

Chapter 2

Envelopment DEA Models

2.1 Introduction

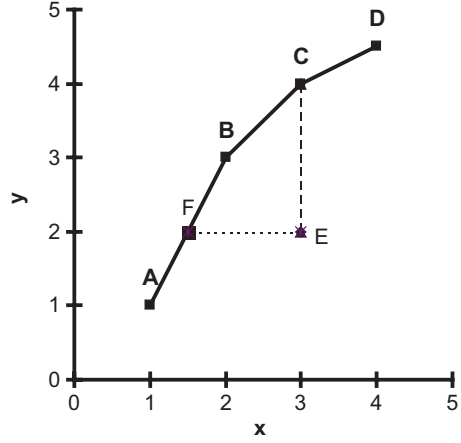
This chapter presents some basic DEA models that are used to determine the best-practice frontier characterized by (Sect. 1.1) in Chap. 1. These models are called envelopment models, because the identified best-practice frontier envelops all the observations (DMUs). The shapes of best-practice (or efficient) frontiers obtained from these models can be associated with the concept of Returns-to-Scale (RTS) which will be discussed in details in Chap. 13. This is because the best-practice (or efficient) frontiers can be viewed as exhibiting of various types of RTS. However, if the inputs and outputs are not related to a “production function”, RTS concept cannot be applied. Under such cases, RTS is merely used to refer to different shapes of frontiers.

Consider Fig. 2.1 where we have 5 DMUs (A, B, C, D, and E) with one input and one output. One possible best-practice frontier consists of DMUs A, B, C, and D. AB exhibits increasing RTS (IRS), B exhibits constant RTS (CRS), and BC and CD exhibit decreasing RTS (DRS). As a result, this best-practice frontier is called Variable RTS (VRS) frontier.

DMU E is not efficient (or best-practice), because it uses too much input and/or it does not produce enough output. In fact, there are two ways to improve the performance of E. One is to reduce its input to reach F on the frontier, and the other to increase its output to reach C on the frontier. As a result, DEA models will have two orientations: input-oriented and output-oriented.

Input-oriented models are used to test if a DMU under evaluation can reduce its inputs while keeping the outputs at their current levels. Output-oriented models are used to test if a DMU under evaluation can increase its outputs while keeping the inputs at their current levels.

Fig. 2.1 Variable returns-to-scale (VRS) frontier



2.2 Variable Returns-to-Scale (VRS) Model

The following DEA model is an input-oriented model where the inputs are minimized and the outputs are kept at their current levels (Banker et al. 1984)

$$\begin{aligned}
 &\theta^* = \min \theta \\
 &\text{subject to} \\
 &\sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io} \quad i = 1, 2, \dots, m; \\
 &\sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s; \\
 &\sum_{j=1}^n \lambda_j = 1 \\
 &\lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{2.1}$$

where DMU_o represents one of the n DMUs under evaluation, and x_{io} and y_{ro} are the i th input and r th output for DMU_o , respectively.

Since $\theta=1$ is a feasible solution to (1.2), the optimal value to (2.1), $\theta^* \leq 1$. If $\theta^* = 1$, then the current input levels cannot be reduced (proportionally), indicating that DMU_o is on the frontier. Otherwise, if $\theta^* < 1$, then DMU_o is dominated by the frontier. θ^* represents the (input-oriented) efficiency score of DMU_o .

Consider a simple numerical example shown in Table 2.1 where we have five DMUs (supply chain operations). Within a week, each DMU generates the same profit of \$ 2,000 with a different combination of supply chain cost and response time.

Table 2.1 Supply chain operations within a week

DMU	Cost (\$ 100)	Response time (days)	Profit (\$ 1,000)
1	1	5	2
2	2	2	2
3	4	1	2
4	6	1	2
5	4	4	2

