

Chapter 2

Harald Cramér and Insurance Mathematics

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Abstract A short history of Harald Cramér's work in insurance mathematics is given. In particular, the early development of the collective risk theory is outlined, starting with the works of F. Lundberg. Also, the so-called zero point method for premium calculations invented by Cramér is described.

2.1 The Early Years

Born in 1893, Harald Cramér grew up in Stockholm and was a student at Stockholms Högskola. In those days, this was a small university which emphasised higher studies and research, mainly sponsored by the city of Stockholm.

He started his scientific career in biochemistry with the famous Hans von Euler, but soon changed to mathematics and wrote his thesis on Dirichlet series in 1917. His research was then devoted to analytic number theory, and he wrote, among other things, the survey article on this topic in the *Encyclopedie der Mathematischen Wissenschaften* with Harald Bohr, who was a close friend of his.

In the decades around 1900 mathematics was a very active field of research at Stockholms Högskola. There were several very able young research associates of Gösta Mittag-Leffler, such as I. Bendixson, I. Fredholm, H. von Koch, S. Kovalevsky, E. Lindelöf and E. Phragmén. Mittag-Leffler was a very ambitious man, both for himself and for the position of mathematics in general. He was also politically active and took part in many of the political and scientific controversies of his time. Among

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other things, he was eager to promote the use of mathematics in insurance to ensure a proper and scientifically sound foundation for the activities of insurance companies. He was active in founding the Swedish Actuarial Society in 1904, and argued in favour of creating a chair for insurance mathematics at Stockholms Högskola.

In those days it was quite common for prominent university professors to work part time as actuaries for life insurance companies. Several such companies had been formed at the end of the last century. Fredholm, Gylden, Mittag-Leffler and Phragmén, for example, were engaged in such work. Fredholm, in particular, was quite active and made a well-known mortality investigation. Phragmén, who was the successor of Sonja Kovalevski in 1892, left his chair after about ten years to become the leader of the newly founded inspectorate of the insurance industry, and he later moved on to a long career as manager of the Allmänna Livförsäkringsaktiebolaget.

This tradition of having highly qualified mathematically trained persons as responsible actuaries and often in other leading positions in insurance companies has persisted since then. Cramér eloquently defended this practice on several occasions.

Cramér started to work as an amanuensis in the inspectorate in 1918 and in 1920 became the actuary of one of the life insurance companies. In 1929, the chair for Insurance Mathematics and Mathematical Statistics at Stockholms Högskola was created with support from the insurance industry, and he was elected as its first holder.

2.2 Career and Theoretical Development

It appears that Cramér's departure from the university and his research in pure mathematics was prompted by a controversy with Mittag-Leffler, who accused him of claiming a result that had already been found by a colleague in Uppsala. Mittag-Leffler vowed that he would ban Cramér from getting a position at the university, so he had to look for another career and thereby took up his research in probability and insurance mathematics. In retrospect, this seems to have been a blessing in disguise for this science, since he came to it at the right time—the beginning of its vigorous development in the twenties, which has continued up to our own time. During the first decades of this century, a very interesting and probably rather unique development in the use of random models for answering questions about the proper risk management of large fluctuations in the results of insurance companies took place in Sweden. (In those days and later, very little consideration was paid by the traditional theory of life insurance to fluctuations in the results).

This development of 'collective risk theory' emanates to a large extent from the pioneering research of Filip Lundberg. Born in 1876, he studied mathematics and science in Uppsala and soon began to work in insurance. In 1903, he wrote his thesis: 'On the approximation of the probability function in the insurance of collective risks'. (He felt that his company was not very safe and wanted to safeguard himself and be qualified to become a lecturer at a gymnasium, which required a Ph.D. in those days.) In this thesis, which was considered very odd at that time, he formulated in an intuitive

way a model for the random variation of the surplus of an insurance business, which we call today a compound Poisson process, but which was then unknown. Like Bachelier's work from the same year, introducing the Wiener process to describe fluctuating share prices, this was one of the first instances where a stochastic process in continuous time was defined.

