

Chapter 2

Solid Mechanics Fundamentals

Abstract In this chapter the mechanical fundamentals behind the numerical applications presented in this work are developed. Firstly it is present a brief exposition of the used continuum formulation, where the solid kinematics and constitutive equations are shown. Following it is presented the used weak form and the consequent generated discrete equation system. Next, the dynamic analysis equations are presented and transient analysis basic concepts are introduced.

2.1 Continuum Formulation

The continuum mechanics is the foundation of the nonlinear numerical analysis. Solids and structures subjected to loads or forces become stressed. The stresses lead to strains, which can be interpreted as deformations or relative displacements. Solid Mechanics and Structural Mechanics deals, for a given solid and boundary condition (external forces and displacements constrains), with the relationship between stress and strain and the relationship between strain and displacements [1–3]. Solids can show different behaviours, depending on the solid material stress-strain curve. In this work only linear elastic materials are considered. In elastic materials the deformation in the solid caused by loading disappears fully with the unloading, in contrast, plastic materials show a residual deformation (which cannot be naturally recovered) that remains after the total unload process. The material properties on the solid can also be anisotropic, i.e., the material property varies with the direction [4]. On an anisotropic material the deformation caused by a load applied in a given direction causes a different deformation if the same load is applied in a distinct direction. Composite laminates are generally constituted by layers of anisotropic material. There are many material constants to be considered and defined in order to fully describe an anisotropic material, which is the reason why so often the engineering problems reduce the analysis to an isotropic material analysis. Isotropic materials are a special case of anisotropic materials, where only two independent material properties need to be known, the

Young modulus and the Poisson ratio. In this chapter the rigid solid motion and deformation are described, with emphasis on rotation, which plays an important role in nonlinear continuum mechanics. Also the concepts of strain and stress in nonlinear mechanics are introduced. The equilibrium and the constitutive equations are presented afterwards.

2.1.1 Kinematics

The general motion of a deformable body is represented in Fig. 2.1. The body, in the initial position $t = 0$, is considered to be an assemblage of material particles, labelled by the coordinates \mathbf{X} , with respect to the Cartesian basis \mathbf{e} . The current position of a particle is defined at time t by the coordinates \mathbf{x} .

The motion can be mathematically described by a mapping function ϕ between initial and current particle positions,

$$\mathbf{x} = \phi(\mathbf{X}, t) \quad (2.1)$$

It is considered the material description, the Lagrangian description, since the variation of the solid deformation is described with respect to the initial coordinates \mathbf{X} , at time t .

2.1.1.1 Deformation Gradient

The deformation gradient \mathbf{F} , is a key quantity in finite deformation analysis, since it is involved in all equations relating quantities before deformation (initial configuration) with the correspondent quantities after the finite deformation (current configuration). The deformation gradient tensor \mathbf{F} can be defined as,

$$\mathbf{F} = \frac{\partial \phi}{\partial \mathbf{X}} = \nabla \phi \quad (2.2)$$

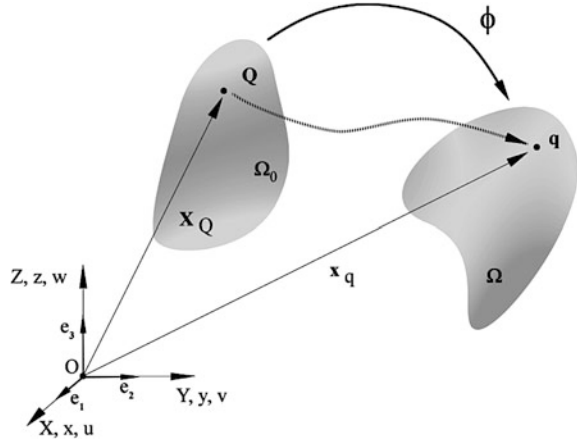
Alternatively to Eq. (2.1) the motion can be expressed by,

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \quad (2.3)$$

which permits the deformation gradient to be written as,

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \quad (2.4)$$

Fig. 2.1 General motion of a deformable body



As so, for a three-dimensional deformation problem the deformation gradient tensor of an initial material position $\mathbf{X} = \{X \ Y \ Z\}$ in respect to a current material position $\mathbf{x} = \{x \ y \ z\}$ can be presented as,

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} \quad (2.5)$$

The determinant of the \mathbf{F} is denoted by J and is called the ‘Jacobian determinant’.

$$J = \det(\mathbf{F}) \quad (2.6)$$

The Jacobian determinant can be used to relate the integral of a given functional f in the current and in the initial configuration by,

$$\int_{\Omega} f(\mathbf{x}, t) d\Omega = \int_{\Omega_0} f(\phi(\mathbf{X}, t), t) J d\Omega_0. \quad (2.7)$$

2.1.1.2 Strain

Consider the change of the scalar product of the two elemental vectors from $d\mathbf{X}_1 = \mathbf{Q}_1 - \mathbf{P}$ and $d\mathbf{X}_2 = \mathbf{Q}_2 - \mathbf{P}$, initial configuration, to $d\mathbf{x}_1 = \mathbf{q}_1 - \mathbf{p}$ and $d\mathbf{x}_2 = \mathbf{q}_2 - \mathbf{p}$, current configuration, as a general measure of deformation. Where \mathbf{Q}_1 and \mathbf{Q}_2 are two material particles in the neighbourhood of a material particle \mathbf{P} for the initial configuration and \mathbf{q}_1 and \mathbf{q}_2 and \mathbf{p} the same respective material

particles in the current configuration. Equation (2.4) permits the following relations $d\mathbf{x}_1 = \mathbf{F} d\mathbf{X}_1$ and $d\mathbf{x}_2 = \mathbf{F} d\mathbf{X}_2$, and the spatial scalar product $d\mathbf{x}_1 \cdot d\mathbf{x}_2$ can be found in terms of the material vectors $d\mathbf{X}_1$ and $d\mathbf{X}_2$ as,

$$d\mathbf{x}_1 \cdot d\mathbf{x}_2 = d\mathbf{X}_1 \cdot \mathbf{F}^T \mathbf{F} d\mathbf{X}_2 \quad (2.8)$$

The right Cauchy-Green deformation tensor is defined by

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} \quad (2.9)$$

Which operates directly on the material vectors $d\mathbf{X}_1$ and $d\mathbf{X}_2$. Alternatively the initial scalar product $d\mathbf{X}_1 \cdot d\mathbf{X}_2$ can be obtained in terms of spatial vectors $d\mathbf{x}_1$ and $d\mathbf{x}_2$ using the left Cauchy-Green deformation tensor \mathbf{b} ,

$$\mathbf{b} = \mathbf{F} \mathbf{F}^T \quad (2.10)$$

The change in scalar product can now be found in terms of the material vectors $d\mathbf{X}_1$ and $d\mathbf{X}_2$ and the Lagrange or Green strain tensor \mathbf{E} can be defined as,

$$\frac{1}{2}(d\mathbf{x}_1 \cdot d\mathbf{x}_2 - d\mathbf{X}_1 \cdot d\mathbf{X}_2) = d\mathbf{X}_1 \cdot \mathbf{E} d\mathbf{X}_2 \quad (2.11)$$

where the strain tensor \mathbf{E} is expressed as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}). \quad (2.12)$$

2.1.1.3 Polar Decomposition

The tensor \mathbf{F} can be expressed as the product of the orthogonal rotation tensor \mathbf{R} by the symmetric stretch tensor \mathbf{U} ,

$$\mathbf{F} = \mathbf{R} \mathbf{U} \quad (2.13)$$

where

$$\mathbf{R}^T \mathbf{R} = \mathbf{I} \quad \text{and} \quad \mathbf{U} = \mathbf{U}^T \quad (2.14)$$