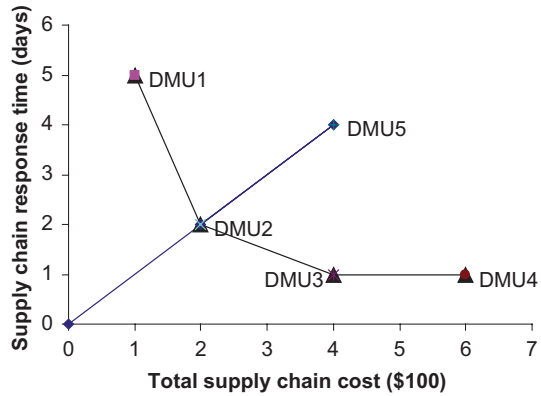
Fig. 2.2 Five supply chain operations

Figure 2.2 presents the five DMUs and the piecewise linear frontier. DMUs 1, 2, 3, and 4 are on the frontier. If we calculate model (2.1) for DMU5,

$$\begin{aligned}
 &\text{Min } \theta \\
 &\text{Subject to} \\
 &1 \lambda_1 + 2\lambda_2 + 4\lambda_3 + 6\lambda_4 + 4\lambda_5 \leq 4\theta \\
 &5 \lambda_1 + 2\lambda_2 + 1\lambda_3 + 1\lambda_4 + 4\lambda_5 \leq 4\theta \\
 &2 \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 \geq 2 \\
 &\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \\
 &\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0
 \end{aligned}$$

we obtain a set of unique optimal solutions of $\theta^* = 0.5$, $\lambda_2^* = 1$ and $\lambda_j^* = 0$ ($j \neq 2$), indicating that DMU2 is the benchmark for DMU5, and DMU5 should reduce its cost and response time to the amounts used by DMU2.

Now, if we calculate model (2.1) for DMU4, we obtain $\theta^* = 1$, $\lambda_4^* = 1$, and $\lambda_j^* = 0$ ($j \neq 4$), indicating that DMU4 is on the frontier. However, Fig. 2.2 indicates that DMU4 can still reduce its total supply chain cost by \$ 200 to reach DMU3. This individual input reduction is called input slack.

In fact, both input and output slack values may exist in model (2.1). Usually, after calculating (2.1), we have

$$\begin{cases} s_i^- = \theta^* x_{io} - \sum_{j=1}^n \lambda_j x_{ij} & i = 1, 2, \dots, m \\ s_r^+ = \sum_{j=1}^n \lambda_j y_{rj} - y_{ro} & r = 1, 2, \dots, s \end{cases} \quad (2.2)$$

where s_i^- and s_r^+ represent input and output slacks, respectively. An alternate optimal solution of $\theta^* = 1$ and $\lambda_3^* = 1$ exists when we calculate model (2.1) for DMU4. This leads to $s_1^- = 2$ for DMU4. However, if we obtain $\theta^* = 1$ and $\lambda_4^* = 1$ from model (2.1), we have all zero slack values. i.e., because of possible multiple optimal solutions, (2.2) may not yield all the non-zero slacks.

Therefore, we use the following linear programming model to determine the possible non-zero slacks after (2.1) is solved.

$$\begin{aligned} & \max \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta^* x_{io} \quad i = 1, 2, \dots, m; \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s; \\ & \sum_{j=1}^n \lambda_j = 1 \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \quad (2.3)$$

For example, applying (2.3) to DMU4 yields

$$\begin{aligned} & \text{Max } s_1^- + s_2^- + s_1^+ \\ & \text{Subject to} \\ & 1 \lambda_1 + 2 \lambda_2 + 4 \lambda_3 + 6 \lambda_4 + 4 \lambda_5 + s_1^- = 6 \theta^* = 6 \\ & 5 \lambda_1 + 2 \lambda_2 + 1 \lambda_3 + 1 \lambda_4 + 4 \lambda_5 + s_2^- = \theta^* = 1 \\ & 2 \lambda_1 + 2 \lambda_2 + 2 \lambda_3 + 2 \lambda_4 + 2 \lambda_5 - s_1^+ = 2 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, s_1^-, s_2^-, s_1^+ \geq 0 \end{aligned}$$

with optimal slacks of $s_1^{*-} = 2$, $s_2^{*-} = s_1^{*+} = 0$.

DMU_o is efficient if and only if $\theta^* = 1$ and $s_i^{*-} = s_r^{*+} = 0$ for all i and r . DMU_o is weakly efficient if $\theta^* = 1$ and $s_i^{*-} \neq 0$ and (or) $s_r^{*+} \neq 0$ for some i and r . In Fig. 2.2, DMUs 1, 2, and 3 are efficient, and DMU 4 is weakly efficient.

