# Chapter 2
# Classical Mechanics

*Une intelligence qui, à un instant donné, connaîtrait toutes les forces dont la nature est animée et la situation respective des êtres qui la compose embrasserait dans la même formule les mouvements des plus grands corps de l'univers et ceux du plus léger atome; rien ne serait incertain pour elle, et l'avenir, comme le passé, serait présent à ses yeux.*

This chapter is meant to provoke curiosity on the topic of symmetries merely by reflecting on conservation laws in good old classical mechanics. Just by asking the deeper question why these laws hold, we arrive at a first understanding of how properties of space and time relate to invariances. And invariances in turn entail symmetry groups, which in case of classical mechanics is the Galilei group. Thus the more appropriate caption of this chapter could be "Galilei Group".

All of us started to learn the concepts of physics along the notions of classical mechanics. It's qualitative formulation began with Galileo Galilei (1564–1642), found its powerful formulation by Isaac Newton (1643–1727) a century later, and got mathematically refined as analytical mechanics by Jean L. Lagrange (1736–1813), William R. Hamilton (1805–1865) and Gustav J. Jacobi (1804–1851) in the 19th century. It received a further mathematical refinement in the 20th century; see e.g. [14]. Classical mechanics was considered as the model of science up to the end of the 19th century. Today we believe that classical mechanics is a threefold limiting case of our world: It is the limit of low velocities–or–a world with an infinite velocity of light, a world with vanishing Planck constant, and a world with weak gravitational fields–meaning that the Schwarzschild radius of an object is much larger than the typical length scale of the object. Nevertheless, this book, although aimed at fundamental physics, starts with a chapter on classical mechanics. It may come as a surprise that classical mechanics is conceptually more complicated than modern quantum field theory–the reason being found in symmetries.

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Pierre-Simon Laplace, Essai philosophique sur les probabilités, Paris, Christian Bourgeois, 1986, pages 32 et 33.

K. Sundermeyer, *Symmetries in Fundamental Physics*,
Fundamental Theories of Physics 176, DOI: 10.1007/978-3-319-06581-6_2,

## 2.1 Newtonian and Analytical Mechanics

The basic laws of mechanics can be expressed either in terms of differential equations (e.g. Newton's, Lagrange's, Hamilton's) or as integral variational principles (e.g. those of Maupertuis or Hamilton). Let us recapitulate the formulation by differential equations first.

- Newton
  The foundation of qualitative and quantitative mechanics in terms of mass points, velocities, accelerations, forces, ... and their mutual dependencies is due to I.Newton and is laid out in the three so-called "Newton's laws". From today's point of view one recognizes a circular definition in these laws: There are (inertial) systems in which Newton's laws are valid.
- Lagrange
  Although from the historical perspective the roles of Lagrange, Hamilton and others are more subtle, every student in physics associates with Lagrange the concept of generalized coordinates, the Lagrange function and the Euler-Lagrange equations derived from the Lagrange function. The Lagrange function may be taken as the starting point to understand and to investigate symmetries in terms of operations and invariants.
- Hamilton
  Notions such as phase space, canonical transformations, Poisson brackets, ... are associated with the name of Hamilton. The phase space approach on the one hand side reveals the "furniture" of classical mechanics and on the other hand paves a canonical way towards quantization.
- Hamilton-Jacobi
  This is a formulation of classical mechanics in terms of action-angle variables, a formulation that directly uncovers conserved quantities. The Hamilton-Jacobi form of physics is most appropriate for describing chaotic systems and for geometrical optics. It also may serve as a bridge to the Schrödinger equation of quantum mechanics.

### *2.1.1 Newtonian Mechanics*

Newton bequeathed us with three laws [385]

N1: Every body continues in its state of rest, or of uniform motion in a right line unless it is compelled to change that state of forces impressed upon it.

N2: The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.

N3: To every action there is always opposed an equal reaction.

Newton points out that his laws do hold in an ever-existing absolute space and in an external absolute time flow[1]. Those frames in which N1 holds are called inertial

[1] A comment about this assumption is made in the concluding remarks of this chapter.

frames. These are at rest in absolute space or move uniformly along straight lines and are interrelated by the transformations

$$x^i \to x'^i = x^i + v^i t + a^i \qquad t \to t' = t + \tau \qquad \text{with constants} \qquad v^i, a^i, \tau, \tag{2.1}$$

also called Galilei translations.

In our modern notation: For a system with $K$ mass points $m_\alpha$ $(\alpha = 1, .., K)$ at (Cartesian) co-ordinates $\vec{x}_\alpha(t)$ and with velocities $d\vec{x}_\alpha/dt =: \dot{\vec{x}}_\alpha$ one defines

$$\begin{aligned}
&\text{momenta :} && \vec{p}_\alpha = m_\alpha \dot{\vec{x}}_\alpha \\
&\text{kinetic energy:} && T = \sum_\alpha \frac{m_\alpha}{2} \dot{\vec{x}}_\alpha^{\,2} \\
&\text{total momentum:} && \vec{P} = \sum_\alpha \vec{p}_\alpha \\
&\text{total angular momentum:} && \vec{J} = \sum_\alpha \vec{x}_\alpha \times \vec{p}_\alpha \\
&\text{center of mass:} && \vec{R} = \frac{1}{M} \sum_\alpha m_\alpha \vec{x}_\alpha \qquad \text{with} \qquad M := \sum_\alpha m_\alpha .
\end{aligned}$$

The force $\vec{F}_\alpha(\vec{x}, \dot{\vec{x}}, t)$ exerted on the mass point $m_\alpha$ is the sum of an external force and internal forces,

$$\vec{F}_\alpha = F_\alpha^{(ext)} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta}$$

where according to Newton's third law $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$. The equations of motion according to Newton's second law are

$$\vec{N}_\alpha := m_\alpha \ddot{\vec{x}}_\alpha - \vec{F}_\alpha \equiv \dot{\vec{p}}_\alpha + \vec{\nabla}_\alpha V = 0, \tag{2.2}$$

where the last identity holds for conservative forces, namely those derivable from a potential function as $\vec{F}_\alpha = -\vec{\nabla}_\alpha V$. In this case (which will be assumed in the sequel) the total energy of the system is defined as $E = T + V$. The Newtonian equations of motion are 3N differential equations of second order for the positions $\vec{x}_\alpha$ as functions of time. It is remarkable that not only the equations of motion of classical mechanics, but all dynamical equations of fundamental physics are of second order. As a matter of fact, higher order differential equations tend to have instabilities as shown by a theorem by Ostrogradski[2].

[2] I suppose that aside from a historian of science, no one will really read the original article in Mem. Ac. St. Petersbourgh VI, 4, 385 (1850); the year is not a misprint!

### 2.1.2 Lagrange Form of Mechanics

The crucial step from Newtonian to analytical mechanics is the recognition that the equations of motion can be derived from a Lagrange function

$$L(\vec{x}_\alpha, \dot{\vec{x}}_\alpha, t) \;=\; T \;-\; V(\vec{x}_\alpha, t)$$

with the kinetic energy $T$ and potential energy $V$. The Lagrange function can also be written in "generalized" coordinates $q^k$ (e.g. curvilinear coordinates) and velocities $\dot{q}^k$ :

$$L(q, \dot{q}, t) \;=\; \sum_{i,k}^{N} a_{ik}(q, t)\, \dot{q}^i \dot{q}^k \;-\; V(q, t), \tag{2.3}$$

where $N$ is the number of degrees of freedom. One easily convinces oneself that the set of Euler-Lagrange equations

$$[L]_k \;:=\; \frac{\partial L}{\partial q^k} - \frac{d}{dt}\,\frac{\partial L}{\partial \dot{q}^k} \;= 0 \tag{2.4}$$

is equivalent to the equations of motion in the Newtonian form (2.2). The Lagrangian is not unique, since adding to the Lagrangian a total derivative $\frac{d}{dt}B(q, t)$ does not change the equations of motion.

It is comprehensible that because of the freedom in choosing the generalized coordinates, the Euler-Lagrange equations remain "structurally" invariant after an (at least locally) invertible coordinate transformation $q \to \hat{q}(q)$. Explicitly

$$\dot{\hat{q}}^k \;= \frac{\partial \hat{q}^k}{\partial q^l}\,\dot{q}^l \qquad \text{from which} \qquad \frac{\partial \dot{\hat{q}}^k}{\partial \dot{q}^j} = \frac{\partial \hat{q}^k}{\partial q^j} \qquad \text{and} \qquad \frac{d}{dt}\,\frac{\partial \hat{q}^k}{\partial q^l} \;=\; \frac{\partial \dot{\hat{q}}^k}{\partial q^l},$$

and similar expressions, where the hatted and the un-hatted variables are exchanged. Now, with $\hat{L}(\hat{q}, \dot{\hat{q}}, t) := L(q, q, t)$ we derive

$$\begin{aligned}\frac{d}{dt}\,\frac{\partial \hat{L}}{\partial \dot{\hat{q}}^k} \;&=\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^l}\,\frac{\partial \dot{q}^l}{\partial \dot{\hat{q}}^k}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^l}\,\frac{\partial q^l}{\partial \hat{q}^k}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^l}\right)\frac{\partial q^l}{\partial \hat{q}^k} \;+\; \frac{\partial L}{\partial \dot{q}^l}\,\frac{d}{dt}\,\frac{\partial q^l}{\partial \hat{q}^k} \\ &= \left(\frac{\partial L}{\partial q^l} \;-\; [L]_l\right)\frac{\partial q^l}{\partial \hat{q}^k} \;+\; \frac{\partial L}{\partial \dot{q}^l}\,\frac{\partial \dot{q}^l}{\partial \hat{q}^k} \;=\; \frac{\partial \hat{L}}{\partial \hat{q}^k} \;-\; [L]_l\,\frac{\partial q^l}{\partial \hat{q}^k}.\end{aligned}$$

This can be rewritten as

$$[\hat{L}]_k \;=\; [L]_l\,\frac{\partial q^l}{\partial \hat{q}^k}, \tag{2.5}$$

and reveals what is meant by structural invariance of the Euler-Lagrange equations under coordinate transformations[3]: Together with the equations $[L]_l = 0$ also

[3] Equation (2.5) expresses that the Euler-Lagrange derivative $[L]_k$ transforms like a covariant vector with respect to the coordinates $q^i$.

$[\hat{L}]_k = 0$. Structural invariance is different from form invariance (or covariance) with respect to particular transformations. The covariance of the equations of motion under a coordinate transformation is the defining property of a Lie symmetry.

It is found to be insightful to write the Euler-Lagrange equations explicitly as

$$[L]_k = \frac{\partial L}{\partial q^k} - \frac{\partial^2 L}{\partial \dot{q}^k \partial q^j}\dot{q}^j - \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j}\ddot{q}^j := V_k - W_{kj}\ddot{q}^j = 0. \tag{2.6}$$

From this we observe that if the *Hessian*

$$W_{kj} := \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j} \tag{2.7}$$

has an inverse $\bar{W}^{jl}$, the $N$ Euler-Lagrange equations can be expressed as

$$\ddot{q}^j = \bar{W}^{ji} V_i = F^j(q, \dot{q}, t).$$

This "normal form" allows us to apply theorems about the existence and uniqueness of solutions of ordinary differential equations. Lagrangians for which $\det W \neq 0$ are called regular, and singular otherwise. Also, a system described by a regular (singular) Lagrangian is called regular (singular). This is justified since regularity of a Lagrangian is a coordinate-independent statement for invertible coordinate transformations $q \to \hat{q}$: The determinant of $\hat{W}$ is the product $\det W (\det J)^2$ with the Jacobian $J$ of the coordinate transformation. Also the addition of a total derivative to a Lagrangian does not change its character of being regular or singular.

The Hessian also plays a role in the inverse problem of the calculus of variations which asks for conditions under which a set of second order equations $\ddot{q}^j = F^j(q, \dot{q}, t)$ can be derived from a Lagrange function. These conditions–in the literature known as the Helmholtz conditions in view of a publication of Hermann von Helmholtz from 1895–require the existence of a nonsingular matrix $(w_{ij})$ that obeys a set of differential equations. As you can guess, if a Lagrangian exists, then taking for $(w_{ij})$ the Hessian, the Helmholtz conditions are fulfilled. On the other hand, if for given $F^j$ a matrix $(w_{ij})$ obeying the Helmholtz conditions can be found, the Lagrange function can be constructed explicitly and uniquely–up to an overall multiplicative constant and up to a total derivative. The solution of the inverse problem is far from trivial: Only in 1941, J. Douglas succeeded in solving completely the two-dimensional case.

Since this book is mainly about variational symmetries, we always assume the existence of a Lagrangian–and "luckily"–or by "basic principles"–for all fundamental interactions Lagrangians are known. But this remark already leads outside classical physics. There are even surprising arguments that quantization requires Lagrangians [279].

### 2.1.3 Hamiltonian Formulation

Mechanics in the Hamiltonian form is couched in terms of generalized positions and momenta instead of generalized positions and velocities, as is the case for the Lagrangian formulation. The generalized momenta are defined by

$$p_k := \frac{\partial L}{\partial \dot{q}^k}. \tag{2.8}$$

This expresses the momenta $p$ in terms of $(q, \dot{q}, t)$. These functions can be inverted for the velocities, i.e. $\dot{q}^i = \dot{q}^i(q, p, t)$ iff the matrix

$$\frac{\partial p_k}{\partial \dot{q}^i} = \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}$$

has non-zero determinant. Again we see the appearance of the Hessian $W_{ij}$. Only for regular systems the subsequent derivation of their Hamiltonian formulation is sound. But as we will see later, fundamental physics is inherently singular due to its symmetries; there the transition to a Hamiltonian is a little tricky–to say the least; see Appendix C.

In the regular case the Hamilton function is defined as

$$H(q, p, t) := p_k \dot{q}^k(p, q) - L(q, \dot{q}(p, q, t), t). \tag{2.9}$$

The transformation from $L(q, \dot{q}, t)$ to $H(q, p, t)$ is a *Legendre transformation*. For the Lagrange function of the form (2.3) the numerical value of the Hamilton function $H$ is calculated from

$$H = p_k \dot{q}^k - L = \frac{\partial L}{\partial \dot{q}^k}\dot{q}^k - (a_{ik}\dot{q}^i\dot{q}^k - V) = 2a_{ik}\dot{q}^i\dot{q}^k - a_{ik}\dot{q}^i\dot{q}^k + V = T + V = E$$

as the total energy of the system.

The equations of motion (2.4) can–in the non-singular case–uniquely be expressed by the Hamilton function. To get these Hamiltonian equations we derive for (2.9)

$$dH = \frac{\partial H}{\partial q^k}dq^k + \frac{\partial H}{\partial p_k}dp_k + \frac{\partial H}{\partial t}dt = (p_k d\dot{q}^k + \dot{q}^k dp_k) - \frac{\partial L}{\partial q^k}dq^k - \frac{\partial L}{\partial \dot{q}^k}d\dot{q}^k - \frac{\partial L}{\partial t}dt.$$

Since with the definition of the generalized momentum (2.8) the Euler-Lagrange equations are equivalent to

$$\dot{p}_k = \frac{\partial L}{\partial q^k}$$

the two terms in the previous expression proportional to $d\dot{q}^k$ cancel, so that

$$\delta H = (\dot{q}^k \delta p_k - \dot{p}_k \delta q^k) - \frac{\partial L}{\partial t}\delta t.$$

Thus the Hamilton equations of motion become

$$\dot{q}^k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k}. \tag{2.10}$$

The $2N$-dimensional space spanned by the $q^k$ and $p_k$ is called the "phase space". In taking the time as an additional variable one arrives at the $(2N+1)$-dimensional "state space". The motion can be pictured as that of a $2N$-dimensional "phase fluid": Each stream-line of the moving fluid represents the motion of the system starting from specific initial conditions. The fluid as a whole represents the complete solution.

A central concept of the phase space formulation of classical mechanics is the Poisson bracket, defined for two phase functions $A, B$ by

$$\{A, B\} := \frac{\partial A}{\partial q^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q^k}\frac{\partial A}{\partial p_k}. \tag{2.11}$$

The Poisson brackets form an algebra in the mathematical sense (see Appendix A.1.1) in identifying the algebra operation $\square$ with the bracket operation {, }. This Poisson bracket algebra obeys additionally the properties of a Lie algebra, and especially the Jacobi identity

$$\{A, \{B, C\} + \{B, \{A, C\} + \{C, \{A, B\} = 0. \tag{2.12}$$

The Poisson brackets of the coordinates and their canonically conjugate momenta become the "fundamental brackets"

$$\{q^k, q^l\} = 0, \quad \{p_k, p_l\} = 0, \quad \{q^k, p_l\} = \delta^k_l. \tag{2.13}$$

The Hamiltonian equations (2.10) can be written in terms of Poisson brackets as

$$\dot{q}^k = \{q^k, H\} = \frac{\partial H}{\partial p^k} \qquad \dot{p}_k = \{p_k, H\} = -\frac{\partial H}{\partial q^k} \tag{2.14}$$

by which the time evolution of a state space function $F(q, p, t)$ can be expressed as

$$\dot{F} := \frac{dF(q, p, t)}{dt} = \{F, H\} + \frac{\partial F}{\partial t}. \tag{2.15}$$

The Poisson bracket (2.11) is not only a nice means to write down the Hamilton equations in a compact form, but it constitutes–so to say–the backbone of the phase-space formulation of classical mechanics. This will be expounded in the sequel.