Such decomposition is called, polar decomposition, and Eq. (2.9) can be expressed as,

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{U}^T \mathbf{R}^T \mathbf{R} \mathbf{U} = \mathbf{U}^T \mathbf{I} \mathbf{U} = \mathbf{U} \mathbf{U} \quad (2.15)$$

In order to actually obtain \mathbf{U} from Eq. (2.15) it is first necessary to evaluate the principal directions of \mathbf{C} , represented by the eigenvectors set \mathbf{W}_i and the

correspondent eigenvalues λ_i , with $i = \{1, 2, 3\}$ for the three-dimensional case. In this manner \mathbf{C} can be defined as,

$$\mathbf{C} = \sum_{i=1}^3 \lambda_i^2 \mathbf{W}_i \otimes \mathbf{W}_i \quad (2.16)$$

since the eigenvectors \mathbf{W}_i are in fact orthogonal unit vectors, because $\mathbf{C} = \mathbf{C}^T$. As so, with Eqs. (2.15) and (2.16) it is possible to write the material stretch tensor \mathbf{U} as,

$$\mathbf{U} = \sum_{i=1}^3 \lambda_i \mathbf{W}_i \otimes \mathbf{W}_i \quad (2.17)$$

Once the stretch tensor \mathbf{U} is known, the rotation tensor \mathbf{R} can be obtained without difficulty from Eq. (2.13).

$$\mathbf{R} = \mathbf{F} \mathbf{U}^{-1}. \quad (2.18)$$

2.1.1.4 Stress

In a large deformation analysis a body can experience a large rotation and/or a large strain. The defined stress terms together with the obtained strain terms enables to express the virtual work as an integral over the known body volume, expressing in this manner the change in the body configuration. Both strain tensor and stress tensor are referred to the same deformed state. The Cauchy stress tensor, here defined as $\mathbf{\Lambda}$, is a symmetric tensor and it represents the stresses of the current configuration. For the three-dimensional case it can be defined as,

$$\mathbf{\Lambda} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (2.19)$$

In this work it is used the Voigt notation, since the development of fourth order tensors is less practical. In Voigt notation the tensors are expressed in column vectors, so the stress tensor $\mathbf{\Lambda}$ is reduced to the stress vector $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} = \{ \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx} \}^T \quad (2.20)$$

and the strain tensor \mathbf{E} to the strain vector $\boldsymbol{\varepsilon}$,

$$\boldsymbol{\varepsilon} = \{ \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \varepsilon_{xy} \quad \varepsilon_{yz} \quad \varepsilon_{zx} \}^T \quad (2.21)$$

The use of vectors is more practical in the programming process.

2.1.1.5 Principal Stress

Another way of describing the Cauchy stress tensor, which completely defines the stress state in an interest point, is through,

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{t}^{(\hat{e}_1)} \\ \mathbf{t}^{(\hat{e}_2)} \\ \mathbf{t}^{(\hat{e}_3)} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (2.22)$$

where \hat{e}_1 , \hat{e}_2 and \hat{e}_3 are the versors of the coordinate system and $\mathbf{t}^{(\hat{e}_i)}$ is the stress vector on a plane normal to \hat{e}_1 passing through the interest point, Fig. 2.2(a). Following Cauchy's stress theorem, if the stress vectors of three orthogonal planes, with a common point, are known, then the stress vector on any other plane passing through that point can be found through the coordinate transformation equations [5]. Thus, the stress vector $\mathbf{t}^{(n)}$ in a point belonging to an inclined plane, Fig. 2.2(b), can be defined by,

$$\mathbf{t}^{(n)} = \mathbf{n} \cdot \sigma_{ij} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (2.23)$$

where \mathbf{n} is the inclined plane normal vector. The relation in Eq. (2.23) leads to the transformation rule of the stress tensor. The initial stress tensor σ_{ij} , defined in the \mathbf{x}_i coordinate system, can be transformed in a new stress tensor σ'_{ij} , defined in another \mathbf{x}'_i coordinate system by the relation,

$$\mathbf{\Lambda}' = \mathbf{A} \mathbf{\Lambda} \mathbf{A}^T \quad (2.24)$$

being \mathbf{A} the rotation matrix. Developing Eq. (2.24),

$$\begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad (2.25)$$

The a_{ij} coefficients can be understood as the projection of the \mathbf{x}'_i coordinate system versors in the \mathbf{x}_i coordinate system versors. Therefore, the angle between the versors of each coordinate system can be defined as,

$$\gamma_{ij} = \cos^{-1}(a_{ij}) \quad (2.26)$$

Through Eq. (2.26) and Fig. 2.3 it is possible to comprehend better the physical meaning of the a_{ij} coefficients and the respective angles.

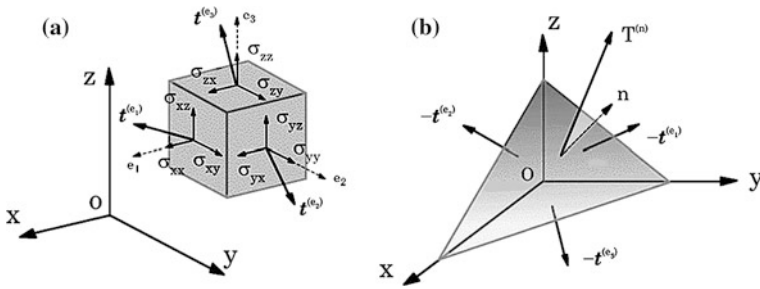


Fig. 2.2 **a** Three-dimensional stress components. **b** Stress vector acting on a plane with normal vector n

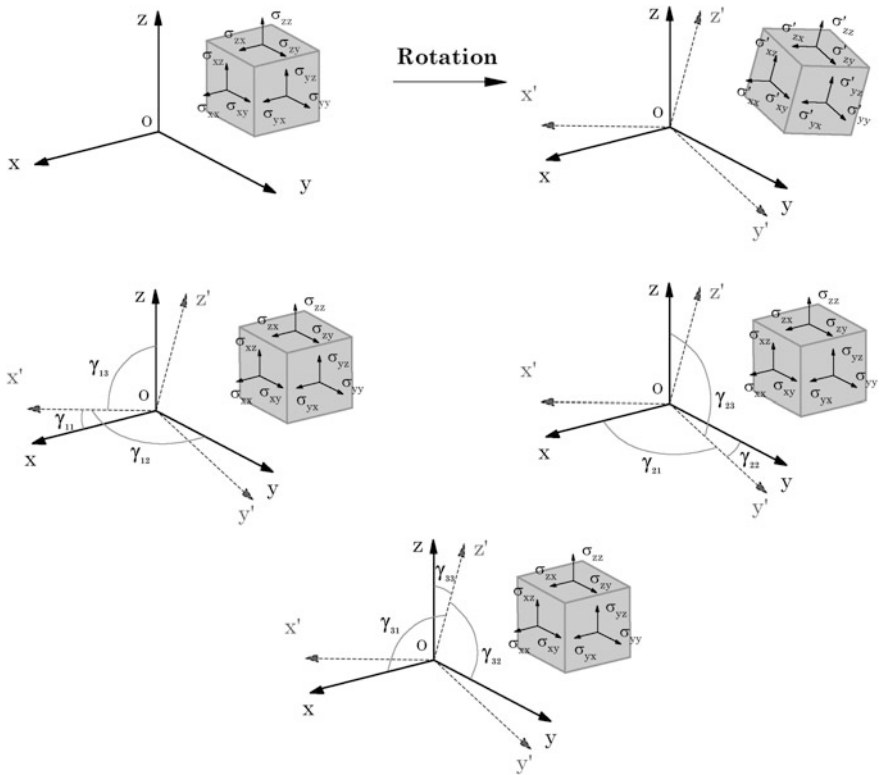


Fig. 2.3 Stress tensor transformation and respective angles

Let P be an interest point of a considered stressed body. There are at least three planes, orthogonal with each other, crossing P where the corresponding stress vector is normal to the plane. These planes are called principal planes and the

normal vectors of each plane are called principal directions. The stress vectors are parallel to the plane normal vectors and are called principal stresses.