Definition 2.1 The slacks obtained by (2.3) are called DEA slacks. Or specifically, slacks calculated from a second-stage DEA calculation are called DEA slacks.

In fact, models (2.1) and (2.3) represent a two-stage DEA process involved in the following DEA model.

$$\begin{aligned}
 & \min \theta - \varepsilon \left(\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right) \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{2.4}$$

The presence of the non-Archimedean ε in the objective function of (2.4) effectively allows the minimization over θ to preempt the optimization involving the slacks, s_i^- and s_r^+ . Thus, (2.4) is calculated in a two-stage process with maximal reduction of inputs being achieved first, via the optimal θ^* in (2.1); then, in the second stage, movement onto the efficient frontier is achieved via optimizing the slack variables in (2.3). It is incorrect if one attempts to solve model (2.4) in a single model/stage by specifying an ε value in the objective function of (2.4).

In fact, the presence of weakly efficient DMUs is the cause of multiple optimal solutions. Thus, if weakly efficient DMUs are not present, the second stage calculation (2.3) is not necessary, and we can obtain the slacks using (2.2). However, prior to calculation, we usually do not know whether weakly efficient DMUs are present or not.

Note that the frontier determined by model (2.1) exhibits variable returns to scale (VRS). Therefore, model (2.1) is called input-oriented VRS envelopment model. (see Chap. 13 for a detailed discussion on DEA and Returns-to-Scale (RTS).)

The output-oriented VRS envelopment model can be expressed as

$$\begin{aligned}
 & \max \phi + \varepsilon \left(\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right) \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = \phi y_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{2.5}$$

Model (2.5) is also calculated in a two-stage process. (One should never try to solve model (2.5) in a single model by specifying an ε value in the objective function of (2.5)).

First, we calculate ϕ^* by ignoring the slacks, namely,

$$\begin{aligned}
 & \max \phi \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq \phi y_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned}$$

Then we optimize the slacks by fixing the ϕ^* in the following linear programming problem.

$$\begin{aligned}
 & \max \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = \phi^* y_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{2.6}$$

DMU_o is efficient if and only if $\phi^* = 1$ and $s_i^- = s_r^+ = 0$ for all i and r . DMU_o is weakly efficient if $\phi^* = 1$ and $s_i^- \neq 0$ and (or) $s_r^+ \neq 0$ for some i and r . If weakly efficient DMUs are not present, then we need not to calculate (2.6), and we can obtain the slacks via

$$\begin{cases} s_i^- = x_{io} - \sum_{j=1}^n \lambda_j x_{ij} & i = 1, 2, \dots, m \\ s_r^+ = \sum_{j=1}^n \lambda_j y_{rj} - \phi^* y_{ro} & r = 1, 2, \dots, s \end{cases}$$

Fig. 2.3 Output efficient frontier

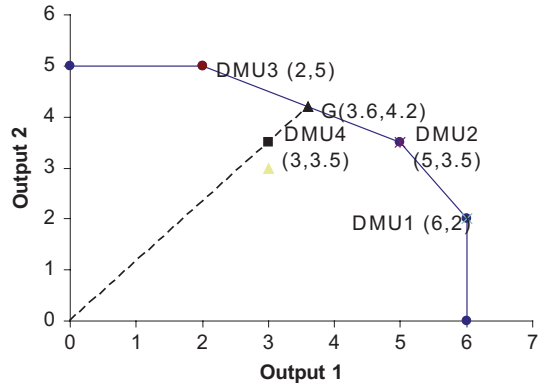


Figure 2.2 shows an input efficient frontier when outputs are fixed at their current levels. Similarly, we can obtain an output efficient frontier when inputs are fixed at their current levels. Consider the four DMUs shown in Fig. 2.3 where we have two outputs.