### Canonical Transformations

We saw that point transformations $q \to \hat{q}(q, t)$ leave the Euler-Lagrange equations invariant. They also leave the Hamiltonian equations (structurally) invariant. There is, however, a larger class of invariance transformations, especially canonical transformations[4]. These are defined as those invertible transformations

$$\hat{q}^j = \hat{q}^j(q, p), \qquad \hat{p}_j = \hat{p}_j(q, p)$$

which leave the fundamental brackets (2.13) invariant. Before deriving properties of canonical transformations, let us introduce another more condensed notation. Since with the canonical transformations the concept of distinct coordinates and momenta fades into oblivion, it makes sense to collect all $2N$ phase space variables into one set $(x^\alpha) = (q^1, ..., q^N, p_1, ..., p_N)$. In this notation the fundamental brackets (2.13) can be written as

$$\{x^\alpha, x^\beta\} = \Gamma^{\alpha\beta} \qquad \text{with} \qquad \Gamma := \begin{pmatrix} 0_N & 1_N \\ -1_N & 0_N \end{pmatrix}. \tag{2.16}$$

In terms of the matrix $\Gamma$, the Poisson bracket for two phase space functions $A$ and $B$ becomes

$$\{A, B\} = \Gamma^{\alpha\beta} \frac{\partial A}{\partial x^\alpha} \frac{\partial B}{\partial x^\beta}.$$

The condition for $\hat{x}(x)$ being a canonical transformation simply is $\{\hat{x}^\alpha, \hat{x}^\beta\} = \Gamma^{\alpha\beta}$. Denoting by $X^{\alpha\beta} := \frac{\partial \hat{x}^\alpha}{\partial x^\beta}$, a canonical transformation obeys

$$X\,\Gamma\,X^T = \Gamma. \tag{2.17}$$

This allows to derive that canonical transformations form a group: Take as group elements two matrices $X_1$ and $X_2$. Then

$$(X_1 X_2)\Gamma(X_1 X_2)^T = X_1 X_2 \Gamma X_2^T X_1^T = X_1 \Gamma X_1^T = \Gamma,$$

and the existence of an identity element and an inverse is obvious. The matrices $X_i$ are a representation of the symplectic group $\mathbf{Sp(2N)}$.

Let us now derive under which conditions a transformation is a canonical transformation. Since in most of this text we are interested in continuous transformations

[4] Some textbooks define canonical transformations by the property of leaving the Hamiltonian equations invariant, but this is not true in general, see ([14]). The canonical transformations can also be defined as contact transformations with respect to the Lagrangian.

we restrict our considerations to infinitesimal transformations $\hat{x}^\alpha = x^\alpha + \delta_c x^\alpha$. Then–to first order–the defining relation (2.17) amounts to

$$\frac{\partial(\delta_c x^\alpha)}{\partial x^{\alpha'}}\Gamma^{\alpha'\beta} + \frac{\partial(\delta_c x^\beta)}{\partial x^{\alpha'}}\Gamma^{\alpha\alpha'} \stackrel{!}{=} 0,$$

which after multiplication and summation by $\Gamma^{\beta\gamma}\Gamma^{\alpha\gamma'}$ (and some renaming of indices) becomes

$$\frac{\partial}{\partial x^\alpha}\left(\Gamma^{\beta\gamma}\,\delta_c x^\gamma\right) - \frac{\partial}{\partial x^\beta}\left(\Gamma^{\alpha\gamma}\,\delta_c x^\gamma\right) \stackrel{!}{=} 0.$$

Hence $\Gamma^{\beta\gamma}\,\delta_c x^\gamma$ is the gradient of an (infinitesimal function) $g$, and

$$\hat{x}^\alpha = x^\alpha + \Gamma^{\alpha\beta}\frac{\partial g}{\partial x^\beta} = x^\alpha + \{x^\alpha, g\} =: x^\alpha + \delta_g x^\alpha. \tag{2.18}$$

This is the most general infinitesimal canonical transformation. The phase space function $g$ is called the generator for infinitesimal canonical transformations. The commutator of two infinitesimal transformations generated by $g$ and $h$ is

$$[\delta_g, \delta_h]x^\alpha = (\delta_g\delta_h - \delta_h\delta_g)x^\alpha = \{\delta_h x^\alpha, g\} - \{\delta_g x^\alpha, h\} = \{\{x^\alpha, h\}, g\} - \{\{x^\alpha, g\}, h\},$$

which due to the Jacobi identity (2.12) can be written as

$$[\delta_g, \delta_h]x^\alpha = \{x^\alpha, \{h, g\}\} = \delta_{\{h,g\}}x^\alpha. \tag{2.19}$$

Finally, for an arbitrary phase space function $F$

$$\delta_g F(x) = F(x + \delta_g x) - F(x) = \frac{\partial F}{\partial x^\alpha}\,\delta_g x^\alpha = \{F, g\}. \tag{2.20}$$

### *2.1.4 Principle of Stationary Action*

The action $S$ is defined as the functional

$$S\{q\} = S[q^k(t_1, t_2)] := \int_{t_1}^{t_2} L(q, \dot{q}, t)dt,$$

that is a mapping of functions $q^k(t)$ to real numbers. Based on previous work of P.L. de Maupertuis, L. Euler and L. Lagrange in the 18th century, W. Hamilton in 1832 formulated the principle of stationary action: “The classical path $q^k_{class}$ between $t_1$ and $t_2$ is the one for which $S$ is stationary”. This indeed seems to be a primary principle of physics. It can be traced back to quantum mechanics: According to the formulation with Feynman path integrals (see Appendix D.1), a particle virtually takes all possible paths between the endpoints. The ones near the classical paths dominate the exponential in the transition amplitude

$$\langle q_b t_b | q_a t_a \rangle = \sum_{path} \exp \frac{i}{\hbar} S_{path}.$$

Nevertheless, it remains still a mystery why Nature knows about an action, to begin with.

**Euler-Lagrange Equations**

Consider different paths $q + \delta q$ and require $\delta S \overset{!}{=} 0$. The variation $\delta S$ is

$$\delta S = \int_{t_1}^{t_2} \{L(q + \delta q, \dot{q} + \delta\dot{q}, t) - L(q, \dot{q}, t)\} dt.$$

For infinitesimal $\delta q^k$

$$L(q + \delta q, \dot{q} + \delta\dot{q}, t) = L(q, \dot{q}, t) + \left( \frac{\partial L}{\partial q^k} \delta q^k + \frac{\partial L}{\partial \dot{q}^k} \delta\dot{q}^k \right) + \cdots$$

The requirement $\delta S \overset{!}{=} 0$ is thus equivalent to the requirement

$$\int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial q^k} \delta q^k + \overbrace{\frac{\partial L}{\partial \dot{q}^k} \delta\dot{q}^k}^{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\delta q^k\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k}\right)\delta q^k} \right) \overset{!}{=} 0$$

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \left[ \left( \frac{\partial L}{\partial q^k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^k} \right) \delta q^k + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^k} \delta q^k \right) \right] \\ &= \int_{t_1}^{t_2} [L]_k \delta q^k + \left( \frac{\partial L}{\partial \dot{q}^k} \delta q^k \right) \Bigg|_{t_1}^{t_2} \overset{!}{=} 0. \end{aligned} \tag{2.21}$$

This must be valid for arbitrary variations $\delta q^k$. Assuming now that the variation at the initial and final points of time vanish ($\delta q^k(t_1) = 0 = \delta q^k(t_2)$) the second term in this identity vanishes, and the remaining equations $[L]_k = 0$ are nothing but the Euler-Lagrange equations (2.4). For later purposes we might characterize the dynamical equations also by requiring

$$\delta L = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^k} \delta q^k \right).$$

This derivation of the equations of motion should not be completely new to you, since it is taught in the basic course of theoretical classical mechanics. But mostly it is mentioned only in passing that this holds true only by assuming appropriate boundary conditions. As you will see later, boundary terms contain additional information of the physical system. They are integral part of Noether currents and play an essential role in theories which are invariant with respect to general coordinate transformations, Einstein's general relativity being a prime example.

### Higher Derivatives

We were assuming that the Lagrangian of the system depends on the coordinates and on at most their first time derivatives. It is obvious how the previous results can be extended to situations in which the Lagrangian depends on higher derivatives. For example in case of $L(q, \dot{q}, \ddot{q})$ one derives straightforwardly

$$\delta L = [L]_k \delta q^k + \frac{d}{dt} B \tag{2.22}$$

where now

$$[L]_k \ := \ \frac{\partial L}{\partial q^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} \ + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}^k} \tag{2.23}$$

and the boundary term is

$$B = \left( \frac{\partial L}{\partial \dot{q}^k} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}^k} \right) \delta q^k + \frac{\partial L}{\partial \ddot{q}^k} \delta \dot{q}^k .$$

From (2.22) the equations of motion $[L]_k = 0$ only follow if both $\delta q^k$ and $\delta \dot{q}^k$ vanish at $t_1$ and $t_2$. Even if we assume that these stringent conditions can be fulfilled, we are still faced with dynamical equations of fourth order due to the last term in (2.23). Nevertheless, there are exceptions. Take for instance the Lagrange function $L(q, \dot{q}, \ddot{q})$ as derived from a function $\bar{L}(q, \dot{q})$ in the form

$$L(q, \dot{q}, \ddot{q}) = \bar{L}(q, \dot{q}) - \frac{d}{dt}\left( q \frac{\partial \bar{L}}{\partial \dot{q}} \right) . \tag{2.24}$$

Instead of plugging this into the previous expressions we repeat the variation (indices suppressed):

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} dt \left[ \frac{\partial \bar{L}}{\partial q} \delta q + \frac{\partial \bar{L}}{\partial \dot{q}} \delta \dot{q} \right] - \delta \left( q \frac{\partial \bar{L}}{\partial \dot{q}} \right) \Bigg|_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} dt \left( \frac{\partial \bar{L}}{\partial q} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}} \right) \delta q + \left( \frac{\partial \bar{L}}{\partial \dot{q}} \delta q \right) \Bigg|_{t_1}^{t_2} - \delta \left( q \frac{\partial \bar{L}}{\partial \dot{q}} \right) \Bigg|_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} dt \, [\bar{L}]_k \delta q^k - q^k \delta p_k \Big|_{t_1}^{t_2} \stackrel{!}{=} 0.
\end{aligned}$$

Thus indeed the Euler derivatives $[\bar{L}]_k$ are of second order. The equations of motion follow if the variation of the momenta $p_k$ vanish at the boundary (rather then the $\delta q^k$). The specific Lagrangian (2.24) is not just a curiosity; instead, it displays the typical structure of the original Hilbert action of general relativity and its relation to the Einstein action; see (7.70).

Moreover we learn that the addition of a total derivative to the action–although not changing the equations of motion–may require different boundary data in order that the variational problem is well-posed. The generic relation between boundary data and second order terms can be found in the following way [375]: Assume that a second order Lagrangian can be written as

$$L_C(q,\dot{q},\ddot{q}) = L_q(q,\dot{q}) - \frac{df(q,\dot{q})}{dt}.$$

Here $L_q$ is a first order Lagrangian for which $q$ has to be kept fixed at the boundaries, and $L_C$ a Lagrangian for which a given function $C(q,\dot{q})$ is kept fixed. $L_C$ and $L_q$ bring about the same (second-order) equations of motion if

$$f(q,C) = \int p(q,C)\,dq + F(C),$$

where $F(C)$ is an arbitrary function. Here, it is assumed that one can solve $C(q,\dot{q})$ as $\dot{q} = \dot{q}(q,C)$, and express the momenta as $p = p(q,C)$. The previous case (2.24) immediately follows with $C = p$.

**Hamilton Equations**

The Hamilton equations can also be derived by variational methods if the action is expressed in terms of phase space coordinates. Rewrite

$$S = \int L(q,\dot{q},t)dt = \int (p_k\dot{q}^k - H)dt = \int (p_k dq^k - H dt).$$

Now

$$\begin{aligned}\delta S &= \int \left\{ dq^k \delta p_k + p_k \delta dq^k - \frac{\partial H}{\partial q^k}\delta q^k dt - \frac{\partial H}{\partial p_k}\delta p_k dt \right\} \\ &= \int \delta p_k \left( dq^k - \frac{\partial H}{\partial p_k} dt \right) - \int \delta q^k \left( dp_k + \frac{\partial H}{\partial q^k} dt \right) + \int d(p_k \delta q^k).\end{aligned}$$

Assuming again that the $\delta q^k$ vanish at the boundary, the otherwise arbitrariness of the variations of the phase space variables leads us to conclude that the two integrands vanish separately, giving rise to the equations

$$dq^k = \frac{\partial H}{\partial p_k} dt \qquad\qquad dp_k = -\frac{\partial H}{\partial q^k} dt$$

which are indeed nothing but the Hamilton equations of motion (2.10). Notice that the boundary term we are dropping here is the same in both the Lagrangian and the Hamiltonian case, namely $d(p_k \delta q^k)$.

### Boundary Terms and Canonical Transformations

Since the action is invariant with respect to canonical transformations $(q, p) \to (\hat{q}, \hat{p})$, it must be the case that

$$p_k dq^k - H dt = \hat{p}_k d\hat{q}^k - \hat{H} dt + dF. \tag{2.25}$$

Writing this as $dF = p_k dq^k - \hat{p}_k d\hat{q}^k + (\hat{H} - H)dt$ we find

$$\frac{\partial F}{\partial q^k} = p^k \qquad \frac{\partial F}{\partial \hat{q}^k} = -\hat{p}^k \qquad \frac{\partial F}{\partial t} = \hat{H} - H$$

where $F$ is considered as depending on the old and the new coordinates, i.e. $F(q, \hat{q}, t)$. It turns out that $F$ generates a canonical transformation. Three other types of generators for canonical transformations can be obtained by taking other ways of distributing partial derivatives in (2.25). By this, in principle one can obtain generating functions depending on any of the combinations $(q, \hat{q}), (q, \hat{p}), (\hat{q}, p), (\hat{q}, \hat{p})$. For instance define

$$G := d(F + \hat{p}_k \hat{q}^k) = p_k dq^k + \hat{q}^k d\hat{p}_k + (\hat{H} - H)dt$$

where now $G(q, \hat{p}, t)$ fulfills

$$\frac{\partial G}{\partial q^k} = p_k \qquad \frac{\partial G}{\partial \hat{p}_k} = \hat{q}^k \qquad \frac{\partial G}{\partial t} = \hat{H} - H. \tag{2.26}$$

The canonical transformations generated by $G$ include the point transformations $\hat{q}^k = \hat{q}^k(q)$ by the choice $G = f^k(q)\hat{p}_k$ and the identity transformations by the further choice $f^k(q) = q^k$. The transformation infinitesimally deviating from the identity transformation can be written as

$$G = q^k \hat{p}_k - g(q, \hat{p}) = q^k \hat{p}_k - g(q, p)$$

where the latter relation is valid, because $g$ is infinitesimal. Then

$$\frac{\partial G}{\partial q^k} = \hat{p}_k - \frac{\partial g}{\partial q^k} = p_k \qquad \frac{\partial G}{\partial \hat{p}_k} = q^k - \frac{\partial g}{\partial \hat{p}_k} = \hat{q}^k,$$

which reproduces (2.18).

**Hamilton-Jacobi Equations**

If the generating function $G(q, \hat{p}, t)$ in (2.26) is chosen in such a way that the transformed Hamiltonian $\hat{H}$ vanishes, it is called Hamilton's principal function $\bar{S} = G(q, \hat{p}, t)$. Then the first and the third relation in (2.26) yield

$$H(q, p, t) + \frac{\partial \bar{S}}{\partial t} = 0 = H\left(q^k, \frac{\partial \bar{S}}{\partial q^k}, t\right) + \frac{\partial \bar{S}}{\partial t} \tag{2.27}$$

the latter expression defining the Hamilton-Jacobi equation. Since by assumption the transformed Hamiltonian is identically zero, the corresponding Hamilton equations are simply $\dot{\hat{q}}^k = 0 = \dot{\hat{p}}_k$. Therefore the new variables are constants of motion: $\hat{p}_k = \alpha_k,\ \hat{q}^k = \beta^k$. Therefore the Hamilton-Jacobi equation (2.27) is a differential equation for $\bar{S}(q, \alpha, t)$. Assuming that we can find a solution of this equation, we may solve the relation $\hat{q}^k = \partial \bar{S}(q, \alpha, t)/\partial \alpha_k$ as $q^i(\alpha, \beta, t)$ and

$$p_i = \left.\frac{\partial \bar{S}(q, \alpha, t)}{\partial q^i}\right|_{q^i = q^i(\alpha, \beta, t)}.$$

Therefore the solutions for the coordinates and the momenta are explicitly expressed by conserved quantities.