The stress tensor is a physical quantity, independent of the coordinate system chosen to represent it. Therefore, there are certain invariants associated with it which are also independent of the coordinate system. Being a second order tensor, the stress tensor has associated three independent invariant quantities. One set of such invariants are the principal stresses of the stress tensor, which are just the eigenvalues of the stress tensor. Their direction vectors are the principal directions or eigenvectors. A stress vector parallel to the normal vector \mathbf{n} is given by,

$$\mathbf{t}^{(n)} = \lambda \mathbf{n} = \sigma_n \mathbf{n} \quad (2.27)$$

being λ a constant of proportionality, and in this particular case the magnitude of σ_n , the principal stress in \mathbf{n} direction. Knowing $t_i^{(n)} = \sigma_{ij}n_j$ and $n_i = \delta_{ij}n_j$, where δ_{ij} is the Kronecker delta, the following development can be performed,

$$t_i^{(n)} = \lambda n_i \Rightarrow \sigma_{ij}n_j = \lambda \delta_{ij}n_j \Rightarrow (\sigma_{ij} - \lambda \delta_{ij})n_j = 0 \quad (2.28)$$

which is a homogeneous system, three linear equations for three n_j unknowns. To obtain the n_j non-zero solution, the matrix determinant must be equal to zero,

$$|\sigma_{ij} - \lambda \delta_{ij}| = \begin{vmatrix} \sigma_{xx} - \lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \lambda & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \lambda \end{vmatrix} = 0 \quad (2.29)$$

which leads to the following cubic equation,

$$|\sigma_{ij} - \lambda \delta_{ij}| = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0 \quad (2.30)$$

being I_1 , I_2 and I_3 the stress invariants,

$$I_1 = \sigma_{kk} \quad (2.31)$$

$$I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \quad (2.32)$$

$$I_3 = \det(\sigma_{ij}) \quad (2.33)$$

The three roots $\lambda_1 = \sigma_1$, $\lambda_2 = \sigma_2$ and $\lambda_3 = \sigma_3$ of Eq. (2.30) are the eigenvalues or principal stresses, which are unique. Therefore the stress invariants have always the same value regardless of the orientation of the chosen coordinate system. For each eigenvalue λ , exists a non-trivial solution \mathbf{n} on Eq. (2.28). These n_j solutions, called eigenvectors, are the principal directions, which defines the plane where the respective stress acts. Applying Eq. (2.25),

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_{11} & n_{21} & n_{31} \\ n_{12} & n_{22} & n_{32} \\ n_{13} & n_{23} & n_{33} \end{bmatrix} \quad (2.34)$$

The principal stresses and principal directions characterize the stress in \mathbf{P} and are independent of the orientation of the coordinate system.

2.1.2 Constitutive Equations

The following relation between the stress rate and the strain rate is assumed,

$$d\boldsymbol{\sigma} = \mathbf{c} \, d\boldsymbol{\varepsilon} \quad (2.35)$$

The material constitutive matrix is defined by \mathbf{c} and if material nonlinear relations exists between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, then $\mathbf{c} = \mathbf{c}^{ep}$. With Eq. (2.35) the following relation can be established,

$$d\boldsymbol{\varepsilon} = \mathbf{c}^{-1} \, d\boldsymbol{\sigma} \quad (2.36)$$

being $\mathbf{s} = \mathbf{c}^{-1}$ and defined for the three-dimensional case as,

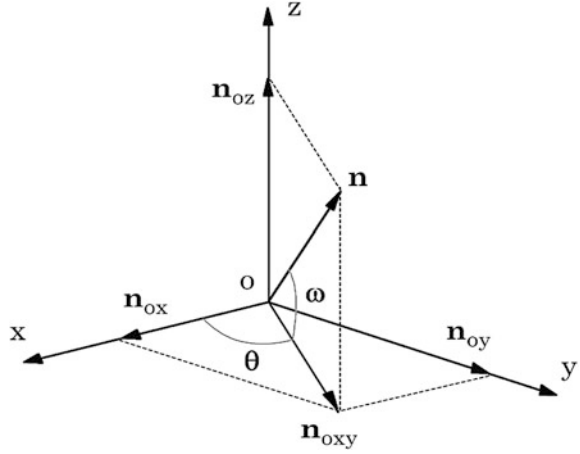
$$\mathbf{s} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} & -\frac{\nu_{zx}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_{xx}} & \frac{1}{E_{yy}} & -\frac{\nu_{zy}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_{xx}} & -\frac{\nu_{yz}}{E_{yy}} & \frac{1}{E_{zz}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{zx}} \end{bmatrix} \quad (2.37)$$

The material constitutive matrix \mathbf{c} is obtained by inverting the material compliance matrix \mathbf{s} , which is here defined for an three-dimensional anisotropic material. The elements on matrix \mathbf{s} are obtained experimentally. E_{ii} is the Young modulus in direction i , ν_{ij} is the Poisson ratio which characterizes the deformation rate in direction j when a force is applied in direction i , G_{ij} is the shear modulus which characterizes the variation angle between directions i and j . Due to symmetry the following relation can be established,

$$E_i \nu_{ji} = E_j \nu_{ij} \quad (2.38)$$

For the two-dimensional case the plane stress and plane strain [5] deformation theory assumptions can be presumed. Considering the plane stress assumptions,

Fig. 2.4 Projection of vector \mathbf{n} in the coordinate axis and in the oxy plane



$\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$, the material compliance matrix \mathbf{s} is obtained directly from the three-dimensional compliance matrix \mathbf{s} ,

$$\mathbf{s} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} & 0 \\ -\frac{\nu_{xy}}{E_{xx}} & \frac{1}{E_{yy}} & 0 \\ 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \quad (2.39)$$

For the plane strain deformation theory it is considered $\varepsilon_{zx} = \varepsilon_{zy} = \varepsilon_{zz} = 0$ and the material compliance matrix \mathbf{s} is defined as,

$$\mathbf{s} = \begin{bmatrix} \frac{1}{E_{xx}} - \frac{\nu_{xx}\nu_{xx}}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} - \frac{\nu_{xx}\nu_{yz}}{E_{yy}} & 0 \\ -\frac{\nu_{xy}}{E_{xx}} - \frac{\nu_{xy}\nu_{yz}}{E_{xx}} & \frac{1}{E_{yy}} - \frac{\nu_{yx}\nu_{yz}}{E_{yy}} & 0 \\ 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \quad (2.40)$$

In the case of an anisotropic material, it is possible to rotate the material constitutive matrix \mathbf{c} and orientate the material directions with a vector. Consider a known vector \mathbf{n} in the Euclidean space \mathbb{R}^3 , Fig. 2.4, and the respective projections on the coordinate axis and in the oxy plane. As it is known,

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_{oxy} \cdot \mathbf{n}_{ox}}{\|\mathbf{n}_{oxy}\| \cdot \|\mathbf{n}_{ox}\|} \right) \quad \text{and} \quad \omega = \cos^{-1} \left(\frac{\mathbf{n}_{oxy} \cdot \mathbf{n}}{\|\mathbf{n}_{oxy}\| \cdot \|\mathbf{n}\|} \right) \quad (2.41)$$

