Hilbert-Type Inequalities Including Some Operators, the Best Possible Constants and Applications: A Survey

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Abstract The present work is a review article about some recent results dealing with Hilbert-type inequalities including certain operators in both integral and discrete case. A particular emphasis is given to inequalities including classical means operators. The constants appearing in all discussed inequalities are the best possible. For an illustration, some proofs are given, as well as some applications.

Keywords Hilbert inequality • Hilbert-type inequality • Hardy inequality • Knopp inequality • Carleman inequality • The best possible constant • Mean operator

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1 Introduction

The Hilbert inequality is one of the most important inequalities in mathematical analysis. Applications of this inequality in diverse fields of mathematics have certainly contributed to its importance. After its discovery, the Hilbert inequality was studied by numerous authors, who either reproved it using various techniques, or applied and generalized it in many different ways. For a comprehensive inspection of the initial development of the Hilbert inequality, the reader is referred to a classical monograph [14].

Nowadays, more than a century after its discovery, this problem area is still of interest to numerous authors. In 2005, Krnić and Pečarić [17] established a unified treatment of Hilbert-type inequalities with a general measurable kernel and weight functions. Here we just refer to Hilbert-type inequalities from [17] regarding a homogeneous kernel and power weight functions. Namely, if p and q are non-negative mutually conjugate exponents, that is, if $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $q > 1$, then

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x, y) f(x) g(y) dx dy$$
$$< L \left[\int_{\mathbb{R}_+} x^{1-\lambda+p(A_1-A_2)} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\mathbb{R}_+} y^{1-\lambda+q(A_2-A_1)} g^q(y) dy \right]^{\frac{1}{q}} \quad (1)$$

and

$$\left[\int_{\mathbb{R}_+} y^{(p-1)(\lambda-1)+p(A_1-A_2)} \left(\int_{\mathbb{R}_+} K_\lambda(x, y) f(x) dx \right)^p dy \right]^{\frac{1}{p}}$$
$$< L \left[\int_{\mathbb{R}_+} x^{1-\lambda+p(A_1-A_2)} f^p(x) dx \right]^{\frac{1}{p}}, \quad (2)$$

where $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, $\lambda > 0$, $f, g : \mathbb{R}_+ \to \mathbb{R}$ are non-negative functions such that $f, g \neq 0$ a.e. on $\mathbb{R}_+$, and $L = k_\lambda^{\frac{1}{p}}(pA_2) k_\lambda^{\frac{1}{q}}(2 - \lambda - qA_1)$, $k_\lambda(\alpha) = \int_0^\infty K_\lambda(1, t) t^{-\alpha} dt$. Of course, A_1 and A_2 are real parameters such that all integrals in above inequalities converge.

Inequalities (1) and (2) are equivalent. Considering (1) with the kernel $K_\lambda(x, y) = (x + y)^{-1}$ and parameters $A_1 = A_2 = \frac{1}{pq}$, it follows that $\lambda = 1$ and $L = \frac{\pi}{\sin \frac{\pi}{p}}$, so (1) reduces to one of the earliest versions of the Hilbert inequality (for more details, see [14]). Hence, inequalities related to (1) are usually referred to as Hilbert-type inequalities. On the other hand, inequality (2) and its consequences are referred to as Hardy–Hilbert-type inequalities, since (2) is a generalization of the classical Hardy inequality (for more details, see [17]). In this paper, for the reason of simplicity, a whole class of inequalities related to (1) and (2) will sometimes be referred to as Hilbert-type inequalities.

In paper [17], authors also derived discrete versions of inequalities (1) and (2), i.e., the relations

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n)a_m b_n$$
$$< L\left[\sum_{m=1}^{\infty} m^{1-\lambda+p(A_1-A_2)}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{1-\lambda+q(A_2-A_1)}b_n^q\right]^{\frac{1}{q}} \tag{3}$$

and

$$\left[\sum_{n=1}^{\infty} n^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)a_m\right)^p\right]^{\frac{1}{p}}$$
$$< L\left[\sum_{m=1}^{\infty} m^{1-\lambda+p(A_1-A_2)}a_m^p\right]^{\frac{1}{p}}, \tag{4}$$

which hold under some stronger conditions. Namely, when dealing with discrete Hilbert-type inequalities, some integral bounds are used for certain sums. Usually, such sums may be recognized as the lower Darboux sums for the corresponding integrals. Therefore, inequalities (3) and (4) hold if in addition K_λ is strictly decreasing in each argument, and parameters A_1 and A_2 are chosen so that $pA_2 \geq 0$ and $2-\lambda-qA_1 \geq 0$. Moreover, $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are non-negative sequences, not identically equal to zero, and we assume convergence of all series appearing in (3) and (4).

Considering inequalities (1)–(4) with parameters A_1 and A_2 fulfilling condition $pA_2 + qA_1 = 2-\lambda$, the constant L reduces to $L = k_\lambda(pA_2)$. It was shown that such constant is the best possible in the corresponding inequalities (for more details, see [18, 24]). Hilbert-type inequalities may also be considered in the setting of non-conjugate exponents (see [10, 11]), but in that case there is no evidence that the constants appearing in the corresponding inequalities are the best possible. For comprehensive accounts on Hilbert inequality including history, different proofs, refinements and diverse applications, we refer to recent monograph [19] and references therein.

In the last few years, considerable attention is given to a class of Hilbert-type inequalities where the functions and sequences are replaced by certain integral or discrete operators. As an example, the classical Hardy operator $f \mapsto \frac{1}{x}\int_0^x f(t)dt$ represents the arithmetic mean in integral case. Such inequalities may be derived by virtue of Hilbert-type inequalities from this Introduction and several well-known classical inequalities, such as the Hardy, the Knopp inequality etc. But the most interesting fact in connection with this topic is that the constants appearing in these inequalities remain the best possible.

The present work is a review article of research of several authors in this area. More precisely, this paper is based on some 15 significant papers dealing with Hilbert-type inequalities including some integral and discrete operators (such as above mentioned classical Hardy operator), published in the course of the last few years. All results that will be discussed refer to homogeneous kernels and involve the best constants on their right-hand sides.

The paper is divided into six sections as follows: After this Introduction, in Sect. 2 we introduce notation and list some important classical inequalities necessary for studying Hilbert-type inequalities including classical means operators. In Sect. 3, we present the recent result about a unified treatment of two-dimensional Hilbert-type inequalities including classical mean operators in both integral and discrete case. To illustrate the technique, some proofs are also given, as well as some applications. Further, Sect. 4 deals with the so-called half-discrete case, while in Sect. 5 we discuss an extension to a multidimensional case. Finally, in Sect. 6, we discuss several new Hilbert-type inequalities involving some differential operators.

Since the present work is based on numerous papers written by different authors, the terminology in the paper is not quite unified. However, to avoid misunderstandings, some extra notation and definitions are presented when it is necessary.

2 Notation and Preliminaries

Throughout this paper $L^p(\mathbb{R}_+)$, $p \geq 1$ denotes the space of all Lebesgue measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $\|f\|_{L^p(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(t)|^p dt\right)^{\frac{1}{p}} < \infty$. Similarly, l^p, $p \geq 1$, denotes the space of all real sequences $a = (a_n)_{n\in\mathbb{N}}$ such that $\|a\|_{l^p} = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} < \infty$. In addition, $L^p(\mathbb{R}_+, \varphi)$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a non-negative measurable function, stands for the weighted Lebesgue space with the norm $\|f\|_{L^p(\mathbb{R}_+,\varphi)} = \left(\int_{\mathbb{R}_+} \varphi(t)|f(t)|^p dt\right)^{\frac{1}{p}} < \infty$.

Hilbert-type inequalities we deal with in this article will often contain constants expressed in terms of some special functions. Throughout this article $B(\cdot,\cdot)$ stands for the usual Beta function $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $a, b > 0$, while $\Gamma(\cdot)$ denotes the usual Gamma function defined by $\Gamma(a) = \int_{\mathbb{R}_+} t^{a-1}e^{-t}\, dt$, $a > 0$.

Besides Hilbert-type inequalities presented in the Introduction, we will need several other important inequalities. The first of them is the well-known Hardy inequality

$$\int_{\mathbb{R}_+} \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}_+} f^p(x)dx, \tag{5}$$

which holds for $p > 1$ and for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$, provided that $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$. Its discrete version asserts that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \tag{6}$$

where $p > 1$ and $a = (a_n)_{n\in\mathbb{N}}$ is a non-negative sequence such that $0 < \|a\|_{l^p} < \infty$. It should be noticed here that the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible in both inequalities. For comprehensive accounts on Hardy inequality including history, different proofs, refinements, and diverse applications, we refer to recent monograph [21] and references therein.

Observe that the Hardy inequality includes arithmetic mean in integral and discrete case. We shall also be occupied with the corresponding inequalities including a geometric mean. The integral version of such inequality is known as the Knopp inequality, i.e.,

$$\int_{\mathbb{R}_+} \exp\left(\frac{1}{x} \int_0^x \log f(t)dt \right) dx < e \int_{\mathbb{R}_+} f(x)dx, \tag{7}$$

while its discrete version is known as the Carleman inequality:

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k \right)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n. \tag{8}$$

The constant e appearing in both inequalities is the best possible (see [23]).

In 2005, Yang [31] derived the corresponding inequalities equipped with a generalized harmonic mean. Namely, integral version asserts that

$$\int_{\mathbb{R}_+} \left(\frac{x}{\int_0^x f^{-r}(t)dt} \right)^{\frac{1}{r}} dx < (1+r)^{\frac{1}{r}} \int_{\mathbb{R}_+} f(x)dx \tag{9}$$

holds for $r > 0$, while its discrete analogue holds for $0 < r \le 1$:

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sum_{k=1}^{n} a_k^{-r}} \right)^{\frac{1}{r}} < (1+r)^{\frac{1}{r}} \sum_{n=1}^{\infty} a_n. \tag{10}$$

Moreover, Yang also proved that inequalities (9) and (10) include the best constant $(1+r)^{\frac{1}{r}}$. In accordance to [31], inequalities (9) and (10) will be referred to as the integral and discrete Hardy–Carleman inequality.

For the reader's convenience, we define integral arithmetic, geometric, and harmonic mean operators $\mathscr{A}, \mathscr{G}, \mathscr{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ by

$$(\mathscr{A} f)(x) = \frac{1}{x}\int_0^x f(t)dt,$$

$$(\mathscr{G} f)(x) = \exp\left(\frac{1}{x}\int_0^x \log f(t)dt\right),$$

$$(\mathscr{H} f)(x) = \frac{x}{\int_0^x f^{-1}(t)dt}.$$

Obviously, the above operators are well-defined since inequalities (5), (7), and (9) may, respectively, be rewritten as

$$\|\mathscr{A} f\|_{L^p(\mathbb{R}_+)} < \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}_+)}, \tag{11}$$

$$\|\mathscr{G} f\|_{L^p(\mathbb{R}_+)} < e^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_+)}, \tag{12}$$

$$\|\mathscr{H} f\|_{L^p(\mathbb{R}_+)} < \left(1+\frac{1}{p}\right)\|f\|_{L^p(\mathbb{R}_+)}. \tag{13}$$

Moreover, since these inequalities include the best constants on their right-hand sides, we are able to compute norms of the corresponding integral operators. Namely, since $\|\mathscr{A}\| = \sup_{f\neq 0}\frac{\|\mathscr{A} f\|_{L^p(\mathbb{R}_+)}}{\|f\|_{L^p(\mathbb{R}_+)}}$, it follows that $\|\mathscr{A}\| = \frac{p}{p-1}$, and similarly $\|\mathscr{G}\| = e^{\frac{1}{p}}$, $\|\mathscr{H}\| = 1+\frac{1}{p}$.

