascade of accumulating bifurcation points (period doublings) is a general phenomenon and not limited to the here reported example of the logistic map. Examples of so-called mixed Hamiltonian systems showing bifurcations of initially stable resonance islands with increasing perturbation are discussed in Sect. 3.8.7.

### 2.2.2 The Dyadic Map

Another important example of a discrete dynamical map showing chaotic behavior is the dyadic map, also known as Bernoulli shift. It is defined by

<sup>4</sup>Also in the chaotic regime there exist fixed points but they are not attractive. Since at each point of a period doubling the fixed point does not vanish but only loses the property of attraction, the fixed points in the chaotic regime form a dense set (yet of Lebesgue measure zero in the interval  $[0, 1]$ ).



**Fig. 2.4** The dyadic map. As indicated by the arrows, the map is not one-to-one

$$T(x) = \begin{cases} 2x & \text{for } 0 \leq x < \frac{1}{2} \\ 2x - 1 & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}. \quad (2.2.10)$$

Its phase space is the unit interval  $\Omega = [0, 1]$ . When we represent the numbers  $x \in \Omega$  in the binary representation  $x = \sum_{i=1}^{\infty} x_i 2^{-i}$ ,  $x_i \in \{0, 1\}$ , the map can be interpreted as a shift of decimals:

$$x = 0.x_1x_2x_3x_4 \dots \rightarrow T(x) = 0.x_2x_3x_4 \dots \quad (2.2.11)$$

This explains also the name (Bernoulli) shift map. The dyadic map is displayed in Fig. 2.4.

It can be shown that it is topologically conjugate<sup>5</sup> to the logistic map with  $r = 4$  (which is chaotic for this parameter), see [13–15].<sup>6</sup>

The dynamics of the dyadic map can be summarized as follows. If the initial condition is irrational, the motion will be non-periodic. Note that this is true for

<sup>5</sup>Two functions  $f$  and  $g$  are said to be topologically conjugate if there exists a homeomorphism  $h$  (continuous and invertible) that conjugates one into the other, in formulas  $g = h^{-1} \circ f \circ h$ . This is important in the theory of dynamical systems because the same must hold for the iterated system  $g_n = h^{-1} \circ f_n \circ h$ . Hence if one can solve one system, the solution of the other one follows immediately.

<sup>6</sup>In the literature one often finds that the logistic map for  $r = 4$  is topologically conjugate to the tent map [14], but the tent map is topologically equivalent to the dyadic map [15], which together gives the wanted equivalence.



almost all<sup>7</sup> initial values. For  $x_0 \in \mathbb{Q}$ , the evolved value converges towards zero if the binary representation of  $x_0$  is non-periodic and hence finite, or towards a periodic orbit if the representation shows periodicity. Hence, as for the logistic map, the fixed points in the chaotic regime form a set of Lebesgue measure zero. Being such a simple system the dyadic map can be solved exactly. Exploiting the connection to the logistic maps, the solution can be used to compute an analytical solution of the dynamics of the logistic map for  $r = 4$  as well. It reads

$$x_{n+1} = \sin^2(2^n \Theta \pi), \quad (2.2.12)$$

with  $\Theta = \frac{1}{\pi} \sin^{-1}(\sqrt{x_0})$  for the initial condition  $x_0$ , see [17, 18].

Analyzing the dyadic map, it is particularly simple to see why the forecast of chaotic systems is so difficult. If one starts with a number that is only precisely known up to  $m$  digits, in the binary representation all information and therewith the predictability of the model is lost after  $m$  iterations. As a consequence, we can easily compute the rate of spreading of initially close points. This rate is known as *Lyapunov exponent* and defined by

$$\sigma = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{1}{n} \ln \left| \frac{x_n(x_0) - x_n(x'_0)}{2^{-m}} \right| = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{1}{n} \ln \left| \frac{2^{-m+n}}{2^{-m}} \right| = \ln 2. \quad (2.2.13)$$

Here the initial conditions  $x_0$  and  $x'_0$  differ only in the  $m$ -th digit in Eq. (2.2.11).  $\sigma$  is called exponent since it characterizes the speed of exponential spreading as time evolves. For Hamiltonian systems, we will discuss Lyapunov exponents in detail in Sect. 3.9.

### 2.2.3 Deterministic Random Number Generators

A somewhat surprising application of chaotic maps is the deterministic generation of so-called pseudo-random numbers. Since the motion of a chaotic map depends sensitively on the initial conditions, a different series of numbers is generated for different initial values. If one does not start the dynamics at one of the fixed points (which form a set of measure zero anyhow), these series will neither be periodic nor be converging to a single point. Unfortunately, this scenario does not work on a computer, which needs to rely on a finite set of numbers,<sup>8</sup> and therefore necessarily produces periodic orbits at some stage. Nevertheless, so-called linear congruential generators, as generalizations of the dyadic map, can be used as low-quality random number generators:

<sup>7</sup>The measure theoretical notion “for almost all” means for all but a set of Lebesgue measure zero [16].