With the obtained angle information it is now possible to rotate the material matrix using the rotational transformation matrix and therefore align the material ox axis with the known vector \mathbf{n} . The rotational transformation matrix that permits an anticlockwise rotation along the ox axis of a known angle β can be defined as,

$$\mathbf{T}_{ox} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2 \beta & \sin^2 \beta & 0 & -\sin 2\beta & 0 \\ 0 & \sin^2 \beta & \cos^2 \beta & 0 & \sin 2\beta & 0 \\ 0 & 0 & 0 & \cos \beta & 0 & \sin \beta \\ 0 & \sin \beta \cdot \cos \beta & -\sin \beta \cdot \cos \beta & 0 & \cos^2 \beta - \sin^2 \beta & 0 \\ 0 & 0 & 0 & -\sin \beta & 0 & \cos \beta \end{bmatrix} \quad (2.42)$$

Along the oy axis,

$$\mathbf{T}_{oy} = \begin{bmatrix} \cos^2 \beta & 0 & \sin^2 \beta & 0 & 0 & \sin 2\beta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin^2 \beta & 0 & \cos^2 \beta & 0 & 0 & -\sin 2\beta \\ 0 & 0 & 0 & \cos \beta & -\sin \beta & 0 \\ 0 & 0 & 0 & \sin \beta & \cos \beta & 0 \\ -\sin \beta \cdot \cos \beta & 0 & \sin \beta \cdot \cos \beta & 0 & 0 & \cos^2 \beta - \sin^2 \beta \end{bmatrix} \quad (2.43)$$

Along the oz axis,

$$\mathbf{T}_{oz} = \begin{bmatrix} \cos^2 \beta & \sin^2 \beta & 0 & -\sin 2\beta & 0 & 0 \\ \sin^2 \beta & \cos^2 \beta & 0 & \sin 2\beta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sin \beta \cdot \cos \beta & -\sin \beta \cdot \cos \beta & 0 & \cos^2 \beta - \sin^2 \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \beta & -\sin \beta \\ 0 & 0 & 0 & 0 & \sin \beta & \cos \beta \end{bmatrix} \quad (2.44)$$

The material matrix after rotation can be defined as,

$$\mathbf{c}^{current} = [\mathbf{T}_{oz}]_{\theta}^T \left[[\mathbf{T}_{oy}]_{\omega}^T \mathbf{c}^{initial} [\mathbf{T}_{oy}]_{\omega} \right] [\mathbf{T}_{oz}]_{\theta}. \quad (2.45)$$

2.2 Weak Form

The strong form system equations are the partial differential system equations governing the studied physic phenomenon. In contrast, the weak form requires a weaker consistency on the adopted approximation (or interpolation) functions. The ideal would be obtaining the exact solution from strong form system equations, however this is usually an extremely difficult task in complex practical engineering problems. Formulations based on weak forms are able to produce stable algebraic system equations and to give a discretized system of equations which leads to more

accurate results. These are the reasons why so many prefer the weak form to obtain the approximated solution.

In this work the discrete equation system is obtained using the Galerkin weak form, which is a variational method [6]. For meshless methods used in this book the discrete system of equations is obtained similarly with the FEM, with some differences inherent to the meshless approach. The discrete equations for the static and the dynamic approach are developed and shown for the basic three-dimensional deformation theory.

2.2.1 Weak Form of Galerkin

Consider the solid with a domain Ω bounded by Γ , Fig. 2.5. The continuous solid surface on which the external forces $\bar{\mathbf{t}}$ are applied is denoted as Γ_t (natural boundary) and the surface where the displacements are constrained is denoted as Γ_u (essential boundary).

The Galerkin weak form is a variational principle based on the energy principle. Of all possible displacement configurations satisfying the compatibility conditions, the essential boundary conditions (kinematical and displacement) and the initial and final time conditions, the real solution correspondent configuration is the one which minimizes the Lagrangian functional L ,

$$L = T - U + W_f \quad (2.46)$$

being T the kinetic energy, U is the strain energy and W_f is the work produced by the external forces. The kinetic energy is defined by,

$$T = \frac{1}{2} \int_{\Omega} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} \, d\Omega \quad (2.47)$$

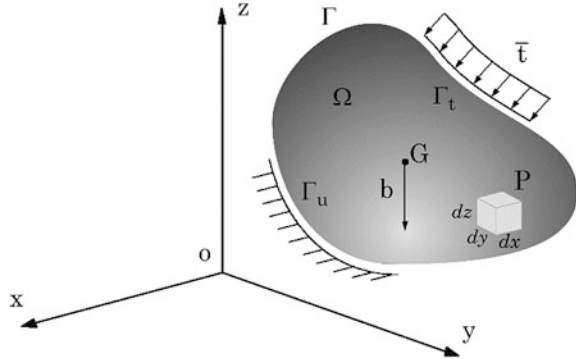
where the solid volume is defined by Ω and $\dot{\mathbf{u}}$ is the displacement first derivative with respect to time, i.e., the velocity. The solid mass density is defined by ρ . The strain energy, for elastic materials, is defined as,

$$U = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, d\Omega \quad (2.48)$$

being $\boldsymbol{\varepsilon}$ the strain vector and $\boldsymbol{\sigma}$ the stress vector. The work produced by the external forces can be expressed as,

$$W_f = \int_{\Omega} \mathbf{u}^T \mathbf{b} \, d\Omega + \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} \, d\Gamma \quad (2.49)$$

Fig. 2.5 Continuous solid subject to volume forces and external forces



where \mathbf{u} represents the displacement, \mathbf{b} the body forces and Γ_t the traction boundary where the external forces $\bar{\mathbf{t}}$ are applied. By substitution the Lagrangian functional L can be rewritten as,

$$L = \frac{1}{2} \int_{\Omega} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} \, d\Omega - \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, d\Omega + \int_{\Omega} \mathbf{u}^T \mathbf{b} \, d\Omega + \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} \, d\Gamma \quad (2.50)$$

and then minimized,

$$\delta \int_{t_1}^{t_2} \left[\frac{1}{2} \int_{\Omega} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} \, d\Omega - \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, d\Omega + \int_{\Omega} \mathbf{u}^T \mathbf{b} \, d\Omega + \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} \, d\Gamma \right] dt = 0 \quad (2.51)$$

Moving the variation operator δ inside the integrals,

$$\int_{t_1}^{t_2} \left[\frac{1}{2} \int_{\Omega} \delta(\rho \dot{\mathbf{u}}^T \dot{\mathbf{u}}) \, d\Omega - \frac{1}{2} \int_{\Omega} \delta(\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}) \, d\Omega + \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} \, d\Omega + \int_{\Gamma_t} \delta \mathbf{u}^T \bar{\mathbf{t}} \, d\Gamma \right] dt = 0 \quad (2.52)$$

Since all operations are linear, changing the order of operation does not affect the result. In the first term of Eq. (2.52) the time integral can be moved inside the spatial integral,

$$\int_{t_1}^{t_2} \left[\frac{1}{2} \int_{\Omega} \delta(\rho \dot{\mathbf{u}}^T \dot{\mathbf{u}}) \, d\Omega \right] dt = \frac{1}{2} \int_{\Omega} \left[\int_{t_1}^{t_2} \delta(\rho \dot{\mathbf{u}}^T \dot{\mathbf{u}}) \, dt \right] d\Omega \quad (2.53)$$

