Chapter 2
Axiomatic Set Theory

ERNST ZERMELO (1871–1953) was the first to find an axiomatization of set theory, and it was later expanded by ABRAHAM FRAENKEL (1891–1965).

2.1 Zermelo–Fraenkel Set Theory

The language of set theory, which we denote by $\mathscr{L}_\in$, is the usual language of first order logic (with one type of variables) equipped with just one binary relation symbol, $\in$. The intended domain of set theoretical discourse (i.e., the range of the variables) is the universe of all sets, and the intended interpretation of $\in$ is "is an element of." We shall use x, y, z, ..., a, b, ..., etc. as variables to range over sets.

The standard axiomatization of set theory, ZFC (ZERMELO–FRAENKEL set theory with choice), has infinitely many axioms. The first one, the *axiom of extensionality*, says that two sets are equal iff they contain the same elements.

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)). \tag{Ext}$$

A set x is a *subset* of y, abbreviated by $x \subset y$, if $\forall z (z \in x \rightarrow z \in y)$. (Ext) is then logically equivalent to $\forall x \forall y (x \subset y \wedge y \subset x \rightarrow x = y)$. We also write $y \supset x$ for $x \subset y$. x is a *proper subset* of y, written $x \subsetneq y$, iff $x \subset y$ and $x \neq y$.

The next axiom, the *axiom of foundation*, says that each nonempty set has an $\in$-minimal member.

$$\forall x (\exists y \; y \in x \rightarrow \exists y (y \in x \wedge \neg \exists z (z \in y \wedge z \in x))). \tag{Fund}$$

This is easier to grasp if we use the following abbreviations: We write $x = \emptyset$ for $\neg \exists y \; y \in x$ (and $x \neq \emptyset$ for $\exists y \; y \in x$), and $x \cap y = \emptyset$ for $\neg \exists z (z \in x \wedge z \in y)$. (Fund) then says that

$$\forall x (x \neq \emptyset \rightarrow \exists y (y \in x \wedge y \cap x = \emptyset)).$$

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(Fund) plays an important technical role in the development of set theory.

Let us write $x = \{y, z\}$ instead of

$$y \in x \wedge z \in x \wedge \forall u(u \in x \rightarrow (u = y \vee u = z)).$$

The *axiom of pairing* runs as follows.

$$\forall x \forall y \exists z\ z = \{x, y\}. \tag{Pair}$$

We also write $\{x\}$ instead of $\{x, x\}$.

In the presence of (Pair), (Fund) implies that there cannot be a set x with $x \in x$: if $x \in x$, then x is the only element of $\{x\}$, but $x \cap \{x\} \neq \emptyset$, as $x \in x \cap \{x\}$. A similar argument shows that there cannot be sets $x_1, x_2, \ldots, x_k$ such that $x_1 \in x_2 \in \cdots \in x_k \in x_1$ (cf. Problem 2.1).

Let us write $x = \bigcup y$ for

$$\forall z(z \in x \leftrightarrow \exists u(u \in y \wedge z \in u)).$$

The *axiom of union* is the following one.

$$\forall x \exists y\ y = \bigcup x. \tag{Union}$$

Writing $z = x \cup y$ for

$$\forall u(u \in z \leftrightarrow u \in x \vee u \in y),$$

(Pair) and (Union) prove that $\forall x \forall y \exists z(z = x \cup y)$, as $x \cup y = \bigcup \{x, y\}$.

The *power set axiom*, (Pow), says that for every set x, the set of all subsets of x exists. We write $x = \mathscr{P}(y)$ for

$$\forall z(z \in x \leftrightarrow z \subset y)$$

and formulate

$$\forall x \exists y\ y = \mathscr{P}(x). \tag{Pow}$$

The *axiom of infinity*, (Inf), tells us that there is a set which contains all of the following sets as members:

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots.$$

To make this precise, we call a set x *inductive* iff

$$\emptyset \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x).$$

We then say:

$$\exists x(x \text{ is inductive}). \tag{Inf}$$

We now need to formulate the separation and replacement schemas.

A *schema* is an infinite set of axioms which is generated in a simple (recursive) way.

Let φ be a formula of $\mathscr{L}_\in$ in which exactly the variables $x, v_1, \ldots, v_p$ (which all differ from b) occur freely. The *axiom of separation*, or "*Aus*sonderung," corresponding to φ runs as follows.

$$\forall v_1 \ldots \forall v_p \forall a \exists b \forall x\, (x \in b \leftrightarrow x \in a \wedge \varphi). \tag{Aus_φ}$$

Let us write $b = \{x \in a : \varphi\}$ for $\forall x(x \in b \leftrightarrow x \in a \wedge \varphi)$. If we suppress $v_1, \ldots, v_p$, (Aus_φ) then says that

$$\forall a \exists b\, b = \{x \in a : \varphi\}.$$

Writing $z = x \cap y$ for

$$\forall u(u \in z \leftrightarrow u \in x \wedge u \in y),$$

($\text{Aus}_{x \in c}$) proves that $\forall a \forall c \exists b\, b = a \cap c$. Writing $z = x \backslash y$ for

$$\forall u(u \in z \leftrightarrow u \in x \wedge \neg u \in y),$$

($\text{Aus}_{\neg x \in c}$) proves that $\forall a \forall c \exists b\, b = a \backslash c$. Also, if we write $x = \bigcap y$ for

$$\forall z(z \in x \leftrightarrow \forall u(u \in y \rightarrow z \in u)),$$

then ($\text{Aus}_{\forall u(u \in y \rightarrow z \in u)}$), applied to any member of y proves that

$$\forall y(y \neq \emptyset \rightarrow \exists x\, x = \bigcap y).$$

The *separation schema* (Aus) is the set of all (Aus_φ). It says that we may separate elements from a given set according to some well-defined device to obtain a new set.