Since $\bar{S}$ is a function only of $q^i$ and $t$ we find due to the first relation in (2.26) and to (2.27)

$$d\bar{S} = \frac{\partial \bar{S}}{\partial q^i} dq^i + \frac{\partial \bar{S}}{\partial t} dt = \left(p_i \frac{dq^i}{dt} - H\right) dt = L dt$$

revealing that $\bar{S}$ is the indefinite time integral of the Lagrangian. Therefore the action can be expressed as $S = \bar{S}(t_2)$ - $\bar{S}(t_1)$. In the case that the Hamiltonian $H$ is independent of time, the principal function becomes $\bar{S}(q, \hat{p}, t) = W(q, \hat{p})$ - $Et$ with the energy $E$ and Hamilton's characteristic function $W$, which obeys

$$H\left(q^k, \frac{\partial W}{\partial q^k}\right) = E.$$

## 2.1.5 *Classical Mechanics in Geometrical Terms

All of the preceding findings for classical mechanics were written in terms of generalized coordinates, velocities, momenta, ... These are local expressions which hide the essential geometric structures of analytical mechanics. Indeed the Lagrangian and the Hamiltonian description can properly be formulated on a tangent and a cotangent bundle. These geometric concepts are in more detail explained in Appendix E.

### Lagrangian Dynamics

The configuration space is assumed to be a manifold $\mathbb{Q}$ (with coordinates $q^i$). In order to describe the dynamics (locally given by $q^i(t)$) we need to consider the configuration-velocity space which is the tangent bundle $T\mathbb{Q}$ with local coordinates $(q^i, \dot{q}^i)$. The time development of any function $f(q, \dot{q}) \in \mathcal{F}(T\mathbb{Q}) : T\mathbb{Q} \to \mathbb{R}$ (let us for simplicity assume that the configuration-velocity space functions do not explicitly depend on time) is defined in terms of the vector field

$$\Delta = \dot{q}^i \frac{\partial}{\partial q^i} + a^i(q, \dot{q})^i \frac{\partial}{\partial \dot{q}^i} \tag{2.28}$$

where the $a^i$, having the meaning of accelerations, will be determined below as

$$\dot{f} = \dot{q}^i \frac{\partial f}{\partial q^i} + a^i(q, \dot{q}) \frac{\partial f}{\partial \dot{q}^i} = \pounds_\Delta f.$$

The Lagrangian is a function on the tangent bundle[5] $L : T\mathbb{Q} \to \mathbb{R}$. Introduce the (Cartan) one-form

$$\theta_L := \frac{\partial L}{\partial \dot{q}^i} dq^i. \tag{2.29}$$

This gives rise to a natural two-form

$$\omega_L := -d\theta_L = \left( \frac{\partial^2 L}{\partial \dot{q}^i \partial q^k} \right) dq^i \wedge dq^k + \left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k} \right) dq^i \wedge d\dot{q}^k. \tag{2.30}$$

Now

$$\begin{aligned} \pounds_\Delta \theta_L &:= \pounds_\Delta \left( \frac{\partial L}{\partial \dot{q}^i} \right) dq^i + \frac{\partial L}{\partial \dot{q}^i} d\pounds_\Delta q^i \\ &= \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \doteq \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i = dL, \end{aligned}$$

where, indicated by the $\doteq$ notation, the coordinate version of the Euler-Lagrange equations was used. Therefore the coordinate-free form of the Lagrange equation is

$$\pounds_\Delta \theta_L = dL. \tag{2.31}$$

[5] Throughout this subsection it is assumed for simplicity that there is no explicit dependence on time in any of the entities involved. Dropping this assumption amounts to investigating the Lagrangian on the contact manifold $T\mathbb{Q} \times \mathbb{R}$.

This becomes explicitly $-i_\Delta d\theta_L = i_\Delta \omega_L = d(i_\Delta \theta_L - L)$ and can therefore be written in terms of the Lagrange energy $E$ as

$$i_\Delta \omega_L = dE \qquad \text{with} \qquad E := i_\Delta \theta_L - L = \frac{\partial L}{\partial \dot{q}^k}\dot{q}^k - L. \tag{2.32}$$

Let us find out under which conditions the equation $i_X \omega_L = dE$ has a unique solution for the vector field $X$. On the one hand,

$$dE = \frac{\partial E}{\partial q^i} dq^i + \frac{\partial E}{\partial \dot{q}^i} d\dot{q}^i = -V_i dq^i + W_{ij}\dot{q}^j d\dot{q}^i \tag{2.33}$$

with the notation as in (2.6).
On the other hand, for a generic vector field $X = A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial \dot{q}^i}$ we find

$$i_X \omega_L = V_{ij}(A^i dq^j - A^j dq^i) + W_{ij}(A^i dq^j - B^i dq^j)$$

with the abbreviation

$$V_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \qquad \text{such that} \qquad V_i = \frac{\partial L}{\partial q^i} - V_{ji}\dot{q}^j.$$

The comparison with (2.33) results in

$$\frac{\partial L}{\partial q^i} - V_{ji}\dot{q}^j = (V_{ij} - V_{ji})A^j + W_{ij}B^j \tag{2.34a}$$

$$W_{ij}\dot{q}^j = W_{ij}A^j. \tag{2.34b}$$

The integral curves of $X$ are given by

$$\frac{dq^i}{d\lambda} = A^i, \qquad \frac{d\dot{q}^i}{d\lambda} = B^i.$$

Thus one could be tempted to identify from (2.34b) $\dot{q}^j$ with $A^j$. But this is only allowed if the Hessian $W_{ij}$ can be inverted. Only in this case (2.34a) do have a unique solution with

$$A^i = \dot{q}^i, \qquad B^i = \bar{W}^{ij} V_j$$

which is indeed the Lagrangian vector field (2.28) with $a^i = B^i = \ddot{q}^i(q, \dot{q})$.

### Hamiltonian Dynamics

Hamiltonian mechanics takes place on the cotangent space $T^*\mathbb{Q}$, coordinized by $(q^i, p_i)$. The Hamiltonian is a function $H : T^*\mathbb{Q} \to \mathbb{R}$. Introduce a vector field

$$\nabla = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i}$$

and define the two-form

$$\omega = dq^i \wedge dp_i$$

which derives from the (Liouville) one-form $\theta_H = p_i dq^i$ as $\omega = -d\theta_H$. In denoting coordinates in $T^*\mathbb{Q}$ as $(x^\alpha)$ we find that the components of $\omega = 1/2\, \omega_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$ are identical to the symplectic matrix $\Gamma$ as in (2.16).

Now calculate

$$i_\nabla \omega = i_\nabla dq^i \wedge dp_i - dq^i \wedge i_\nabla dp_i = \dot{q}^i dp_i - \dot{p}_i dq^i \doteq \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i = dH.$$

This reveals that the coordinate independent Hamilton equations of motion are

$$i_\nabla \omega = dH. \tag{2.35}$$

Indeed they are formulated completely in terms of the vector field $\nabla$ and the two-form $\omega$ (both being defined within the cotangent space geometry), and the cotangent space function $H$.

The two-form $\omega$ is a representation of the Poisson bracket structure of Hamiltonian dynamics in the following sense: Associate to every function $f \in \mathcal{F}(T\mathbb{Q})$ a vector field $X_f$ by

$$i_{X_f} \omega = df, \qquad \text{in coordinates} \qquad X_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}. \tag{2.36}$$

A vector field $X_f$ associated to a cotangent space function $f$ is called a Hamiltonian vector field[6] or a Hamiltonian with respect to $f$. The equations (2.35) show that the vector field $\nabla$ is the Hamiltonian vector field with respect to the Hamilton function.

Calculate $\pounds_{X_g} f$ and find that in coordinates this reproduces the Poisson bracket $\{f, g\}$, thus

$$\{f, g\} = \pounds_{X_g} f = i_{X_g} df = i_{X_g} i_{X_f} \omega, \tag{2.37}$$

and for two Hamiltonian vector fields:

$$[X_f, X_g] = X_{\{f,g\}}. \tag{2.38}$$

[6] Not all vector fields are Hamiltonian, which tells that there are more conceivable motions on $T^*\mathbb{Q}$ beyond those described by Hamiltonian dynamical systems.

The dynamical evolution of a space function can be written as $\pounds_{\nabla} f = \{f, H\}$.

Canonical transformations are a subset of diffeomorphisms $F : T^*\mathbb{Q} \to T^*\mathbb{Q}$, namely those for which the symplectic form is preserved: $F^*\omega = \omega$. Indeed the pull-back of $i_{\nabla}\omega$ then yields

$$F^* i_{\nabla}\omega = i_{F^*\nabla} F^*\omega = i_{F^*\nabla}\omega = dF^* H$$

showing that the Hamilton equations (2.35) are preserved. Further taking the exterior derivative of the relation $i_{X_f}\omega = df$ that defines the Hamiltonian vector field with respect to the function $f$,

$$0 = d(i_{X_f}\omega) = \pounds_{X_f}\omega.$$

This demonstrates that the symplectic form is invariant under any Hamiltonian flow. In using the relation (E.3) between a diffeomorhism and a vector field and the relation (2.36) between a vector field and a function one can associate to any function $f$ a one-parameter group of transformations $\Phi^f$. In the language of analytical mechanics the function $f$ is called the infinitesimal generator of the canonical transformation.

#### Legendre Transformation

In the previous two subsections, the Lagrangian and the Hamiltonian dynamics were described in purely geometric terms, but separately of each other. What is known as the Legendre transformation from the one description to the other, needs to be grasped in geometric terms as a specific mapping

$$\begin{aligned} \mathcal{F}L : T\mathbb{Q} &\to T^*\mathbb{Q} \\ (q, \dot{q}) \mapsto \mathcal{F}L(q, \dot{q}) &= \left(q, \hat{p} = \frac{\partial L}{\partial \dot{q}}\right). \end{aligned} \tag{2.39}$$

This is a mapping of the vector with components $\dot{q}^k$ in $T_q\mathbb{Q}$ onto its covector $\hat{p}_k = \partial L/\partial \dot{q}^k$ in $T_q^*\mathbb{Q}$. In other words, it maps the one form $\theta_L = (\partial L/\partial \dot{q}^k)\, dq^k \in TQ$ onto the one-form $\hat{p}_i dq^i \in T^*\mathbb{Q}$. On the other hand, independently of any Lagrangian, in the cotangent bundle there exists a Liouville form $\theta_H = p_i dq^i$. Obviously both $\theta_L$ and the Legendre mapping depend on the Lagrangian. It is not at all obvious that these dependencies act in such a way that $\theta_L$ is always sent to the same canonical one-form $\theta_H$. And indeed, for this to happen one can show that a necessary condition is–lo and behold–the nonsingularity of the Hessian $W_{ij}$. This is also seen in that the pull-back $\omega_L = \mathcal{F}L^*\omega$ given by (2.30) has components which in matrix form can be written as

$$\begin{pmatrix} A & W \\ -W & 0 \end{pmatrix}$$

(with $A = V\text{-}V^T$ and $V_{ij}$ as in (2.34a)) since all components proportional to $d\dot{q}^i \wedge d\dot{q}^k$ do vanish. This matrix is nonsingular only for det $W \neq 0$. Only in this case is the two-form $\omega_L$ symplectic (non-degenerate and closed). The non-singularity of $W$ is of course also the condition necessary to solve $\hat{p}_k = (\partial L/\partial \dot{q}^k)(q, \dot{q})$ uniquely as $\dot{q}^i = \dot{q}^i(q, \hat{p})$ and to identify $\hat{p}_k \equiv p_k$. For regular systems, the Hamiltonian is the projection of the Lagrangian energy $H = \mathcal{F}L(E)$. Furthermore, the connection between the Lagrangian and the Hamiltonian dynamics becomes

$$\dot{q}^k = \mathcal{F}L^* \left(\frac{\partial H}{\partial p_k}\right) \qquad\qquad \frac{\partial L}{\partial q^k} = -\mathcal{F}L^* \left(\frac{\partial H}{\partial q^k}\right). \tag{2.40}$$

To emphasize again: only with the existence of a regular Lagrangian one can define this canonical isomorphism between $T\mathbb{Q}$ and $T^*\mathbb{Q}$. This is reminiscent to what is known from differential geometry: If a manifold is equipped with a metric, this metric mediates maps between the tangent and the cotangent bundle, see Appendix E.5.4. In regular classical mechanics, this metric is visible in the kinetic energy term $T = g_{ij}\dot{q}^i\dot{q}^j$.[7]

## 2.2 Symmetries and Conservation Laws

### *2.2.1 Conservation Laws*

Physic was successful–or even possible–because one was able to find "laws of nature". In a very broad sense, a law of physics encodes many observations (experiments and their results) on a physical system in a compact mathematical relation [184]. The more measurements on one and the same system are encoded the better the law is established experimentally (and may be sanctioned as a theory). The more systems are encoded by one and the same law, the more profound the unification. The least one requires is that "under the same circumstances" it does not matter whether the experiment takes place:

- TODAY or TOMORROW
- in BERLIN or in NEW YORK
- in the NORTH-SOUTH or the EAST-WEST direction
- on the COAST of the Baltic Sea or on a SAILBOAT moving uniformly with respect to the coast.

The term "under the same circumstances" needs a comment. If we investigate a physical system at another time, or in another position, orientation, relative movement, we must make sure that everything surrounding and possibly influencing the system must also be transformed correspondingly. In Chap. 11 of [181] R.P. Feynman describes a grandfather's clock as an example. If you do not place it standing upright, its pendulum will hit the case, and the clock will not work at all. So this system seems not to work in the same manner, independently of its orientation. But, if you include the earth in your consideration, you are again in the position to formulate

[7] The analogy goes even further, in that the Liouville one-form acts as a soldering form.

direction-independent statements about the system. This is related to the distinction of objects in terms of background and dynamical structures already mentioned in the Introduction.

### Homogeneity of Time and Energy Conservation

If the outcome of an experiment on a physical system does not depend on when it is performed ("TODAY or TOMORROW"), the Lagrange function can not depend on time explicitly. Therefore

$$\frac{dL}{dt} = \underbrace{\frac{\partial L}{\partial q^i}}_{\left([L]_i + \frac{d}{dt}\frac{dL}{d\dot{q}^i}\right),} \dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i$$

by the use of the Euler derivatives $[L]_i$. Thus

$$\frac{dL}{dt} = [L]_i\dot{q}^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)$$

or

$$[L]_i\dot{q}^i = \frac{d}{dt}\left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) = \frac{d}{dt}(-E) \doteq 0,$$

in other words, for trajectories or "on-shell" (that is, for solutions of the equations of motion) the energy $E$ is conserved.

### Homogeneity of Space and Momentum Conservation

Start from

$$L = \sum_\alpha \frac{m_\alpha \dot{\vec{x}}_\alpha}{2} - V(\vec{x})$$

and consider a translation of all position vectors by a constant vector $\vec{d}$:

$$\vec{x}_\alpha \to \vec{x}_\alpha + \vec{d},$$

i.e. $\delta\vec{x}_\alpha = \vec{d}, \quad \delta\dot{\vec{x}}_\alpha = 0$. Thus the variation of the Lagrange function becomes

$$\delta L = \sum_\alpha \frac{\partial L}{\partial \vec{x}_\alpha} \cdot \delta\vec{x}_\alpha = \vec{d} \cdot \sum_\alpha \frac{\partial L}{\partial \vec{x}_\alpha}.$$

If the measurement on the system at two different positions ("in BERLIN or in NEW YORK") leads to the same result, the Lagrange function can only depend on relative positions $\vec{r}_{\alpha\beta} = \vec{x}_\alpha - \vec{x}_\beta$. Therefore

$$\sum_\alpha \frac{\partial L}{\partial \vec{x}_\alpha} = 0.$$

On the other hand $\partial L/\partial \vec{x}_\alpha$ can be expressed by the Euler-derivatives and the momenta in the form

$$\frac{\partial L}{\partial \vec{x}_\alpha} = [\vec{L}]_\alpha + \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}_\alpha} = [\vec{L}]_\alpha + \dot{\vec{p}}_\alpha.$$

From the fact that

$$\sum_\alpha([\vec{L}]_\alpha + \dot{\vec{p}}_\alpha) = \dot{\vec{P}} + \sum_\alpha [\vec{L}]_\alpha = 0$$

one derives, that on trajectories $([\vec{L}]_\alpha \doteq 0)$ the total momentum $\vec{P}$ is conserved.

A special case is present if the Lagrange function does not depend on a specific coordinate (called a cyclic coordinate) $\bar{q}^\kappa$. In this case the momentum $\bar{p}_\kappa$ conjugate to $\bar{q}^\kappa$ is itself a conserved quantity.