Using the chain rule of variation and then the scalar property, the integral can be rewritten as,

$$\int_{t_1}^{t_2} \delta(\rho \dot{\mathbf{u}}^T \dot{\mathbf{u}}) dt = \rho \int_{t_1}^{t_2} (\delta \dot{\mathbf{u}}^T \dot{\mathbf{u}} + \dot{\mathbf{u}}^T \delta \dot{\mathbf{u}}) dt = 2\rho \int_{t_1}^{t_2} (\delta \dot{\mathbf{u}}^T \dot{\mathbf{u}}) dt \quad (2.54)$$

And knowing that $\dot{\mathbf{u}}^T \dot{\mathbf{u}}$ is a scalar and $\dot{\mathbf{u}} = \partial \mathbf{u} / \partial t$,

$$\int_{t_1}^{t_2} (\delta \dot{\mathbf{u}}^T \dot{\mathbf{u}}) dt = \int_{t_1}^{t_2} \left(\frac{\partial \delta \mathbf{u}^T}{\partial t} \frac{\partial \mathbf{u}}{\partial t} \right) dt \quad (2.55)$$

Then integrating by parts, with respect to time,

$$\int_{t_1}^{t_2} \left(\frac{\partial \delta \mathbf{u}^T}{\partial t} \frac{\partial \mathbf{u}}{\partial t} \right) dt = \int_{t_1}^{t_2} \left(-\delta \mathbf{u}^T \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) dt + \left(\delta \mathbf{u}^T \frac{\partial \mathbf{u}}{\partial t} \right) \Big|_{t_1}^{t_2} \quad (2.56)$$

Notice that \mathbf{u} satisfies, by imposition, the conditions at the initial time, t_1 , and final time, t_2 , leading to a null $\delta \mathbf{u}$ at t_1 and t_2 . Therefore the last term in Eq. (2.56) vanishes. Considering the last development and switching the integration order again, Eq. (2.53) becomes,

$$\int_{t_1}^{t_2} \left[\frac{1}{2} \int_{\Omega} \delta(\rho \dot{\mathbf{u}}^T \dot{\mathbf{u}}) d\Omega \right] dt = - \int_{t_1}^{t_2} \left[\rho \int_{\Omega} (\delta \mathbf{u}^T \ddot{\mathbf{u}}) dt \right] \quad (2.57)$$

being $\ddot{\mathbf{u}} = \partial^2 \mathbf{u} / \partial t^2$ the acceleration. The second term on Eq. (2.52) can also be developed. The integrand function in the second integral term can be written as follows,

$$\delta(\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}) = \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} + \boldsymbol{\varepsilon}^T \delta \boldsymbol{\sigma} \quad (2.58)$$

as the two terms in Eq. (2.58) are in fact scalars, the transpose does not affect the result, as so,

$$\boldsymbol{\varepsilon}^T \delta \boldsymbol{\sigma} = (\boldsymbol{\varepsilon}^T \delta \boldsymbol{\sigma})^T = \delta \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} \quad (2.59)$$

Using the constitutive equation $\boldsymbol{\sigma} = \mathbf{c} \boldsymbol{\varepsilon}$ and the symmetric property of the material matrix, $\mathbf{c}^T = \mathbf{c}$, it is possible to write,

$$\delta \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} = \delta(\mathbf{c} \boldsymbol{\varepsilon})^T \boldsymbol{\varepsilon} = \delta \boldsymbol{\varepsilon}^T \mathbf{c}^T \boldsymbol{\varepsilon} = \delta \boldsymbol{\varepsilon}^T \mathbf{c} \boldsymbol{\varepsilon} = \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \quad (2.60)$$

Therefore Eq. (2.58) becomes,

$$\delta(\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}) = 2\delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \quad (2.61)$$

simplifying the second term in Eq. (2.52),

$$\int_{t_1}^{t_2} \left[\frac{1}{2} \int_{\Omega} \delta(\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}) d\Omega \right] dt = \int_{t_1}^{t_2} \left[\int_{\Omega} \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega \right] dt \quad (2.62)$$

Equation (2.52) now becomes,

$$\int_{t_1}^{t_2} \left[-\rho \int_{\Omega} (\delta\mathbf{u}^T \ddot{\mathbf{u}}) d\Omega - \int_{\Omega} \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega + \int_{\Omega} \delta\mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \delta\mathbf{u}^T \bar{\mathbf{t}} d\Gamma \right] dt = 0 \quad (2.63)$$

To satisfy Eq. (2.63) for all possible choices of the integrand of the time integration has to be null, leading to the following expression,

$$-\rho \int_{\Omega} (\delta\mathbf{u}^T \ddot{\mathbf{u}}) d\Omega - \int_{\Omega} \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega + \int_{\Omega} \delta\mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \delta\mathbf{u}^T \bar{\mathbf{t}} d\Gamma = 0 \quad (2.64)$$

This last equation is known as the ‘Galerkin weak form’, which can also be viewed as the principle of virtual work. The principle of virtual work states that if a solid body is in equilibrium, the virtual work produced by the body inner stresses and the body applied external forces should vanish when the body experiments a virtual displacement. Considering the stress-strain relation, $\boldsymbol{\sigma} = \mathbf{c} \boldsymbol{\varepsilon}$, and the strain-displacement relation, $\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u}$, Eq. (2.64) can be rearranged in the following expression,

$$\int_{\Omega} \delta(\mathbf{L}\mathbf{u})^T \mathbf{c}(\mathbf{L}\mathbf{u}) d\Omega - \int_{\Omega} \delta\mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma_t} \delta\mathbf{u}^T \bar{\mathbf{t}} d\Gamma + \int_{\Omega} \rho(\delta\mathbf{u}^T \ddot{\mathbf{u}}) d\Omega = 0 \quad (2.65)$$

which is the generic Galerkin weak form written in terms of displacement, very useful in solid mechanical problems. In static problems the fourth term of Eq. (2.65) disappears.

2.3 Discrete System of Equations

The discrete equations for meshless methods are obtained from the principle of virtual work by using the meshless shape functions as trial and test functions. The domain Ω is discretized in a nodal distribution, and each node possesses an

“influence-domain”, which imposes the nodal connectivity between the neighbouring nodes. The meshless trial function $\mathbf{u}(\mathbf{x}_I)$ is given by,

$$\mathbf{u}(\mathbf{x}_I) = \sum_{i=1}^n \varphi_i(\mathbf{x}_I) u_i \quad (2.66)$$

being $\varphi_i(\mathbf{x}_I)$ the meshless approximation or interpolation function and u_i are the nodal displacements of the n nodes belonging to the influence-domain of interest node \mathbf{x}_I . Considering the NNRPIM, it is known that the NNRPIM interpolation function satisfies the condition,

$$\varphi_i(\mathbf{x}_j) = \delta_{ij} \quad (2.67)$$

where δ_{ij} is the Kronecker delta, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Following Eq. (2.66), the test functions (or virtual displacements) are defined as,

$$d\mathbf{u}(\mathbf{x}_I) = \sum_{i=1}^n \varphi_i(\mathbf{x}_I) du_i \quad (2.68)$$

where du_i are the nodal values for the test function.

