Kinematic Mapping of SE(4) and the Hypersphere Condition

Georg Nawratil

Abstract In this chapter we present a novel kinematic mapping for the Euclidean 4-space E^4. We show that there is a bijection between the group SE(4) of Euclidean displacements of E^4 and points on a quadric (sliced along a 3-dimensional generator-space), which is located in a projective 11-dimensional space. These 12 homogeneous motion parameters can be seen as a natural extension of the Blaschke-Grünwald parameters for E^2 and the Study parameters for E^3, respectively. In addition we also study the constraint that a point is located on a hypersphere of E^4. We prove that this hypersphere condition is a homogeneous quadratic equation in the 12 homogeneous motion parameters.

Keywords Kinematic mapping · Euclidean 4-space · Quaternion · Hypersphere

1 Motivation

The study of displacements of the Euclidean 4-space is motivated by Stewart Gough (SG) platforms. These are 6-dof $S_3\underline{P}S_3$ parallel manipulators, as the platform is connected with the base via six $S_3\underline{P}S_3$-legs, where $\underline{P}$ denotes an active prismatic joint and S_3 a passive spherical[1] one. If the centers of the S_3-joints located at the base (resp. platform) are coplanar, then the base (resp. platform) is called planar. A SG manipulator with planar platform and planar base is called planar SG platform. These manipulators are geometrically a lot better understood than their non-planar counterparts (e.g. attachment of additional legs without changing the direct

[1] S_n denote the spherical joint, which admits the group of spherical motions SO(n) of E^n. Note that a S_2-joint equals a rotational joint (R-joint).

G. Nawratil (✉)
Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria
e-mail: nawratil@geometrie.tuwien.ac.at

J. Lenarčič and O. Khatib (eds.), *Advances in Robot Kinematics*,
DOI: 10.1007/978-3-319-06698-1_2,

kinematics [1] and singularity set [2], self-motions [3] and Duporcq's theorem [4], …).

We hope to gain a deeper geometric insight into the nature of non-planar SG platforms by studying the analogs of planar SG platforms in E^4, which are so-called hyperplanar 10-dof $S_4\underline{P}S_4$ parallel manipulators.[2]

The basic equation for an algebraic kinematical study of 10-dof $S_4\underline{P}S_4$ parallel manipulators is the so-called hypersphere condition which means that the center of the platform S_4-joint is located on a hypersphere centered in the corresponding base S_4-joint. For the formulation of this equation, we need a proper kinematic mapping of SE(4), which is given in Sect. 2. Based on the presented kinematic mapping, we study the hypersphere condition and its derivative (infinitesimal direct kinematics) in Sect. 3. Finally we close the chapter with conclusions and an outlook.

2 Kinematic Mappings of SE(n)

A kinematic mapping of SE(n) is a bijective mapping between the group of displacements of E^n and a set $\mathscr{S}$ of points in a certain space. Well known examples of these mappings are the one of Blaschke [5] and Grünwald [6] for E^2 and the one of Study [7] for E^3, which are reviewed within the next two subsections.

2.1 Study Mapping of SE(3)

$\mathfrak{Q} := q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ with $q_0, \ldots, q_3 \in \mathbb{R}$ is an element of the skew field of quaternions $\mathbb{H}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the so-called quaternion units. The conjugated quaternion to $\mathfrak{Q}$ is given by $\widetilde{\mathfrak{Q}} := q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. Moreover $\mathfrak{Q}$ is called a pure quaternion for $q_0 = 0$ and a unit quaternion for $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. Finally we can embed the points X of E^3 with Cartesian coordinates (x_1, x_2, x_3) into the set of pure quaternions by the following mapping:

$$\iota_3 : \mathbb{R}^3 \to \mathbb{H} \quad \text{with} \quad (x_1, x_2, x_3) \mapsto \mathfrak{X} := x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}. \tag{1}$$

Classically the Study mapping is introduced by the usage of dual quaternions $\mathbb{H} + \varepsilon\mathbb{H}$, where ε is the dual unit with the property $\varepsilon^2 = 0$. An element $\mathfrak{E} + \varepsilon\mathfrak{T}$ of $\mathbb{H} + \varepsilon\mathbb{H}$ is called unit dual quaternion if $\mathfrak{E}$ is an unit quaternion and following condition holds:

$$e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3 = 0. \tag{2}$$

[2] As there are 10 dofs in E^4 (six rotational ones and four translatory dofs), the hyperplanar platform (moving 3-space) and the hyperplanar base (fixed 3-space) have to be connected via ten $S_4\underline{P}S_4$-legs (cf. footnote 1). In this context it should be noted that the lower-dimensional counterparts of planar SG platforms are 3-dof $R\underline{P}R$ parallel manipulators with collinear base points and platform points.