The process consists of an infinite succession of independent elementary games. In each game there is a 'risk mass distribution' defining the risks of claims of various sizes. From these assumptions he derived a law for the compound Poisson distribution of the total amount of claims in a finite time interval, and then he proceeded to derive approximations to it for large times. This amounts to proving the central limit theorem with methods used only much later by others. Later he studied the probability of ruin of a company, i.e. the probability that the accumulated surplus will ever become negative as a function of the initial capital. In this context, he derived his famous approximation

$$P(\text{ruin}) \approx c \exp(-Ru),$$

where R is Lundberg's constant and u the initial capital. He wrote in a very obscure manner, and his works were considered very theoretical by his contemporaries in the insurance world.

Lundberg's works were a challenge and a rich source of inspiration for Cramér, who was probably one of the few who really tried to penetrate them. He devoted much work to explaining and developing the theory in an intelligible and rigorous way.

In a charming, very positive, but also critical review of Lundberg's risk theory [5], gave the following vivid description of his style:

A member of the Swedish Actuarial Society once compared the main work by the famous mathematician Stieltjes to "a not excessively pruned luxuriantly flourishing herb garden". If we want to find a similar image of Lundberg's works we have to go to a southerly climate, to the tropical forests, where the researcher only with great effort, step by step forces his way through the vegetation without being able to see far, neither forwards nor backwards. An expedition into the jungle requires months, perhaps years, but it is rewarding! If one opens up a path through brushes and obstacles in Lundberg's theory one is gradually, perhaps grudgingly, forced to admit that there is order in the apparent mess. One proceeds further, and vistas open, and finally one is seized by admiration for the power and originality of the brain that created this whole world of its own. On many points one can disagree on what is right and has any purpose, one can often understand nothing of the form chosen for the exposition, but I do not think that anyone who has really understood Lundberg's line of thought, and compared it to what had been achieved by others in this field, can deny him his wholehearted admiration.

In 1930 he published an exposition, *On the Mathematical Theory of Risk* [9], in a jubilee volume of the Skandia life insurance company. It is a brilliant example of his great talent for giving clear expositions, where one sees the important points and is not lost among the technicalities, even if they are there. It is still a very readable introduction to the stochastic aspects of insurance, which have only recently come into prominence. In this chapter the distinction between 'individual' and 'collective' risk theory is clarified. The former is perhaps the most natural one. A whole portfolio

of insurances is considered during a sample year. Each has a certain probability p_i of having an accident or death, for example, and in this case a sum, s_i , has to be paid by the company. The payments are hence independent random variables S_i with a two point distribution: $P(S_i = s_i) = p_i$, $P(S_i = 0) = 1 - p_i$. The problem is to calculate the distribution of the total claim amount $S = \sum_1^n S_i$. It has a ‘compound binomial distribution’, and is the sum of many independent terms. Hence one can expect that the central limit theorem should give a good approximation to the distribution of the sum. However, as he often points out, the amounts are of such varying sizes that the terms are not uniformly small, and the approximation is not good enough. Much of the motivation for his work on asymptotic expansions improving the Gaussian approximation in the central limit theorem comes from these problems. Also, the important first work on ‘large deviations’ giving an approximation of the form

$$P(S > nx) \approx c(x)(1/n)^{1/2} \exp[nh(x)],$$

which he published in 1938 [9], is inspired by the ‘Esscher approximation’ invented in 1932 by the actuary Fredrik Esscher for the same purpose.

It has turned out to be the starting point of a systematic theory of such estimates, which has been developed much further in recent decades, and has found many applications in statistics and applied probability.