2.3.1 Weak Form of Galerkin

The meshless formulation can be established in terms of a weak form of the differential equation under consideration, Eq. (2.64). In the solid mechanics context this implies the use of the virtual work equation.

$$L = \int_{\Omega} \sigma \, d\epsilon \, d\Omega - \int_{\Omega} \mathbf{b} \cdot d\mathbf{u} \, d\Omega - \int_{\Gamma} \bar{\mathbf{t}} \cdot d\mathbf{u} \, d\Gamma + \rho \int_{\Omega} (d\mathbf{u}^T \ddot{\mathbf{u}}) \, d\Omega = 0 \quad (2.69)$$

The virtual deformation $d\epsilon$ is defined by,

$$d\epsilon = \bar{\mathbf{B}} d\mathbf{u} \quad (2.70)$$

where $\bar{\mathbf{B}}$ is the deformation matrix. Thus, the virtual work of the first term in Eq. (2.69), using Eq. (2.70), can be expressed as,

$$L_1 = \int_{\Omega} d\mathbf{u}^T \bar{\mathbf{B}}^T \boldsymbol{\sigma} \, d\Omega \quad (2.71)$$

The strain vector can be divided in two parts, the linear part and the nonlinear part,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + \boldsymbol{\varepsilon}_{NL} \quad (2.72)$$

which can also be presented as,

$$\boldsymbol{\varepsilon} = \underbrace{\mathbf{L}\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}_0} + \underbrace{\frac{1}{2}\mathbf{A}\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}_{NL}} = \left(\mathbf{L} + \frac{1}{2}\mathbf{A}\right)\boldsymbol{\theta} \quad (2.73)$$

Matrix \mathbf{L} is defined as,

$$\mathbf{L} = \begin{bmatrix} \mathbf{e}_1^T & \mathbf{0} & \mathbf{0} \\ [1 \times 3] & [1 \times 3] & [1 \times 3] \\ \mathbf{0} & \mathbf{e}_2^T & \mathbf{0} \\ [1 \times 3] & [1 \times 3] & [1 \times 3] \\ \mathbf{0} & \mathbf{0} & \mathbf{e}_3^T \\ [1 \times 3] & [1 \times 3] & [1 \times 3] \\ \mathbf{e}_2^T & \mathbf{e}_1^T & \mathbf{0} \\ [1 \times 3] & [1 \times 3] & [1 \times 3] \\ \mathbf{0} & \mathbf{e}_3^T & \mathbf{e}_2^T \\ [1 \times 3] & [1 \times 3] & [1 \times 3] \\ \mathbf{e}_3^T & \mathbf{0} & \mathbf{e}_1^T \\ [1 \times 3] & [1 \times 3] & [1 \times 3] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (2.74)$$

Being \mathbf{e}_i the coordinate i director column vector,

$$\mathbf{I} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \quad (2.75)$$

The column vector $\boldsymbol{\theta}$ is defined by,

$$\boldsymbol{\theta} = \mathbf{G}\mathbf{u} \quad (2.76)$$

The geometric matrix \mathbf{G} is defined by,

$$\mathbf{G}^T = \begin{bmatrix} \frac{\partial \varphi}{\partial x} & 0 & 0 & \frac{\partial \varphi}{\partial y} & 0 & 0 & \frac{\partial \varphi}{\partial z} & 0 & 0 \\ 0 & \frac{\partial \varphi}{\partial x} & 0 & 0 & \frac{\partial \varphi}{\partial y} & 0 & 0 & \frac{\partial \varphi}{\partial z} & 0 \\ 0 & 0 & \frac{\partial \varphi}{\partial x} & 0 & 0 & \frac{\partial \varphi}{\partial y} & 0 & 0 & \frac{\partial \varphi}{\partial z} \end{bmatrix} \quad (2.77)$$

which produces the following column vector $\boldsymbol{\theta}$,

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \quad \text{being} \quad \theta_\xi = \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial w}{\partial \xi} \end{bmatrix} \quad (2.78)$$

The current configuration displacement is considered in matrix \mathbf{A} , which corresponds in Eq. (2.73) to the actualized component.

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\theta}_x^T & \mathbf{0} & \mathbf{0} \\ & [1 \times 3] & [1 \times 3] \\ \mathbf{0} & \boldsymbol{\theta}_y^T & \mathbf{0} \\ [1 \times 3] & & [1 \times 3] \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\theta}_z^T \\ [1 \times 3] & [1 \times 3] & \\ \boldsymbol{\theta}_y^T & \boldsymbol{\theta}_x^T & \mathbf{0} \\ & & [1 \times 3] \\ \mathbf{0} & \boldsymbol{\theta}_z^T & \boldsymbol{\theta}_y^T \\ [1 \times 3] & & \\ \boldsymbol{\theta}_z^T & \mathbf{0} & \boldsymbol{\theta}_x^T \\ & [1 \times 3] & \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} & 0 & 0 & 0 & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \end{bmatrix} \quad (2.79)$$

The deformation matrix $\bar{\mathbf{B}}$, dependent of \mathbf{u} , can be defined as,

$$\bar{\mathbf{B}} = \mathbf{B}_0 + \mathbf{B}_{NL}(\mathbf{u}) \quad (2.80)$$

since it varies with the deformation of the solid. The linear part of the deformation matrix is represented by \mathbf{B}_0 and the nonlinear contribution by \mathbf{B}_{NL} . For the three-dimensional case,

$$\mathbf{B}_0^T = \begin{bmatrix} \frac{\partial \varphi}{\partial x} & 0 & 0 & \frac{\partial \varphi}{\partial y} & 0 & \frac{\partial \varphi}{\partial z} \\ 0 & \frac{\partial \varphi}{\partial y} & 0 & \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial z} & 0 \\ 0 & 0 & \frac{\partial \varphi}{\partial z} & 0 & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial x} \end{bmatrix} \quad (2.81)$$

and

$$\mathbf{B}_{NL} = \mathbf{A} \mathbf{G} \quad (2.82)$$

The nonlinear deformation is actualized through matrix \mathbf{A} , which contains the displacement current configuration.

2.3.2 Stiffness Matrix

The tangential stiffness matrix \mathbf{K}_T can be determined considering the variation of the virtual work of Eq. (2.71), in order to the generalized displacements $d\mathbf{u}$,

$$dL_1 = d \left[\int_{\Omega} \bar{\mathbf{B}}^T \boldsymbol{\sigma} d\Omega \right] \quad (2.83)$$

which can be developed as,

$$dL_1 = \int_{\Omega} d\bar{\mathbf{B}}^T \boldsymbol{\sigma} d\Omega + \int_{\Omega} \bar{\mathbf{B}}^T d\boldsymbol{\sigma} d\Omega = \mathbf{K}_T du \quad (2.84)$$

Using Eqs. (2.35) and (2.70) the following relation is obtained,

$$d\boldsymbol{\sigma} = \mathbf{c} \bar{\mathbf{B}} du \quad (2.85)$$

In the deformation matrix $\bar{\mathbf{B}}$ only the nonlinear part $d\mathbf{B}_{NL}$ is dependent of \mathbf{u} , Eq. (2.80), thus $d\bar{\mathbf{B}} = d\mathbf{B}_{NL}$ and therefore,

$$dL_1 = \int_{\Omega} d\mathbf{B}_{NL}^T \boldsymbol{\sigma} d\Omega + \int_{\Omega} \bar{\mathbf{B}}^T \mathbf{c} \bar{\mathbf{B}} d\Omega \quad (2.86)$$

Where the stiffness matrix can be presented as,

$$\mathbf{K}_T = \mathbf{K}_{\sigma} + \mathbf{K}_0 + \mathbf{K}_{NL} \quad (2.87)$$

Being,

$$\mathbf{K}_{\sigma} = \int_{\Omega} d\mathbf{B}_{NL}^T \boldsymbol{\sigma} d\Omega \quad (2.88)$$

$$\mathbf{K}_0 = \int_{\Omega} \mathbf{B}_0^T \mathbf{c} \mathbf{B}_0 d\Omega \quad (2.89)$$

$$\mathbf{K}_{NL} = \int_{\Omega} (\mathbf{B}_0^T \mathbf{c} \mathbf{B}_{NL} + \mathbf{B}_{NL}^T \mathbf{c} \mathbf{B}_{NL} + \mathbf{B}_{NL}^T \mathbf{c} \mathbf{B}_0) d\Omega \quad (2.90)$$