Discrete versions of means operators $\mathscr{A}, \mathscr{G}, \mathscr{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$, i.e., the operators $\overline{\mathscr{A}}, \overline{\mathscr{G}}, \overline{\mathscr{H}} : l^p \to l^p$ are defined by

$$(\overline{\mathscr{A}} a)_n = \frac{\sum_{k=1}^n a_k}{n},$$

$$(\overline{\mathscr{G}} a)_n = \left(\prod_{k=1}^n a_k\right)^{\frac{1}{n}},$$

$$(\overline{\mathscr{H}} a)_n = \frac{n}{\sum_{k=1}^n a_k^{-1}}.$$

With this notation, discrete inequalities (6), (8), and (10), respectively, read

$$\|\overline{\mathscr{A}} a\|_{l^p} < \frac{p}{p-1}\|a\|_{l^p}, \tag{14}$$

$$\|\overline{\mathscr{G}} a\|_{l^p} < e^{\frac{1}{p}}\|a\|_{l^p}, \tag{15}$$

$$\|\overline{\mathscr{H}} a\|_{l^p} < \left(1+\frac{1}{p}\right)\|a\|_{l^p}. \tag{16}$$

Clearly, due to the best constants, above inequalities provide norms of the corresponding operators, that is, $\|\overline{\mathscr{A}}\| = \frac{p}{p-1}$, $\|\overline{\mathscr{G}}\| = e^{\frac{1}{p}}$, and $\|\overline{\mathscr{H}}\| = 1 + \frac{1}{p}$.

3 Hilbert-Type Inequalities Involving Means Operators

In this section we deal with two-dimensional Hilbert-type inequalities, in both integral and discrete case, involving arithmetic, geometric, harmonic, as well as some related operators. It should be noticed here that the inequalities appearing in this section refer to non-negative conjugate parameters p and q, i.e., to parameters such that $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, and $q > 1$. A pair of non-negative conjugate parameters will be denoted in this way throughout the whole paper.

We start this overview with some particular results involving arithmetic mean operators $\mathscr{A}$ and $\overline{\mathscr{A}}$.

3.1 Some Particular Results

In 2010, based on the Hardy integral inequality, Das and Sahoo [12] obtained the following pair of Hilbert-type inequalities involving the arithmetic mean operator $\mathscr{A}$.

Theorem 1 ([12]). *If r, s, λ are positive real parameters such that $\lambda = r + s$, then the inequalities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{x^{r-\frac{1}{q}} y^{s-\frac{1}{p}}}{(x+y)^\lambda} (\mathscr{A} f)(x)(\mathscr{A} g)(y)\,dx\,dy < pqB(r,s)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{17}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} \frac{x^{r-\frac{1}{q}}}{(x+y)^\lambda}(\mathscr{A} f)(x)\,dx\right)^p dy\right]^{\frac{1}{p}} < qB(r,s)\|f\|_{L^p(\mathbb{R}_+)} \tag{18}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ such that $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$. In addition, the constants $pqB(r,s)$ and $qB(r,s)$ are the best possible in the corresponding inequalities.

It should be noticed here that some particular cases of inequality (17) were studied in [28], few years earlier. Furthermore, with the assumption $\lambda > 2$, Das and Sahoo also proved a discrete version of Theorem 1.

Theorem 2 ([12]). *Let $r, s > 0$ and $\lambda > 2$ be real parameters such that $\lambda = r + s$. Then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^{r-\frac{1}{q}} n^{s-\frac{1}{p}}}{(m+n)^{\lambda}} (\overline{\mathscr{A}}a)_m (\overline{\mathscr{A}}b)_n < pqB(r,s)\|a\|_{l^p}\|b\|_{l^q} \tag{19}$$

and

$$\left[\sum_{n=1}^{\infty} n^{ps-1}\left(\sum_{m=1}^{\infty} \frac{m^{r-\frac{1}{q}}}{(m+n)^{\lambda}} (\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < qB(r,s)\|a\|_{l^p} \tag{20}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ satisfying $0 < \|a\|_{l^p} < \infty$ and $0 < \|b\|_{l^q} < \infty$. In addition, the constants $pqB(r, s)$ and $qB(r, s)$ are the best possible in the corresponding inequalities.

Observe also that reference [13] provides the corresponding result for the kernel $1/\max\{x^{\lambda}, y^{\lambda}\}$, with the best possible constant. Moreover, Adiyasuren and Batbold [3] also obtained some related inequalities:

Theorem 3 ([3]). *Let α and β be such that $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \le 1$, and let $\lambda, s, r > 0$ with $s + r = \lambda$. Then the inequalities*

$$\begin{aligned} &\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{x^{r-\frac{1}{q}} y^{s-\frac{1}{p}}}{\max\{x^{\lambda}, y^{\lambda}\}} (\mathscr{A} f)^{\alpha}(x)(\mathscr{A} g)^{\beta}(y)\,dx\,dy \\ &\quad < \frac{\lambda}{rs}\left(\frac{\alpha p}{\alpha p - 1}\right)^{\alpha}\left(\frac{\beta q}{\beta q - 1}\right)^{\beta} \|f^{\alpha}\|_{L^p(\mathbb{R}_+)}\|g^{\beta}\|_{L^q(\mathbb{R}_+)} \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} \frac{x^{r-\frac{1}{q}}}{\max\{x^{\lambda}, y^{\lambda}\}} (\mathscr{A} f)^{\alpha}(x)\,dx\right)^p dy\right]^{\frac{1}{p}} \\ &\quad < \left(\frac{\lambda}{rs}\right)\left(\frac{\alpha p}{\alpha p - 1}\right)^{\alpha} \|f^{\alpha}\|_{L^p(\mathbb{R}_+)} \end{aligned} \tag{22}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ such that $0 < \|f^{\alpha}\|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|g^{\beta}\|_{L^q(\mathbb{R}_+)} < \infty$. In addition, the constants appearing on the right-hand sides of (21) and (22) are the best possible.

Theorem 4 ([3]). *With the assumptions of Theorem 3, inequalities*

$$\begin{aligned} &\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{x^{r-\frac{1}{q}} y^{s-\frac{1}{p}}}{|x-y|^{\lambda}} (\mathscr{A} f)^{\alpha}(x)(\mathscr{A} g)^{\beta}(y)\,dx\,dy < (B(s, 1-\lambda) + B(r, 1-\lambda)) \\ &\quad \times \left(\frac{\alpha p}{\alpha p - 1}\right)^{\alpha}\left(\frac{\beta q}{\beta q - 1}\right)^{\beta} \|f^{\alpha}\|_{L^p(\mathbb{R}_+)}\|g^{\beta}\|_{L^q(\mathbb{R}_+)} \end{aligned} \tag{23}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+}\frac{x^{r-\frac{1}{q}}}{|x-y|^{\lambda}}(\mathscr{A}f)^{\alpha}(x)dx\right)^{p}dy\right]^{\frac{1}{p}}$$
$$< (B(s,1-\lambda)+B(r,1-\lambda))\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\|f^{\alpha}\|_{L^p(\mathbb{R}_+)} \tag{24}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f^{\alpha}\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g^{\beta}\|_{L^q(\mathbb{R}_+)} < \infty$. *Moreover, the constants appearing on the right-hand sides of* (23) *and* (24) *are the best possible.*

All inequalities in this subsection are simple consequences of Hilbert-type inequalities (1)–(4) and the Hardy inequality. For the proofs of the best possible constants, the reader is referred to the corresponding references.

3.2 A General Homogeneous Kernel

Observe that all results in the previous subsection have a homogeneity in common. Now, we give an extension of Theorems 1–4 to a general homogeneous case. Throughout the whole paper, we deal with the constant

$$c_{\lambda}(s) = \int_{\mathbb{R}_+} K_{\lambda}(1,t)t^{s-1}dt,$$

where $K_{\lambda} : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$ and s is a non-negative real parameter. Observe that $c_{\lambda}(s) = k_{\lambda}(1-s)$, where $k_{\lambda}(\cdot)$ is the constant appearing in relations (1)–(4).

It should be noticed here that Sulaiman (see [29, 30]) investigated some related results with a homogeneous kernel, without considering the problem of the best constants.

In already mentioned reference [3], Adiyasuren and Batbold derived a pair of Hilbert-type inequalities with the arithmetic mean operator $\mathscr{A}$, referring to a general homogeneous kernel.

Theorem 5 ([3]). *Let* α *and* β *be such that* $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leq 1$, *and let* $s + r = \lambda$, *where* $\lambda, s, r > 0$. *Further, let* $K_{\lambda} : \mathbb{R}_+^2 \to \mathbb{R}$ *be a non-negative homogeneous function of degree* $-\lambda$, *provided that*

$$0<c_{\lambda}(s) < \infty, 0<\int_{\mathbb{R}_+} K_{\lambda}(1,u)u^{s-\frac{1}{p}-\beta}du<\infty, 0<\int_{\mathbb{R}_+} K_{\lambda}(1,u)u^{r-\frac{1}{q}-\alpha}du<\infty.$$

Then the inequalities

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{A}f)^\alpha(x)(\mathscr{A}g)^\beta(y)dxdy$$
$$< \left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta c_\lambda(s)\|f^\alpha\|_{L^p(\mathbb{R}_+)}\|g^\beta\|_{L^q(\mathbb{R}_+)} \tag{25}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}(\mathscr{A}f)^\alpha(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< \left(\frac{\alpha p}{\alpha p-1}\right)^\alpha c_\lambda(s)\|f^\alpha\|_{L^p(\mathbb{R}_+)} \tag{26}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f^\alpha\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g^\beta\|_{L^q(\mathbb{R}_+)} < \infty$. *In addition, the constants* $\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta c_\lambda(s)$ *and* $\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha c_\lambda(s)$ *appearing in* (25) *and* (26) *are the best possible.*

Remark 1. Clearly, if $K_\lambda(x,y) = (x+y)^{-\lambda}$ and $\alpha = \beta = 1$, Theorem 5 reduces to Theorem 1. In a similar manner, Theorem 5 also represents an extension of Theorems 3 and 4.

Recently, Adiyasuren et al. [5] derived discrete analogues of relations (25) and (26), as well as the corresponding analogues with geometric and harmonic mean operators in both integral and discrete case. We first give the corresponding integral results including operators $\mathscr{G}$ and $\mathscr{H}$.