### Isotropy of Space and Angular Momentum Conservation

The variation of the Lagrangean $L(\vec{x}, \dot{\vec{x}})$, namely

$$\delta L = \frac{\partial L}{\partial \vec{x}_\alpha} \cdot \delta\vec{x}_\alpha + \frac{\partial L}{\partial \dot{\vec{x}}_\alpha} \cdot \delta\dot{\vec{x}}_\alpha$$

can be expressed–like in the previous section–through the Euler-derivatives and the momenta as

$$\delta L = ([\vec{L}]_\alpha + \dot{\vec{p}}_\alpha) \cdot \delta\vec{x}_\alpha + \vec{p}_\alpha \cdot \delta\dot{\vec{x}}_\alpha.$$

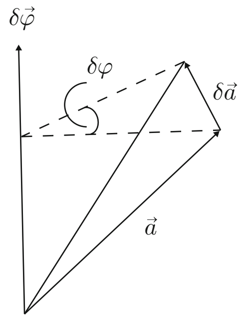
Consider an infinitesimal rotation: $\delta\vec{\varphi}$ points into the direction of the rotation axis, $\delta\varphi = |\delta\vec{\varphi}|$ is the magnitude of the rotation, see Fig. 2.1. For each vector $\vec{a}$ we have because of $\delta\vec{a} \perp \vec{a}$ and $\delta\vec{a} \perp \delta\vec{\varphi}$

$$\delta\vec{a} = \gamma\, \delta\vec{\varphi} \times \vec{a} \qquad \text{where} \quad \gamma = \text{const}.$$

Therefore,

$$\delta L = \gamma \sum_\alpha ([\vec{L}]_\alpha + \dot{\vec{p}}_\alpha) \cdot (\delta\vec{\varphi} \times \vec{x}_\alpha) + \vec{p}_\alpha \cdot (\delta\vec{\varphi} \times \dot{\vec{x}}_\alpha).$$

**Fig. 2.1** Vector conventions for infinitesimal rotations




We can extract from this expression $\delta\vec{\varphi}$ if we make use of the relation $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$:

$$\begin{aligned}\delta L &= \gamma\ \delta\vec{\varphi}\cdot\sum_\alpha\left\{\vec{x}_\alpha\times([\vec{L}]_\alpha + \dot{\vec{p}}_\alpha) + \dot{\vec{x}}_\alpha\times\vec{p}_\alpha\right\}\\ &= \gamma\ \delta\vec{\varphi}\cdot\sum_\alpha\left\{x_a\times[\vec{L}]_\alpha + \frac{d}{dt}(\vec{x}_\alpha\times\vec{p}_\alpha)\right\}.\end{aligned}$$

If the variation $\delta L$ vanishes for arbitrary $\delta\vec{\varphi}$ we conclude that the total angular momentum $\vec{J} = \sum_\alpha(\vec{x}_\alpha\times\vec{p}_\alpha)$ is a constant for solutions of the dynamical equations. The argument can be extended to finite (non-infinitesimal) rotations, exemplifying that if the experiment reveals the same observations "in the NORTH-SOUTH and the EAST-WEST direction" necessarily the total angular momentum is conserved.

**Galilei Relativity and Uniform Center-of-Mass Velocity**

Galilei relativity is the invariance of classical mechanics with respect to Galilei translations. These are transformations from one system of coordinates to another system which moves with a constant velocity $\vec{v}$ with respect to the former. An example is a "SAILBOAT moving uniformly with respect to a COAST". Quantitatively

$$\vec{x}'_\alpha = \vec{x}_\alpha + \vec{v}t.$$

Introduce the quantity $\vec{Q} = \sum_\alpha(m_\alpha\vec{x}_\alpha - \vec{p}_\alpha t)$ and consider the expression

$$\frac{d}{dt}(\vec{v}\cdot\vec{Q}) = \vec{v}\cdot\frac{d}{dt}\left(\sum_\alpha m_\alpha\vec{x}_\alpha - \vec{p}_\alpha t\right) = \vec{v}\cdot\sum_\alpha(m_\alpha\dot{\vec{x}}_\alpha - \dot{\vec{p}}_\alpha t - \vec{p}_\alpha).$$

Now by definition, $(m_a\dot{\vec{x}}_\alpha - \vec{p}_\alpha)\equiv 0$, and $\dot{\vec{p}}_\alpha$ can be expressed by the Newtonian derivatives (2.2) so that

$$\frac{d}{dt}(\vec{v}\cdot\vec{Q}) = \vec{v}\cdot\sum_\alpha(\vec{\nabla}_\alpha V - \vec{N}_\alpha).$$

If $V$ depends on the coordinate distances $r_{ij} = |\vec{x}_i - \vec{x}_j|$ only, $\sum_\alpha \vec{\nabla}_\alpha V$ vanishes. Therefore, again on trajectories ($\vec{N}_\alpha \doteq 0$) the complete left-hand side vanishes. Since $\vec{v}$ is arbitrary we deduce $\frac{d}{dt}\vec{Q} = 0$. Now

$$\vec{Q} = M\vec{R} - \vec{P}t \tag{2.41}$$

with the center of mass $\vec{R}$ ($M = \sum_\alpha m_\alpha$) and the total momentum $\vec{P}$. Since the total momentum is conserved, we find that the center of mass moves with uniform velocity.

#### "Deriving" the Lagrangian from Properties of Space-Time and from Galilei Relativity

The next consideration is not directly related to conservation laws. But in order to derive the standard conserved quantities in previous subsections we used characteristics of space and time and assumed that the Lagrangian has the form $L = T\text{-}V$. Indeed, one may turn the arguments around to derive the structure of the Lagrangian[8]. To begin with, consider a free mass point. Because of homogeneity of space and time, the Lagrangian cannot depend on the position $\vec{x}$ and on the time variable $t$. Therefore it can only depend on the velocity $\dot{\vec{x}}$. Furthermore, because of the isotropy of space, the Lagrangian cannot depend on the direction of velocity. Therefore $L = L(\dot{x}^2)$. The change of the Lagrangian under an infinitesimal transformation to another inertial system with $\dot{\vec{x}} \to \dot{\vec{x}}' = \dot{\vec{x}} + \vec{\epsilon}$ is

$$L' = L(\dot{x}'^2) = L(\dot{x}^2 + 2\dot{\vec{x}} \cdot \vec{\epsilon} + \epsilon^2) = L(\dot{x}^2) + \frac{\partial L}{\partial \dot{x}^2} 2\dot{\vec{x}} \cdot \vec{\epsilon} + \mathcal{O}(\epsilon^2).$$

Requiring that in both inertial systems the dynamics stays the same, the Lagrangians $L$ and $L'$ are allowed to only differ by a total derivative. Therefore $\frac{\partial L}{\partial \dot{x}^2}$ must be a constant, or $L = a\dot{x}^2$; the constant $a$ is identified as $a = m/2$, with $m$ being the mass. (But be aware that this identification can only be justified by including a second point particle.) In any case we derived the standard kinetic energy part $T$. This can be generalized to a system of free mass points, and we notice that the kinetic part of the Lagrangian transforms under Galilei translations (2.1) as

$$\begin{aligned} L(\dot{\vec{x}}) \;\to\; L' &= \sum_\alpha \frac{m_\alpha}{2} \frac{d\vec{x}'^{\,2}_\alpha}{dt'} = \sum_\alpha \frac{m_\alpha}{2} \left[ \dot{\vec{x}}^{\,2}_\alpha + 2\dot{\vec{x}}_\alpha \vec{v} + v^2 \right] \\ &= L + \frac{d}{dt} \sum_\alpha \frac{m_\alpha}{2} \left[ 2\vec{x}_\alpha \vec{v} + v^2 t \right]. \end{aligned}$$

The last term is a total time derivative and we say that the Lagrangian for a system of free mass points is quasi-invariant by going from inertial system to another one.

[8] In the sequel I follow [332], but you can find this line of reasoning in other textbooks on classical mechanics as well.

For a system of mass points without external forces the potential $V(\vec{x})$ only depends on the relative distances $r_{\alpha\beta} = |\vec{x}_\alpha - \vec{x}_\beta|$, so that this term is itself invariant under Galilei translations.

Instead of assuming from the very start the homogeneity of space and time one can arrive at the same result [459] by considering the Euler-Lagrange equations for a free particle as derived from a Lagrangian $L(\dot{x}, x, t)$:

$$\frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta}\, \ddot{x}^\beta + \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial x^\beta}\, \dot{x}^\beta + \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial t} - \frac{\partial L}{\partial x^\alpha} = 0.$$

Let us demand that this dynamical equation is invariant under Galilei transformations. Now, only the first term depends on $\ddot{x}$, and $\ddot{x}$ does not change under Galilei transformations. Therefore the coefficient must be a constant

$$\frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} = k_{\alpha\beta}.$$

This can be integrated to

$$L = \frac{1}{2} k_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta + \dot{x}^\beta F_\beta(x, t) + G(x, t).$$

The Euler-Lagrange equations then become

$$k_{\alpha\beta}\, \ddot{x}^\beta + \dot{x}^\beta \left[ \frac{\partial F_\alpha}{\partial x^\beta} - \frac{\partial F_\beta}{\partial x^\alpha} \right] + \frac{\partial F_\alpha}{\partial t} - \frac{\partial G}{\partial x^\alpha} = 0.$$

By the previous argumentation, the first term is invariant with respect to Galilei transformations. The second term is the only one depending on velocities. This is required to vanish, and therefore $[F_{\alpha,\beta} - F_{\beta,\alpha}] = 0$ or

$$F_\alpha = \Phi_{,\alpha} \qquad \text{and} \qquad F_{\alpha,t} - G_{,\alpha} = K_\alpha$$

with a constant $K_\alpha$. The latter condition is solved by $G = \Phi_{,t} - K_\alpha x^\alpha + C(t)$. Inserted into the previous Lagrangian this becomes

$$\begin{aligned} L &= \frac{1}{2} k_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta + \dot{x}^\beta \Phi_{,\beta} + \Phi_{,t} - K_\alpha x^\alpha + C(t) \\ &= \frac{1}{2} k_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta - K_\alpha x^\alpha + \frac{d}{dt}\left( \Phi + \int C\, dt \right). \end{aligned}$$

The last term, being a total derivative, can be dropped. The equations of motion become $k_{\alpha\beta}\, \ddot{x}^\beta + K_\alpha = 0$. Next we require that these equations are invariant with respect to rotations $x^\alpha \to R^\alpha_{\ \beta} x^\beta$. The transformed Lagrangian and equations of motion become

$$L' = \frac{1}{2} k_{\alpha\beta} R^\alpha_{\ \gamma} R^\beta_{\ \delta}\, \dot{x}^\gamma \dot{x}^\delta - K_\alpha R^\alpha_{\ \beta} x^\beta \qquad k_{\alpha\beta} R^\alpha_{\ \delta} R^\beta_{\ \gamma}\, \ddot{x}^\gamma + K_\alpha R^\alpha_{\ \delta} = 0.$$

Invariance under rotations is ensured if $K_\alpha R^\alpha_{\ \delta} = K_\delta$ and $k_{\alpha\beta} R^\alpha_{\ \delta} R^\beta_{\ \gamma} = k_{\delta\gamma}$. And this is the case if $K_\alpha = 0$ and $k_{\alpha\beta} = k\delta_{\alpha\beta}$, that is if $L = \frac{k}{2}\,\dot{x}^2$.

### 2.2.2 *Noether Theorem–A First Glimpse*

In the previous section we became acquainted with relations between certain invariance properties of the Lagrange function and conservation laws. In each case, we argued in a different manner, but common to all the cases is the fact that the conservation law holds only "on-shell", that is only for solutions of the equations of motion. Emmy Noether[9] (1882-1935) was able, amongst others, to relate conservation laws directly to the underlying symmetry group in the case of continuous symmetries.

The Noether theorems (actually two theorems are usually distinguished) relate to continuous symmetries of an action. Let us assume that we are dealing with point transformations[10]

$$q^k \mapsto \hat{q}^k(q,t) \qquad\qquad t \mapsto \hat{t}(t,q).$$

The action functional

$$S[q] = \int_{t_1}^{t_2} dt L(q], \qquad L[q] := L(q^k, \dot{q}^k, t)$$

becomes in the new variables

$$S[q] = \int_{\hat{t}(t_1)}^{\hat{t}(t_2)} d\hat{t} \ \left(\frac{dt}{d\hat{t}}\right) L[q(\hat{q},\hat{t})] := \hat{S}[\hat{q}].$$

A variational symmetry fulfills $S[\hat{q}] \stackrel{!}{=} S[q]$ , i.e.

$$\underbrace{\int_{\hat{t}(t_1)}^{\hat{t}(t_2)} d\hat{t} L[\hat{q}]}_{\displaystyle \int_{t_1}^{t_2} dt \frac{d\hat{t}}{dt} L[\hat{q}],} \stackrel{!}{=} \int_{t_1}^{t_2} dt L[q]$$

so that

$$\left(\frac{d\hat{t}}{dt}\right) L[\hat{q}] \stackrel{!}{=} L[q] + \frac{d}{dt}\Sigma_S(q,t). \tag{2.42}$$

[9] In the literature you can spot people who show their acquaintance with German by knowing that in many words the "oe" means "ö". But this is not always true, especially for proper names.

[10] The theorem also holds for generalized transformations in which the "new" variables also depend on the "old" velocities $\dot{q}^i$. Later it will be shown that if velocity-dependent transformations are allowed one can derive the conservation of the Runge-Lentz vector of the Kepler problem from a Noether theorem.

The Lagrange function is called invariant iff $\Sigma_S = 0$, and quasi-invariant otherwise. By notation, the boundary term depends on the symmetry transformation.

The Noether theorems only hold for continuous symmetry transformations, namely those which are continuously attainable from the identity transformations. This is for instance not true for time reversal and space inversion. In case of continuous symmetries, we can restrict ourself to transformations near the identity:

$$\hat{q}^k = q^k + \delta_S q^k(q,t) = q^k + \epsilon\eta^k(q,t) \qquad \hat{t} = t + \delta_S(t,q) = t + \epsilon\xi(t,q)$$

with the understanding that every one-parameter continuous set of symmetry transformations is characterized by the parameter $\epsilon$ and functions $\eta^k(q,t)$ and $\xi(t,q)$. For these infinitesimal transformations the requirement (2.42) reads

$$\left(1 + \frac{d}{dt}\delta_S t\right) L[\hat{q}] \stackrel{!}{=} L[q] + \frac{d}{dt}\sigma_S(q,t),$$

where the notation $\sigma_S$ indicates that we are arguing infinitesimally. With the definition

$$\delta_S L := L[\hat{q}] - L[q]$$

the previous expression can be rewritten as

$$\delta_S L + L[q]\frac{d}{dt}\delta_S t \stackrel{!}{=} \frac{d}{dt}\sigma_S(q,t).$$

After applying the chain rule it becomes

$$\delta_S L - \frac{dL}{dt}\delta_S t \stackrel{!}{=} \frac{d}{dt}(\sigma_S - L\delta_S t). \tag{2.43}$$

The terms on the left-hand side are explicitly

$$\delta_S L = \frac{\partial L}{\partial q}\delta_S q + \frac{\partial L}{\partial \dot{q}}\delta_S \dot{q} + \frac{\partial L}{\partial t}\delta_S t.$$
$$\frac{dL}{dt}\delta_S t = \left(\frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t}\right)\delta_S t.$$

(Here the coordinate indices are dropped for convenience–or–laziness; they will be re-introduced at the end result.) The entire left-hand side of (2.43) becomes

$$\delta_S L - \frac{dL}{dt}\delta_S t = \frac{\partial L}{\partial q}\bar{\delta}_S q + \frac{\partial L}{\partial \dot{q}}\bar{\delta}_S \dot{q}, \tag{2.44}$$

with

$$\bar{\delta}_S q := \delta_S q - \dot{q}\delta_S t = \epsilon(\eta - \dot{q}\xi) \qquad \bar{\delta}_S \dot{q} := \delta_S \dot{q} - \ddot{q}\delta_S t. \tag{2.45}$$

This so-called form variation arises here as a convenient abbreviation. As a matter of fact it has a geometric meaning, denoting the difference between the coordinates at two moments of time separated by $\delta t$:

$$\bar{\delta} q := \hat{q}(t) - q(t).$$

In contrast, the total variation $\delta q$ is the difference between the new and the original coordinate compared at the same point of time: $\delta q = \hat{q}(\hat{t})$ - $q(t)$. The $\bar{\delta}$-variation[11] commutes–in contrast to the $\delta$-variation–with the derivatives:

$$\bar{\delta} \frac{d}{dt} q = \frac{d}{dt} \bar{\delta} q.$$

Let's see how this comes about: From $\delta q = \hat{q}(\hat{t})$ - $q(t)$

$$\begin{aligned}(\delta q)^{\cdot} = \frac{d}{dt}\delta q &= \frac{d\hat{q}}{d\hat{t}}\frac{d\hat{t}}{dt} - \frac{dq}{dt} = \frac{d\hat{q}}{d\hat{t}} + \frac{d\hat{q}}{d\hat{t}}\frac{d}{dt}\delta t - \frac{dq}{dt} \\ &= \delta \frac{d}{dt} q + \frac{d\hat{q}}{d\hat{t}}\frac{d}{dt}\delta t = \delta \dot{q} + \frac{d\hat{q}}{d\hat{t}}(\delta t)^{\cdot}.\end{aligned}$$

On the other hand, from (2.45)

$$\frac{d}{dt}\bar{\delta} q = (\bar{\delta} q)^{\cdot} = (\delta q)^{\cdot} - (\dot{q}\delta t)^{\cdot} \qquad \bar{\delta}\frac{d}{dt} q = (\bar{\delta}\dot{q}) = \delta\dot{q} - \ddot{q}\delta t$$

so that

$$\frac{d}{dt}\bar{\delta} q - \bar{\delta}\frac{d}{dt} q = (\delta q)^{\cdot} - (\dot{q}\delta t)^{\cdot} - \delta\dot{q} + \ddot{q}\delta t = \frac{d\hat{q}}{d\hat{t}}(\delta t)^{\cdot} - \dot{q}(\delta t)^{\cdot} = \delta\dot{q}(\delta t)^{\cdot} \sim 0$$

and this vanishes because it is a term of second order in the infinitesimal transformations considered here.