The initial stress matrix or geometric matrix \mathbf{K}_{σ} is defined as,

$$\mathbf{K}_{\sigma} du = \int_{\Omega} \mathbf{G}^T d\mathbf{A}^T \boldsymbol{\sigma} d\Omega \quad (2.91)$$

The variation of matrix \mathbf{A} in order to \mathbf{u} can be defined as,

$$d\mathbf{A}^T = \begin{bmatrix} d\theta_x & \mathbf{0}_{[3 \times 1]} & \mathbf{0}_{[3 \times 1]} & d\theta_y & \mathbf{0}_{[3 \times 1]} & d\theta_z \\ \mathbf{0}_{[3 \times 1]} & d\theta_y & \mathbf{0}_{[3 \times 1]} & d\theta_x & d\theta_z & \mathbf{0}_{[3 \times 1]} \\ \mathbf{0}_{[3 \times 1]} & \mathbf{0}_{[3 \times 1]} & d\theta_z & \mathbf{0}_{[3 \times 1]} & d\theta_y & d\theta_x \end{bmatrix} \quad (2.92)$$

As so, the term $d\mathbf{A}^T \boldsymbol{\sigma}$ can be represented as,

$$d\mathbf{A}^T \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} & \tau_{xy} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} & \tau_{xz} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} \\ \tau_{yx} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} & \sigma_{yy} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} & \tau_{yz} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} \\ \tau_{zx} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} & \tau_{zy} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} & \sigma_{zz} \begin{smallmatrix} I \\ [3 \times 3] \end{smallmatrix} \end{bmatrix} d\boldsymbol{\theta} = \mathbf{Z} d\boldsymbol{\theta} = \mathbf{ZG} du \quad (2.93)$$

and therefore Eq. (2.88) can be presented as,

$$\mathbf{K}_\sigma = \int_{\Omega} \mathbf{G}^T \mathbf{ZG} d\Omega \quad (2.94)$$

Therefore, the initial stress matrix \mathbf{K}_σ takes into consideration the actualized stress field.

2.3.3 Mass Matrix

The virtual work of the last term in Eq. (2.69) can be expressed and developed as,

$$dL_4 = d \left[\rho \int_{\Omega} (du^T \ddot{u}) d\Omega \right] = \mathbf{M} d\ddot{u} \quad (2.95)$$

where the mass matrix \mathbf{M} can be defined as,

$$\mathbf{M} = \int_{\Omega} \mathbf{H}^T \boldsymbol{\rho} \mathbf{H} d\Omega \quad (2.96)$$

being \mathbf{H} the interpolation function matrix for the interest point i defined as,

$$\mathbf{H}_i = \varphi_i \mathbf{I} \quad (2.97)$$

Where φ_i is the interpolation function for interest node i and \mathbf{I} is the identity matrix defined in Eq. (2.75). The density diagonal matrix can be defined as,

$$\boldsymbol{\rho} = \rho \mathbf{I} \quad (2.98)$$

being ρ the solid material density.

2.3.4 Force Vector

The virtual work of the middle terms in Eq. (2.69) can be expressed and developed as,

$$dL_2 = d \left[\int_{\Omega} \mathbf{b} \cdot d\mathbf{u} \, d\Omega \right] = \mathbf{f}_b \quad (2.99)$$

and

$$dL_3 = d \left[\int_{\Gamma} \bar{\mathbf{t}} \cdot d\mathbf{u} \, d\Gamma \right] = \mathbf{f}_{\bar{t}} \quad (2.100)$$

being the total force vector \mathbf{f} defined as,

$$\mathbf{f}_b + \mathbf{f}_{\bar{t}} = \mathbf{f} \quad (2.101)$$

Thus, the total force vector \mathbf{f} can be developed in a matrix form,

$$\mathbf{f} = \int_{\Omega} \mathbf{H}^T \mathbf{b} \, d\Omega + \int_{\Gamma} \mathbf{H}^T \bar{\mathbf{t}} \, d\Gamma. \quad (2.102)$$

2.3.5 Essential Boundary Conditions Imposition

If the shape functions of the meshless method possess the Kronecker delta property, then the boundary conditions can be imposed directly as in the FEM. The continuum analysis involves two types of boundary conditions, the essential boundary conditions (displacement related) and the natural boundary conditions (force related). Neglecting damping effects and assuming that the matricial form of the equilibrium equations resulting from virtual work expression, Eq. (2.69), can be presented as,

$$\mathbf{K}\mathbf{u} + \mathbf{M}\ddot{\mathbf{u}} = \mathbf{f} \quad (2.103)$$

Such equation can be rewritten as,

$$\begin{bmatrix} \mathbf{K}_{cc} & \mathbf{K}_{cd} \\ \mathbf{K}_{dc} & \mathbf{K}_{dd} \end{bmatrix} \begin{bmatrix} \mathbf{u}_c \\ \mathbf{u}_d \end{bmatrix} + \begin{bmatrix} \mathbf{M}_{cc} & \mathbf{M}_{cd} \\ \mathbf{M}_{dc} & \mathbf{M}_{dd} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_c \\ \ddot{\mathbf{u}}_d \end{bmatrix} = \begin{bmatrix} \mathbf{f}_c \\ \mathbf{f}_d \end{bmatrix} \quad (2.104)$$

where \mathbf{u}_c are the unknown displacements and \mathbf{u}_d the known, or prescribed, displacements. The vectors \mathbf{f}_c and \mathbf{f}_d correspond respectively to the known applied

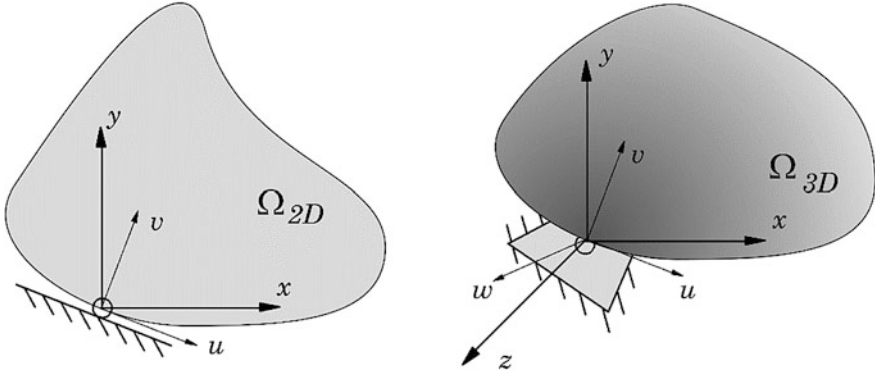


Fig. 2.6 Essential boundary condition nonaligned with the global axis

loads (external and body forces) and to the unknown reactions due the imposed displacement constrains. With the Eq. (2.104) it is assumed that the displacement components considered are axial aligned with the prescribed displacements. If this is not the case it is required the identification of all prescribed displacement orientations and transform locally the discrete equilibrium equations to correspond to the global axis. Thus,

$$\mathbf{u} = \mathbf{T} \bar{\mathbf{u}} \quad (2.105)$$

where $\bar{\mathbf{u}}$ is the vector of nodal point degrees of freedom in the required directions. The transformation matrix \mathbf{T} is defined by Eq. (2.106) and Fig. 2.6, which is a typical representation of the constrained displacements in 2D and 3D analysis.

$$\mathbf{T}_{2D} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \quad \text{and} \quad \mathbf{T}_{3D} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \quad (2.106)$$

Using Eqs. (2.105) and (2.106) it is possible to write,

$$\bar{\mathbf{K}} \bar{\mathbf{u}} + \bar{\mathbf{M}} \ddot{\bar{\mathbf{u}}} = \bar{\mathbf{f}} \quad (2.107)$$

where,

$$\bar{\mathbf{M}} = \mathbf{T}^T \mathbf{M} \mathbf{T} \quad (2.108)$$

$$\bar{\mathbf{K}} = \mathbf{T}^T \mathbf{K} \mathbf{T} \quad (2.109)$$

$$\bar{\mathbf{f}} = \mathbf{T}^T \mathbf{f} \quad (2.110)$$

Notice that the matrix multiplications in Eqs. (2.108), (2.109) and (2.110) involve changes only in those columns and rows of \mathbf{M} , \mathbf{K} and \mathbf{f} that are actually

affected by the prescribed displacement. In practice, the transformation can be effectively carried out on the local level just prior to adding the local matrices to the global assembled matrices.

