Chapter 2
Unitarity of Gravity Coupled to Models of Particle Physics

As discussed in the opening chapter, one of the best ways to understand the realm of validity for an effective theory is to calculate the energy scale where perturbative unitarity breaks down. In the first section of this chapter we do exactly this for the effective theory of gravity coupled to matter as given by the action (1.1.10). In the second section we apply the bound to various grand unified theories. In the third section we incorporate renormalisation group (RG) effects into the bounds and are then able to compare the scale at which unitarity breaks down with the scale of strong coupling. We discuss the consequences of the RG improved bounds for various models of particle physics and introduce two models which can lower the scale of quantum gravity in four dimensions. The unitarity bound derived here will also provide an important basis for later chapters.

2.1 Unitarity of Linearised General Relativity

In this section we calculate the unitarity bound for the effective theory of gravity coupled to matter as given by the action (1.1.10). The calculation was first performed by Han and Willenbrock [5]. We have verified their calculation using `FeynCalc` and `Mathematica`.

The first step is to calculate $2 \to 2$ graviton exchange amplitudes for tree level scattering of complex scalars s, Weyl fermions ψ and vector bosons V in the high energy (massless) limit. We restrict ourselves to the case where initial and final states consist of different particles. This simplifies the calculations tremendously since only s-channel processes need to be considered. The amplitudes for all possible such processes are given in Table 2.1 and agree with those obtained in Ref. [5]. Note that a factor of $-\frac{1}{4}sM_P^{-2}$ has been extracted from each of the amplitudes. We have used the helicity basis[1] for the 'in' and 'out' states and the subscripts $+$ and $-$ refer to

[1] Note that as in the case for WW scattering in the standard model (see Sect. 1.2.1) we might think that including longitudinally polarised vector bosons in the external states may lead to the largest

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Table 2.1 Scattering amplitudes for scalars, fermions and vector bosons via s-channel graviton exchange in terms of the Wigner d-functions in the massless limit

$\rightarrow$	$s'\bar{s}'$	$\psi'_+\bar{\psi}'_-$	$\psi'_-\bar{\psi}'_+$	$V'_+V'_-$	$V'_-V'_+$
$s\bar{s}$	$2/3\ d^2_{0,0} - 2/3(1+6\xi)^2\ d^0_{0,0}$	$\sqrt{2/3}\ d^2_{0,1}$	$\sqrt{2/3}\ d^2_{0,-1}$	$2\sqrt{2/3}\ d^2_{0,2}$	$2\sqrt{2/3}\ d^2_{0,-2}$
$\psi_+\bar{\psi}_-$	$\sqrt{2/3}\ d^2_{1,0}$	$d^2_{1,1}$	$d^2_{1,-1}$	$2\ d^2_{1,2}$	$2\ d^2_{1,-2}$
$\psi_-\bar{\psi}_+$	$\sqrt{2/3}\ d^2_{-1,0}$	$d^2_{-1,1}$	$d^2_{-1,-1}$	$2\ d^2_{-1,2}$	$2\ d^2_{-1,-2}$
V_+V_-	$2\sqrt{2/3}\ d^2_{2,0}$	$2\ d^2_{2,1}$	$2\ d^2_{2,-1}$	$4\ d^2_{2,2}$	$4\ d^2_{2,-2}$
V_-V_+	$2\sqrt{2/3}\ d^2_{-2,0}$	$2\ d^2_{-2,1}$	$2\ d^2_{-2,-1}$	$4\ d^2_{-2,2}$	$4\ d^2_{-2,-2}$

A factor of $-\frac{1}{4}sM_P^{-2}$ has been extracted from each of the amplitudes

helicity. The spinors and polarisation vectors in this basis are given in Appendix A. We also use the Feynman rules of Ref. [4] which are reproduced in Appendix E.

2.1.1 $j = 2$ Partial Wave Amplitude

The partial wave amplitudes are found using Eq. (1.2.6) and so are simply proportional to the entries in Table 2.1 with the Wigner d-functions removed. To obtain the lowest unitarity bound we wish to find the eigenvalues of the matrix of partial wave amplitudes for N_s complex scalars, N_ψ fermions and N_V vector bosons. Since all entries contain a $j = 2$ partial wave, this is what will be focussed on here. The $j = 0$ partial wave will be considered separately later. Because the partial waves for opposite helicity processes are identical, the matrix can be simplified by only considering the $+, -$ helicity combinations and not $-, +$. With N_φ degrees of freedom we may consider the normalised state obtained by including all N_φ particles in the initial and final states: $(1/N_\varphi)\sum \varphi_+\varphi_-$. The matrix of partial waves thus obtained is given in Table 2.2.