In terms of the form variation $\bar{\delta}$ the condition (2.43) becomes

$$\Delta_S L := \bar{\delta}_S L - \frac{d}{dt}(\sigma_S - L\delta_S t) \stackrel{!}{=} 0. \tag{2.46}$$

This is one way to state the invariance of a Lagrangian with respect to a symmetry transformation $\delta_S$. Another form, useful as a further step towards Noether's theorems is to rewrite (2.44) in terms of the Euler-derivative of the Lagrangian:

$$\delta L - \frac{dL}{dt}\delta t = \underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)}_{[L]_q} \bar{\delta} q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\bar{\delta} q\right).$$

[11] The $\bar{\delta}$ notation was already introduced by E. Noether; today it is most often called the "active" variation, mathematicians denote it as Lie variation.

Then (2.43) can be written in the form

$$[L]_q \bar{\delta}_S q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\bar{\delta}_S q + L\delta_S t - \sigma_S\right) \equiv 0,$$

or, if the indexed position variables $q^k$ are re-introduced,

$$[L]_k \bar{\delta}_S q^k + \frac{d}{dt} J_S \equiv 0 \tag{2.47}$$

with

$$J_S(t, q, \dot{q}) := \frac{\partial L}{\partial \dot{q}^k}\bar{\delta}_S q^k + L\delta_S t - \sigma_S. \tag{2.48}$$

In the expression for $J_S$ we identify three terms, which can easily be understood by comparing this with the variation procedure leading to the Euler-Lagrange equations (2.21). In that derivation the time $t$ was not varied, and thus the last term did not show up. Therefore also $\bar{\delta} = \delta$ and (2.48) boils down to (2.21), specialized to $\delta = \delta_S$. The current (2.48) can alternatively be expressed as

$$J_S(t, q, \hat{p}) := \hat{p}_k \delta_S q^k + (L - \hat{p}_k \dot{q}^k)\delta_S t - \sigma_S = \hat{p}_k \delta_S q^k - H_C \delta_S t - \sigma_S \tag{2.49}$$

with the generalized momenta $\hat{p}_k := \frac{\partial L}{\partial \dot{q}^k}$. Here $H_C$ is the Hamiltonian in the case of regular Lagrangians and the "canonical" Hamiltonian in the case of singular systems. As shown in Appendix C, the canonical Hamiltonian is a function of the momenta and the coordinates, despite the fact that the Legendre transformation from the Lagrangian to the Hamiltonian is not invertible.

If in the identity (2.47) we insert the Euler-Lagrange equations in the form (2.6)

$$\left(V_i(q, \dot{q}) - \ddot{q}^k W_{ki}(q, \dot{q})\right) \bar{\delta}_S q^i + \frac{dJ_S(q, \dot{q}, t)}{dt} = 0$$

and observe the dependence on the second derivatives $\ddot{q}$, this splits into the identities

$$V_i \bar{\delta}_S q^i + \dot{q}^i \frac{\partial J_S}{\partial q^i} + \frac{\partial J_S}{\partial t} \equiv 0 \qquad \frac{\partial J_S}{\partial \dot{q}^i} - W_{ik}\bar{\delta}_S q^k \equiv 0. \tag{2.50}$$

For regular systems the symmetry transformations are formally related to the currents by

$$\bar{\delta}_S q^i = \bar{W}^{ik} \frac{\partial J_S}{\partial \dot{q}^k}. \tag{2.51}$$

Inserting these into the first part of (2.50) returns the on-shell conservation of $J_S$ because of $V_i \bar{W}^{ik} = \ddot{q}^k$. Things become different for singular systems; see Appendix C.5. Equation (2.51) has advantages in formal proofs. Inserting into it the explicit form (2.48) of the current, it becomes $\frac{\partial \sigma_S}{\partial \dot{q}^i} = 0$ in the present case of velocity-independent transformations.

It is advantageous to rephrase the Noether condition in terms of the functions $\eta$ and $\xi$ as

$$\delta_\epsilon t = \epsilon\,\xi(t,q) \qquad \delta_\epsilon q^k = \epsilon\,\eta^k(t,q) \qquad \bar{\delta}_\epsilon q^k = \epsilon\,(\eta^k - \dot{q}^k\xi) =: \epsilon\,\chi^k. \quad (2.52)$$

The infinitesimal transformations are generated by the differential operator

$$X = \xi(t,q)\frac{\partial}{\partial t} + \eta^k(t,q)\frac{\partial}{\partial q^k}$$

in that $Xt = \xi$ and $Xq^k = \eta^k$. (On the other hand, the one-parameter subgroup is recovered by integrating the vector field $X$; this is made explicit in Sect. 2.2.4.) For a function $F(t,q)$ we obtain its variation as $\delta_\epsilon F = \epsilon XF$. This can be extended to functions $G(t,q,\dot{q},\ddot{q},\cdots)$ as $\delta_\epsilon G = \epsilon\overline{X}G$ with the vector field

$$\overline{X} = X + \eta^k_{(1)}(t,q,\dot{q})\,\frac{\partial}{\partial \dot{q}^k} + \eta^k_{(2)}(t,q,\dot{q},\ddot{q})\,\frac{\partial}{\partial \ddot{q}^k} + \cdots \quad (2.53)$$

provided the coefficient functions $\eta^k_{(a)}$ are chosen consistently: Consider at first

$$\frac{d\hat{q}}{d\hat{t}} = \frac{dq + \epsilon d\eta}{dt + \epsilon d\xi} = \frac{\dot{q} + \epsilon\dot{\eta}}{1 + \epsilon\dot{\xi}} = (\dot{q} + \epsilon\dot{\eta})(1 - \epsilon\dot{\xi} + \mathcal{O}(\epsilon^2)) = \dot{q} + \epsilon\dot{\eta} - \epsilon\dot{q}\dot{\xi} + \mathcal{O}(\epsilon^2).$$

Thus if $(q^k, t)$ transform according to (2.52) the transformation of $\dot{q}^k$ is fixed to

$$\delta_\epsilon \dot{q}^k = \epsilon\,\eta^k_{(1)}(t,q) \qquad \text{with} \qquad \eta^k_{(1)} = \dot{\eta}^k - \dot{q}^k\dot{\xi}.$$

Notice that for all quantities $g$ the $\dot{g}$ always stands for the total $t$-derivative. Although a little superfluous at this stage, I introduce the differential operator

$$d_t g := \left(\frac{\partial}{\partial t} + \dot{q}^k\frac{\partial}{\partial q^k} + \ddot{q}^k\frac{\partial}{\partial \dot{q}^k} + \cdots\right) g. \quad (2.54)$$

For the higher terms in (2.53) one finds

$$\eta^k_{(I)} = d_t\eta^k_{(I-1)} - (d_t^I q^k)\,(d_t\xi) \qquad \text{with} \qquad \eta^k_{(0)} = \eta^k.$$

In using this expression one is able to show that (2.53) can be written as

$$\overline{X} = \xi d_t + \chi^k\frac{\partial}{\partial q^k} + (d_t\chi^k)\frac{\partial}{\partial \dot{q}^k} + (d_t d_t\chi^k)\frac{\partial}{\partial \ddot{q}^k} + \cdots \quad (2.55)$$

where $\chi^k = \eta^k - \dot{q}^k\xi$ is the expression that became introduced together with the $\bar{\delta}$-transformation; see (2.52). In the mathematics literature the generator $\overline{X}$ is called the "prolongation" of the symmetry generating operator $X$. All those objects $I(t,q,\dot{q},\ddot{q},...)$ are called invariant for which $\overline{X}I = 0$. The concept of using infinitesimal generators for the investigation of symmetries was an essential discovery of Sophus Lie in his attempt to give a systematics to solution methods of differential equations.

After this mathematical prologue let us get back to the Noether theorem: In a Taylor expansion of the right-hand-side in (2.42) one obtains the identity

$$\xi\frac{\partial L}{\partial t}+\eta^k\frac{\partial L}{\partial q^k}+\eta^k_{(1)}\frac{\partial L}{\partial \dot q^k}+L\frac{d}{dt}\xi=\overline{X}L+L\,d_t\xi=d_t\sigma \tag{2.56}$$

(with $\sigma_S=\epsilon\sigma$). This can be interpreted as a partial differential equation to be fulfilled by the functions $\eta^k$ and $\xi$ in a Noether symmetry transformation. You can also read it as a condition which must be fulfilled by the surface part $\sigma$ for given $\eta$ and $\xi$, and gives a clue to derive a possible surface part. Notice that together with $\overline{X}$ also $\hat{X}=\overline{X}+\lambda d_t$ fulfills (2.56) as long $d_t\lambda=0$. Also notice that (2.56) holds as well if the $\xi$ and $\eta^k$ depend on velocities $\dot q$, that is for non-point transformations.

**Noether Charges**

In writing $J_\epsilon=\epsilon\, C$ we have

$$C(q,\dot q,t)=\frac{\partial L}{\partial \dot q^k}\chi^k+L\xi-\sigma,$$

and because of (2.47) this is on-shell conserved, in other words, it is a constant of motion; nowadays also called “Noether charge”. This can compactly be derived by means of the symmetry generator $X$: The condition (2.56) for a Noether symmetry becomes with (2.55)

$$\begin{aligned}0&=\left[\xi\, d_tL+\chi^k\frac{\partial L}{\partial q^k}+(d_t\chi^k)\frac{\partial L}{\partial \dot q^k}\right]+Ld_t\xi-d_t\sigma\\&=d_t\left(L\xi-\sigma\right)+\chi^k\frac{\partial L}{\partial q^k}+d_t\left(\chi^k\frac{\partial L}{\partial \dot q^k}\right)-\chi^k\,d_t\frac{\partial L}{\partial \dot q^k}\\&=d_t\left(L\xi-\sigma+\chi^k\frac{\partial L}{\partial \dot q^k}\right)+\chi^k\,[L]_k\doteq d_tC.\end{aligned}$$

Notice, that the Noether charge may be void of any physical meaning. There are examples where the charge is simply a numerical constant or where the charge vanishes on-shell.

The previous expression of the conserved charge for a one-parameter group of transformations can be generalized to the case of $r$ symmetry transformations

- Write the $(\delta_S t,\delta_S q)$ generically like

  $$\delta_\epsilon t=\epsilon^a\xi_a(t,q)\qquad\qquad \delta_\epsilon q^k=\epsilon^a\eta^k_a(t,q)$$

  with $r$ constant parameters $\epsilon^a$ $(a=1,\ldots,r)$. Correspondingly there are the generators of infinitesimal symmetry transformations $X_a=\xi_a\frac{\partial}{\partial t}+\eta^k_a\frac{\partial}{\partial q^k}$. The functions $\eta^k_a$ and $\xi_a$ are not arbitrary since we assumed that the $\delta_S$ are symmetry transformations. As such they must constitute a group. In infinitesimal form this is expressed in that the commutator of two infinitesimal transformations is another infinitesimal transformation:

$$[X_a, X_b] = X_a X_b - X_b X_a = \Upsilon_{abc} X_c.$$

(In principle we must allow for further terms being proportional to the equations of motion; these are dropped here. We will reconsider them again in 3.3.4.) If the $\Upsilon_{abc}$ are constants, the symmetry group is an $r$-dimensional Lie group. The $\Upsilon_{abc}$ are the structure constants of the associated Lie algebra. With $X_a q^k = \eta_a^k$ and $X_a t = \xi_a$ this becomes the system of differential equations

$$\begin{aligned}(\xi_a \frac{\partial}{\partial t}\eta_b^k + \eta_a^j \frac{\partial}{\partial q^j}\eta_b^k) - (a \leftrightarrow b) &= \Upsilon_{abc}\eta_c^k \\ (\xi_a \frac{\partial}{\partial t}\xi_b + \eta_a^j \frac{\partial}{\partial q^j}\xi_b) - (a \leftrightarrow b) &= \Upsilon_{abc}\xi_c. \end{aligned} \tag{2.57}$$

It can be proven (see e.g. [485]) that the prolongations of the infinitesimal generators obey the same algebra, that is $[\overline{X}_a, \overline{X}_b] = \Upsilon_{abc}\overline{X}_c$, and therefore (2.57) are the only conditions that need to be fulfilled.

- Also, the conserved currents $J_S$ from (2.48) can be expanded with the infinitesimal parameters $\epsilon^a$: Write $\sigma_S = \epsilon^a \sigma_a$ and $J = \epsilon^a C_a$ . Then

$$C_a(t, q, \dot{q}) := \frac{\partial L}{\partial \dot{q} k}\chi_a^k + L\xi_a - \sigma_a \tag{2.58}$$

where the $C_a$ constitute $r$ conserved Noether charges. Observe, that these need not to be independent; as a matter of fact there might be less than $r$ independent conserved charges. One can verify that $C_a$ is an invariant of the Noether symmetry itself: $\overline{X}_a C_a = 0$.

- The symmetry variation of the coordinates can be expressed by the Noether charges

$$\bar{\delta}_\epsilon q^k = \{q^k, \epsilon^a C_a\} \tag{2.59}$$

where $\{,\}$ is the Poisson bracket. This can directly be verified by writing the conserved charge as $C_a(t, q, p) = \hat{p}_j \eta_a^j - H_C \xi_a - \sigma_a$:

$$\{q^k, \epsilon^a C_a\} = \epsilon^a \{q^k, \hat{p}_j\}\eta_a^j - \epsilon^a \{q^k, H_C\}\xi_a,$$

this indeed leading to $\bar{\delta}_\epsilon q^k = \epsilon^a(\eta_a^k - \dot{q}^k \xi_a)$ at least in the regular case for which the $\hat{p}_j$ can be identified with the canonical phase space momenta $p_j$, and for which the Hamilton equations of motion hold with $H(t, q, p) = H_C$.

- The commutator of two $\bar{\delta}$-transformations becomes by (2.59) and the Jacobi identity for the Poisson brackets

$$(\bar{\delta}_2\bar{\delta}_1 - \bar{\delta}_1\bar{\delta}_2)q^k = \epsilon_1^a\epsilon_2^b\Big[\{\{q^k, C_a\}, C_b\} - \{\{q^k, C_b\}, C_a\}\Big] = -\epsilon_1^a\epsilon_2^b\{\{C_a, C_b\}, q^k\}.$$

Since we require that this is again a symmetry transformation $\bar{\delta}_3 q^k = \{q^k, \epsilon_3^c C_c\}$ we obtain

$$\{C_a, C_b\} = \Upsilon_{abc}C_c + Z_{ab}.$$

The constants $Z_{ab}$ are related to the central charges (a term explained in the group theory appendix.)