Theorem 6 ([5]). *Let* r, s, λ *be non-negative real parameters such that* $\lambda = r + s$. *Further, suppose* $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ *is a non-negative homogeneous function of degree* $-\lambda$ *such that* $0 < c_\lambda(s) < \infty$. *Then the inequalities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{G}f)(x)(\mathscr{G}g)(y)dxdy$$
$$< e\cdot c_\lambda(s)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{27}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}(\mathscr{G}f)(x)dx\right)^p dy\right]^{\frac{1}{p}} < e^{\frac{1}{p}}c_\lambda(s)\|f\|_{L^p(\mathbb{R}_+)} \tag{28}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$. *In addition, the constants* $e \cdot c_\lambda(s)$ *and* $e^{\frac{1}{p}} c_\lambda(s)$ *are the best possible in* (27) *and* (28).

Theorem 7 ([5]). *With the assumptions of Theorem 6, the inequalities*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x, y) x^{r-\frac{1}{q}} y^{s-\frac{1}{p}} (\mathscr{H} f)(x)(\mathscr{H} g)(y) dx dy$$
$$< \left(2 + \frac{1}{pq}\right) c_\lambda(s) \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)} \tag{29}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1} \left(\int_{\mathbb{R}_+} K_\lambda(x, y) x^{r-\frac{1}{q}} (\mathscr{H} f)(x) dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< \left(1 + \frac{1}{p}\right) c_\lambda(s) \|f\|_{L^p(\mathbb{R}_+)} \tag{30}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$. *In addition, the constants* $\left(2 + \frac{1}{pq}\right) c_\lambda(s)$ *and* $\left(1 + \frac{1}{p}\right) c_\lambda(s)$ *are the best possible in the corresponding inequalities.*

The methods of proving Theorems 5–7 are quite similar. For an illustration, we give the proof of Theorem 6.

Proof (Proof of Theorem 6). The starting point in the proof is inequality (1) with parameters $A_1 = \frac{1-r}{q}$, $A_2 = \frac{1-s}{p}$, and with functions f and g, respectively, replaced by $x^{r-\frac{1}{q}} (\mathscr{G} f)(x)$ and $y^{s-\frac{1}{p}} (\mathscr{G} g)(y)$, that is, the inequality

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x, y) x^{r-\frac{1}{q}} y^{s-\frac{1}{p}} (\mathscr{G} f)(x)(\mathscr{G} g)(y) dx dy$$
$$< k_\lambda(1 - s) \|\mathscr{G} f\|_{L^p(\mathbb{R}_+)} \|\mathscr{G} g\|_{L^q(\mathbb{R}_+)} = c_\lambda(s) \|\mathscr{G} f\|_{L^p(\mathbb{R}_+)} \|\mathscr{G} g\|_{L^q(\mathbb{R}_+)}.$$

Now, due to the Knopp inequality (12), it follows that $\|\mathscr{G} f\|_{L^p(\mathbb{R}_+)} < e^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathscr{G} g\|_{L^q(\mathbb{R}_+)} < e^{\frac{1}{q}} \|g\|_{L^q(\mathbb{R}_+)}$, which yields inequality (27). Similarly, inequality (28) follows from Hardy–Hilbert-type inequality (2) and the Knopp inequality.

In order to prove that inequalities (27) and (28) involve the best constants on their right-hand sides, we first suppose that there exists a positive constant C, smaller than $e \cdot c_\lambda(s)$, such that the inequality

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{G}f)(x)(\mathscr{G}g)(y)dxdy < C\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{31}$$

holds for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ with $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$.

Considering the above inequality with functions $\tilde{f}, \tilde{g} : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} 1, & 0 < x < 1 \\ e^{-\frac{1}{p}}x^{\frac{-\varepsilon-1}{p}}, & x \geq 1 \end{cases}, \quad \tilde{g}(y) = \begin{cases} 1, & 0 < y < 1 \\ e^{-\frac{1}{q}}y^{\frac{-\varepsilon-1}{q}}, & y \geq 1 \end{cases},$$

where $\varepsilon > 0$ is sufficiently small number, its right-hand side reduces to

$$C\|\tilde{f}\|_{L^p(\mathbb{R}_+)}\|\tilde{g}\|_{L^q(\mathbb{R}_+)} = \frac{C}{\varepsilon}\left(\varepsilon + \frac{1}{e}\right). \tag{32}$$

On the other hand, since

$$(\mathscr{G}\tilde{f})(x) = \begin{cases} 1, & 0 < x < 1 \\ e^{\frac{\varepsilon}{p}-\frac{\varepsilon}{xp}}x^{\frac{-\varepsilon-1}{p}}, & x \geq 1 \end{cases}$$

and

$$(\mathscr{G}\tilde{g})(y) = \begin{cases} 1, & 0 < y < 1 \\ e^{\frac{\varepsilon}{q}-\frac{\varepsilon}{yq}}y^{\frac{-\varepsilon-1}{q}}, & y \geq 1 \end{cases},$$

the well-known Fubini theorem and the change of variables $t = \frac{y}{x}$ imply the following series of relations:

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}\big(\mathscr{G}\tilde{f}\big)(x)\big(\mathscr{G}\tilde{g}\big)(y)dxdy \\
&\quad > \int_1^\infty\int_1^\infty K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}\big(\mathscr{G}\tilde{f}\big)(x)\big(\mathscr{G}\tilde{g}\big)(y)dxdy \\
&\quad = \int_1^\infty\int_1^\infty K_\lambda(x,y)x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}e^{\varepsilon-\frac{\varepsilon}{xp}-\frac{\varepsilon}{yq}}dxdy \\
&\quad \geq \int_1^\infty\int_1^\infty K_\lambda(x,y)x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}dxdy
\end{aligned}$$

$$= \int_1^\infty x^{-\varepsilon-1} \int_{\frac{1}{x}}^\infty K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt dx$$

$$= \frac{1}{\varepsilon} \int_1^\infty K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt + \int_1^\infty x^{-\varepsilon-1} \int_{\frac{1}{x}}^1 K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt dx$$

$$= \frac{1}{\varepsilon} \left(\int_1^\infty K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt + \int_0^1 K_\lambda(1,t) t^{s+\frac{\varepsilon}{p}-1} dt \right). \tag{33}$$

Now, multiplying both sides of inequality (31) by ε, relations (32) and (33) yield inequality

$$\int_1^\infty K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt + \int_0^1 K_\lambda(1,t) t^{s+\frac{\varepsilon}{p}-1} dt < C \left(\varepsilon + \frac{1}{e} \right).$$

Finally, when ε goes to 0, it follows that $e \cdot c_\lambda(s) \leq C$, which is in contrast to our hypothesis. Therefore, the constant $e \cdot c_\lambda(s)$ is the best possible in (27).

It remains to show that $e^{\frac{1}{p}} c_\lambda(s)$ is the best possible constant in (28). Similarly to above discussion, suppose that there exists a constant C' smaller than $e^{\frac{1}{p}} c_\lambda(s)$ such that inequality

$$\left[\int_{\mathbb{R}_+} y^{ps-1} \left(\int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} (\mathscr{G} f)(x) dx \right)^p dy \right]^{\frac{1}{p}} < C' \|f\|_{L^p(\mathbb{R}_+)}$$

holds for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$. Then, utilizing the well-known Hölder and the Knopp inequality, we have

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} y^{s-\frac{1}{p}} (\mathscr{G} f)(x) (\mathscr{G} g)(y) dx dy$$

$$= \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} y^{s-\frac{1}{p}} (\mathscr{G} f)(x) dx \right) (\mathscr{G} g)(y) dy$$

$$\leq \left[\int_{\mathbb{R}_+} y^{ps-1} \left(\int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} (\mathscr{G} f)(x) dx \right)^p dy \right]^{\frac{1}{p}} \|\mathscr{G} g\|_{L^q(\mathbb{R}_+)}$$

$$< C' e^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)},$$

which results that the constant $e \cdot c_\lambda(s)$ is not the best possible in (27), since $C' e^{\frac{1}{q}} < c_\lambda(s) e^{\frac{1}{p}} e^{\frac{1}{q}} = e \cdot c_\lambda(s)$. This contradiction completes the proof.

In order to prove Theorem 5, it is necessary to use Hardy integral inequality (5), while the proof of Theorem 7 is accompanied with Hardy–Carleman integral inequality (9). Of course, to establish the best constants in these theorems, it is necessary to find suitable functions to obtain contradiction, as in the proof of Theorem 6 (for more details, see [5]).

Paper [5] also provides discrete versions of Theorems 5–7, including discrete means operators $\overline{\mathscr{A}}$, $\overline{\mathscr{G}}$, and $\overline{\mathscr{H}}$. Discrete Hilbert-type inequalities are more complicated than the integral ones. Namely, in order to derive discrete forms of the corresponding integral inequalities, it is necessary to estimate certain sums by integrals, which requires some extra conditions regarding a kernel and the weight functions.

Theorem 8 ([5]). *Let r, s, λ be real parameters such that $0 < r, s \leq 1$ and $\lambda = r + s$, and let $K_\lambda : \mathbb{R}^2_+ \to \mathbb{R}$ be a non-negative homogeneous function of degree $-\lambda$, strictly decreasing in each argument, such that $0 < c_\lambda(s) < \infty$. Then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} n^{s-\frac{1}{p}} (\overline{\mathscr{A}} a)_m (\overline{\mathscr{A}} b)_n < pq c_\lambda(s) \|a\|_{l^p} \|b\|_{l^q} \tag{34}$$

and

$$\left[\sum_{n=1}^{\infty} n^{ps-1} \left(\sum_{m=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} (\overline{\mathscr{A}} a)_m\right)^p\right]^{\frac{1}{p}} < q c_\lambda(s) \|a\|_{l^p} \tag{35}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ satisfying $0 < \|a\|_{l^p} < \infty$ and $0 < \|b\|_{l^q} < \infty$. In addition, the constants $pqc_\lambda(s)$ and $qc_\lambda(s)$ are the best possible in the corresponding inequalities.

Theorem 9 ([5]). *With the assumptions as in Theorem 8, the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} n^{s-\frac{1}{p}} (\overline{\mathscr{G}} a)_m (\overline{\mathscr{G}} b)_n < e \cdot c_\lambda(s) \|a\|_{l^p} \|b\|_{l^q} \tag{36}$$

and

$$\left[\sum_{n=1}^{\infty} n^{ps-1} \left(\sum_{m=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} (\overline{\mathscr{G}} a)_m\right)^p\right]^{\frac{1}{p}} < e^{\frac{1}{p}} c_\lambda(s) \|a\|_{l^p} \tag{37}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$, $0 < \|a\|_{l^p} < \infty$, $0 < \|b\|_{l^q} < \infty$. In addition, the constants $e \cdot c_\lambda(s)$ and $e^{\frac{1}{p}} c_\lambda(s)$ are the best possible in the corresponding inequalities.