Based on the usage of unit dual quaternions $\mathfrak{E} + \varepsilon\mathfrak{T}$ it can be shown (e.g. Sect. 3.3.2.2 of [8]) that the mapping of points $\mathsf{X} \in E^3$ to $\mathsf{X}' \in E^3$ induced by any element of SE(3), can be written as follows (by using ι_3):

$$\mathfrak{X} \mapsto \mathfrak{X}' \quad \text{with} \quad \mathfrak{X}' := \mathfrak{E} \circ \mathfrak{X} \circ \widetilde{\mathfrak{E}} + (\mathfrak{T} \circ \widetilde{\mathfrak{E}} - \mathfrak{E} \circ \widetilde{\mathfrak{T}}), \tag{3}$$

where $\circ$ denotes the well-known quaternion multiplication. Moreover it can be shown that the mapping of Eq. (3) is an element of SE(3) for any unit dual quaternion $\mathfrak{E} + \varepsilon\mathfrak{T}$. Note that $\mathfrak{X}'$ is again a pure quaternion, where the first summand $\mathfrak{E} \circ \mathfrak{X} \circ \widetilde{\mathfrak{E}}$ is the rotational component, which can be written in vector-representation as $(x_1', x_2', x_3')^T = \mathbf{R}_3(x_1, x_2, x_3)^T$ with

$$\mathbf{R}_3 = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}, \tag{4}$$

where $det\mathbf{R}_3 = (e_0^2 + e_1^2 + e_2^2 + e_3^2)^3 = 1$ holds. As the remaining part of Eq. (3) does not depend on X, it corresponds to a translation with vector $\mathbf{s}_3 := (s_1, s_2, s_3)^T$ and

$$\begin{aligned} &s_1 = 2(e_0t_1 - e_1t_0 + e_2t_3 - e_3t_2), \quad s_2 = 2(e_0t_2 - e_1t_3 - e_2t_0 + e_3t_1), \\ &s_3 = 2(e_0t_3 + e_1t_2 - e_2t_1 - e_3t_0). \end{aligned} \tag{5}$$

As both unit dual quaternions $\pm(\mathfrak{E} + \varepsilon\mathfrak{T})$ correspond to the same Euclidean motion of E^3, we consider the homogeneous 8-tuple $(e_0 : \ldots : e_3 : t_0 : \ldots : t_3)$. These so-called Study parameters can be interpreted as a point of a projective 7-dimensional space P^7. Therefore there is a bijection between SE(3) and all real points $\mathscr{S}$ of P^7 located on the so-called Study quadric $\Phi \subset P^7$, which is given by Eq. (2) ($\Rightarrow$ the signature of Φ is $(4_+, 4_-, 0_0)$) and is sliced along the 3-dimensional generator-space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathfrak{E}$ cannot be normalized.

2.2 Blaschke-Grünwald Mapping of SE(2)

The Blaschke-Grünwald mapping can be obtained from the Study mapping by restricting ourselves to planar Euclidean displacements within a plane α, which corresponds to a 3-dimensional generator-space of Φ. If we choose α as the plane given by $x_1 = 0$, it can easily be seen (cf. Remark 3.38 of [8]) that the corresponding generator-space of Φ is given by $e_2 = e_3 = t_0 = t_1 = 0$. Therefore there is a bijection between SE(2) and all real points $(e_0 : e_1 : t_2 : t_3)$ of a projective 3-dimensional space P^3, with exception of the points located on the line $e_0 = e_1 = 0$.