Due to the symmetric nature of the matrix it only has a single eigenvalue, given by the trace

$$a_2 = -\frac{1}{320\pi}\frac{s}{M_P^2}N, \tag{2.1.1}$$

where [5]

$$N = \frac{2}{3}N_s + N_\psi + 4N_V. \tag{2.1.2}$$

(Footnote 1 continued)
high energy behaviour of the scattering amplitudes. However, as can be seen by considering the Goldstone boson equivalence theorem, there should be cancellations that happen in the calculation of such amplitudes so that the high energy behaviour is no stronger than for transversely polarised vector bosons. Indeed this is the case and we have verified it for the scattering amplitudes presented here.

Table 2.2 $j = 2$ partial wave amplitudes for N_s scalars, N_ψ fermions and N_V vector bosons via s-channel graviton exchange

$\rightarrow$	$\frac{1}{\sqrt{N_s}}\Sigma s'\bar{s}'$	$\frac{1}{\sqrt{N_\psi}}\Sigma\psi'_+\bar{\psi}'_-$	$\frac{1}{\sqrt{N_V}}\Sigma V'_+V'_-$
$\frac{1}{\sqrt{N_s}}\Sigma s\bar{s}$	$2/3N_s$	$\sqrt{2/3}\sqrt{N_S N_\psi}$	$2\sqrt{2/3}\sqrt{N_S N_V}$
$\frac{1}{\sqrt{N_\psi}}\Sigma\psi_+\bar{\psi}_-$	$\sqrt{2/3}\sqrt{N_S N_\psi}$	N_ψ	$2\sqrt{N_\psi N_V}$
$\frac{1}{\sqrt{N_V}}\Sigma V_+V_-$	$2\sqrt{2/3}\sqrt{N_s N_V}$	$2\sqrt{N_\psi N_V}$	$4N_V$

A factor of $-\frac{1}{320\pi}sM_P^{-2}$ has been extracted from each of the amplitudes

This amplitude is the main result of this chapter. Using this amplitude it is possible to test where tree level unitarity breaks down in models of particle physics coupled to linearised general relativity. Requiring that $|Re(a_2)| \leq \frac{1}{2}$ leads to the unitarity bound

$$\sqrt{s} \leq M_P\sqrt{\frac{160\pi}{N}}. \tag{2.1.3}$$

2.1.2 $j = 0$ Partial Wave Amplitude

For models with large numbers of scalar fields or large non-minimal couplings, it may also be of interest to consider the unitarity bound obtained from from the $j = 0$ partial wave amplitude. First, consider the scattering of N_s scalar fields, all with identical non-minimal coupling ξ. The partial wave amplitude for this process can be read off from Table 2.1 giving

$$a_0 = \frac{(1+6\xi)^2}{96\pi}\frac{s}{M_P^2}N_s. \tag{2.1.4}$$

Applying the unitarity bound to this amplitude, $|\text{Re}(a_0)| \leq \frac{1}{2}$ for complex scalars or $|\text{Re}(a_0)| \leq 1$ for real scalars gives

$$\sqrt{s} \leq \frac{M_P}{1+6\xi}\sqrt{\frac{96\pi}{N_s}} \tag{2.1.5}$$

where N_s is the number of complex scalar fields (or twice the number of real scalar fields).

2.2 Unitarity of Models of Particle Physics

Given the unitarity bounds (2.1.3) and (2.1.5) it is possible to find where tree level unitarity breaks down for any model by considering its matter content. For example, in the standard model, $N_s = 2$, $N_\psi = 45$ (we only include left handed neutrinos),

Table 2.3 Different grand unified models which have been considered in the literature