The ‘collective’ description of S can be obtained as an approximation to the ‘individual’ one when the probabilities are small by introducing $S(t) = \sum_i N_i(t)s_i$. Here, the $N_i(t)$ are independent Poisson processes with means tp_i . Then the distribution of $S(t)$ is a compound Poisson process defined by the ‘risk mass distribution’ $P(ds)$, which is the distribution with masses p_i at the positions s_i on the s -axis:

$$E\{\exp(zS(t))\} = \exp\left\{\sum_t \{\exp(zs_i) - 1\}tp_i\right\} = \exp\left[t \int \{\exp(zs) - 1\}P(ds)\right].$$

$S(1)$ is then an approximation to S , which can be seen by comparing the generating functions

$$\begin{aligned} E\{\exp(zS)\} &= \prod_i \{1 - p_i + p_i \exp(zs_i)\} \\ &= \exp \sum_i \log\{1 + p_i(\exp(zs_i) - 1)\} \\ &\approx \exp \left[\sum_i p_i(\exp(zs_i) - 1) \right] = E\{\exp(zS(1))\}. \end{aligned}$$

One considers $S(t)$ as $t > 0$, with $P(ds)$ approximately constant in time and disregards the fact that, at least in life insurance, it is changed when people die and new customers arrive.

The ruin problem, i.e. the problem of finding the probability $r(u)$ that the surplus $U(t) = u + pt - S(t)$ becomes negative for some $t > 0$, is much easier for the compound Poisson process than for the original ‘individual’ model. Here u is the initial capital and p the rate of inflowing premiums.

Risk theory, which studies this ruin problem, became a favourite subject of research in Cramér’s institute in Stockholm in the 1930, 1940 and 1950s. Several papers in the Scand. Act. J. from this period by Segerdahl [13, 14], Täcklind [15] and Arfwedson [1, 2] consider $r(u)$ and $r(u, T)$, the probability of ruin within a finite time T . These works all consider the natural integral equations for $r(u)$ and $r(u, T)$ which are obtained by considering what can happen in the first short time interval dt , and then the process starts from scratch with a different initial capital.

Already in the chapter [9] from 1930 Cramér gives a nice account of the most natural case $p > 0$ and $P(ds) > 0$ only for $s > 0$. Then he obtains a Volterra equation for $r(u)$ whose Laplace transform can be explicitly written down. From this he rigorously derives the Lundberg approximation

$$r(u) \approx c \exp(-Ru) \text{ as } u \rightarrow \infty.$$

His formula for the Laplace transform turns out to be the same as the so-called Pollaczek–Khinchin [8, 11] formula from 1930 for the Laplace transform of the waiting time distribution in an $M/G/1$ queueing system. This fact was realised much later in the fifties on the basis of a natural connection between the process $U(t)$ and the virtual waiting time process for the queueing system. So we should perhaps change the name to the Cramér–Pollaczek–Khinchin formula. To find the solution of the integral equation for $r(u, T)$, Cramér and Täcklind applied the so-called Wiener–Hopf technique, and gave explicit formulae for the Laplace transform in terms of the so-called Wiener–Hopf factors associated with the risk process. This theory is entirely ‘analytical’ and uses analytic function theory to a high degree. One obtains explicit formulae for the ruin probabilities in a few cases and asymptotic approximations when u and T are large. This theory is very completely explained in Cramér’s monograph *Risk Theory* [9] from 1955 in the 100-year jubilee volume of the Skandia insurance company.

In the 1950s and 1960s, the theory of passage times for random walks and their relation to the Wiener–Hopf factors was very strongly developed using more probabilistic methods. Then it was realised that a great unification of the results in queueing, storage, and risk theory could be achieved, and one could probabilistically, by simple transformations, see why these theories solve more or less the same problems. Also, the asymptotic formulae can be obtained directly by probabilistic methods. The treatments of Feller [7] and von Bahr [18] give a very good insight into this, and Prabhu [12] gives a more complete account. The present author has himself shown [10] that the asymptotic estimates can be obtained in a direct way using the methods of large deviation theory.

A detailed numerical study of risk theory and various approximation methods for the calculation of the distribution of $S(t)$ was carried out at the beginning of the 1960s by a ‘convolution committee’, of which Cramér, Grenander, Esscher and Bohman

were members. Their results have been well documented by Bohman and Esscher [3, 4], and rely on delicate and good methods for numerical Fourier inversion. This line of work was subsequently generalised and rounded off by Thorin, who considers the ruin problem when $S(t)$ is a ‘compound renewal process’. His work has been well documented by Thorin [16], and the numerical aspects by Thorin [17] and Wikstad [19].