The vector-representation of planar displacements in dependency of the Blaschke-Grünwald parameters $(e_0 : e_1 : t_2 : t_3)$ can immediately be obtained from Eqs. (4)

and (5) and reads as $(x_2', x_3')^T = \mathbf{R}_2(x_2, x_3)^T + \mathbf{s}_2$ with:

$$\mathbf{R}_2 = \begin{pmatrix} e_0^2 - e_1^2 & -2e_0e_1 \\ 2e_0e_1 & e_0^2 - e_1^2 \end{pmatrix}, \qquad \mathbf{s}_2 = \begin{pmatrix} 2(e_0t_2 - e_1t_3) \\ 2(e_0t_3 + e_1t_2) \end{pmatrix}, \tag{6}$$

where $det\mathbf{R}_2 = (e_0^2 + e_1^2)^2 = 1$ holds.

2.3 Kinematic Mappings of SE(4)

Until now the author is only aware of one kinematic mapping of SE(4), which was given by Klawitter and Hagemann [9]. They presented an unified concept based on Clifford algebras, for constructing kinematic mappings for certain Cayley-Klein geometries. Especially for E^2 and E^3, they demonstrated that their approach yields the Blaschke-Grünwald mapping and the Study mapping, respectively.

According to Sects. 6.3 and 7 of [9], displacements of SE(4) are mapped onto points of a real 15-dimensional projective space P^{15}, which are located in the intersection of nine quadrics R_i $(i = 1, \ldots, 9)$ sliced along the quadric N_1. For the explicit equations of $R_1, \ldots, R_9$ and N_1 we refer to Sect. 7 of [9].

Due to the large number of homogeneous motion parameters as well as the resulting set of quadratic constraints, the Klawitter-Hagemann mapping is not suited for performing computational algebraic kinematics in E^4. Therefore we are interested in a simplified kinematic mapping of SE(4), which is constructed next.

We start by embedding the points X of E^4 with Cartesian coordinates (x_0, x_1, x_2, x_3) into the set of quaternions by the mapping:

$$\iota_4 : \mathbb{R}^4 \to \mathbb{H} \quad \text{with} \quad (x_0, x_1, x_2, x_3) \mapsto \mathfrak{X} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}. \tag{7}$$

Moreover we need the classical quaternion representation theorem for SO(4), which has many fathers (Euler, Cayley, Salmon, Elfrinkhof, Stringham, Bouman) according to Mebius [10] and states the following:

Theorem 1 *The mapping of points $\mathsf{X} \in E^4$ to $\mathsf{X}' \in E^4$ induced by any element of SO(4), can be written as follows (by using ι_4):*

$$\mathfrak{X} \mapsto \mathfrak{X}' \quad \textit{with} \quad \mathfrak{X}' := \mathfrak{E} \circ \mathfrak{X} \circ \mathfrak{F}, \tag{8}$$

where $\mathfrak{E}$ and $\mathfrak{F}$ is a pair of unit quaternions, which is determined uniquely up to the sign. Moreover the mapping of Eq. (8) is an element of SO(4) for any pair of unit quaternions $\mathfrak{E}$ and $\mathfrak{F}$.

Direct computation shows that the mapping given in Eq. (8) can be written in vector-representation as $(x_0', x_1', x_2', x_3')^T = \mathbf{R}_4(x_0, x_1, x_2, x_3)^T$ with $\mathbf{R}_4 = \mathbf{EF}$ and

$$\mathbf{E} = \begin{pmatrix} e_0 & -e_1 & -e_2 & -e_3 \\ e_1 & e_0 & -e_3 & e_2 \\ e_2 & e_3 & e_0 & -e_1 \\ e_3 & -e_2 & e_1 & e_0 \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} f_0 & -f_1 & -f_2 & -f_3 \\ f_1 & f_0 & f_3 & -f_2 \\ f_2 & -f_3 & f_0 & f_1 \\ f_3 & f_2 & -f_1 & f_0 \end{pmatrix}, \tag{9}$$

where $det\mathbf{R}_4 = det\mathbf{E}\, det\mathbf{F} = (e_0^2 + e_1^2 + e_2^2 + e_3^2)^2 (f_0^2 + f_1^2 + f_2^2 + f_3^2)^2 = 1$ holds.