Theorem 10 ([5]). *With the assumptions of Theorem 8, the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} n^{s-\frac{1}{p}} (\overline{\mathscr{H}} a)_m (\overline{\mathscr{H}} b)_n < \left(2 + \frac{1}{pq}\right) c_\lambda(s) \|a\|_{l^p} \|b\|_{l^q} \tag{38}$$

and

$$\left[\sum_{n=1}^{\infty} n^{ps-1}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}(\overline{\mathscr{H}}a)_m\right)^p\right]^{\frac{1}{p}} < \left(1+\frac{1}{p}\right)c_\lambda(s)\|a\|_{l^p} \quad (39)$$

hold for all non-negative sequences $a=(a_m)_{m\in\mathbb{N}}$ *and* $b=(b_n)_{n\in\mathbb{N}}$*, provided that* $0<\|a\|_{l^p}<\infty$ *and* $0<\|b\|_{l^q}<\infty$*. In addition, the constants* $\left(2+\frac{1}{pq}\right)c_\lambda(s)$ *and* $\left(1+\frac{1}{p}\right)c_\lambda(s)$ *are the best possible in the corresponding inequalities.*

To illustrate the discrete case, we provide the proof of Theorem 8. For the proofs of the remaining theorems, the reader is referred to [5].

Proof (Proof of Theorem 8). Utilizing discrete Hilbert-type inequality (3) with sequences $m^{r-\frac{1}{q}}(\overline{\mathscr{A}}a)_m$, $n^{s-\frac{1}{p}}(\overline{\mathscr{A}}b)_n$, and with parameters $A_1=\frac{1-r}{q}$, $A_2=\frac{1-s}{p}$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < c_\lambda(s)\|\overline{\mathscr{A}}a\|_{l^p}\|\overline{\mathscr{A}}b\|_{l^q}.$$

Now, double use of discrete Hardy inequality (14) yields (34). Similarly, inequality (35) follows by virtue of discrete Hardy–Hilbert-type inequality (4).

Now, we prove that the constants appearing in (34) and (35) are the best possible. First, suppose that there exists a positive constant $0<K<pqc_\lambda(s)$ so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < K\|a\|_{l^p}\|b\|_{l^q} \quad (40)$$

holds for $0<\|a\|_{l^p}<\infty$ and $0<\|b\|_{l^q}<\infty$. Let $\tilde{L}$ and $\tilde{R}$, respectively, denote the left-hand side and the right-hand side of (40) equipped with the sequences

$$\tilde{a}_m=\begin{cases} m^{-\frac{1}{p}}, & m\le N\\ 0, & \text{otherwise}\end{cases} \quad \text{and} \quad \tilde{b}_n=\begin{cases} n^{-\frac{1}{q}}, & n\le N\\ 0, & \text{otherwise}\end{cases}, \quad (41)$$

where $N\in\mathbb{N}$ is fixed. Then, the right-hand side of (40) may be bounded from above by a natural logarithm function:

$$\begin{aligned}\tilde{R}=K\|\tilde{a}\|_{l^p}\|\tilde{b}\|_{l^q}&=K\left(\sum_{m=1}^{N}\frac{1}{m}\right)=K\left(1+\sum_{m=2}^{N}\frac{1}{m}\right)\\ &< K\left(1+\int_1^N\frac{dx}{x}\right)=K(1+\log N). \quad (42)\end{aligned}$$

Our next intention is to estimate the left-hand side of inequality (40) from below. More precisely, considering $\sum_{k=1}^{m} k^{-\frac{1}{p}}$ as the upper Darboux sum for the function $h(x) = x^{-\frac{1}{p}}$ on the segment $[1, m+1]$, we have

$$\sum_{k=1}^{m} k^{-\frac{1}{p}} > \int_1^{m+1} x^{\frac{1}{p}} dx > \int_1^{m} x^{-\frac{1}{p}} dx = q(m^{\frac{1}{q}} - 1),$$

and consequently,

$$(\overline{\mathscr{A}}\tilde{a})_m > \frac{q(m^{\frac{1}{q}} - 1)}{m} = qm^{-\frac{1}{p}}(1 - m^{-\frac{1}{q}}), \quad m \leq N,$$
$$(\overline{\mathscr{A}}\tilde{b})_n > \frac{p(n^{\frac{1}{p}} - 1)}{n} = pn^{-\frac{1}{q}}(1 - n^{-\frac{1}{p}}), \quad n \leq N.$$

Therefore, $\tilde{L}$ may be estimated as follows:

$$\tilde{L} > pq \sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m,n) m^{r-1} n^{s-1} (1 - m^{-\frac{1}{q}})(1 - n^{-\frac{1}{p}}).$$

Moreover, since $(1 - m^{-\frac{1}{q}})(1 - n^{-\frac{1}{p}}) > 1 - m^{-\frac{1}{q}} - n^{-\frac{1}{p}}$, the above relation implies inequality

$$\begin{aligned} \frac{\tilde{L}}{pq} > &\sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m,n) m^{r-1} n^{s-1} \\ &- \sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m,n) m^{r-1-\frac{1}{q}} n^{s-1} \\ &- \sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m,n) m^{r-1} n^{s-1-\frac{1}{p}}. \end{aligned} \tag{43}$$

The next goal is to establish suitable estimates for double sums on the right-hand side of inequality (43). The first double sum may be regarded as the upper Darboux sum for the function $K_\lambda(x,y) x^{r-1} y^{s-1}$ defined on square $[1, N+1] \times [1, N+1]$, since this two-variable function is strictly decreasing in each argument. Hence, utilizing suitable variable changes and the well-known Fubini theorem, we have

$$\begin{aligned} &\sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m,n) m^{r-1} n^{s-1} \\ &> \int_1^{N} \int_1^{N} K_\lambda(x,y) x^{r-1} y^{s-1} dx dy \end{aligned}$$

$$
\begin{aligned}
&= \int_1^N \frac{dx}{x} \int_{\frac{1}{x}}^{\frac{N}{x}} K_\lambda(1,t) t^{s-1} dt \\
&= \int_{\frac{1}{N}}^1 \left(\int_{\frac{1}{t}}^N \frac{dx}{x} \right) K_\lambda(1,t) t^{s-1} dt + \int_1^N \left(\int_1^{\frac{N}{t}} \frac{dx}{x} \right) K_\lambda(1,t) t^{s-1} dt \\
&= \log N \int_{\frac{1}{N}}^1 K_\lambda(1,t) t^{s-1} \left(1 + \frac{\log t}{\log N} \right) dt \\
&\quad + \log N \int_1^N K_\lambda(1,t) t^{s-1} \left(1 - \frac{\log t}{\log N} \right) dt. \qquad (44)
\end{aligned}
$$

The second sum on the right-hand side of (43) may be rewritten as

$$
\sum_{m=1}^N \sum_{n=1}^N K_\lambda(m,n) m^{r-1-\frac{1}{q}} n^{s-1} = \sum_{n=1}^N K_\lambda(1,n) n^{s-1} + \sum_{m=2}^N \sum_{n=1}^N K_\lambda(m,n) m^{r-1-\frac{1}{q}} n^{s-1},
$$

and both sums on the right-hand side of this relation may be regarded as the lower Darboux sums for the corresponding functions. More precisely, we have

$$
\sum_{n=1}^N K_\lambda(1,n) n^{s-1} < \int_0^N K_\lambda(1,t) t^{s-1} dt < \int_0^\infty K_\lambda(1,t) t^{s-1} dt = c_\lambda(s)
$$

and

$$
\begin{aligned}
\sum_{m=2}^N \sum_{n=1}^N K_\lambda(m,n) m^{r-1-\frac{1}{q}} n^{s-1} &< \int_1^N \int_0^N K_\lambda(x,y) x^{r-1-\frac{1}{q}} y^{s-1} dx dy \\
&= \int_1^N \frac{dx}{x^{1+\frac{1}{q}}} \int_0^{\frac{N}{x}} K_\lambda(1,t) t^{s-1} dt \\
&< \int_1^N \frac{dx}{x^{1+\frac{1}{q}}} \int_0^\infty K_\lambda(1,t) t^{s-1} dt \\
&= \left(q - \frac{q}{N^{\frac{1}{q}}} \right) c_\lambda(s),
\end{aligned}
$$

so that

$$
\sum_{m=1}^N \sum_{n=1}^N K_\lambda(m,n) m^{r-1-\frac{1}{q}} n^{s-1} < \left(1 + q - \frac{q}{N^{\frac{1}{q}}} \right) c_\lambda(s). \qquad (45)
$$

In a similar manner, it follows that

$$\sum_{m=1}^{N}\sum_{n=1}^{N} K_\lambda(m,n)m^{r-1}n^{s-1-\frac{1}{p}} < \left(1 + p - \frac{p}{N^{\frac{1}{p}}}\right) c_\lambda(s). \tag{46}$$

Now, relations (40), (42)–(46) yield inequality

$$\begin{aligned}\frac{K(1+\log N)}{pq} &> \log N \int_{\frac{1}{N}}^{1} K_\lambda(1,t)t^{s-1}\left(1+\frac{\log t}{\log N}\right)dt\\ &\quad + \log N \int_{1}^{N} K_\lambda(1,t)t^{s-1}\left(1-\frac{\log t}{\log N}\right)dt\\ &\quad - \left(2 + pq - \frac{p}{N^{\frac{1}{p}}} - \frac{q}{N^{\frac{1}{q}}}\right) c_\lambda(s).\end{aligned} \tag{47}$$

Dividing inequality (47) by $\log N$ and letting N to infinity, it follows that $\frac{K}{pq} \geq c_\lambda(s)$, which contradicts with the assumption that K is smaller than $pqc_\lambda(s)$. Therefore, the constant $pqc_\lambda(s)$ is the best possible in (34).

It remains to prove that $qc_\lambda(s)$ is the best constant in (35). For this reason, suppose that there exists a positive constant $0 < K' < qc_\lambda(s)$ such that inequality

$$\left[\sum_{n=1}^{\infty} n^{ps-1}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < K'\|a\|_{l^p}$$

holds for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$, provided that $0 < \|a\|_{l^p} < \infty$. Then, utilizing the Hölder and the Hardy inequality, we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n\\ &\quad= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}(\overline{\mathscr{A}}a)_m\right)(\overline{\mathscr{A}}b)_n\\ &\quad\leq \left[\sum_{n=1}^{\infty} n^{ps-1}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} \|\overline{\mathscr{A}}b\|_{l^q}\\ &\quad< K'p\|a\|_{l^p}\|b\|_{l^q},\end{aligned}$$

which is impossible since $K'p < pqc_\lambda(s)$ and $pqc_\lambda(s)$ is the best constant in (34).