Particle physics model	N	N_S	$j = 2$ bound	$j = 0$ bound
Standard model	$283/3$	4	2.3	8.7
MSSM	$425/3$	98	1.9	1.8
SU(5) w/ **5**, **24**	$457/3$	34	1.8	3.0
SU(5) w/ **5**, **200**	211	210	1.5	1.2
SU(5) w/ **5**, **24**, **75**	$532/3$	109	1.7	1.7
SU(5) w/ **5**, **24**, **75**, **200**	244	309	1.4	0.99*
SO(10) w/ **10**, **16**, **45**	$781/3$	97	1.4	1.8
SO(10) w/ **10**, **16**, **210**	$946/3$	262	1.3	1.1
SO(10) w/ **10**, **16**, **770**	502	822	1.0	0.61*
SUSY-SU(5) w/ **5**, $\bar{\mathbf{5}}$, **24**	$755/3$	158	1.4	1.4
SUSY-SU(5) w/ **5**, $\bar{\mathbf{5}}$, **24**, **75**	$1,130/3$	308	1.2	0.99*
SUSY-SU(5) w/ **5**, $\bar{\mathbf{5}}$, **200**	545	510	0.96*	0.77*
SUSY-SO(10) w/ **10**, **16**, $\overline{\mathbf{16}}$, **45**, **54**	540	378	0.96*	0.89*
SUSY-SO(10) w/ **10**, **16**, $\overline{\mathbf{16}}$, **210**	725	600	0.83*	0.71*
SUSY-SO(10) w/ **10**, **16**, $\overline{\mathbf{16}}$, **770**	$4,975/3$	$1,720$	0.55*	0.42*

The last two columns show the scale at which tree level unitarity breaks down as a fraction of M_P in each model due to the bound from the $j = 2$ partial wave bound (2.1.3) or the $j = 0$ partial wave bound (2.1.5). It is assumed that $\xi = 0$. Entries marked with $*$ highlight where tree level unitarity breaks down below M_P. This can be compared with the approximate scale at which one expects strong coupling, see for example Eq. (2.3.9)

$N_V = 12$ and so we find $N = 283/3$ and we find unitarity breaks down at a scale $E_\star = 2.3 M_P$ from the $j = 2$ partial wave amplitude.

In Ref. [1] we calculated the scale at which unitarity breaks down in a variety of models. The results are presented in Table 2.3. The last two columns show the scale at which unitarity is violated for both the $j = 0$ and $j = 2$ partial wave amplitudes as a fraction of the Planck mass M_P. It is assumed that $\xi = 0$ in all models. Note that in some models unitarity breaks down below M_P. Naively, one may expect gravitational effects to become strongly coupled at M_P, so it may be a surprise to see unitarity problems appearing below this scale. However, to properly interpret these results we need to analyse more carefully the scale at which we expect gravity to become strongly coupled. This subject will be taken up in the next section using the techniques of the renormalisation group.

2.3 Running of the Planck Mass and Renormalisation Group Improved Unitarity Bound

Within the effective field theory framework of gravity, it is possible to define the Planck mass as a coupling that runs under renormalisation group (RG) effects, analogously to the well established RG running of the gauge couplings in the standard model. For example, based on calculations of the renormalisation of Newton's

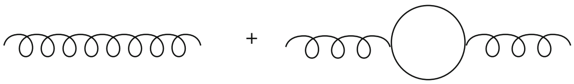

Fig. 2.1 The one loop contribution to the running Planck mass. The curly line represents the graviton and the straight lines represents the matter particles running in the loop

constant by Larsen and Wilczek [7] (see also Ref. [6]), Calmet, Hsu and Reeb defined a running Planck mass which depends on the RG scale μ in the following way [3]:

$$M_P(\mu)^2 = M_P(0)^2 - \frac{1}{96\pi^2}\mu^2 N_l, \tag{2.3.1}$$

where

$$N_l = N_s + N_\psi - 4N_V. \tag{2.3.2}$$

Note that N_l is not the same as N and noticeably the sign for the contribution of vector bosons is opposite in the two cases.

This result has been rigorously derived using heat kernel techniques (see Refs. [3, 7] for more details). Here, we give a brief illustration of how this effect can be seen to arise. Consider the one loop self energy correction to the graviton propagator, Fig. 2.1, where matter particles may run in the loop. Neglecting the index structure, this correction to the propagator is

$$\Delta(q^2) \sim \frac{i}{M_P^2 q^2} + \frac{i}{M_P^2 q^2}\Sigma\frac{i}{M_P^2 q^2} + \cdots, \tag{2.3.3}$$

where Σ is the self energy insertion. Σ can be estimated from the Feynman diagram:

$$\Sigma \sim -iq^2 \int^\Lambda d^4p\Delta_m(p^2)^2 p^2 + \cdots, \tag{2.3.4}$$

where Δ_m is the propagator for the matter particle running in the loop and Λ is the ultraviolet cutoff of the loop. For scalar fields, the loop integral is quadratically divergent, and by absorbing the divergence in the redefinition of M_P we obtain

$$M^2_{P(ren)} = M^2_{P(bare)} + c\Lambda^2. \tag{2.3.5}$$

Taking $\Lambda = \mu$ we recover the form of the rigorously derived running Planck mass Eq. (2.3.1).[2]