Due to the free choice of sign in Theorem 1, the decomposition into a left unit quaternion $\mathfrak{E}$ and a right unit quaternion $\mathfrak{F}$ yields a double cover of SO(4). Therefore we consider again the homogeneous 8-tuple $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3)$, which can be seen as a point in P^7. Hence there is a bijection between SO(4) and all real points $\mathscr{S}$ of P^7, which are located on the quadric $\Psi \subset P^7$ given by

$$(e_0^2 + e_1^2 + e_2^2 + e_3^2) - (f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0, \tag{10}$$

($\Rightarrow$ the signature of Ψ is $(4_+, 4_-, 0_0)$) sliced along the 3-dimensional space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathfrak{E}$ cannot be normalized. But this 3-space does not have a real intersection with Ψ and therefore no point of Ψ has to be removed. Note that Eq. (10) expresses the fact that $\mathfrak{F}$ is also normalized if $\mathfrak{E}$ is.

Remark 1 If we identify E^3 with the hyperplane $x_0 = 0$, all points of the 3-dimensional generator-space $f_0 = e_0$, $f_i = -e_i$ for $i = 1, 2, 3$ ($\Leftrightarrow \mathfrak{F} = \widetilde{\mathfrak{E}}$) of Ψ, map the hyperplane $x_0 = 0$ onto itself. Therefore this 3-dimensional generator-space is the well-known Euler-Rodrigues parameter space $(e_0 : \ldots : e_3)$ of SO(3).

The extension of this kinematic mapping of SO(4) with respect to translations of E^4 can be done as follows:

Theorem 2 *The mapping of points* $\mathsf{X} \in E^4$ *to* $\mathsf{X}' \in E^4$ *induced by any element of SE(4), can be written as follows (by using* ι_4*):*

$$\mathfrak{X} \mapsto \mathfrak{X}' \quad with \quad \mathfrak{X}' := \mathfrak{E} \circ \mathfrak{X} \circ \mathfrak{F} - (\mathfrak{T} \circ \mathfrak{E} + \mathfrak{E} \circ \widetilde{\mathfrak{T}}). \tag{11}$$

Moreover the mapping of Eq. (11) is an element of SE(4) for any triple of quaternions $\mathfrak{E}, \mathfrak{F}, \mathfrak{T}$, *where* $\mathfrak{E}$ *and* $\mathfrak{F}$ *are unit quaternions.*

Proof Due to Theorem 1, we only have to show that there is a bijection between the coordinates of the translation vector $\mathbf{s}_4 = (s_0, s_1, s_2, s_3)^T$ and the entries $t_0, \ldots, t_3$ of $\mathfrak{T}$ for a given unit quaternion $\mathfrak{E}$. It can easily be seen that s_1, s_2, s_3 equal the expressions given in Eq. (5) and that $s_0 = -2(e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3)$ holds. Now we solve these four equations for $t_0, \ldots, t_3$ which yields:

$$t_0 = -(e_0s_0 + e_1s_1 + e_2s_2 + e_3s_3)/2, \qquad t_1 = (e_0s_1 - e_1s_0 - e_2s_3 + e_3s_2)/2,$$
$$t_2 = (e_0s_2 + e_1s_3 - e_2s_0 - e_3s_1)/2, \qquad t_3 = (e_0s_3 - e_1s_2 + e_2s_1 - e_3s_0)/2.$$

This already proves the theorem. □

As both triples of quaternions $\pm(\mathfrak{E}, \mathfrak{F}, \mathfrak{T})$, where $\mathfrak{E}$ and $\mathfrak{F}$ are unit quaternions, correspond to the same Euclidean motion of E^4, we consider the homogeneous 12-tuple $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3 : t_0 : \ldots : t_3)$. These 12 homogeneous motion parameters for E^4, which are called the *new parameters* for short, can be interpreted as a point of a projective 11-dimensional space P^{11}. Therefore there is a bijection between SE(4) and all real points $\mathscr{S}$ of P^{11} located on the cylinder Ξ over Ψ, which is also given by Eq. (10) ($\Rightarrow$ the signature of Ξ is $(4_+, 4_-, 4_0)$) and is sliced along the 7-dimensional space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathfrak{E}$ cannot be normalized. But the real intersection of this 7-space and Ξ equals the 3-dimensional generator-space $e_0 = e_1 = e_2 = e_3 = f_0 = f_1 = f_2 = f_3 = 0$ of Ξ. Therefore only the points of this 3-space have to be removed from Ξ.