As the application of Theorems 8–10, we consider the function $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$, defined by $K_\lambda(x,y) = \frac{\ln y - \ln x}{y-x}$. Evidently, it is homogeneous of degree -1 and strictly decreasing in both arguments, $c_\lambda(s)$ converges for all $s \in (0,1)$, and we have $c_\lambda(s) = \frac{\pi^2}{\sin^2 \pi s}$, (see [1, 5]). Now, Theorems 8–10 equipped with this kernel and parameters $r = \frac{1}{q}$, $s = \frac{1}{p}$ read as follows:

Corollary 1 ([5]). *The series of inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{A}} a)_m (\overline{\mathscr{A}} b)_n < \frac{pq\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p} \|b\|_{l^q},$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{G}} a)_m (\overline{\mathscr{G}} b)_n < \frac{e\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p} \|b\|_{l^q},$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{H}} a)_m (\overline{\mathscr{H}} b)_n < \left(2 + \frac{1}{pq}\right) \frac{\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p} \|b\|_{l^q},$$

and

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{A}} a)_m\right)^p\right]^{\frac{1}{p}} < \frac{q\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p},$$

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{G}} a)_m\right)^p\right]^{\frac{1}{p}} < \frac{e^{\frac{1}{p}}\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p},$$

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{H}} a)_m\right)^p\right]^{\frac{1}{p}} < \left(1 + \frac{1}{p}\right) \frac{\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$, provided that $0 < \|a\|_{l^p} < \infty$ and $0 < \|b\|_{l^q} < \infty$. Moreover, above inequalities include the best constants on their right-hand sides.

3.3 Inequalities with Some Related Integral Operators

We continue our discussion with a few related Hilbert-type inequalities involving some other integral operators. In 2012, Adiyasuren and Batbold [2] gave the following analogue of Theorem 5, where the arithmetic mean operator $\mathscr{A}$ is replaced by integral operator $(\mathscr{A}_1 f)(x) = \frac{1}{x}\int_0^x (x-t) f(t)dt$.

Theorem 11 ([2]). *Let $s+r = \lambda$, $\lambda, s, r > 0$, and $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative homogeneous function of degree $-\lambda$, provided that*

$$0<c_\lambda(s)<\infty, 0<\int_{\mathbb{R}_+} K_\lambda(1,u) u^{s-\frac{1}{p}-1} du<\infty, 0<\int_{\mathbb{R}_+} K_\lambda(1,u) u^{r-\frac{1}{q}-1} du<\infty.$$

Then the inequalities

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}(\mathscr{A}_1 f)(x)(\mathscr{A}_1 g)(y)\,dx\,dy$$
$$< \frac{(pq)^2}{1+2pq}c_\lambda(s)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{48}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}-1}(\mathscr{A}_1 f)(x)\,dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< \frac{p^2}{1+p}c_\lambda(s)\|f\|_{L^p(\mathbb{R}_+)}, \tag{49}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$. *The constants* $\frac{(pq)^2}{1+2pq}c_\lambda(s)$ *and* $\frac{p^2}{1+p}c_\lambda(s)$ *are the best possible in* (48) *and* (49).

It should be noticed here that Theorem 11 follows from inequalities (1), (2), and general version of the Hardy integral inequality (5). On the other hand, Liu and Yang [22] obtained a pair of inequalities, based on the so-called dual Hardy inequality (see relation (83), for more details see also [21]). The following result deals with an integral operator $\mathscr{A}_\lambda^*$ defined by $(\mathscr{A}_\lambda^* f)(x) = \frac{1}{x}\int_x^\infty \frac{f(t)}{t^\lambda}dt$.

Theorem 12 ([22]). *Let* $\lambda_1 + \lambda_2 = \lambda < 2$, *and let* $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ *be a non-negative homogeneous function of degree* $-\lambda$, *provided that* $0 < c_\lambda(\lambda_1) < \infty$ *for any* $\lambda_1 \in (\lambda - 1, 1)$. *Then, for* $\varphi(x) = x^{p(2-\lambda-\lambda_1)-1}$, $\psi(y) = y^{q(1-\lambda_2)-1}$, $0 < \|f\|_{L^p(\mathbb{R}_+,\varphi)} < \infty$, *and* $0 < \|g\|_{L^q(\mathbb{R}_+,\psi)} < \infty$, *the inequalities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)xy(\mathscr{A}_\lambda^* f)(x)(\mathscr{A}_\lambda^* g)(y)\,dx\,dy$$
$$< \frac{c_\lambda(\lambda_1)}{(1-\lambda_1)(1-\lambda_2)}\|f\|_{L^p(\mathbb{R}_+,\varphi)}\|g\|_{L^q(\mathbb{R}_+,\psi)} \tag{50}$$

and

$$\left[\int_{\mathbb{R}_+}\psi^{1-p}(y)\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x(\mathscr{A}_\lambda^* f)(x)\,dx\right)^p dy\right]^{\frac{1}{p}} < \frac{c_\lambda(\lambda_1)}{1-\lambda_1}\|f\|_{L^p(\mathbb{R}_+,\varphi)} \tag{51}$$

hold and the constants $\frac{c_\lambda(\lambda_1)}{(1-\lambda_1)(1-\lambda_2)}$ *and* $\frac{c_\lambda(\lambda_1)}{1-\lambda_1}$ *are the best possible.*

Observe also that Yang and Xie proved discrete versions of inequalities (50) and (51) in [34].

3.4 Applications

In Sect. 2 we have defined a class of operators representing arithmetic, geometric, and harmonic mean in both integral and discrete case. Their norms were determined as a simple consequences of the corresponding inequalities. With the same reasoning, Hardy–Hilbert-type inequalities established in this section enable us to define another class of integral and discrete operators and to determine their norms.

Regarding notations from this section and Sect. 2, we define integral operators $\mathbf{A}, \mathbf{G}, \mathbf{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ by

$$(\mathbf{A}f)(y) = y^{s-\frac{1}{p}} \int_0^\infty K_\lambda(x, y) x^{r-\frac{1}{q}} (\mathscr{A} f)(x) dx,$$

$$(\mathbf{G}f)(y) = y^{s-\frac{1}{p}} \int_0^\infty K_\lambda(x, y) x^{r-\frac{1}{q}} (\mathscr{G} f)(x) dx,$$

$$(\mathbf{H}f)(y) = y^{s-\frac{1}{p}} \int_0^\infty K_\lambda(x, y) x^{r-\frac{1}{q}} (\mathscr{H} f)(x) dx.$$

Due to inequalities (26), (28), and (30), the above operators are well-defined. Moreover, since the corresponding inequalities include the best constants, it follows that $\|\mathbf{A}\| = qc_\lambda(s)$, $\|\mathbf{G}\| = e^{\frac{1}{p}} c_\lambda(s)$, and $\|\mathbf{H}\| = \big(1 + \frac{1}{p}\big)c_\lambda(s)$.

Similarly to integral case, we also define discrete operators $\overline{\mathbf{A}}, \overline{\mathbf{G}}, \overline{\mathbf{H}} : l^p \to l^p$ by

$$(\overline{\mathbf{A}}a)_n = n^{s-\frac{1}{p}} \sum_{m=1}^\infty K_\lambda(m, n) m^{r-\frac{1}{q}} (\overline{\mathscr{A}} a)_m,$$

$$(\overline{\mathbf{G}}a)_n = n^{s-\frac{1}{p}} \sum_{m=1}^\infty K_\lambda(m, n) m^{r-\frac{1}{q}} (\overline{\mathscr{G}} a)_m,$$

$$(\overline{\mathbf{H}}a)_n = n^{s-\frac{1}{p}} \sum_{m=1}^\infty K_\lambda(m, n) m^{r-\frac{1}{q}} (\overline{\mathscr{H}} a)_m.$$

Due to inequalities (35), (37), and (39), these operators are well-defined. Moreover, by virtue of the best constants, it follows that $\|\overline{\mathbf{A}}\| = qc_\lambda(s)$, $\|\overline{\mathbf{G}}\| = e^{\frac{1}{p}} c_\lambda(s)$, and $\|\overline{\mathbf{H}}\| = \big(1 + \frac{1}{p}\big)c_\lambda(s)$.

4 Half-Discrete Versions

Nowadays, considerable attention is given to the so-called half-discrete Hilbert-type inequalities, that is, to inequalities which include both integral and sum. Recently, Krnić et al. [20] provided a unified treatment of half-discrete Hilbert-type inequalities with a homogeneous kernel and in the setting with non-conjugate exponents. In this article we only refer to a conjugate version, since in this case one may obtain the best constants.

More precisely, if $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, $\lambda > 0$, Krnić et al. [20] have showed that the following triple of half-discrete Hilbert-type inequalities

$$\sum_{n=1}^{\infty} a_n \int_{\mathbb{R}_+} K_\lambda(x,n) f(x) dx = \int_{\mathbb{R}_+} f(x) \left(\sum_{n=1}^{\infty} K_\lambda(x,n) a_n \right) dx$$
$$< L \left[\int_{\mathbb{R}_+} x^{1-\lambda+p(\alpha_1-\alpha_2)} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{1-\lambda+q(\alpha_2-\alpha_1)} a_n^q \right]^{\frac{1}{q}}, \quad (52)$$

$$\left[\sum_{n=1}^{\infty} n^{(p-1)(\lambda-1)+p(\alpha_1-\alpha_2)} \left(\int_{\mathbb{R}_+} K_\lambda(x,n) f(x) dx \right)^p \right]^{\frac{1}{p}}$$
$$< L \left[\int_{\mathbb{R}_+} x^{1-\lambda+p(\alpha_1-\alpha_2)} f^p(x) dx \right]^{\frac{1}{p}}, \quad (53)$$

and

$$\left[\int_{\mathbb{R}_+} x^{(q-1)(\lambda-1)+q(\alpha_2-\alpha_1)} \left(\sum_{n=1}^{\infty} K_\lambda(x,n) a_n \right)^q dx \right]^{\frac{1}{q}}$$
$$< L \left[\sum_{n=1}^{\infty} n^{1-\lambda+q(\alpha_2-\alpha_1)} a_n^q \right]^{\frac{1}{q}}, \quad (54)$$

where $L = k_\lambda^{\frac{1}{p}}(p\alpha_2) k_\lambda^{\frac{1}{q}}(2-s-q\alpha_1)$, $k_\lambda(\alpha) = \int_{\mathbb{R}_+} K_\lambda(1,t) t^{-\alpha} dt$, and α_1, α_2 are real parameters such that the function $K(x,y) y^{-q'\alpha_2}$ is decreasing on $\mathbb{R}_+$ for any $x \in \mathbb{R}_+$, holds for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$. Clearly, in the above inequalities all integrals and sums are assumed to be convergent, and the function and the sequence are not equal to zero. For some related half-discrete Hilbert-type inequalities, regarding some particular classes of kernels and weight functions, the reader is referred to the following references: [15, 26, 32, 33].

Based on the above half-discrete inequalities, Adiyasuren et al. [8] derived half-discrete versions of inequalities from Sect. 3. Clearly, the following set of inequalities include both integral and discrete mean operators.