[2] We remark here that despite the rigorous heat kernel derivation of Eq. (2.3.1), a recent publication [11] has criticised attempts to define a running Planck mass. The main argument is that a precise definition of the running is not independent of the process from which it was derived. This therefore leads to difficulty in defining a universally applicable running. If true, these criticisms could cast doubt on the validity of our arguments here. However, we only consider a single

It is simple to incorporate a running Planck mass into the tree level unitarity bounds Eqs. (2.1.1) and (2.1.4) in order to give an 'RG improved' unitarity bound[3]:

$$\sqrt{s} \leq M_P(\sqrt{s})\sqrt{160\pi/N}, \tag{2.3.6}$$

$$\sqrt{s} \leq M_P(\sqrt{s})\sqrt{96\pi/N_s}. \tag{2.3.7}$$

In Ref. [1], we argued that since the running Planck mass incorporates quantum effects into the definition of the coupling constant, a running Planck mass gives a good indication of when quantum gravitational effects become strong. The scale at which quantum gravitational effects become strong is therefore defined as $\mu_\star$, where

$$\mu_\star^2 \simeq M_P(\mu_\star)^2. \tag{2.3.8}$$

This criteria ensures that the scale $\mu_\star$ is the scale at which the expansion parameter for the effective theory, $E/M_P(E)$, is equal to one, i.e. the theory becomes strongly coupled. Since loop effects are normally accompanied by a factor of $1/16\pi^2$ (coming from the integral over unconstrained loop momenta), it could even be argued that the criteria (2.3.8) is rather conservative and in fact the scale at which gravitational effects are expected to become strong could easily be an order of magnitude higher than $\mu_\star$. For example in Ref. [8] (see also Ref. [10]) a careful power counting analysis is carried out and it is shown that a generic sufficient condition for successive loops of interactions to be smaller than preceding ones is

$$\frac{E}{4\pi M_P} < 1. \tag{2.3.9}$$

Despite this, we will retain the criteria (2.3.8) as the scale at which we expect gravitational effects to become strongly coupled, safe in the knowledge that this is a conservative estimate.

Accepting that $\mu_\star$ is the scale at which gravitational effects are expected to become strong, we argued in Ref. [1] that if either of the RG improved unitarity bounds, Eq. (2.3.6) or (2.3.7), showed unitarity problems below $\mu_\star$, then the unitarity problem could not be fixed by strong coupling effects. We are still in the weakly coupled regime and so higher order effects can not be sizeable enough to counteract the rapid growth with energy of the amplitudes. Additionally, higher order effects coming from the graviton self energy have already been incorporated into the bound via the RG. The interpretation is then that the unitarity problem is a clear sign that either new physics

(Footnote 2 continued)
process, s-channel scattering via graviton exchange, and so we need not worry about universality of the definition of the running. The running we employ is defined from exactly the process we wish to consider and should therefore be applicable everywhere we have used it, even if it were not applicable for other processes.

[3] A similar procedure of defining an RG improved unitarity bound was given in Ref. [9] for the bound on the Higgs boson mass from WW scattering as outlined in Sect. 1.2.1.

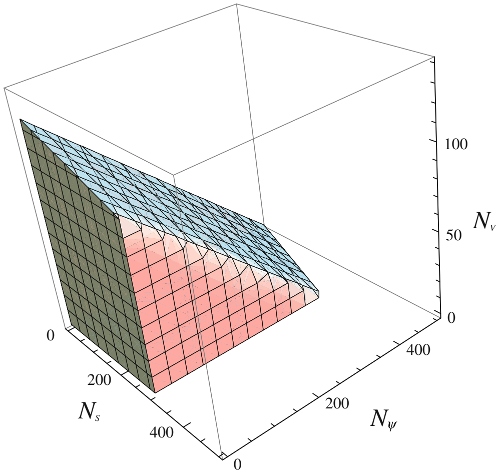

Fig. 2.2 The parameter space for models in which unitarity is maintained up to the scale $\mu_\star$

that fixes the unitarity problem must enter at or below the unitarity violation scale, or the model would be inconsistent (suffer from an incurable unitarity problem). The requirement to distinguish such cases is therefore whether or not the theory remains unitary up to the point when $\sqrt{s} = M_P(\sqrt{s})$. Clearly this will occur if

$$N \leq 160\pi \tag{2.3.10}$$

and

$$N_s \leq 96\pi. \tag{2.3.11}$$

Note that these criteria are completely independent of the specific running of the Planck mass.[4]