Remark 2 If we identify E^3 with the hyperplane $x_0 = 0$, all points of the 7-dimensional generator-space $f_0 = e_0$, $f_i = -e_i$ for $i = 1, 2, 3$ (cf. Remark 1) of Ξ, which additionally fulfill the condition that no translation is done in direction of x_0 ($\Leftrightarrow s_0 = 0$), map the hyperplane $x_0 = 0$ onto itself. As the condition $s_0 = 0$ equals the Study condition, the 7-dimensional generator-space of Ξ is the Study parameter space of SE(3). This shows that the Study parameters and subsequently the Blaschke-Grünwald parameters can be obtained from the *new parameters*.

Finally it should be noted that the exceptional quadric of the *new parameter space* is given by $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 0$ and therefore it is also quasi-elliptic (cf. [11]) like the kinematic image spaces named after Study and Blaschke-Grünwald.

3 The Hypersphere Condition

In the following we study the hypersphere condition Ω_n of E^n for $n = 4$, but we formulate everything in a way that it is also valid for the lower-dimensional counterparts, i.e. the sphere condition for $n = 3$ and the circle condition for $n = 2$.

Based on our results of Sect. 2, the mapping $\mathsf{X} \mapsto \mathsf{X}'$ implied by an element of SE(n) can be written in vector-representation as follows:

$$\begin{pmatrix} x'_{4-n} \\ \ldots \\ x'_3 \end{pmatrix} = \frac{1}{N_n} \left[\mathbf{R}_n \begin{pmatrix} x_{4-n} \\ \ldots \\ x_3 \end{pmatrix} + \mathbf{s}_n \right], \tag{12}$$

with $N_2 = e_0^2 + e_1^2$ and $N_3 = N_4 = e_0^2 + e_1^2 + e_2^2 + e_3^2$, respectively, if we neglect the normalizing condition $N_n = 1$. Note that the factor N_n^{-1}, which corresponds to the division by 1, is inserted in order to homogenize Eq. (12).

Now we can write the constraint Ω_n that the point X is located on a hypersphere of E^n with midpoint M and radius ρ as follows:

$$\Omega_n : \quad (x'_{4-n} - m_{4-n})^2 + \ldots + (x'_3 - m_3)^2 - \rho^2 = 0, \tag{13}$$

where $m_{4-n}, \ldots, m_3$ are the coordinates of M and with $x'_{4-n}, \ldots, x'_3$ of Eq. (12). As $N_n \neq 0$ holds the denominator of Ω_n cannot vanish and so we can focus on the nominator, which is a homogeneous polynomial P_n of degree 4 in the motion parameters.

- $n = 2$: A closer look at P_2 shows that N_2 factors out and we remain with a homogeneous quadratic equation in the Blaschke-Grünwald parameters, which is the so-called circle equation Q_2.
- $n = 3$: Interestingly P_3 does not behave like P_2, but Husty [12] showed that N_3 factors out if we add four times the squared Study condition to P_3. The remaining homogeneous quadratic equation in the Study parameters is the so-called sphere equation Q_3, which is the key for solving the direct kinematics of SG platforms (cf. [12]). Note that according to Sect. 2.2, we can obtain Q_2 from Q_3 by setting $m_1 = x_1 = e_2 = e_3 = t_0 = t_1 = 0$.
- $n = 4$: It is not difficult to see that P_4 factors into N_4 and a homogeneous quadratic equation in the *new parameters*. This is the so-called hypersphere equation Q_4, which is the base of any algebraic kinematical study (e.g. solution of the direct kinematics) of 10-dof $S_4\underline{P}S_4$ parallel manipulators.
  According to Remark 2 we can obtain Q_3 from Q_4 by setting $m_0 = x_0 = 0$, $f_0 = e_0$, $f_i = -e_i$ for $i = 1, 2, 3$. This also sheds light onto Husty's tricky addition, as it corresponds to the summand s_0^2 within the *new parameter* approach.