Theorem 13 ([8]). *Let $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative homogeneous function of degree $-\lambda, \lambda > 0$, and let α_1 and α_2 be real parameters fulfilling condition $p\alpha_2 + q\alpha_1 = 2 - \lambda$. If the function $K_\lambda(x, y)y^{-p\alpha_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, then the inequalities*

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{A}} a)_n \int_{\mathbb{R}_+} K_\lambda(x,n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{A} f)(x) dx \\
&\quad = \int_{\mathbb{R}_+} x^{\frac{1-pq\alpha_1}{p}} (\mathscr{A} f)(x) \left(\sum_{n=1}^{\infty} K_\lambda(x,n) n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{A}} a)_n \right) dx \\
&\quad < c_\lambda(1 - p\alpha_2) pq \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},
\end{aligned} \tag{55}$$

$$\left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pq\alpha_2}{q}} \int_{\mathbb{R}_+} K_\lambda(x,n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{A} f)(x) dx \right)^p \right]^{\frac{1}{p}} < c_\lambda(1-p\alpha_2) q \|f\|_{L^p(\mathbb{R}_+)}, \tag{56}$$

and

$$\left[\int_{\mathbb{R}_+} \left(x^{\frac{1-pq\alpha_1}{p}} \sum_{n=1}^{\infty} K_\lambda(x,n) n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{A}} a)_n \right)^q dx \right]^{\frac{1}{q}} < c_\lambda(1 - p\alpha_2) p \|a\|_{l^q} \tag{57}$$

hold for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$, provided $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|a\|_{l^q} < \infty$. In addition, the constants $c_\lambda(1 - p\alpha_2)pq$, $c_\lambda(1 - p\alpha_2)q$, and $c_\lambda(1 - p\alpha_2)p$ are the best possible in the corresponding inequalities.

Theorem 14 ([8]). *Under the same assumptions as in Theorem 13, inequalities*

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{G}} a)_n \int_{\mathbb{R}_+} K_\lambda(x,n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{G} f)(x) dx \\
&\quad = \int_{\mathbb{R}_+} x^{\frac{1-pq\alpha_1}{p}} (\mathscr{G} f)(x) \left(\sum_{n=1}^{\infty} K_\lambda(x,n) n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{G}} a)_n \right) dx \\
&\quad < c_\lambda(1 - p\alpha_2) e \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},
\end{aligned} \tag{58}$$

$$\left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pq\alpha_2}{q}} \int_{\mathbb{R}_+} K_\lambda(x,n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{G} f)(x) dx \right)^p \right]^{\frac{1}{p}} < c_\lambda(1-p\alpha_2) e^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}_+)}, \tag{59}$$

and

$$\left[\int_{\mathbb{R}_+}\left(x^{\frac{1-pq\alpha_1}{p}}\sum_{n=1}^{\infty}K_\lambda(x,n)n^{\frac{1-pq\alpha_2}{q}}(\overline{\mathscr{G}}a)_n\right)^q dx\right]^{\frac{1}{q}} < c_\lambda(1-p\alpha_2)e^{\frac{1}{q}}\|a\|_{l^q} \quad (60)$$

hold and the constants appearing on their right-hand sides are the best possible.

Theorem 15 ([8]). *With the assumptions of Theorem 13, inequalities*

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{\frac{1-pq\alpha_2}{q}}(\overline{\mathscr{H}}a)_n\int_{\mathbb{R}_+}K_\lambda(x,n)x^{\frac{1-pq\alpha_1}{p}}(\mathscr{H}f)(x)dx\\ &\quad=\int_{\mathbb{R}_+}x^{\frac{1-pq\alpha_1}{p}}(\mathscr{H}f)(x)\left(\sum_{n=1}^{\infty}K_\lambda(x,n)n^{\frac{1-pq\alpha_2}{q}}(\overline{\mathscr{H}}a)_n\right)dx\\ &\quad< c_\lambda(1-p\alpha_2)\left(2+\frac{1}{pq}\right)\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q}, \end{aligned} \quad (61)$$

$$\begin{aligned}&\left[\sum_{n=1}^{\infty}\left(n^{\frac{1-pq\alpha_2}{q}}\int_{\mathbb{R}_+}K_\lambda(x,n)x^{\frac{1-pq\alpha_1}{p}}(\mathscr{H}f)(x)dx\right)^p\right]^{\frac{1}{p}}\\ &\quad< c_\lambda(1-p\alpha_2)\big(1+\frac{1}{p}\big)\|f\|_{L^p(\mathbb{R}_+)}, \end{aligned} \quad (62)$$

and

$$\left[\int_{\mathbb{R}_+}\left(x^{\frac{1-pq\alpha_1}{p}}\sum_{n=1}^{\infty}K_\lambda(x,n)n^{\frac{1-pq\alpha_2}{q}}(\overline{\mathscr{H}}a)_n\right)^q dx\right]^{\frac{1}{q}} < c_\lambda(1-p\alpha_2)\big(1+\frac{1}{q}\big)\|a\|_{l^q} \quad (63)$$

hold and the constants appearing on their right-hand sides are the best possible.

The idea of proving Theorems 13–15 is quite similar to theorems from Sect. 3, except that we utilize half-discrete inequalities (52)–(54) instead of integral and discrete Hilbert-type inequalities. Moreover, to obtain the best constants, we simultaneously plug the appropriate function and the sequence in the corresponding inequality. For detailed proofs of these theorems, the reader is referred to [8].

Remark 2. Similarly to Sect. 3.4, by virtue of Hardy–Hilbert-type inequalities from Theorems 13–15, one can define certain half-discrete operators and determine their norms.

Namely, with the assumptions of Theorem 13, it follows from (56) and (57), that a pair of half-discrete arithmetic operators $\mathbf{A_1} : L^p(\mathbb{R}_+) \to l^p$ and $\mathbf{A_2} : l^q \to L^q(\mathbb{R}_+)$,

$$(\mathbf{A_1} f)_n = n^{\frac{1-pq\alpha_2}{q}} \int_{\mathbb{R}_+} K(x,n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{A} f)(x) dx,$$

$$(\mathbf{A_2} a)(x) = x^{\frac{1-pq\alpha_1}{p}} \sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{A}} a)_n,$$

is well-defined. Moreover, inequalities (56) and (57) may be rewritten as $\|\mathbf{A_1} f\|_{l^p} < c_\lambda(1-p\alpha_2)q\|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathbf{A_2} a\|_{L^q(\mathbb{R}_+)} < c_\lambda(1-p\alpha_2)p\|a\|_{l^q}$. Due to the best constants, it follows that $\|\mathbf{A_1}\| = c_\lambda(1-p\alpha_2)q$ and $\|\mathbf{A_2}\| = c_\lambda(1-p\alpha_2)p$.

In the same way, Theorems 14 and 15 are utilized to define the corresponding half-discrete geometric and harmonic operators. For more details, the reader is referred to [8].

5 Extension to a Multidimensional Case

The main goal of this section is to present extensions of Theorems 5–7 to a multidimensional case. Such results are consequences of multidimensional Hilbert-type inequalities.

In 2005, Brnetić et al. [10] (see also [11]) provided a unified treatment of multidimensional Hilbert-type inequalities with non-conjugate exponents, with a basic result including a general non-negative measurable kernel and weight functions. Moreover, Perić and Vuković [25] studied the latter inequalities for the case of a homogeneous kernel. For some related multidimensional Hilbert-type inequalities, regarding some particular classes of kernels and weight functions, the reader is referred to the following references: [19, 27].

Before we state the basic result, we need some conventions. Recall that the function $K_\lambda : \mathbb{R}^n_+ \to \mathbb{R}$ is said to be homogeneous of degree $-\lambda$, $\lambda > 0$, if $K(t\mathbf{x}) = t^{-\lambda} K(\mathbf{x})$ for all $t > 0$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$. If $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}^{n-1}_+} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^{n} u_j^{a_j} \hat{d}^i \mathbf{u}, \quad i = 1, 2, \ldots, n, \tag{64}$$

where $\hat{\mathbf{u}}^i = (u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n)$, $\hat{d}^i \mathbf{u} = du_1 \ldots du_{i-1} du_{i+1} \ldots du_n$, and provided that the above integral converges. Further, in the sequel $d\mathbf{u}$ is the abbreviation for $du_1 du_2 \ldots du_n$.

Although the general Hilbert-type inequalities are derived in the setting with non-conjugate exponents, we consider here only the conjugate case. More precisely, in this section $\{p_1, p_2, \ldots, p_n\}$ represents the set of non-negative conjugate parameters, that is, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \ldots, n$. The parameters p_i' are defined as associated conjugates, that is, $\frac{1}{p_i} + \frac{1}{p_i'} = 1$.

Here we refer to a pair of inequalities derived in [25], regarding a homogeneous kernel K_λ and some particular parameters. More precisely, the authors obtained inequalities

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mathbf{x} \le k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n} \| f_i \|_{L^{p_i}(\mathbb{R}_+, x_i^{-1-p_i \tilde{A}_i})} \tag{65}$$

and

$$\left[\int_{\mathbb{R}_+} x_n{}^{(p_n'-1)(1+p_n \tilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K_\lambda(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) \hat{d}^n \mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$
$$\le k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n-1} \| f_i \|_{L^{p_i}(\mathbb{R}_+, x_i^{-1-p_i \tilde{A}_i})}, \tag{66}$$

where the parameters $\tilde{A}_i$, $i = 1, \ldots, n$, fulfill conditions

$$k_1(\tilde{\mathbf{A}}) < \infty \text{ for } \tilde{A}_2, \ldots, \tilde{A}_n > -1, \sum_{i=2}^{n} \tilde{A}_i < \lambda - n + 1, \text{and} \sum_{i=1}^{n} \tilde{A}_i = \lambda - n. \tag{67}$$

In addition, the constant $k_1(\tilde{\mathbf{A}})$, appearing in (65) and (66) is the best possible in both inequalities.

Utilizing the above two inequalities, Krnić [16] obtained the following multidimensional version of Theorem 5.

Theorem 16 ([16]). *Let $K_\lambda : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-\lambda$, $\lambda > 0$, such that for every $i = 2, 3, \ldots, n$,*

$$K_\lambda(1, t_2, \ldots, t_i, \ldots, t_n) \le C_K K_\lambda(1, t_2, \ldots, 0, \ldots, t_n),\ 0 \le t_i \le 1,$$

where C_K is a positive constant. Further, let $1/p_i < \mu_i \le 1$, $i = 1, 2, \ldots, n$, and let the parameters $\tilde{A}_i$, $i = 1, 2, \ldots, n$, fulfill conditions as in (67). If $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are non-negative measurable functions, then

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^{n} x_i^{\frac{1}{p_i} + \tilde{A}_i} (\mathscr{A} f_i)^{\mu_i}(x_i) d\mathbf{x} \le \overline{m}_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\mu}) \prod_{i=1}^{n} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+)}, \tag{68}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\tilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K_\lambda(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\frac{1}{p_i}+\tilde{A}_i} (\mathscr{A} f_i)^{\mu_i} (x_i) \hat{d}^n \mathbf{x}\right)^{p_n'} dx_n\right]^{\frac{1}{p_n'}}$$

$$\leq \overline{m}_{n-1}^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\mu}) \prod_{i=1}^{n-1} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+)}, \tag{69}$$

where the constants

$$\overline{m}_n^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\mu}) = k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n} \left(\frac{p_i \mu_i}{p_i \mu_i - 1}\right)^{\mu_i},$$

and

$$\overline{m}_{n-1}^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\mu}) = k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n-1} \left(\frac{p_i \mu_i}{p_i \mu_i - 1}\right)^{\mu_i}$$

are the best possible in the corresponding inequalities.