**Example: Galilei Symmetry Group and its Noether Charges**

The conservation laws derived in the previous section are attributable to Noether symmetries of Newtonian mechanics with respect to the Galilei group. A Lagrange function of the form

$$L = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{\vec{x}}_\alpha^2 - \sum_{\alpha<\beta} V(|\vec{x}_\alpha - \vec{x}_\beta|) \tag{2.60}$$

is (quasi-)invariant with respect to the independent transformations

$$\begin{array}{llr}
\delta_\tau t = \tau & \delta_\tau \vec{x}_\alpha = 0 & \text{time translations} \\
\delta_a t = 0 & \delta_a \vec{x}_\alpha = \vec{a} & \text{space translations} \\
\delta_R t = 0 & \delta_R x_\alpha^i = R^i{}_j x_\alpha^j \quad (R^i{}_j = -R^j{}_i) & \text{space rotation} \\
\delta_v t = 0 & \delta_v x_\alpha^i = v^i t & \text{Galilei boost}
\end{array}$$

where $(\tau, \vec{a}, R, \vec{v})$ are ten infinitesimal constants corresponding to the $\epsilon_a$ above. The generators corresponding to these infinitesimal transformations are

$$H = \partial_t \tag{2.61}$$
$$T_i = \partial_i \tag{2.62}$$
$$M_{ij} = x^i\partial_j - x^j\partial_i \tag{2.63}$$
$$G_{0i} = t\partial_i . \tag{2.64}$$

These form the Galilei algebra (2.75a, 2.75b, 2.75c). You may convince yourself that the Lagrangian (2.60) is invariant with respect to time translations, space translations and rotations. This means that the surface terms $\sigma_\tau, \sigma_a, \sigma_R$ in (2.58) are zero. The Lagrangian is only quasi-invariant with respect to Galileian boosts, namely

$$\delta_v L = \sum m_\alpha\, \delta_v \dot{\vec{x}}_\alpha \cdot \dot{\vec{x}}_\alpha = \sum m_\alpha\, \vec{v} \cdot \dot{\vec{x}}_\alpha$$

such that $\sigma_v = m_\alpha \vec{v} \cdot \vec{x}_\alpha$. The (quasi)-invariances of the Lagrangian (2.60) lead to the following conserved objects:

$$\tau(\sum \vec{p}_\alpha \cdot \dot{\vec{x}}_\alpha - L) = \tau E \tag{2.65a}$$

$$\vec{a} \cdot \sum \vec{p}_\alpha = \vec{a} \cdot \vec{P} \tag{2.65b}$$

$$R \sum \vec{x}_\alpha \times \vec{p}_\alpha = R^i{}_k J^k \tag{2.65c}$$

$$\vec{v} \cdot (\sum m_\alpha \vec{x}_\alpha - \vec{p}_\alpha t) = \vec{v} \cdot \vec{Q}. \tag{2.65d}$$

Here we identify the conserved energy $C_\tau = E$, the conserved (total) linear momentum $\vec{C}_a = \vec{P}$ and angular momentum $\vec{C}_R = \vec{J}$ in the first three relations. The conserved $\vec{C}_v = \vec{Q}$ in the last expression is related to the center of mass $\vec{R}$ and the total momentum $\vec{P}$ as in (2.41). One should be aware that the existence of the ten "standard" conservation laws (2.65a, 2.65b, 2.65c, 2.65d) is tightly bound to the form of the Lagrangian (2.60). Specifically it is essential that the forces on any of the particles depend on the mutual distances $r_{\alpha\beta} = |\vec{x}^i_\alpha - \vec{x}^i_\beta|$.

Covariance with respect to Galilei transformations is not only a characteristic of non-relativistic classical mechanics but also of non-relativistic classical field theory (e.g. fluid dynamics) and of non-relativistic quantum mechanics [446]. More about the latter is dealt with in Subsect. 4.3.4.

You may verify that the equations of motion following from the Lagrange function (2.60) are form-invariant (covariant) with respect to Galilei transformations. But, in general there are more symmetries in the equations of motion than in the action; more about this later when we will consider specifically the Kepler problem.

### 2.2.3 Symmetry and Canonical Transformations

The relation (2.51), valid for regular systems, is an identity in the tangent-bundle. It can be written as a cotangent-bundle expression by defining $G(q, p(q, \dot{q}), t) := J_S(q, \dot{q}, t)$, namely

$$\bar{\delta}_S q^k = \bar{W}^{ki} \frac{\partial G(q, p(q, \dot{q}))}{\partial \dot{q}^i} = \bar{W}^{ki} \frac{\partial p_j}{\partial \dot{q}^i} \frac{\partial G}{\partial p_j} = \frac{\partial G}{\partial p_k} = \{q^k, G\}.$$

The variation of the momenta are calculated as

$$\bar{\delta}_S \hat{p}_k = \bar{\delta}_S \left( \frac{\partial L}{\partial \dot{q}^k} \right) = V_{kj} \delta q^j + W_{kj} \delta \dot{q}^j = -\frac{\partial G}{\partial q^k} - [L]_j W_{ki} \frac{\partial^2 G}{\partial p_i \partial p_j} \doteq \{p_k, G\}.$$

Therefore (at least for regular systems) the Noether symmetry transformations can on-shell be written as infinitesimal canonical transformations. Off-shell it reproduces the canonical transformation of the momenta if

$$W_{ki} \frac{\partial^2 G}{\partial p_i \partial p_j} = \frac{\partial(\bar{\delta}_S q^j)}{\partial \dot{q}^k} = 0.$$

The interrelation of symmetry transformations and canonical transformations is visible also from the following line reasoning: In subsection 2.1.3, we derived the variation of an arbitrary phase-space function under an infinitesimal canonical transformation $\hat{x}^\alpha = x^\alpha + \delta_g x^\alpha$ in the form of (2.20):

$$\delta_g A(x) = \{A, g\}. \tag{2.66}$$

By this, $A$ is invariant under $g$-transformations if $\{A, g\} = 0$. In the special case where $A$ is the Hamiltonian we derive

$$\delta_g H = 0 \quad \leftrightarrow \quad \{H, g\} = 0 \quad \leftrightarrow \quad \frac{d}{dt} g \doteq 0,$$

where the last double arrow holds if $g$ does not depend on time explicitly. Thus: Every infinitesimal canonical generator that has a vanishing Poisson bracket with the Hamiltonian leaves it invariant and is a conserved quantity. Expressed in another way, the generating function of an infinitesimal canonical transformation is a conserved quantity for those systems, whose Hamilton function is invariant with respect to this transformation.

This observation allows us to derive the ten conserved quantities in classical mechanics from homogeneity and isotropy of the space-time continuum:

- The momentum components $p_x, p_y, p_z$ are canonically conjugate to the $x, y, z$-components of the space coordinates and therefore the generating functions of infinitesimal spatial translations in the $x, y, z$-directions. The total momentum is thus conserved for those systems whose Hamilton function is invariant with respect to infinitesimal spatial translations.
- The angular momentum components $J_x, J_y, J_z$ are canonically conjugate to the rotation angles $\alpha_x, \alpha_y, \alpha_z$ and therefore the generating functions of infinitesimal rotations around the $x, y, z$-axis. The total angular momentum is thus conserved for those systems whose Hamilton function is invariant with respect to infinitesimal rotations.
- The object $\vec{G} = \sum_\alpha (m_\alpha \vec{x}_\alpha - \vec{p}_\alpha t)$ is the generator of infinitesimal Galilei-boosts

  $$\hat{\vec{x}}_\alpha = \vec{x}_\alpha + \vec{v}t \qquad \hat{\vec{p}}_\alpha = \vec{p}_\alpha + m_\alpha \vec{v},$$

  with an infinitesimal velocity $\vec{v}$. If the Hamilton function is invariant with respect to infinitesimal Galilei-boosts, $\vec{G}$ is conserved in time, implying together with the conservation of the total momentum that the center of mass moves uniformly.
- The Hamilton function $H$ is canonically conjugate to the time variable $t$ and therefore the generating function of infinitesimal time translations $\hat{t} = t + \tau$. The total energy is conserved for those systems whose Hamilton function is invariant with respect to time translations. And this is the case if the Hamilton function does not depend on time explicitly.

The conserved quantities obey an algebra: If together with $g$ also $g'$ is a Noether charge, the Jacobi identity

$$\big\{\{H, g\}, g'\big\} + \big\{\{g', H\}, g\big\} + \big\{\{g, g'\}, H\big\} = 0$$

reveals that $\{g, g'\}$ is a Noether charge too. If there are finitely many independent conserved quantities, each commutator can be written as a conserved quantity again. In case of classical mechanics, this leads again to the algebra of the Galilei group.

### *2.2.4 Conservation Laws and Symmetries*

**Conservation Laws and Symmetries in Which Sense?**

This Sect. 2.2. carries the title "Symmetries and Conservation Laws", and indeed as derived previously, the Noether symmetries lead to conserved quantities, the Noether charges. To observe that conserved quantities follow from symmetries of an action is quite intriguing. But it also raises many questions such as: Does a symmetry necessarily imply the existence of a conserved quantity? Is there a way to find (all) symmetries of the action? Can one find all conservation laws by Noether's theorem? Is every conservation law a consequence of a symmetry? As we will see, posed this way, the questions cannot really be answered–or put another way–we need to be more precise in questioning[12]. The Noether theorem itself only tells that for an r-parameter Lie group, r linear combinations of Euler-derivatives are expressible as total time-derivatives (or–in field theories–as divergences).

- The mere existence of a symmetry is not at all sufficient to establish a conservation law. Take for instance the equation of motion for a particle under the influence of a frictional force
$$m\ddot{\vec{x}} = -k\dot{\vec{x}}.$$
This equation is invariant under space and time translations, but neither the momentum $m\dot{\vec{x}}$ nor the energy $\frac{m}{2}\dot{x}^2$ are conserved. This is not a counter-example to Noether's theorem, since we are arguing with the equations of motion and thus with Lie symmetries. In any case, the equations of motion can be derived from the Lagrangian
$$L = \frac{1}{2}m\dot{\vec{x}}^2 \exp\left[\left(\frac{k}{m}\right)t\right].$$
This Lagrangian is covariant with respect to space translations, and–in agreement with Noether's first theorem–indeed the generalized momentum $\vec{p} = \partial L/\partial\dot{\vec{x}}$ is

[12] E. Wigner warned against a "facile identification" of symmetry and conservation principles [557] and P. Havas wrote a "Folklore, Fiction, and Fact" article on this topic [255].

on-shell conserved. And although the Lagrangian is not form invariant with respect to time translations, there is a conserved quantity

$$E = \frac{\vec{p}^2}{2m} = \frac{1}{2} m \dot{\vec{x}}^2 \exp\left[2\left(\frac{k}{m}\right) t\right]$$

which, however, is not the numerical value of the Hamiltonian, although for $k = 0$ it becomes identical to the "usual" energy. For a more elaborate analysis of this example see [255].

- When asking for all symmetries of either the action or the equations of motion we need to specify whether we have in mind symmetries with respect to point transformations $\delta q^k(q, t)$, $\delta t(t, q)$ only, or whether more general transformations are to be considered.
- Even if all variational symmetries of a Lagrangian were known, does the Noether theorem guarantee that all conservation laws are found? Not at all: For a system with $N$ degrees of freedom there are $N$ equations of motion, that is $N$ second order differential equations for functions $q^i(t)$. In their solutions there are $2N$ free constants, which can be determined by the initial values of the coordinates and the velocities. If the system exhibits symmetry under Galilei transformations, at most 10 of these constants may be expressed through the standard Noether charges (2.65a, 2.65b, 2.65c, 2.65d). The authors of [332] emphasize that the distinctive role of the Galilei symmetry group induced constants of motion may be seen in that these do have the important property of being additive.
- It also may happen that the conserved charges are algebraically related, although being due to different symmetry transformations (as will be shown later on examples). In still another case a conservation law might be identically fulfilled; sometimes called a "strong" conservation law in order to distinguish it from a "weak" conservation law being valid on the solutions of the dynamical equations only.
- There are examples where two different Lagrangians–not related by a total derivative–lead to the same equations-of-motion, and where two completely different Noether symmetry transformations lead to the same conserved charge. Take the standard Lagrangian for the two-dimensional oscillator

$$L = \frac{1}{2}\left[(\dot{q}_1^2 + \dot{q}_2^2) - \omega^2(q_1^2 + q_2^2)\right]$$

which is invariant under rotations, giving rise to the conservation of angular momentum $q_1\dot{q}_2 - q_2\dot{q}_1$. The same dynamics is derived from the Lagrangian

$$L' = \dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2.$$

This Lagrangian is not invariant under rotations, but instead under $(q_1, q_2) \rightarrow (e^{\alpha}q_1, e^{-\alpha}q_2)$, from which again the conservation of angular momentum results.

- The "inverse Noether theorem" deals with the question under which circumstances the existence of a conserved quantity relates to a Noether symmetry[13]. The problem of the inverse Noether theorem relates to generalization of Noether's theorem to velocity-dependent transformations [463]. Again, the Hessian (2.7) plays a crucial role here. It was proven [67], that if $C(q, \dot{q}, t)$ is a constant of motion and if the Hessian $W^{ij}$ is invertible, the infinitesimal transformation associated with $C$, namely
$$\bar{\delta} q^i = \epsilon \bar{W}^{ij} \frac{\partial C}{\partial \dot{q}^j} \qquad \delta t = L^{-1}\left[\epsilon C - \frac{\partial L}{\partial \dot{q}^j} \bar{\delta} q^j\right]$$
is a symmetry transformation for the Lagrangian $L$. Assume, for example that the energy $E = (\partial L/\partial \dot{q}^j)\dot{q}^j - L$ is conserved. Then the previous defining equations result in $\bar{\delta} q^i = \epsilon \dot{q}^i$ and $\delta t = -\epsilon$ (and thus $\delta q^i = 0$).

### Lie Symmetries and Noether Symmetries

The previous observations about the (non)-relation between conservation laws and symmetry transformations seems to leave us behind in a hard-to-reach landscape. However, as it turns out, the investigation of symmetries of differential equations is more feasible than the investigation of the Lagrangian from which they are possibly derived as Euler-Lagrange equations; see the comprehensive [399]. Remember that Lie symmetries are symmetries of the equations of motion in the sense that if $q^k(t)$ is a solution, the transformed

$$\hat{q}^k(\hat{t}) = q^k(t) + \epsilon \eta^k(t, q) + \mathcal{O}(\epsilon^2) \qquad \hat{t} = t + \epsilon \xi(t, q) + \mathcal{O}(\epsilon^2)$$

is a solution, too ("mapping solutions to solutions"). Although the following expressions can be defined for a system of differential equations $\Delta^\beta(x_1, ..x_p; u^\alpha(x), u^\alpha_{,i}(x), ..., u^\alpha_{,i_1 \dots i_q}(x)) = 0$ with $p$ independent variables and derivatives of the dependent variables $u^\alpha(x)$ of arbitrary order, I will take the example of classical mechanics with one independent variable $t$ and functions $q^k$ appearing up to the second derivative, that is
$$\Delta^k(t; q, \dot{q}, \ddot{q}) \equiv \Delta^k(\hat{t}; \hat{q}, \hat{\dot{q}}, \hat{\ddot{q}}) = 0.$$

Then we can directly use the operator $\overline{X}$ as given by (2.53) or alternatively by (2.55) to define a Lie symmetry group by transformations which obey

$$\overline{X} \Delta^k{}_{|\Delta^j = 0} = 0. \tag{2.67}$$

All independent vector fields that fulfill this relation are point symmetries of the equations of motion. Furthermore, one is dealing with a Noether-Bessel-Hagen point

[13] Do not get confused here: Noether herself proved that her theorems do have a converse in the sense that the existence of r Euler derivatives which are divergences implies the invariance of an action.

symmetry associated to a Lagrangian $L$, if a "boundary term" $\sigma(t, q)$ exists such that

$$\overline{X}L + L d_t \xi = d_t \sigma. \tag{2.68}$$

Thus, if one knows all Lie symmetries (with respect to a class of transformations) admitted by the Euler-Lagrange equations then one can find all variational symmetries by checking which of these Lie symmetries are symmetries of the action.

### One-Dimensional Free Particle and Harmonic Oscillator

Rather unexpected results do follow from the previous considerations for the most simple system in classical mechanics, namely the one-dimensional free particle and the harmonic oscillator

$$L = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2 \qquad\qquad \Delta = \ddot{q} + \omega^2 q = 0$$

(after an appropriate rescaling such that $m = 1$). In this case, after a straightforward calculation the condition (2.67) becomes explicitly

$$\eta_{tt} + (2\eta_{tq} - \xi_{tt})\dot{q} + (\eta_{qq} - 2\xi_{qt})\dot{q}^2 - \xi_{qq}\dot{q}^3 + \omega^2\eta - \omega^2 q(\eta_q - 2\xi_t - 3\xi_q\dot{q}) = 0.$$

Here the indices on $\eta$ and $\xi$ denote derivatives with respect to $t$ and $q$. The previous condition contains terms proportional to $\dot{q}^n$ $(n = 0, ..., 3)$. These must vanish separately, and thus we get a system of differential equations for $\eta$ and $\xi$. These are called the 'defining equations' for the symmetry generators.