Based on the hypersphere condition, we prove in the next theorem that singular (infinitesimal movable) poses of 10-dof $S_4\underline{P}S_4$ manipulators have an analogous line-geometric characterization as those of their lower-dimensional counterparts.

Theorem 3 *A 10-dof $S_4\underline{P}S_4$ manipulator is in a singular configuration $\mathscr{C}$ if and only if the carrier lines of the ten $\underline{P}$-joints belong to a linear complex of lines of E^4.*

Proof For the proof we consider an arbitrarily given configuration $\mathscr{C}$ of the 10-dof $S_4\underline{P}S_4$ manipulator. Without loss of generality we can assume that the coordinates of the platform anchor points X and the base anchor points M are given with respect to the same reference frame. Therefore $\mathscr{C}$ is given by the identity map of SE(4), which corresponds to the point $I = (1 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0)$ on Ξ.

We study an arbitrary motion through $\mathscr{C}$, which depends on the time τ, where $\mathscr{C}$ is passed at $\tau = 0$. Now the partial derivative of the normalizing condition $N_4 = 1$ and the equation of the cylinder Ξ with respect to τ evaluated in I yields $\dot{e}_0 = \dot{f}_0 = 0$, where the superior dot denotes the time derivative. Under consideration of this result the partial derivative of Q_4 with respect to τ simplifies to:

$$\frac{1}{2\rho}\left(\sum_{i=1}^{3} Q_{4,e_i}\dot{e}_i + \sum_{i=1}^{3} Q_{4,f_i}\dot{f}_i + \sum_{i=0}^{3} Q_{4,t_i}\dot{t}_i\right) = \dot{\rho} \quad \text{with} \quad Q_{4,v_i} := \frac{\partial Q_4}{\partial v_i} \tag{14}$$

for $v \in \{e, f, t\}$ and

$$\begin{aligned}
Q_{4,e_1}+Q_{4,f_1}&=4(m_0x_1-m_1x_0)=:4g_{01} & Q_{4,e_1}-Q_{4,f_1}&=4(m_2x_3-m_3x_2)=:4g_{23}\\
Q_{4,e_2}+Q_{4,f_2}&=4(m_0x_2-m_2x_0)=:4g_{02} & Q_{4,e_2}-Q_{4,f_2}&=4(m_3x_1-m_1x_3)=:4g_{31}\\
Q_{4,e_3}+Q_{4,f_3}&=4(m_0x_3-m_3x_0)=:4g_{03} & Q_{4,e_3}-Q_{4,f_3}&=4(m_1x_2-m_2x_1)=:4g_{12}\\
Q_{4,t_0}&=4(m_0-x_0)=:4g_{04} & Q_{4,t_j}&=4(x_j-m_j)=:4g_{4j}
\end{aligned}$$

for $j = 1, 2, 3$. This is a linear relation between the instantaneous motion of the platform and the velocity $\dot{\rho}$ of the $\underline{P}$-joint. Therefore the coefficient matrix with respect to $\dot{e}_1, \dot{e}_2, \dot{e}_3, \dot{f}_1, \dot{f}_2, \dot{f}_3, \dot{t}_0, \dot{t}_1, \dot{t}_2, \dot{t}_3$ of the system of ten linear equations (14), which are induced by the ten $S_4\underline{P}S_4$-legs, is the 10×10 Jacobian matrix $\mathbf{J}$. As a consequence the given configuration $\mathscr{C}$ is singular for $det\mathbf{J} = 0$.

In the following, we consider the projective point coordinates of X and M, i.e.

$$(x_0^\star : \ldots : x_3^\star : x_4^\star) := (x_0 : \ldots : x_3 : 1),\ (m_0^\star : \ldots : m_3^\star : m_4^\star) := (m_0 : \ldots : m_3 : 1).$$

With this notation the ten Grassmann coordinates $l_{ij} = -l_{ji}$ of the line $[\mathsf{M}, \mathsf{X}] \in E^4$, which are the analogue to the six Plücker coordinates of lines in E^3, can be computed as $l_{ij} := m_i^\star x_j^\star - m_j^\star x_i^\star$ for $i \neq j$ and $i, j \in \{0, 1, 2, 3, 4\}$ (cf. [13]). As the g_{ij}'s defined in Eq. (14) equal the l_{ij}'s, the theorem is proven. □

4 Conclusion and Outlook

In this chapter we developed all basics (kinematic mapping, hypersphere condition, Jacobian matrix, line-geometric characterization of singular configurations) for a future, deeper study of hyperplanar 10-dof $S_4\underline{P}S_4$ parallel manipulators of E^4, which aims to improve the geometric understanding of non-planar SG platforms.