<sup>8</sup>This set is normally composed exclusively of rational numbers, leading e.g. to non-chaotic behavior for the dyadic map.

$$a_{n+1} = N_1 a_n \bmod N_2, \quad (2.2.14)$$

$$b_{n+1} = \frac{a_{n+1}}{N_2}, \quad (2.2.15)$$

where  $N_1, N_2, a_n \in \mathbb{N}$  [19]. This procedure generates “uncorrelated” and uniformly distributed numbers  $b_i$  in the unit interval, but the constants  $N_1$  and  $N_2$  should be chosen very carefully in order to maximize the periodicity as well as to minimize correlations between subsequent numbers [20]. While this algorithm is very fast and easy to implement, it has many problems (see chapter 7.1 of [20]) and today other methods like the Mersenne Twister [21] have widely replaced it due to their high period ( $2^{19937} - 1$ , not a mistake!) and efficient implementations [22]. It should be stressed that the performance of these random number generators needs to be analyzed using number theoretical tools, not the ones presented in this chapter.

Therewith, we conclude the examination of one-dimensional discrete maps which have been introduced as toy models to exemplify chaotic behavior. In the next Chapter we come to “real” physical applications of classical mechanics. Since even systems with one degree of freedom have a two-dimensional phase space and a continuous time evolution a priori, we expect a more complicated dynamical behavior for them. In what follows in this book, we will restrict ourselves exclusively now to so-called Hamiltonian systems without friction or dissipation, whose time evolution is invertible.

## Problems

2.1. Prove the following theorem:

Let  $T$  be a dynamical map, as introduced in Sect. 2.1, with a fixed point  $x^*$ , and continuously differentiable with respect to  $x$  close to the fixed point. If its derivative with respect to  $x$  is  $|T'(x)|_{x=x^*} < 1$ , then  $x^*$  is attractive.

2.2. Find the fixed points of order two of the logistic map from Eq. (2.2.2) with  $r$  as a free parameter. Check also the stability of the fixed points found.

2.3. Find all 1st and 2nd order fixed points of the dyadic shift map from Eq. (2.2.10).

## References

1. Shirley, J.H.: Phys. Rev. **138**, B979 (1965)
2. Sakurai, J.J.: Modern Quantum Mechanics. Addison-Wesley Publishing Company, Reading (1994)
3. Teschl, G.: Ordinary differential equations and dynamical systems (Graduate Studies in Mathematics, Amer. Math. Soc., Providence, (to appear), <http://www.mat.univie.ac.at/~erald/ftp/book-ode/index.html>, 2011)
4. Breuer, P., Petruccione, F.: The Theory of Open Quantum Systems. Oxford University Press, Oxford (2007)
5. Walls, D.F., Milburn, G.J.: Quantum Optics. Springer, Berlin (2008)

6. Risken, H.: The Fokker-Planck Equation: Methods of Solutions and Applications. Springer Series in Synergetics. Springer, Heidelberg (1996)
7. Schuster, H.G.: Deterministic Chaos. VCH, Weinheim (1988)
8. Ott, E.: Chaos in Dynamical Systems. Cambridge University Press, Cambridge (2002)
9. Metzler, W.: Nichtlineare Dynamik und Chaos. Teubner, Stuttgart (1998)
10. Verhulst, P.F.: *Corr. Math. Phys.* **10**, 113 (1838)
11. Weisstein, E.W.: Logistic map. MathWorld—a wolfram web resource. <http://mathworld.wolfram.com/LogisticMap.html>
12. Feigenbaum, M.J.: *J. Stat. Phys.* **21**, 669 (1979)
13. Driebe, D.J.: Fully Chaotic Maps and Broken Time Symmetry. Kluwer Academic Publishers, Dordrecht (1999)
14. Berglund, N.: Preprint. [arXiv:math/0111177v1](https://arxiv.org/abs/math/0111177v1) (2001)
15. Yoshioka, D., Tsuneda, A., Inoue, T.: *Jeice Trans. Fundam.* **E88-A**(10), 2745–2761 (2005)
16. Walter, W.: *Analysis 2*. Springer, Berlin (2002)
17. Little, M., Heesch, D.: *J. Differ. Equ. Appl.* **10**(11), 949 (2004)
18. Schröder, E.: *Math. Ann.* **3**, 296 (1870)
19. Castiglione, P., Falcioni, M., Lesne, A., Vulpiani, A.: *Chaos and Coarse Graining in Statistical Mechanics*. Cambridge University Press, Cambridge (2008)
20. Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P.: *Numerical Recipes 3rd Edition: The Art of Scientific Computing*, 3rd edn. Cambridge University Press, New York (2007)
21. Matsumoto, M., Nishimura, T.: *ACM Trans. Model. Comput. Simul.* **8**(1), 3 (1998)
22. Saito, M., Matsumoto, M.: SIMD-oriented fast Mersenne Twister. In: Keller, A., Heinrich, S., Niederreiter, H. (eds.) *Monte Carlo and Quasi-Monte Carlo Methods 2006*, pp. 607–622. Springer, Berlin Heidelberg (2008)

<http://www.springer.com/978-3-319-06342-3>

Nonlinear Dynamics and Quantum Chaos

An Introduction

Wimberger, S.

2014, XIII, 206 p. 75 illus., 68 illus. in color., Hardcover

ISBN: 978-3-319-06342-3