Remark 3. Considering inequality (68) with the kernel $K_\lambda(\mathbf{x}) = (x_1 + x_2 + \cdots + x_n)^{-\lambda}$, $\lambda > 0$, and the parameters $\tilde{A}_i = s_i - 1$, $i = 1, 2, \ldots, n$, the constant on its right-hand side reduces to $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(s_i) \prod_{i=1}^{n} \left(\frac{p_i \mu_i}{p_i \mu_i - 1}\right)^{\mu_i}$, where Γ stands for the usual Gamma function. This particular result was obtained by Adiyasuren and Batbold [4], in 2012.

Recently, Adiyasuren et al. [6] gave analogues of inequalities (68) and (69), with the weighted geometric and harmonic mean operators, instead of the arithmetic operator. The weighted geometric mean operator $\mathscr{G}_\alpha$, $\alpha > 0$ is defined by

$$(\mathscr{G}_\alpha f)(x) = \exp\left[\frac{\alpha}{x^\alpha} \int_0^x t^{\alpha-1} \log f(t) dt\right], \tag{70}$$

while the weighted harmonic operator $\mathscr{H}_\alpha$, $\alpha > 0$ is given by

$$(\mathscr{H}_\alpha f)(x) = \frac{x^\alpha}{\int_0^x t^{\alpha-1} f^{-1}(t) dt}. \tag{71}$$

Theorem 17 ([6]). *Let $K_\lambda : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-\lambda$, $\lambda > 0$. Further, let ν_i, μ_i, and $\alpha > 0$ be real parameters such that $\tilde{A}_i \leq \nu_i \leq \frac{\alpha}{p_i} + \tilde{A}_i$, $i = 1, 2, \ldots, n$, where the parameters $\tilde{A}_i$, $i = 1, 2, \ldots, n$, fulfill conditions as in (67). If $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \ldots, n$ are non-negative measurable functions, then the following two inequalities hold*

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} \, (\mathscr{G}_\alpha f_i)^{\mu_i} \, (x_i) d\mathbf{x} \leq m_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}) \prod_{i=1}^{n} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))} \tag{72}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\tilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K_\lambda(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} \, (\mathscr{G}_\alpha f_i)^{\mu_i} \, (x_i) \hat{d}^n \mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$
$$\leq m_{n-1}^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}) \prod_{i=1}^{n-1} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))}, \tag{73}$$

where $\varphi_i(x_i) = x_i^{p_i\nu_i - p_i\tilde{A}_i - 1}$*, and the constants*

$$m_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}) = k_1(\tilde{\mathbf{A}}) e^{\frac{1}{\alpha}\left[-\lambda + n + \sum_{i=1}^{n} \nu_i\right]}$$

and

$$m_{n-1}^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}) = k_1(\tilde{\mathbf{A}}) e^{\frac{1}{\alpha}\left[-\lambda + n + \tilde{A}_n + \sum_{i=1}^{n-1} \nu_i\right]}$$

are the best possible in the corresponding inequalities.

Theorem 18 ([6]). *Suppose* $K_\lambda : \mathbb{R}_+^n \to \mathbb{R}$ *is a non-negative measurable homogeneous function of degree* $-\lambda$, $\lambda > 0$*, and let* α, ν_i*, and* $\mu_i > 0$ *be real parameters such that* $\alpha + \frac{1}{\mu_i}(\nu_i - \tilde{A}_i) > 0$, $i = 1, 2, \ldots, n$*, where the parameters* $\tilde{A}_i$, $i = 1, 2, \ldots, n$*, fulfill conditions as in* (67)*. If* $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \ldots, n$ *are non-negative measurable functions, then the following inequalities hold*

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} \, (\mathscr{H}_\alpha f_i)^{\mu_i} \, (x_i) d\mathbf{x} \leq \tilde{m}_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \prod_{i=1}^{n} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))} \tag{74}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\tilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K_\lambda(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} \, (\mathscr{H}_\alpha f_i)^{\mu_i} \, (x_i) \hat{d}^n \mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$
$$\leq \tilde{m}_{n-1}^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \prod_{i=1}^{n-1} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))}, \tag{75}$$

where $\varphi_i(x_i) = x_i^{p_i \nu_i - p_i \tilde{A}_i - 1}$, *and the constants*

$$\tilde{m}_n^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) = k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n} \left[\alpha + \frac{1}{\mu_i}\left(\nu_i - \tilde{A}_i\right)\right]^{\mu_i}$$

and

$$\tilde{m}_{n-1}^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) = k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n-1} \left[\alpha + \frac{1}{\mu_i}\left(\nu_i - \tilde{A}_i\right)\right]^{\mu_i}$$

are the best possible in the corresponding inequalities.

The methodology of proving Theorems 16–18 follows the lines of proofs of Theorems 5–7, except that multidimensional Hilbert-type inequalities (65) and (66) are utilized instead of two-dimensional ones. In addition, in Theorems 16–18 one deals with the weighted versions of the Hardy, Knopp, and Hardy–Carleman inequality (for more details, see [6]). It should also be noticed here that the condition regarding a homogeneous kernel in Theorem 16 may be omitted (see [16]). However, the proofs of these multidimensional theorems are technically more complicated. As an illustration, we give the part of the proof of Theorem 18 referring to the best constant.

Proof (Proof of the Best Constant in (74)*).* Suppose that the inequality

$$\int_{\mathbb{R}_+^n} K_{\lambda}(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} \left(\mathscr{H}_{\alpha} f_i\right)^{\mu_i}(x_i)\, d\mathbf{x} \le C_n \prod_{i=1}^{n} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))}, \tag{76}$$

holds with the constant $0 < C_n < \tilde{m}_n^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$. Considering this inequality with the functions

$$f_i^{\varepsilon}(x_i) = \begin{cases} x_i^{\frac{\tilde{A}_i - \nu_i}{\mu_i} + \frac{\varepsilon}{p_i \mu_i}}, & 0 < x_i \le 1, \\ 0, & x_i > 1, \end{cases}$$

where ε is sufficiently small number, its right-hand side reduces to

$$C_n \prod_{i=1}^{n} \| (f_i^{\varepsilon})^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))} = \frac{C_n}{\varepsilon}. \tag{77}$$

Moreover, since

$$(\mathscr{H}_{\alpha} f_i^{\varepsilon})(x_i) = \begin{cases} \left[\alpha + \frac{\nu_i - \tilde{A}_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}\right] x_i^{\frac{\tilde{A}_i - \nu_i}{\mu_i} + \frac{\varepsilon}{\mu_i p_i}}, & 0 < x_i \le 1, \\ 0, & x_i > 1, \end{cases}$$

the left-hand side of (76), denoted here by L, reads

$$\begin{aligned} L &= \int_{\mathbb{R}^n_+} K_\lambda(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} \left(\mathscr{H}_\alpha f_i^\varepsilon\right)^{\mu_i} (x_i) d\mathbf{x} \\ &= \varphi(\varepsilon) \cdot I, \end{aligned}$$

where

$$\varphi(\varepsilon) = \prod_{i=1}^n \left[\alpha + \frac{\nu_i - \tilde{A}_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}\right]^{\mu_i}$$

and

$$I = \int_{(0,1]^n} K_\lambda(\mathbf{x}) \prod_{i=1}^n x_i^{\tilde{A}_i + \frac{\varepsilon}{p_i}} d\mathbf{x}.$$

Obviously, the integral I can be rewritten as

$$I = \int_0^1 x_1^{\varepsilon-1} \left[\int_{(0,1/x_1]^{n-1}} K_\lambda(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\tilde{A}_i + \varepsilon/p_i} \hat{d}^1\mathbf{u}\right] dx_1,$$

providing the estimate

$$\begin{aligned} I &\geq \int_0^1 x_1^{\varepsilon-1} \left[\int_{\mathbb{R}^{n-1}_+} K_\lambda(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\tilde{A}_i + \varepsilon/p_i} \hat{d}^1\mathbf{u}\right] dx_1 \\ &\quad - \int_0^1 x_1^{\varepsilon-1} \left[\sum_{i=2}^n \int_{\mathbb{E}_i} K_\lambda(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\tilde{A}_j + \varepsilon/p_j} \hat{d}^1\mathbf{u}\right] dx_1 \\ &\geq \frac{1}{\varepsilon} \int_{\mathbb{R}^{n-1}_+} K_\lambda(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\tilde{A}_i + \varepsilon/p_i} \hat{d}^1\mathbf{u} \\ &\quad - \int_0^1 x_1^{-1} \left[\sum_{i=2}^n \int_{\mathbb{E}_i} K_\lambda(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\tilde{A}_j + \varepsilon/p_j} \hat{d}^1\mathbf{u}\right] dx_1, \end{aligned} \tag{78}$$

where $\mathbb{E}_i = \{(u_2, u_3, \ldots, u_n); 1/x_1 \leq u_i < \infty, u_j > 0, j \neq i\}$, $\mathbf{1/p} = (1/p_1, \ldots, 1/p_n)$.

Clearly, it suffices to estimate the integral $\int_{\mathbb{E}_2} K_\lambda(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\tilde{A}_j + \varepsilon/p_j} \hat{d}^1\mathbf{u}$. Namely, choosing $\alpha > 0$ so that $\tilde{A}_2 + 1 > -\varepsilon/p_2 - \alpha$, since $-u_2^{-\alpha} \log \frac{1}{u_2} \to 0$ $(u_2 \to \infty)$, there exists $M \geq 0$ such that $-u_2^{-\alpha} \log \frac{1}{u_2} \leq M$ $(u_2 \in [1, \infty))$. Further, considering the parameters $a_2 = \tilde{A}_2+(\varepsilon/p_2+\alpha)$ and $a_i = \tilde{A}_i+\varepsilon/p_i,\ i = 3,\dots,n$, we have

$$\int_0^1 x_1^{-1} \int_{\mathbb{E}_2} K_\lambda(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\tilde{A}_j+\varepsilon/p_j} \hat{d}^1\mathbf{u}dx_1$$
$$\le M \cdot k_1(\tilde{A}_2 + (\varepsilon/p_2 + \alpha), \tilde{A}_3 + \varepsilon/p_3, \dots, \tilde{A}_n + \varepsilon/p_n) < \infty,$$

and utilizing (78), it follows that

$$L \ge \varphi(\varepsilon) \cdot \left(\frac{1}{\varepsilon}k_1\left(\tilde{\mathbf{A}} + \varepsilon\mathbf{1}/\mathbf{p}\right) - O(1)\right). \tag{79}$$

Finally, taking into account (77) and (79), we have that $\tilde{m}_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \le C_n$ when $\varepsilon \to 0^+$, which is in contrast to our hypothesis.

6 Hilbert-Type Inequalities with Differential Operators

So far, we have discussed Hilbert-type inequalities with certain operators on their left-hand sides. To conclude the paper, in this section we deal with some related inequalities accompanied with operators on their right-hand sides.

Recently, Azar [9] derived several new forms of Hilbert-type inequalities accompanied with some operators on their right-hand sides. The constants appearing in these inequalities are also the best possible.