For the free particle with $\omega = 0$ the most general solution of the defining equations result in

$$\xi(t, q) = \alpha_0 + \alpha_1 t + b_1 t^2 + \beta_0 q + \beta_1 tq, \qquad \eta(t, q) = a_0 + a_1 q + \beta_1 q^2 + b_0 t + b_1 qt$$

with eight constants $\alpha_i, \beta_i, a_i, b_i$. Thus, for the one-dimensional free particle equation there are eight independent point transformations defined by the vector fields

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t} && X_2 = t\frac{\partial}{\partial t} && X_3 = q\frac{\partial}{\partial t} \\
&X_4 = t^2\frac{\partial}{\partial t} + tq\frac{\partial}{\partial q} && X_5 = qt\frac{\partial}{\partial t} + q^2\frac{\partial}{\partial q} && \\
&X_6 = \frac{\partial}{\partial q} && X_7 = q\frac{\partial}{\partial q} && X_8 = t\frac{\partial}{\partial q}.
\end{aligned}$$

Although of course this example is not at all relevant for fundamental physics it is quite instructive to immediately interpret the effect of the vector fields in terms of finite transformations. These can of course be found by using the exponential map:

$$\hat{q} = e^{aX} \cdot q \qquad\qquad \hat{t} = e^{aX} \cdot t.$$

Sometimes there is an easier procedure: Remember that the components of the vector fields are defined as

$$\frac{d\hat{q}}{d\epsilon}\Big|_{\epsilon=0} = q + \epsilon\eta + \epsilon^2 \cdots = \eta \qquad\qquad \frac{d\hat{t}}{d\epsilon}\Big|_{\epsilon=0} = t + \epsilon\xi + \epsilon^2 \cdots = \xi.$$

Therefore–knowing $\xi$ and $\eta$ from the vector fields $X$, we need to solve the first order equations

$$\frac{d\hat{q}}{da} = \eta(\hat{t}(a), \hat{q}(a)) \qquad\qquad \frac{d\hat{t}}{da} = \xi(\hat{t}(a), \hat{q}(a))$$

with initial values $\hat{q} = q$ and $\hat{t} = t$ for $a = 0$. For the generators $X_1$ and $X_6$ the finite transformations are easily found as

$$T_1 : \hat{q} = q; \quad \hat{t} = t + a \qquad\qquad T_6 : \hat{q} = q + a; \quad \hat{t} = t,$$

that is constant translations of $t$ and $q$, respectively[14]. For $X_2$ and $X_7$ the transformations are rescalings:

$$T_2 : \hat{q} = q; \quad \hat{t} = e^a t \qquad\qquad T_7 : \hat{q} = e^a q; \quad \hat{t} = t.$$

The finite transformations corresponding to $X_3$ and $X_8$ are

$$T_3 : \hat{q} = q; \quad \hat{t} = t + aq \qquad\qquad T_8 : \hat{q} = q + at; \quad \hat{t} = t$$

which we might call Galilei boosts (although only the latter is a genuine one). More tricky are the integrations of the differential equations implicated by $X_4$ and $X_5$. Let us instead exponentiate the infinitesimal transformations

$$\begin{aligned}
\hat{q} = e^{aX_4} \cdot q &= \left[1 + atq\partial_q + \frac{1}{2}(atq\partial_q)(atq\partial_q) + ...\right] q \\
&= q + atq + ... + (at)^k q + ... = \frac{q}{1 - at} \\
\hat{t} = e^{aX_4} \cdot t &= \left[1 + at^2\partial_t + \frac{1}{2}(at^2\partial_t)(at^2\partial_t) + ...\right] t \\
&= t + at^2 + ... + a^k t^{k+1} + ... = \frac{t}{1 - at}.
\end{aligned}$$

[14] Of course every generator has its own constant $a$. Thus, for example the constant in $T_1$ has the dimension of time ($a = t_0$) and for $T_6$ it has dimension length ($a = q_0$).

As a result we observe that $X_4$ and $X_5$ generate the specific projective transformations

$$T_4 : \hat{q} = \frac{q}{1-at}; \quad \hat{t} = \frac{t}{1-at} \qquad T_5 : \hat{q} = \frac{q}{1-aq}; \quad \hat{t} = \frac{t}{1-aq}.$$

It can be shown that the generators $X_1, ...X_8$ obey an algebra which is isomorphic to $sl(3, \mathbb{R})$ [357]. It can also be shown that the finite transformations are the projective transformations

$$\hat{t} = \frac{At + Bq + C}{Lt + Mq + N} \qquad \hat{q} = \frac{Dt + Eq + F}{Lt + Mq + N}.$$

These are all those transformations that map a straight line in a plane into a straight line. Further it can be proven that a Lie group of order eight is the largest symmetry group for point transformations for second order equations of motion [357].

Which of the generators above are Noether symmetries? In calculating according to (2.68) the expressions $X_\alpha L + LD\xi_\alpha$ one finds that $X_1$, $X_4$, $X_6$, and $X_8$ are Noether symmetries with $\sigma_\gamma = \{0, \frac{1}{2}q^2, 0, q\}$. Further the linear combination $X_+ = 2X_2 + X_7$ is found to be a generator of a Noether symmetry. Thus there are five independent Noether symmetry transformations for the free particle in one-dimension. This should come as a surprise since–knowing about the Galilei group–we could expect only three symmetries related to time translation, 1D space translations and 1D Galilei boosts. In [357] it is shown, that five is the maximal number of independent generators for the variational symmetries of a one-dimensional system. Which are the conserved charges according to (2.58)? One finds for instance $C_6 = \dot{q}$, $C_8 = (\dot{q}t - q)$, which are indeed obviously on-shell conserved. The other charges depend algebraically on these: $C_1 = -{}^1\!/\!{}_2(C_6)^2$, $C_4 = -{}^1\!/\!{}_2(C_8)^2$, $C_+ = -C_6 C_8$. We should be surprised to get more than two independent constants of motion since the solution $q(t) = v(t - t_0)$ has two constants; we find $C_6 = v$, $C_8 = vt_0$. Let me remark here that already Noether observed that the action of the generator $X_8$ leading to a quasi-invariance can be realized by the generator

$$\tilde{X}_8 = -2\frac{q}{\dot{q}^2}\frac{\partial}{\partial t} + \left(t - 2\frac{q}{\dot{q}}\right)\frac{\partial}{\partial q}.$$

This is no longer a point transformation, since now $\tilde{\xi} = \tilde{\xi}(t, q, \dot{q})$, $\tilde{\eta} = \tilde{\eta}(t, q, \dot{q})$ depend on velocities. In [285] you find a constructive proof that any quasi-invariant point transformation is equivalent to a velocity-dependent transformation that leaves the Lagrangian strictly invariant.

If you track the previous considerations for the harmonic oscillator, you will find that there are seven independent symmetry generators for the equations of motion, five Noether symmetries, and–not a surprise–two independent constants of motion, for instance

$$E = \frac{1}{2}(\dot{q}^2 + \omega^2 q^2) \doteq \frac{1}{2} k Q^2 \qquad G = q\omega \cos\omega t - \dot{q} \sin\omega t \doteq \sin\omega t_0\, \omega Q$$

given the solution $q(t) = Q \sin\omega(t + t_0)$ with two constants $Q$ and $t_0$. Without going into the full calculations observe that energy conservation can be traced to two different transformations: The Lagrangian is invariant with respect to $\delta_\tau\{q, t\} = \{0, \tau\}$ and thus the energy $E$ is conserved with $\dot{E} = -\dot{q}\, [L]_q$. However, the Lagrangian is also quasi-invariant with respect to $\delta_\epsilon\{q, t\} = \{\epsilon \sin\omega t, 0\}$ with $\sigma = \omega q \cos\omega t$ . Therefore the Noether theorem delivers the constant of motion $G$ with $\dot{G} = \sin\omega t\; [L]_q \doteq 0$. Thus, although the two constants are simply related as $E \propto G^2$, the respective symmetry transformations are completely unrelated.

### Free Point Particle in Three Dimensions

We know that the action of the free point particle is invariant under transformations of the 10-dimensional Galilei group. But is this the maximal symmetry group? To find this maximal group we first determine the Lie point symmetries, and then check which of these are Noether point symmetries. (In [300] these are straightforwardly derived from the quest that $\int dt (\frac{dq^i}{dt})^2 = \int d\hat{t} (\frac{d\hat{q}^i}{d\hat{t}})^2$.)

The defining equations for the vector field components of $X = \xi\partial_t + \eta^i \partial_i$ are found to be

$$\xi_{kj} = 0 \qquad \eta^i_{jk} = \delta^i_j\, \xi_{kt} + \delta^i_k\, \xi_{jt} \qquad 2\eta^i_{tk} = \delta^i_k \xi_{tt} \qquad \eta^i_{tt} = 0.$$

The most general solution contains 24 parameter. The 24 independent vector fields are

$$\partial_t \qquad t\partial_t \qquad t^2\partial_t + tq^i\partial_i \qquad q^i\partial_t \qquad tq^j\partial_t + q^j q^i \partial_i \qquad \partial_i \qquad t\partial_i \qquad q^K \partial_I .$$

Again it turns out that only a subset of these generators are generators of variational symmetries, namely the 12 generators

$$\begin{aligned} &H = \partial_t \qquad T_i = \partial_i \qquad L_k = \epsilon_{kij} q^i \partial_j \\ &G_i = t\partial_i \qquad S = 2t\partial_t + q^i \qquad \partial_i C = t^2\partial_t + tq^i\partial_i . \end{aligned}$$

One verifies that $\{H, T_i, L_i\}$ fulfill the Euclid algebra

$$\begin{aligned} &[H, T_i] = 0 \qquad [H, J_i] = 0 \qquad [H, T_i] = 0 \\ &\left[T_i, T_j\right] = 0 \qquad \left[T_i, L_j\right] = \epsilon_{ijk} T_k \qquad \left[L_i, L_j\right] = \epsilon_{ijk} L_k . \end{aligned}$$

Further $\{H, T_i, L_i, G_i\}$ fulfill the Galilei algebra with

$$[H, G_i] = T_i \qquad \left[T_i, G_j\right] = 0 \qquad \left[G_i, L_j\right] = \epsilon_{ijk} G_k \qquad \left[G_i, G_j\right] = 0$$

and the two additional generators $S$ and $C$ obey

$$\begin{array}{ccccc} & [H,S]=2H & [T_i,S]=T_i & \left[L_i,S\right]=0 & \left[G_i,S\right]=-G_i \\ [H,C]=S & \left[T_i,C\right]=G_i & \left[L_i,C\right]=0 & \left[G_i,C\right]=0 & [S,C]=2C. \end{array}$$

This algebra is also known as Schrödinger algebra $\mathfrak{sch}(1,3)$ (without central extension), since prior to its discovery for the classical free particle it was found for the free Schrödinger equation. The Schrödinger algebra plays a similar role in non-relativistic physics as the conformal algebra plays in relativistic physics; see also Sect. 4.3.4.

The Lagrangian $L = \frac{m}{2}(\dot{q}^i)^2$ is invariant with respect to the transformations generated by $H, T_i, L_k, S$. It is quasi-invariant for $G_i$ (with $\Sigma_{G_i} = mq^i$) and for $C$ (with $\Sigma_C = \frac{m}{2}q^2$). Which are the charges? The

$$C_\alpha = m\dot{q}_i\eta^i_\alpha - \frac{1}{2}m\xi_\alpha q^2 - \Sigma_\alpha$$

become explicitly

$$\begin{aligned} &C_H = -\frac{m}{2}\dot{q}^2 =: -E \qquad C_{T_i} = m\dot{q}_i =: P_i \qquad C_{L_i} = m\epsilon_{kli}q^l\dot{q}^i =: J_i \\ &C_{G_i} = m(t\dot{q}_i - q_i) =: Q_i \\ &C_S = m\dot{q}_i(q_i - t\dot{q}_i) = D \qquad C_C = \frac{m}{2}(t\dot{q}_i - q_i)(t\dot{q}^i - q^i) = R. \end{aligned}$$

These 12 charges are not independent but can all be expressed algebraically by $P_i$ and $Q_i$:

$$E = \frac{1}{2m}P^2 \qquad J_i = \frac{1}{m}\epsilon_{ijk}P_jQ_k \qquad D = -\frac{1}{m}P_iQ^i \qquad R = \frac{1}{2m}Q^2.$$

The very fact, that there are only six independent constants of motion agrees with the fact that the solution of the system of differential equations is $q^i(t) = v^it + q^i_0$ which has six integration constants $v^i$ and $q^i_0$. The solution can as well be expressed by the six conserved charges as $q^i(t) = \frac{1}{m}(P^it - Q^i)$.

The finite transformations for the 12-parameter symmetry group $\mathbf{G_{12}}$ are

$$\hat{q}^i = \frac{R^{ij}q^j + a^i + v^it}{\gamma t + \delta} \qquad \hat{t} = \frac{\alpha t + \beta}{\gamma t + \delta} \qquad \text{with } \alpha\delta - \beta\gamma = 1, \qquad R^TR = 1.$$

There are two special cases, namely (i) $\beta = 0 = \gamma$, $\alpha = 1 = \delta$ with group transformations

$$g: \qquad \hat{\vec{q}} = R\vec{q} + \vec{a} + \vec{v}t \qquad \hat{t} = t$$

constituting the 9-parameter static Galilei group $\mathbf{G_9}$, and (ii) $\vec{a} = 0 = \vec{v}$, $R = 1$ with

$$\sigma: \qquad \hat{\vec{q}} = \frac{\vec{q}}{\gamma t + \delta} \qquad \hat{t} = \frac{\alpha t + \beta}{\gamma t + \delta} \qquad \text{with} \qquad \alpha\delta - \beta\gamma = 1$$

which is the 3-parameter group $\mathbf{SL(2, R)}$ containing time translations ($\gamma = 0$, $\alpha = 1 = \delta$), scale transformations ($\beta = 0 = \gamma$) and so-called expansions ($\beta = 0$, $\alpha = 1 = \delta$). From the group composition one finds ([300]) that $\mathbf{G_9}$ is an invariant subgroup of the full group, and that $\mathbf{G_{12}} = \mathbf{G_9} \ltimes \mathbf{SL(2, R)}$.

### The Kepler Problem

Given that the explanation of Kepler's three laws by Newton was the very success story of classical mechanics, we dare to ask how this is related to both Lie and Noether symmetries. The two-body Kepler problem is defined by the Lagrangian

$$L = \frac{1}{2} M \dot{\vec{x}}^2 + \frac{\alpha}{|\vec{x}|}$$

where $\vec{x} = \vec{x}_1 - \vec{x}_2$ and $M = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass (see e.g. [332], sect.13). For the three degrees of freedom $q^k$, the equations of motion are

$$\Delta^k(q, \ddot{q}) = M\ddot{q}^k + \frac{\alpha q^k}{r^3} = 0 \qquad \text{with} \qquad r^2 = \sum_{k=1}^{3} q^k q^k.$$

The Lie point symmetry transformations, found from determining the vector field $\overline{X}$ for which on-shell $\overline{X}\Delta^k = 0$, are

$$\overline{X}_k = \epsilon_{kij}\, q^j \frac{\partial}{\partial q^i} + \epsilon_{kij} \dot{q}^j \frac{\partial}{\partial \dot{q}^i} \tag{2.69a}$$

$$\overline{X}_4 = \frac{\partial}{\partial t} \tag{2.69b}$$

$$\overline{X}_5 = t \frac{\partial}{\partial t} + \frac{2}{3} q^j \frac{\partial}{\partial q^j} - \frac{1}{3} \dot{q}^j \frac{\partial}{\partial \dot{q}^j}. \tag{2.69c}$$

The three generators $X_k$ are recognized as those of the three-dimensional rotation group; indeed $r^2$ is an invariant under these generators: $X_k r^2 = 0$. The generator $X_4$ corresponds to time translations. But what is the meaning of $X_5$? We verify that this generator leaves invariant the quantity $s = t^2/r^3$:

$$X_5\left(\frac{t^2}{r^3}\right) = \frac{1}{r^3} t \frac{\partial t^2}{\partial t} + t \frac{2}{3} q^j \frac{\partial}{\partial q^j} \frac{1}{r^3} = \frac{2t^2}{r^3} - \frac{2t^2}{3} q^j \frac{3}{r^4} \frac{\partial r}{\partial q^j} = 0.$$

This reflects a scale invariance: If $t \to \lambda t$, $q^k \to \lambda^{2/3} q^k$ the quantity $s$ is left invariant. And indeed with $\lambda = (1+\epsilon)$

$$\hat{q}^k = \lambda^{2/3} q^k = (1+\epsilon)^{2/3} q^k \sim q^k + \frac{2}{3}\epsilon q^k \qquad\qquad \hat{t} = (1+\epsilon)t = t + \epsilon t$$

we confirm (2.69c). Although $s$ is an invariant of $X_5$, it is not a generator of a Noether symmetry. One finds $X_5 L + D\xi_5 = (1/3)L$, and this cannot be written as a term $D\sigma$. The other four generators (2.69a, 2.69b) describe Noether symmetries. Their conserved charges are the angular momentum and the energy:

$$J_k = M\epsilon_{kij} q^i \dot{q}^j \qquad E = \frac{M}{2}\dot{q}^2 - \frac{\alpha}{r}.$$

Now, you may be aware that the Kepler problem has a further conserved quantity, known as the Laplace-Runge-Lenz vector. For a long time it was unknown whether this can be derived from a Noether symmetry[15]. Indeed this is possible if one goes beyond point transformations. I remarked already that the Noether-Bessel-Hagen identity (2.56) holds even for the case that the $\xi$ and $\eta^k$ depend on velocities $\dot{q}$. In fact, this identity now splits into two identities, namely one which contains second derivatives $\ddot{q}$, and the rest. The second derivatives appear linearly, and thus the first part yields three identities

$$L\frac{\partial \xi}{\partial \dot{q}^j} + \frac{\partial L}{\partial \dot{q}^k}\left(\frac{\partial \eta^k}{\partial \dot{q}^j} - \dot{q}^k \frac{\partial \xi}{\partial \dot{q}^j}\right) = \frac{\partial \sigma}{\partial \dot{q}^j}.$$

In case of the Kepler problem, it turns out that transformations with $\xi = 0$ and

$$\eta^{(i)k}(q,\dot{q}) = 2q^i \dot{q}^k - q^k \dot{q}^i - \delta^{ki}(q^j \dot{q}^j)$$

render the Lagrangian quasi-invariant. The Noether charges belonging to this variational symmetry are found to be

$$A^k = -M^2\left[\dot{q}^2 q^k - (q^j \dot{q}^j)\dot{q}^k\right] + M\alpha\frac{q^k}{r}.$$

These are the components of the Laplace-Runge-Lenz vector $\vec{A} = -\vec{p} \times \vec{J} + M\alpha\,\vec{x}/|\vec{x}|$. Its conservation has as the consequence that the two masses revolve on elliptic orbits (first Kepler law). The conservation of angular momenta can be expressed as the second Kepler law, and the invariance of $s = t^2/r^3$ amounts to Kepler's third law. In conclusion, Kepler's laws do have their origin in variational and in Lie symmetries.