In Fig. 2.3, DMUs 1, 2 and 3 are efficient. If we calculate model (2.5) for DMU4, we have

$$\begin{aligned}
 & \text{Max } \phi \\
 & \text{Subject to} \\
 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1 \\
 & 6\lambda_1 + 5\lambda_2 + 2\lambda_3 + 3\lambda_4 \geq 3\phi \\
 & 2\lambda_1 + 3.5\lambda_2 + 5\lambda_3 + 3.5\lambda_4 \geq 3.5\phi \\
 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
 & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0
 \end{aligned}$$

The optimal solution is $\phi^* = 1.2$, $\lambda_2^* = 8/15$, and $\lambda_3^* = 7/15$. i.e., DMU4 is inefficient and is compared to G in Fig. 2.3, or DMU4 should increase its two output levels to G. In this case, if we calculate model (2.6), all slacks will be zero.

2.3 DEA Slacks

In this section, we provide two numerical examples to further show the concept of DEA slacks calculated from the second stage DEA calculation in model (2.3) or (2.6).

Consider Fig. 2.4 with one input and one output and input-oriented VRS model. Any DMUs in the shaded area will have an output DEA slack after it is moved onto the VRS frontier by input reduction. For example, DMU H is moved onto G which is a frontier point. However, we can still increase G's output to point A.

Fig. 2.4 DEA slack under input orientation

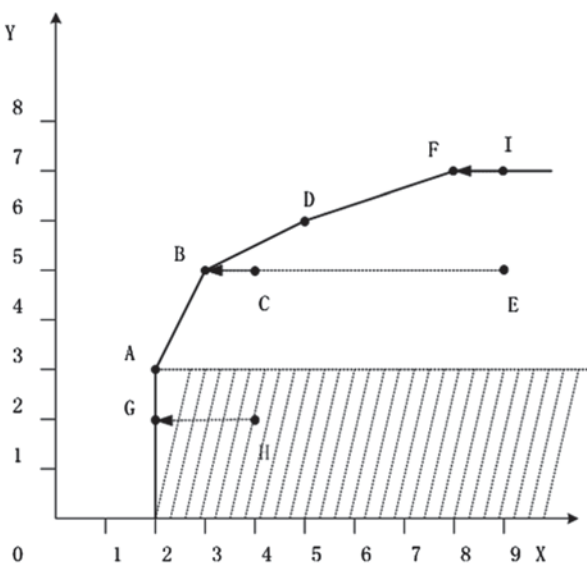
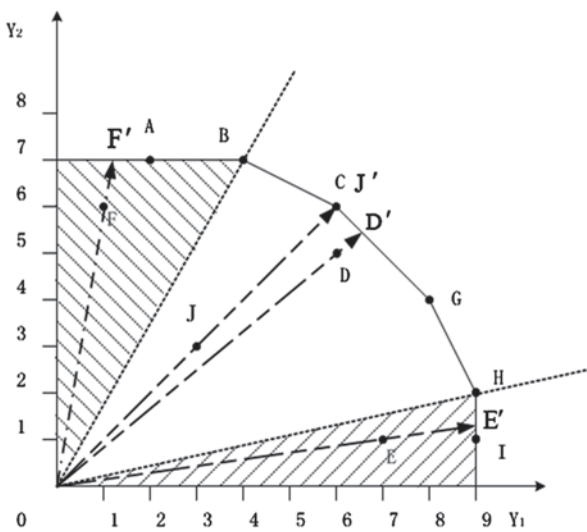
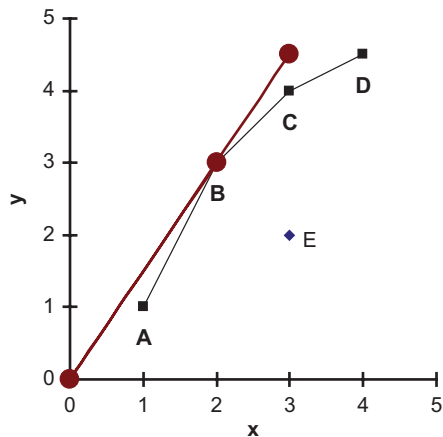


Fig. 2.5 DEA slack under output orientation



DMUs I and E are inefficient under the input-oriented VRS model. Note that DMU I is actually on the VRS frontier, and is a weakly efficient DMU under the output-oriented VRS model. Because under model (2.5), the efficiency score for DMU I is equal to one. Yet, DMU I can further reduce its input to point F.