His first result refers to the homogeneous kernel $K_\lambda(x, y) = (x + y)^{-\lambda}$, $\lambda > 0$ and the Hardy (or integration) operator $(\mathscr{H} f)(x) = \int_0^x f(t)dt$.

Theorem 19 ([9]). *If* $\lambda > 0$, $\|\mathscr{H} f\|_{L^p(\mathbb{R}_+, x^{-\lambda-1})} < \infty$, *and* $\|\mathscr{H} g\|_{L^q(\mathbb{R}_+, y^{-\lambda-1})} < \infty$, *then*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{f(x)g(y)}{(x + y)^\lambda} dxdy$$
$$\le \frac{\lambda^2}{pq} B\left(\frac{\lambda}{q}, \frac{\lambda}{p}\right) \|\mathscr{H} f\|_{L^p(\mathbb{R}_+, x^{-\lambda-1})} \|\mathscr{H} g\|_{L^q(\mathbb{R}_+, y^{-\lambda-1})}, \tag{80}$$

where the constant $\frac{\lambda^2}{pq} B\left(\frac{\lambda}{q}, \frac{\lambda}{p}\right)$ *is the best possible.*

In the same paper, Azar also obtained an analogue of Theorem 19, with a differential operator instead of the Hardy integration operator. Moreover, Adiyasuren et al. [7], extended that result to hold for an arbitrary homogeneous kernel. Before we state the corresponding pair of Hilbert-type inequalities, we first introduce some notation.

We denote by $\mathscr{D}_+^n$, $n \geq 0$, a differential operator defined by $\mathscr{D}_+^n f(x) = f^{(n)}(x)$, where $f^{(n)}$ stands for the n-th derivative of a function $f : \mathbb{R}_+ \to \mathbb{R}$. In addition, throughout this section, Λ_+^n denotes the set of non-negative measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $f^{(k)}(x) > 0$, $k = 0, 1, 2, \ldots, n$, a.e. on $\mathbb{R}_+$, and $f^{(k)}(0) = 0$, $k = 0, 1, 2, \ldots, n-1$.

The following theorem deals with a homogeneous kernel $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$, of degree $-\lambda$, $\lambda > 0$, such that the integral

$$k(\alpha) = \int_{\mathbb{R}_+} K_\lambda(1,t)t^\alpha dt,$$

converges for $-1 < \alpha < \lambda - 1$.

Theorem 20 ([7]). *Let α_1, α_2 be real parameters such that $\alpha_1, \alpha_2 \in (n-1, \lambda-1)$ and $\alpha_1 + \alpha_2 = \lambda - 2$, where n is a fixed non-negative integer and $\lambda > n$, and let $\varphi(x) = x^{p(n-\alpha_1)-1}, \psi(y) = y^{q(n-\alpha_2)-1}$. If $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-\lambda$, then the inequalities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy < M\|\mathscr{D}_+^n f\|_{L^p(\mathbb{R}_+,\varphi(x))}\|\mathscr{D}_+^n g\|_{L^q(\mathbb{R}_+,\psi(y))} \tag{81}$$

and

$$\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} < m\|\mathscr{D}_+^n f\|_{L^p(\mathbb{R}_+,\varphi(x))} \tag{82}$$

hold for all non-negative functions $f, g \in \Lambda_+^n$. In addition, the constants $M = k(\alpha_2)\frac{\Gamma(\alpha_1-n+1)\Gamma(\alpha_2-n+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}$ and $m = k(\alpha_2)\frac{\Gamma(\alpha_1-n+1)}{\Gamma(\alpha_1+1)}$ are the best possible in the corresponding inequalities.

Inequalities (81) and (82) are consequences of Hilbert-type inequalities (1) and (2), equipped with the weighted Hardy inequality. The idea of proving the best constants in (81) and (82) is similar to the proofs presented in this article. Namely, starting from the opposite assumption, it is necessary to plug suitable functions in inequality to obtain a contradiction (for more details, see [7]).

Remark 4. Considering (81) with a homogeneous kernel $K_\lambda(x,y) = (x+y)^{-\lambda}$, $\lambda > 0$, and the parameters $\alpha_1 = \frac{\lambda}{p} - 1$, $\alpha_2 = \frac{\lambda}{q} - 1$, where $\lambda > n\max\{p,q\}$, the above constant M reduce to $\frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$. This particular case was studied by Azar [9], and it was derived by a different technique.

Observe that Theorem 20 covers the case when the degree of homogeneity of the kernel, i.e. $-\lambda$ is less than $-n$, for a fixed non-negative integer n. The next result from [7], that is in some way complementary to Theorem 20, covers the case $0 < \lambda \leq 1$, and it follows by virtue of the weighted dual Hardy inequality.

The dual Hardy inequality, accompanied with the dual integration operator or the dual Hardy operator $\mathscr{H}^* f(x) = \int_x^\infty f(t)dt$, asserts that

$$\int_{\mathbb{R}_+} x^{-r} \left(\mathscr{H}^* f(x)\right)^p dx < \left(\frac{p}{1-r}\right)^p \int_{\mathbb{R}_+} x^{p-r} f^p(x)dx, \tag{83}$$

holds for $p > 1$ and $r < 1$, provided that $0 < \int_{\mathbb{R}_+} x^{p-r} f^p(x)dx < \infty$. We define a differential operator $\mathscr{D}_\pm^n$ by $\mathscr{D}_\pm^n f(x) = (-1)^n f^{(n)}(x)$, where n is a non-negative integer. Moreover, the following theorem holds for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that the n-th derivative $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $\mathscr{D}_\pm^k f(x) > 0$, $k = 0, 1, 2, \ldots, n$, a.e. on $\mathbb{R}_+$, and $\lim_{x\to\infty} f^{(k)}(x) = 0$ for $k = 0, 1, 2, \ldots, n-1$. This set of functions will be denoted by $\Lambda_\pm^n$.

Theorem 21 ([7]). *Suppose that α_1, α_2 are real parameters such that $\alpha_1, \alpha_2 \in (-1, \lambda - 1)$ and $\alpha_1 + \alpha_2 = \lambda - 2$, where $0 < \lambda \leq 1$, and let $\varphi(x) = x^{p(n-\alpha_1)-1}$, $\psi(y) = y^{q(n-\alpha_2)-1}$. If $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, then the inequalities*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x, y) f(x) g(y) dx dy$$
$$< M^* \|\mathscr{D}_\pm^n f\|_{L^p(\mathbb{R}_+,\varphi(x))} \|\mathscr{D}_\pm^n g\|_{L^q(\mathbb{R}_+,\psi(y))} \tag{84}$$

and

$$\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)} \left(\int_{\mathbb{R}_+} K_\lambda(x, y) f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< m^* \|\mathscr{D}_\pm^n f\|_{L^p(\mathbb{R}_+,\varphi(x))} \tag{85}$$

hold for all non-negative functions $f, g \in \Lambda_\pm^n$, where n is a fixed non-negative integer. In addition, the constants $M^* = k(\alpha_2)\frac{\Gamma(-\alpha_1)\Gamma(-\alpha_2)}{\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)}$ and $m^* = k(\alpha_2)\frac{\Gamma(-\alpha_1)}{\Gamma(n-\alpha_1)}$, appearing in (84) and (85) are the best possible.

For an illustration, we only give the proof of inequality (84).

Proof (Proof of Inequality (84)). The starting point is inequality (1) accompanied with the dual Hardy inequality (83). Namely, utilizing (1) with parameters $A_1 = -\frac{\alpha_1}{q}$ and $A_2 = -\frac{\alpha_2}{p}$, its right-hand side may be rewritten as

$$k(\alpha_2)\left[\int_{\mathbb{R}_+} x^{-p\alpha_1-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{-q\alpha_2-1} g^q(y)dy\right]^{\frac{1}{q}}$$
$$= k(\alpha_2)\left[\int_{\mathbb{R}_+} x^{-(p\alpha_1+1)}(\mathscr{H}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_{\mathbb{R}_+} y^{-(q\alpha_2+1)}(\mathscr{H}^*(\mathscr{D}_\pm g)(y))^q dy\right]^{\frac{1}{q}}, \tag{86}$$

since $\mathscr{H}^*(\mathscr{D}_\pm f)(x) = -\int_x^\infty f'(t)dt = f(x)$. Moreover, applying the dual Hardy inequality to the expressions on right-hand side of (86) n times, it follows that

$$\left[\int_{\mathbb{R}_+} x^{-(p\alpha_1+1)}(\mathscr{H}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}}$$
$$< \frac{1}{(-\alpha_1)^{\overline{n}}}\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}(\mathscr{D}_\pm^n f(x))^p dx\right]^{\frac{1}{p}} \tag{87}$$

and

$$\left[\int_{\mathbb{R}_+} y^{-(q\alpha_2+1)}(\mathscr{H}^*(\mathscr{D}_\pm g)(y))^q dy\right]^{\frac{1}{q}}$$
$$< \frac{1}{(-\alpha_2)^{\overline{n}}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}(\mathscr{D}_\pm^n g(y))^q dy\right]^{\frac{1}{q}}, \tag{88}$$

where $x^{\overline{n}}$ stands for a rising factorial power or a Pochhammer symbol, that is, $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$. Now, since $(-\alpha_1)^{\overline{n}} = \frac{\Gamma(n-\alpha_1)}{\Gamma(-\alpha_1)}$ and $(-\alpha_2)^{\overline{n}} = \frac{\Gamma(n-\alpha_2)}{\Gamma(-\alpha_2)}$, the inequality (84) holds due to (1), (86)–(88).

Remark 5. Considering dual inequalities (84) and (85) accompanied with the kernel $K_\lambda(x,y) = (x+y)^{-\lambda}$, $\lambda > 0$, the constants M^* and m^* become, respectively,

$$M_1^* = \frac{\pi^2}{\sin(\alpha_1\pi)\sin(\alpha_2\pi)}\cdot\frac{1}{\Gamma(\lambda)\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)}$$
$$m_1^* = -\frac{\pi}{\sin(\alpha_1\pi)}\cdot\frac{\Gamma(\alpha_2+1)}{\Gamma(\lambda)\Gamma(n-\alpha_1)}, \quad \alpha_1,\alpha_2\in(-1,\lambda-1), 0<\lambda\le 1.$$

In addition, if $K_\lambda(x,y) = \max\{x,y\}^{-\lambda}$, $\lambda > 0$, these constants M^* and m^* reduce to

$$M_2^* = \frac{\lambda}{(\alpha_1+1)(\alpha_2+1)} \cdot \frac{\Gamma(-\alpha_1)\,\Gamma(-\alpha_2)}{\Gamma(n-\alpha_1)\,\Gamma(n-\alpha_2)}$$
$$m_2^* = \frac{\lambda}{(\alpha_1+1)(\alpha_2+1)} \cdot \frac{\Gamma(-\alpha_1)}{\Gamma(n-\alpha_1)}, \quad \alpha_1, \alpha_2 \in (-1, \lambda-1), 0 < \lambda \le 1.$$

For some other applications of Theorems 20 and 21, the reader is referred to [7].

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