2.3.6 Dynamic Equations

The equilibrium equations governing the linear dynamic response can be represented as in Eq. (2.103). The fundamental mathematical method used to solve Eq. (2.103) is the separation of variables. In order to change the equilibrium equations to the modal generalized displacements [7] it is proposed the following transformation:

$$\mathbf{u}(t) = \mathbf{\Phi} \mathbf{x}(t) \quad (2.111)$$

where $\mathbf{\Phi}$ is a $m \times m$ square matrix containing m spatial vectors independent of the time variable t , $\mathbf{x}(t)$ is a time dependent vector and $m = 2N$ for the 2D case and $m = 3N$ for the 3D case, being N the total number of nodes in the problem domain. From Eq. (2.111) also follows that $\dot{\mathbf{u}}(t) = \mathbf{\Phi} \dot{\mathbf{x}}(t)$ and $\ddot{\mathbf{u}}(t) = \mathbf{\Phi} \ddot{\mathbf{x}}(t)$. The components of $\mathbf{u}(t)$ are called generalized displacements. For which the solution can be presented in the form,

$$\mathbf{u}(t) = \mathbf{\phi} \sin(\omega(t - t_0)) \quad (2.112)$$

being $\mathbf{\phi}$ the vector of order m , t the time variable, the constant initial time is defined by t_0 and ω is the vibration frequency vector. Substituting Eqs. (2.112) into (2.103) the generalized eigenproblem is obtained, from which $\mathbf{\phi}$ and ω must be determined,

$$\mathbf{K} \mathbf{\phi} = \omega^2 \mathbf{M} \mathbf{\phi} \quad (2.113)$$

Equation (2.113) yields the m eigensolutions,

$$\begin{cases} \mathbf{K} \mathbf{\phi}_1 = \omega_1^2 \mathbf{M} \mathbf{\phi}_1 \\ \mathbf{K} \mathbf{\phi}_2 = \omega_2^2 \mathbf{M} \mathbf{\phi}_2 \\ \vdots \\ \mathbf{K} \mathbf{\phi}_m = \omega_m^2 \mathbf{M} \mathbf{\phi}_m \end{cases} \quad (2.114)$$

The vector $\mathbf{\phi}_i$ is called the i th mode shape vector and ω_i is the corresponding frequency of vibration. Defining a matrix $\mathbf{\Phi}$ whose columns are the eigenvectors $\mathbf{\phi}_i$,

$$\mathbf{\Phi} = [\mathbf{\phi}_1 \quad \mathbf{\phi}_2 \quad \dots \quad \mathbf{\phi}_m] \quad (2.115)$$

and a diagonal matrix $\mathbf{\Omega}$ which stores the eigenvalues ω_i ,

$$\mathbf{\Omega} = \begin{bmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_m^2 \end{bmatrix} \quad (2.116)$$

the m solutions can be written as,

$$\mathbf{K} \mathbf{\Phi} = \mathbf{M} \mathbf{\Phi} \mathbf{\Omega} \quad (2.117)$$

It is required that the space functions satisfy the following stiffness and mass orthogonality conditions,

$$\mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \mathbf{\Omega} \quad (2.118)$$

and

$$\mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = \mathbf{I} \quad (2.119)$$

After substituting Eq. (2.111) and its time derivatives into Eq. (2.103) and pre-multiplying it by $\mathbf{\Phi}^T$, the equilibrium equation that corresponds to the modal generalized displacement is obtained,

$$\ddot{\mathbf{x}}(t) + \mathbf{\Omega} \mathbf{x}(t) = \mathbf{\Phi}^T \mathbf{F}(t) \quad (2.120)$$

The initial conditions on $\mathbf{x}(t)$ are obtained using Eq. (2.111) and considering the the \mathbf{M} -orthonormality of $\mathbf{\Phi}^T$ at time $t = 0$,

$$\begin{bmatrix} \mathbf{x}_0 = \mathbf{\Phi}^T \mathbf{M} \mathbf{u}_0 \\ \dot{\mathbf{x}}_0 = \mathbf{\Phi}^T \mathbf{M} \dot{\mathbf{u}}_0 \end{bmatrix} \quad (2.121)$$

Equation (2.120) can be represented as m individual equations of the form,

$$\begin{bmatrix} \ddot{x}_i(t) + \omega_i^2 x_i(t) = f_i(t) \\ f_i(t) = \phi_i^T \mathbf{F}(t) \end{bmatrix} \quad (2.122)$$

with the initial conditions,

$$\begin{bmatrix} x_i^{t=0} = \phi_i^T \mathbf{M} \mathbf{u}_0 \\ \dot{x}_i^{t=0} = \phi_i^T \mathbf{M} \dot{\mathbf{u}}_0 \end{bmatrix} \quad (2.123)$$

For the complete response, the solution to all m equations in Eq. (2.122) must be calculated and then the modal point displacements are obtained by superposition of the response in each mode.

$$\mathbf{u}(t) = \sum_{i=1}^m \Phi_i x_i(t) \quad (2.124)$$

Therefore the response analysis requires, first, the solution of the eigenvalues and eigenvectors of the problem, Eq. (2.113), then the solution of the decoupled equilibrium equations in Eq. (2.122) and, finally, the superposition of the response in each eigenvector as expressed in Eq. (2.124).

2.3.7 Forced Vibrations

In this book when forced vibrations are imposed only three different time-dependent loading conditions are considered, $\mathbf{f}(t) = \mathbf{f} \times g(t)$. A time constant load—load case A,

$$g_A(t) = 1 \quad (2.125)$$

A transient load—load case B,

$$\begin{cases} g_B(t) = 1 & \text{if } t \leq t_i \\ g_B(t) = 0 & \text{if } t > t_i \end{cases} \quad (2.126)$$

And a harmonic load—load case C,

$$g_C(t) = \sin(\gamma t) \quad (2.127)$$

The solution of each equation in Eq. (2.123) can be calculated using the Duhamel integral,

$$x_i(t) = \frac{1}{\omega_i} \int_0^t f_i(\tau) \sin(\omega_i(t - \tau)) d\tau + \alpha_i \sin(\omega_i t) + \beta_i \cos(\omega_i t) \quad (2.128)$$

where α_i and β_i are determined from the initial conditions: Eq. (2.123) and $f_i(t) = \Phi_i^T \mathbf{f}(t)$. For load case A and load case B the obtained solution is defined as,

$$x_i(t) = \frac{f_i(t)}{\omega_i^2} (1 - \cos(\omega_i t)) + \frac{\dot{x}_i^{t=0}}{\omega_i} \sin(\omega_i t) + x_i^{t=0} \cos(\omega_i t) \quad (2.129)$$

For load case C the obtained solution is,

$$x_i(t) = \frac{f_i(t)}{\omega_i^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega_i} \sin(\omega_i t) \right) \quad (2.130)$$

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