During the 1930s and 1940s Cramér was also heavily engaged in more practical work in the insurance world. He was the actuary of the reinsurance company Sverige for the Swedish life insurance companies, and he was a member of several committees and the government commission for creating a new insurance law around 1940. In the 1940s the position of private insurance was a hot political subject, and from the left there were strong proposals for socialisation in order to guarantee ‘just and cheap’ insurance for the people. These proposals were, however, rejected after a prolonged debate. At the end of the 1930s, new common technical bases for the Swedish life insurance companies had to be worked out. This was partly prompted by the ‘interest crises’ caused by the fact that the interest rate sank below the level used in the premium calculations, so the reserves were insufficient and had to be increased, but the law prohibited premium increases for policies already in force. This detailed technical work leading to the new technical bases from 1938 was carried out by Cramér. He also worked with mortality predictions using the very detailed Swedish records from the period 1800–1930 in co-operation with Wold [9].

In his work on new bases, Cramér strongly advocated the use of the so-called zero point method for premium calculations. This concerns the problem of how to calculate a safe premium taking into account the uncertainty in the value of the mortality. If one deliberately uses a high mortality then one gets a safe premium for the insurances with benefits at death, whereas one gets too small a premium for insurances with life benefits, and vice versa. There are, however, mixed insurances with both life and death benefits, whose character changes during the lifetime. In the beginning they are, for example, ‘death like’, and later in life ‘life like’, and it is not obvious what a safe mortality assumption is for them. Cramér proposed the use of a mixed mortality as follows: in calculating the reserve using Thiele’s differential equation, one should use a high mortality when the so-called sum at risk is positive and a low mortality when it is negative. That this rule gives an upper bound to the true premium can be seen by considering the following optimal control problem: consider a general life insurance to be paid at the start when $t = 0$. Its reserve $V(t)$ is determined by Thiele’s differential equation

$$\frac{dV(t)}{dt} = \delta V(t) - L(t) - \mu(t)(D(t) - V(t))dt.$$

Here $L(t)$ and $D(t)$ are the life and death benefits, δ the interest rate, and $\mu(t)$ the mortality. The boundary conditions are $V(\infty) = 0$ and $V(0) =$ the premium sought.

Suppose now that $\mu(t)$ is not known exactly, but only upper and lower bounds, $\mu_L \leq \mu(t) \leq \mu_D(t)$. Subject to these restrictions, we want to maximise $V(0)$. The solution to this optimal control problem is obtained if $\mu(t)(D(t) - V(t))$ is maximal

for each t , i.e. if $\mu(t) = \mu_D(t)$ when $D(t) - V(t) > 0$ and $\mu(t) = \mu_L(t)$ when $D(t) - V(t) < 0$. $R(t) = D(t) - V(t)$ is the sum at risk. This is Cramér's solution, and the corresponding $V(0)$ is a safe premium. Before the computer era this was considered to be a computationally complicated method, and was therefore not used in later revisions of the technical bases.

To summarise Cramér's work in insurance, it is fair to say that it shows his ability to work with difficult purely mathematical problems, which were, however, clearly motivated by the applications, and also his practical side, where his eminent ability to analyse and explain the essential features of often complex matters rendered him an undisputed authority in the actuarial community.

I think his own words characterising Edward Phragmén, which he wrote in a biography of him in 1958, express very eloquently his mathematical ideas:

Edward Phragmén belonged to a generation of mathematicians for whom it was self evident that mathematics is one of the highest forms of human thought, perhaps even the highest. For these mathematicians, numbers were a necessary form for human thought, and the science of numbers was a central humanistic discipline with a cultural value completely independent of its role as an auxiliary science in technical or other areas. This does not however mean that they underestimated the importance of "applying theoretical knowledge to obtain practical knowhow", as Phragmén once characterised the task of the actuarial mathematician.

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