In Ref. [1] we distinguish the models analysed in Table 2.3 according to the above criteria. Since the criteria are independent on the details of the running Planck mass, the models in Table 2.3 can be distinguished by whether or not either of the two unitarity bounds are below M_P. Entries in Table 2.3 where unitarity breaks down below M_P have been marked with $*$. All the models for which the unitarity bound is below M_P are therefore classified as being inconsistent without the addition of new physics below the scale at which gravity becomes strong. The parameter space for all models for which unitarity is maintained up to the scale $\mu_\star$ is plotted in Fig 2.2.

[4] We remark again that the concerns raised in Ref. [9] about defining a universal running Planck mass are not relevant here since the argument given in this section turns out to be independent of the specific running employed.

2.3.1 Model with Large Number of Fields

The main motivation for investigating the running of the Planck mass in Ref. [3] was to utilise the running to propose a model which could offer a solution to the seemingly unnatural hierarchy between the electroweak and quantum gravity scales. By introducing an extremely large number of scalar or fermion fields the scale at which gravity becomes strong can be significantly lowered (note that due to the sign of the vector boson contribution to N_l, a large number of spin one particles will act to increase the scale of strong coupling). If the scale $\mu_\star$ is identified as the scale of quantum gravity, then the hierarchy problem will not exist if $\mu_\star$ can be lowered to the electroweak scale.

If $M_P(\mu_\star) \ll M_P(0)$ then we find the value of N_l required to have $M_P(\mu_\star) = \mu_\star$ is given by

$$N_l = 96\pi^2 \frac{M_P(0)^2}{\mu_\star^2}. \tag{2.3.12}$$

In order to have $M_P(\mu_\star) = \mu_\star = 1$ TeV we require $N_l \simeq 5 \times 10^{33}$. Assuming that the entire contribution to N_l is made up of scalars, i.e. $N_l = N_s$, we find unitarity is violated (using the $j = 0$ bound) at a scale

$$E_\star = \frac{\mu_\star}{\sqrt{1+\pi}} \simeq 0.5\mu_\star. \tag{2.3.13}$$

This is below the scale at which gravity is expected to become strong and so new physics will need to enter at this scale in order to fix the unitarity problem.

2.3.1.1 Model with a Large Non-minimal Coupling

In Ref. [2], we noted that one could also define a running Planck mass based on the results of Ref. [7] for models with non-minimally coupled scalar fields. The running Planck mass defined in this way for N_s scalar fields with non-minimal coupling ξ is given by

$$M_P(\mu)^2 = M_P(0)^2 - \frac{(1+6\xi)}{96\pi^2}\mu^2 N_s. \tag{2.3.14}$$

As a result of this running, it was observed that not only could one lower the Planck mass by introducing a large number of fields, one could also achieve this by introducing one or more scalar fields with very large non-minimal couplings. This opened the door to yet another model offering a solution to the hierarchy problem. If we require $M_P(\mu_\star) = \mu_\star$ and assuming that $M_P(0) \gg M_P(\mu_\star)$ and $\xi \gg 1$ we find

$$\xi N_s = 16\pi^2 \frac{M_P(0)^2}{\mu_\star^2}. \tag{2.3.15}$$

In order to have $M_P(\mu_\star) = \mu_\star = 1$ TeV, we would require $\xi N_s \simeq 9 \times 10^{32}$. Assuming $N_s > 2$, so that the s-channel unitarity bound is valid, we then find unitarity is violated (using the $j = 0$ bound) at a scale

$$E_\star = \frac{\mu_\star}{\sqrt{1 + 6\pi\xi}} \ll \mu_\star. \tag{2.3.16}$$

Again this is below the scale at which gravity is expected to become strong and so new physics will need to enter at this scale in order to fix the unitarity problem.

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Atkins, M.

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