We next consider Fig. 2.5 where we have two outputs. Under the output-oriented VRS model, all DMUs (e.g., E and F) within the shaded areas will have (non-zero) DEA slack values. DMUs A and I are weakly efficient. DMUs J and D are

Fig. 2.6 CRS frontier

inefficient and do not have DEA slacks, as their outputs are proportionally increased to J' and D' , respectively.

2.4 Other Envelopment Models

The constraint on $\sum_{j=1}^n \lambda_j$ in model (2.1) actually determines the RTS type of an efficient frontier. If we remove $\sum_{j=1}^n \lambda_j = 1$ from models (2.1) and (2.5), we obtain CRS (Constant RTS) envelopment models where the frontier exhibits CRS (Charnes et al. 1978). Figure 2.6 shows a CRS frontier—ray OB. Based upon this CRS frontier, only B is efficient.

If we replace $\sum_{j=1}^n \lambda_j = 1$ with $\sum_{j=1}^n \lambda_j \leq 1$, then we obtain non-increasing RTS (NIRS) envelopment models. In Fig. 2.7, the NIRS frontier consists of DMUs B, C, D and the origin.

If we replace $\sum_{j=1}^n \lambda_j = 1$ with $\sum_{j=1}^n \lambda_j \geq 1$, then we obtain non-decreasing RTS (NDRS) envelopment models. In Fig. 2.8, the NDRS frontier consists of DMUs A, B, and the section starting with B on ray OB.

Table 2.2 summarizes the envelopment models with respect to the orientations and frontier types. The last row presents the efficient target (DEA projection) of a specific DMU under evaluation.

The interpretation of the envelopment model results can be summarized as

- I. If $\theta^* = 1$ or $\phi^* = 1$, then the DMU under evaluation is a frontier point. i.e., there is no other DMUs that are operating more efficiently than this DMU. Otherwise, if $\theta^* < 1$ or $\phi^* > 1$, then the DMU under evaluation is inefficient. i.e., this DMU can either increase its output levels or decrease its input levels.
- II. The left-hand-side of the envelopment models is usually called the “Reference Set”, and the right-hand-side represents a specific DMU under evaluation. The

Fig. 2.7 NIRS frontier

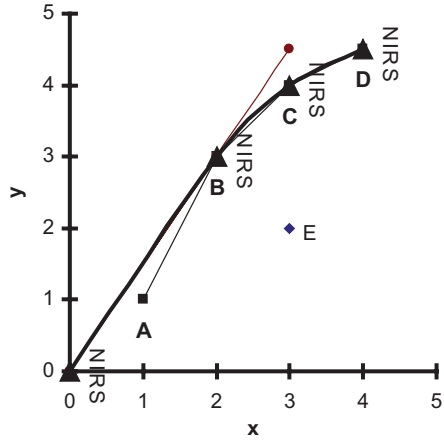
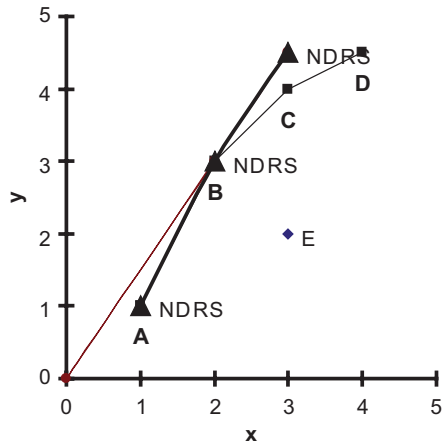


Fig. 2.8 NDRS frontier



non-zero optimal λ_j^* represent the benchmarks for a specific DMU under evaluation. The Reference Set provides coefficients (λ_j^*) to define the hypothetical efficient DMU. The Reference Set or the efficient target shows how inputs can be decreased and outputs increased to make the DMU under evaluation efficient.

- III. The “Efficient Target” in Table 2.2 is a result of two stage DEA calculation. However, sometimes a DEA user may ignore the second stage slack calculation and is only interested in the efficiency scores. In that case, $\theta^* x_{io}$ or $\phi^* y_{ro}$ can be regarded as Target on the frontier.

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