Let me point out a peculiarity here, which makes one to understand, that variational symmetries need not to be Noether symmetries. With regard to a compact notation let me modify the generator $X_4$ to $H := X_4 - D$ (remember that adding to an infinitesimal symmetry generator a multiple of the operator $D$ is again a symmetry

[15] This symmetry is in the literature also designated as a "hidden" or a "dynamical" symmetry.

generator). Then the algebra of the seven Noether generators $\{X_k, H, Y^i = \eta^{(i)k}\frac{\partial}{\partial q^k}\}$ becomes

$$\begin{aligned} \left[X_i, X_j\right] &= \epsilon_{ijk} X_k & \left[X_i, Y^k\right] &= \epsilon^k{}_{ij} Y^j \quad \left[X_j, H\right] = 0 \\ \left[Y^i, Y^j\right] &= -2E\,\epsilon^{ijk} X_k + 2J_k\,\epsilon^{ijk} H & \left[H, Y^j\right] &= 0. \end{aligned}$$

The generators form an algebra, but not a Lie algebra because the structure coefficients in $\left[Y^i, Y^j\right]$ are not numerical constants.

### 2.2.5 *Noether–Geometrically*

In this subsection, the geometrical description of the first Noether theorem will be given under the simplifying assumption that there is no explicit time dependence[16]. Thus we will deal with strict invariance of a Lagrangian and exclude quasi-invariance from the considerations.

**The Noether Theorem in Lagrangian Form**

Point transformations $q^k \to \hat{q}^k(q)$ are diffeomorphism $\mathbb{Q} \to \mathbb{Q}$ inducing transformations

$$\dot{q}^k \to \frac{d}{dt}\hat{q}^k(q) = \frac{\partial \hat{q}^k}{\partial q^j}\dot{q}^k.$$

Therefore, in general, an infinitesimal point transformation is represented by a vector field $X \in \mathfrak{X}(T\mathbb{Q})$ of the form

$$X = X^i(q)\frac{\partial}{\partial q^i} + \left(\frac{d}{dt}X^i(q)\right)\frac{\partial}{\partial \dot{q}^i}. \tag{2.70}$$

If the Lagrangian is invariant under a transformation mediated by this vector field (that is $\delta q^i = X^i$), we have $\delta L := \pounds_X L = 0$; then, in coordinates

$$\begin{aligned} 0 = \pounds_X L = X^i \frac{\partial L}{\partial q^i} + \left(\frac{d}{dt}X^i\right)\frac{\partial L}{\partial \dot{q}^i} &= X^i[L]_i + X^i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \\ + \left(\frac{d}{dt}X^i\right)\frac{\partial L}{\partial \dot{q}^i} &= X^i[L]_i + \frac{d}{dt}\left(X^i \frac{\partial L}{\partial \dot{q}^i}\right) \end{aligned} \tag{2.71}$$

[16] As for generalizations beyond point transformation in a coordinate-independent way there is by now a rich literature beginning with A. Trautman [510]; see also [14, 192].

showing that on-shell $([L]_i = 0)$ the quantity $\Sigma := X^i(\partial L/\partial \dot{q}^i)$ is conserved. In a chart-independent way this becomes

$$\pounds_\Delta(i_X \theta_L) = 0,$$

where $\theta_L$ is the Cartan one-form defined by (2.29).

**The Noether Theorem in Hamiltonian Form**

The Hamiltonian Noether theorem is an immediate consequence of (2.37): Let $H$ be invariant under the one-parameter flow $\Phi^g$ mediated by a vector field $X_g$ according to (E.3):

$$0 = \pounds_{X_g} H = \{H, g\} = -\pounds_\nabla g.$$

Thus $g$ is a constant of motion. The canonical transformation generated by $X_g$ preserves the equations of motion since together with (2.38) $[X_g, \nabla] = X_{\{g,H\}} = 0$. The reverse is also true: If the phase space function $g$ is a constant of motion its associated vector field $X_g$ generates an infinitesimal symmetry. Further, if two infinitesimal symmetries are given with constants of motion $(g, g')$, then the commutator is also a symmetry with an associated constant of motion $\{g, g'\}$. This can be shown by (2.38) and the Jacobi identity for Poisson brackets.

Notice that in terms of geometry the Hamiltonian version of the Noether theorem is more straightforward than in the Lagrangian version. (It also leads more directly into the quantum version.) If the Lagrangian is regular, the constants of motion are related by $\mathcal{F}L^*(g) = i_{X_g}\theta_L$.

## 2.3 Galilei Group

### *2.3.1 Transformations and Invariants of Classical Mechanics*

Each of the following $\mathbf{G_i}$ constitutes a symmetry group:

| | | |
|---|---|---|
| Time translations: | $\mathbf{G}_{\emptyset} : \hat{t} = t + \tau$ | (1 parameter) |
| Space translations: | $\mathbf{G_a} : \hat{\vec{x}} = \vec{x} + \vec{a}$ | (3 parameter) |
| Rotations: | $\mathbf{G}_R : \hat{\vec{x}} = \mathbf{R} \cdot \vec{x}$ | (3 parameter since $\mathbf{R} \in \mathbf{SO(3)}$) |
| Galilei boosts: | $\mathbf{G}_v : \hat{\vec{x}} = \vec{x} + \vec{v}t$ | (3 parameter) |

Furthermore

| | | |
|---|---|---|
| Time reversal: | $\mathbf{G}_T = \{1, T\}$ | $T : t \to -t$ |
| | is a symmetry if | $L = a^{ik}\dot{q}_i\dot{q}_k - U(q)$. |
| Space inversion: | $\mathbf{G}_P = \{1, P\}$ | $P : q_i \to -q_i$ |
| | is a symmetry if | $U = U(\lvert q_i - q_k \rvert)$. |

The largest symmetry group of classical mechanics is the union of all $\mathbf{G}_i$. This is the Galilei group **Gal**. The elements of this group leave distances $\Delta_x := \sqrt{|\vec{x}_i - \vec{x}_j|^2}$ and time intervals $\Delta_t := |t_i - t_j|$ invariant. $\Delta_x$ are the invariants of the three-dimensional Euclidean space $E^3$, $\Delta_t$ is the invariant of the line $E^1$. The product $A := E^1 \times E^3$ is called Aristotelian space-time. Since the Galilei boosts transform in a space "mixing" a time dependence into translations they are symmetries of a space which locally is an Aristotelian space-time, but globally a fibre bundle (with base space $E^1$ and fibre $E^3$), which constitutes Galilean space-time [410]. This exemplifies that the symmetries of classical mechanics are related to an underlying geometry. In the next chapter we will see that the symmetries of relativistic physics are also related to some specific geometry, namely Minkowski spacetime. Since $\mathbf{G}_T$ and $\mathbf{G}_P$ are discrete groups the full Galilei group invariance group is both discrete and continuous. The continuous part $\mathbf{Gal}_c$ contains the rotations, Galilei boosts, space translations and time translations. The elements of this 10-parameter Lie group can be denoted as

$$\mathbf{Gal}_c \ni g = (\tau,\ \vec{a},\ \vec{v},\ \mathbf{R})$$

with the composition law

$$g' \circ g = (\tau' + \tau,\ \vec{a}' + \mathbf{R}'\vec{a} + \vec{v}'\tau,\ \vec{v}' + \mathbf{R}'\vec{v},\ \mathbf{R}'\mathbf{R}). \tag{2.72}$$

The neutral element of $\mathbf{Gal_c}$ is $g_0 := (0, \vec{0}, \vec{0}, \mathbf{1})$. The inverse to g is

$$g^{-1} = (-\tau,\ \mathbf{R}^{-1}(\vec{v}\tau - \vec{a}),\ -\mathbf{R}^{-1}\vec{v},\ \mathbf{R}^{-1}). \tag{2.73}$$

The group $\mathbf{Gal}_c$ is non-Abelian. Observe that the group composition (2.72) and the expression for the inverse of a group element (2.73) look rather weird. As a matter of fact, the Galilei group is an awkward group, especially compared to the Poincaré group of which it is a limiting case (speed of light going to infinity)–or, in mathematical terms–a group contraction, as explained in the next chapter.

### *2.3.2 Structure of the Galilei Group*

Some specific subgroups of $\mathbf{Gal}_c$ are

- As a set $\mathbf{Gal}_c = \{\mathbf{G}_R, \mathbf{G}_v, \mathbf{G}_a, \mathbf{G}_\tau\}$, and each $\mathbf{G}_I$ is a subgroup (with further subgroups). Of these subgroups only $\mathbf{G}_R$ is non-Abelian (generically). Further non-trivial subgroups are the sets

$$\begin{gathered}\{\mathbf{G_ø}, \mathbf{G_a}\}, \quad \{\mathbf{G_ø}, \mathbf{G_R}\}\\ \{\mathbf{G_a}, \mathbf{G_R}\} \quad \{\mathbf{G_a}, \mathbf{G_v}\}\\ \{\mathbf{G_R}, \mathbf{G_v}\}\\ \{\mathbf{G_ø}, \mathbf{G_a}, \mathbf{G_R}\}, \quad \{\mathbf{G_ø}, \mathbf{G_a}, \mathbf{G_v}\}.\end{gathered}$$

- Rotations and translations are the automorphisms of $E^3$. They form the Euclidean group $\mathbf{G_E} = \mathbf{G_R} \ltimes \mathbf{G}_a$.
- The proper orthochronous[17] Galilei group $\mathbf{Gal}_+^\uparrow$ is generated by rotations and Galilei boosts

$$\begin{aligned}\vec{x} &\mapsto \vec{x}' = \mathbf{R}(\vec{\alpha}) \cdot \vec{x} \qquad \mathbf{R} \in \mathbf{SO(3)}\\ \vec{x} &\mapsto \vec{x}' = \vec{x} + \vec{v}t.\end{aligned}$$

  We may compile a group element $(R, v) \in \mathbf{Gal}_+^\uparrow \subset \mathbf{GL(4, R)}$ from a rotation and a boost as

$$(\vec{v}, \mathbf{R}) = (\vec{v}, \mathbf{1}) \circ (\vec{0}, \mathbf{R}) = \begin{pmatrix}\mathbf{1} & \vec{v}^T\\ \vec{0} & 1\end{pmatrix} \cdot \begin{pmatrix}\mathbf{R} & \vec{0}^T\\ \vec{0} & 1\end{pmatrix} = \begin{pmatrix}\mathbf{R} & \vec{v}^T\\ \vec{0} & 1\end{pmatrix}$$

  which is isomorphic to $\mathbf{IG_R}$: $\mathbf{Gal}_+^\uparrow \cong \mathbf{G_R} \ltimes \mathbf{R}^3 \cong \mathbf{SO(3)} \ltimes \mathbf{R}^3$. The group $\mathbf{Gal}_+^\uparrow$ is not simple, and not even semi-simple since it contains a non-trivial Abelian subgroup, namely the Galilei boosts. As we will see, in contrast, the Lorentz group pendant $\mathbf{Lor}_+^\uparrow$, is simple. This seemingly minor difference manifests in quite different representations of these groups.
- The group elements of the inhomogeneous proper Galilei group including space and time translations $(\tau, \vec{a})$ can be built as

$$(\tau,\, \vec{a},\, \vec{v},\, \mathbf{R}(\vec{\alpha})) = \begin{pmatrix}\mathbf{R}(\vec{\alpha}) & \vec{v}^T & \vec{a}^T\\ \vec{0} & 1 & \tau\\ \vec{0} & 0 & 1\end{pmatrix}. \tag{2.74}$$

The unit element $g_0$ of the group is identical to the $5 \times 5$ unit matrix, and you may verify that these matrices do have the inverse (2.73) and obey the multiplication rule (2.72).

### 2.3.3 Lie Algebra of the Galilei Group

The Lie algebra associated to $\mathbf{Gal_c}$ is spanned by ten generators. These generators and the algebra can be determined directly from the group elements (2.74) by taking their partial derivatives with respect to the group parameters at the unit element (according to (A.1)). Instead of calling the generators generically $X_a$, let us rename

[17] These two attributes have an intuitive meaning in case of the Lorentz group; see Chap. 3.4.

them in a way that exhibits their relation to physical quantities[18]:

$$(\tau,\ \vec{a},\ \vec{v},\ \mathbf{R}(\vec{\alpha})) = \exp i\left[\tau H + a^k T^k + v^k G^k + \alpha^k J^k\right].$$

The Lie algebra of the generators $\{H, T^k, G^k, J^k\}$ is

$$[J^j, J^k] = i\epsilon^{jkl} J^l \quad [J^j, T^k] = i\epsilon^{jkl} T_l \quad [J^j, G^k] = i\epsilon^{ijkl} G_l \quad [J_j, H] = 0 \tag{2.75a}$$

$$[H, T^i] = 0 \qquad [H, G^i] = iT^i \tag{2.75b}$$

$$[T^i, T^j] = 0 \qquad [T^i, G^j] = 0 \qquad [G^i, G^j] = 0. \tag{2.75c}$$

This algebra reflects of course the group/subgroup properties stated in the previous subsection. For example $\mathbf{Gal}_+^\uparrow$ is visible in the subalgebra consisting of the $J^i$ and the $G^i$. Further the first commutator in (2.75a) displays the algebra of the $\mathbf{SO(3)}$ generators, the others show that the $T^j$ and $G^j$ transform as vectors and that $H$ transforms as a scalar with respect to $\mathbf{SO(3)}$. The Galilei algebra can be realized in terms of differential operators as in (2.61).

## 2.4 Concluding Remarks and Bibliographical Notes

Already in this chapter we have caught a glimpse of how symmetries in an established theory (even if it is "only" classical mechanics) lead to important connections to conservation laws–those known to every high school student–and to group theory, here the Galilei group. Maybe this is an unusual perspective on classical mechanics, but it prepares the ground for things to become substantial in our current understanding of the "world".

As mentioned in the Preface, the heading of this chapter s hould be read as "Symmetries in Classical Mechanics". There are of course good text books on classical mechanics. But only few deal explicitly with the origin of conservation laws from symmetries, one example being the classic [332]. But Landau and Lifschitz do not mention the Noether theorems. These are derived in the widely-used textbooks [230, 305]. The book by José and Saletan is in its level between the book by Goldstein et al. and the more abstract text by Arnold [14].

The connection between the ten classical conservation laws and the corresponding space-time symmetries was already stated by G. Herglotz in 1911. (Herglotz was in the Göttingen group of mathematicians with D. Hilbert, F. Klein, later also joined by E. Noether.) However a proof was provided only later by E. Noether in 1918–except for the Galilei boosts. These could be derived after the extension of Noether's theorem due to E. Bessel-Hagen allowing quasi-invariance of a Lagrange function. For the history of this topic, see [311]. The classic text on variational principles is

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[18] Here I distinguish the Lie algebra generators from the conserved quantities that are entailed by the symmetry; except for rotations and angular momenta which receive the same symbol $J$

[331], in which C. Lanzcos discusses in detail the different forms of what now is called "the principle of least action" in terms of related principles introduced by Maupertuis, d'Alembert, Euler, Hamilton, Jacobi; see also [573]. Most of the topics of this chapter are treated with even more details in [490], although these authors–strange enough–do not refer to E. Noether at all. For the meaning of the geometric notions with respect to the Galilei group consult [220].

Finally a remark about Newton's assumption about the notions of absolute time and space. Already at his time this was heavily criticized by G.W. Leibniz, and later also for instance by E. Mach. It seems that in light of the success of Newtonian mechanics this criticism did not find the attention which it deserves. The subject matter, however, became topical in recent decades in the context of reconciling general relativity and quantum physics; see e.g. Sect. 2.4 in [451]. In the quantum gravity community more and more attention is given to a relational understanding of space and time, termed with the catchword "background independence". What does this mean for classical mechanics? As elaborated by J.B. Barbour and B. Bertotti [26] a Leibniz/Mach conception of relational space-time means to replace the Galilei symmetry group by the Leibniz group transformations

$$\vec{x} \;\to\; \vec{x} + \mathbf{A}(\lambda)\vec{x} + \vec{g}(\lambda) \qquad\qquad \lambda \to f(\lambda),$$

where $\mathbf{A}(\lambda)$ is an orthogonal matrix, and $\vec{g}(\lambda)$ and $f(\lambda)$ are arbitrary functions (with the additional condition $\dot{f} > 0$). Observe that this does not define a (finite parameter) Lie group but that it requires for its specification arbitrary functions of the parameter $\lambda$. This is the isotropy group of what is investigated as Leibniz spacetime in [172]. Julian Barbour is even more consequent, in denying that time has any meaning as a basic notion of physics; see his [25].

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