Moreover the hypersphere condition, written in the novel 12 homogeneous motion parameters for E^4, yielded a deeper insight into Husty's tricky addition for the generation of the sphere condition in terms of Study parameters (cf. [12]), which is the central equation for (computational) algebraic kinematics of SG manipulators.

Klawitter and Hagemann showed in Sect. 5.1 (resp. 5.2) of [9] (see also Sect. 9 of [14]) that the algebraic structure of the Study parameters (resp. Blaschke-Grünwald parameters) corresponds to the Spin group of the even part of the Clifford Algebra with signature $(3_+, 0_-, 1_0)$ (resp. $(2_+, 0_-, 1_0)$), which is isomorph to the group of unit dual quaternions (cf. Sect. 2.1). The algebraic structure behind the *new parameters* of SE(4) is still unknown and dedicated to future research.

Acknowledgments This research is supported by Grant No. I 408-N13 and Grant No. P 24927-N25 of the Austrian Science Fund FWF. Moreover the author would like to thank the reviewers for their suggestions and comments, which have helped to improve the quality of the chapter.

References

1. Mielczarek, S., Husty, M.L., Hiller, M.: Designing a redundant Stewart-Gough platform with a maximal forward kinematics solution set. In: Proceedings of the International Symposion of Multibody Simulation and Mechatronics (MUSME), Mexico City, Mexico (2002)
2. Borras, J., Thomas, F., Torras, C.: Singularity-invariant leg rearrangements in doubly-planar Stewart-Gough platforms. In: Proceedings of Robotics Science and Systems, Zaragoza, Spain (2010)
3. Nawratil, G.: Review and recent results on Stewart Gough platforms with self-motions. Appl. Mech. Mater. **162**, 151–160 (2012)
4. Nawratil, G.: Correcting Duporcq's theorem. Mech. Mach. Theory **73**, 282–295 (2014)
5. Blaschke, W.: Euklidische Kinematik und nichteuklidische Geometrie. Z. Math. Phys. **60**, 61–91, 203–204 (1911)
6. Grünwald, J.: Ein Abbildungsprinzip, welches die ebene Geometrie und Kinematik mit der räumlichen Geometrie verknüpft. Sitz.-Ber. der math.-nat. Klasse der kaiserlichen Akademie der Wissenschaften Wien **120**, 677–741 (1911)
7. Study, E.: Geometrie der Dynamen. Teubner, Leipzig (1903)
8. Husty, M., Karger, A., Sachs, H., Steinhilper, W.: Kinematik und Robotik. Springer, Berlin (1997)
9. Klawitter, D., Hagemann, M.: Kinematic mappings for Cayley-Klein geometries via Clifford algebras. Beiträge zur Algebra und Geometrie **54**(2), 737–761 (2013)
10. Mebius, J.E.: History of the quaternion representation theorem for four-dimensional rotations. http://www.jemebius.home.xs4all.nl/So4hist.htm
11. Giering, O.: Vorlesungen über höhere Geometrie. Vieweg, Braunschweig (1982)
12. Husty, M.: An algorithm for solving the direct kinematics of general Stewart-Gough platforms. Mech. Mach. Theory **31**(4), 365–380 (1996)
13. Pottmann, H, Wallner, J.: Computational Line Geometry. Springer, Berlin (2001)
14. Selig, J.M.: Geometric Fundamentals of Robotics. Springer, New York (2005)

http://www.springer.com/978-3-319-06697-4

Advances in Robot Kinematics
Lenarcic, J.; Khatib, O. (Eds.)
2014, XII, 561 p. 229 illus., 180 illus. in color.,
Hardcover
ISBN: 978-3-319-06697-4