Now let φ be a formula of $\mathscr{L}_\in$ in which exactly the variables $x, y, v_1, \ldots, v_p$ (all different from b) occur freely. The *replacement axiom* corresponding to φ runs as follows.

$$\forall v_1 \ldots \forall v_p\, (\forall x \exists y' \forall y(y = y' \leftrightarrow \varphi) \rightarrow \forall a \exists b \forall y(y \in b \leftrightarrow \exists x(x \in a \wedge \varphi))). \tag{Rep_φ}$$

The *replacement schema* (Rep) is the set of all (Rep_φ). It says that we may replace elements from a given set according to some well-defined device by other sets to obtain a new set.

We could not have crossed out "$x \in a$" in (Aus_φ). If we did cross it out in (Aus_φ) and let φ be $\neg x \in x$, then we would get

$$\exists b \forall x (x \in b \leftrightarrow \neg x \in x),$$

which is a *false* statement, because it gives $b \in b \leftrightarrow \neg b \in b$. This observation sometimes runs under the title of "RUSSELL*'s Antinomy*."

In what follows we shall write $x \notin y$ instead of $\neg x \in y$, and we shall write $x \neq y$ instead of $\neg x = y$.

A trivial application of the separation schema is the existence of the empty set $\emptyset$ which may be obtained from any set a by separating using the formula $x \neq x$ as φ, in other words,

$$\exists b \forall x (x \in b \leftrightarrow x \neq x).$$

With the help of (Pair) and (Union) we can then prove the existence of each of the following sets:

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \ldots.$$

In particular, we will be able to prove the existence of each member of the intersection of all inductive sets. This will be discussed in the next chapter.

The *axiom of choice* finally says that for each family of pairwise disjoint non-empty sets there is a "choice set," i.e.

$$\forall x (\forall y (y \in x \to y \neq \emptyset) \wedge \forall y \forall y' (y \in x \wedge y' \in x \wedge y \neq y' \to y \cap y' = \emptyset)$$
$$\to \exists z \forall y (y \in x \to \exists u \forall u' (u' = u \leftrightarrow u' \in z \cap y))). \tag{AC}$$

In what follows we shall always abbreviate $\forall y (y \in x \to \varphi)$ by $\forall y \in x\ \varphi$ and $\exists y (y \in x \wedge \varphi)$ by $\exists y \in x\ \varphi$. We may then also formulate (AC) as

$$\forall x (\forall y \in x\ y \neq \emptyset \wedge \forall y \in x \forall y' \in x (y \neq y' \to y \cap y' = \emptyset)$$
$$\to \exists z \forall y \in x \exists u\ z \cap y = \{u\}),$$

i.e., for each member of x, z contains exactly one "representative."

One may also formulate (AC) in terms of the existence of choice functions (cf. Problem 2.6).

The theory which is given by the axioms (Ext), (Fund), (Pair), (Union), (Pow), (Inf) and (Aus$_\varphi$) for all φ is called ZERMELO'*s set theory*, abbreviated by Z. The theory which is given by the axioms of Z together with (Rep$_\varphi$) for all φ is called ZERMELO–FRAENKEL *set theory*, abbreviated by ZF. The theory which is given by the axioms of ZF together with (AC) is called ZERMELO–FRAENKEL *set theory with choice*, abbreviated by ZFC. This system, ZFC, is the standard axiomatization of set theory. Most questions of mathematics can be decided in ZFC, but many questions of set theory and other branches of mathematics are independent from ZFC. The theory which is given by the axioms of Z together with (AC) is called ZERMELO *set theory with choice* and is often abbreviated by ZC. We also use ZFC^- to denote ZFC without (Pow), and we use $\mathsf{ZFC}^{-\infty}$ to denote ZFC without (Inf).

Modulo ZF, (AC) has many equivalent formulations. In order to formulate some of them, we first have to introduce basic notations of axiomatic set theory, though.

For sets x, y we write (x, y) for $\{\{x\}, \{x, y\}\}$. It is easy to verify that for all x, y, x', y', if $(x, y) = (x', y')$, then $x = x'$ and $y = y'$. The set (x, y) can be shown to exist for every x, y by applying the pairing axiom three times; (x, y) is called the *ordered pair* of x and y.

We also write $\{x_1, \ldots, x_{n+1}\}$ for $\{x_1, \ldots, x_n\} \cup \{x_{n+1}\}$ and $(x_1, \ldots, x_{n+1})$ for $((x_1, \ldots, x_n), x_{n+1})$. If $(x_1, \ldots, x_{n+1}) = (x'_1, \ldots, x'_{n+1})$, then $x_1 = x'_1$, …, and $x_{n+1} = x'_{n+1}$.

The *Cartesian product* of two sets a, b is defined to be

$$a \times b = \{(x, y) \colon x \in a \wedge y \in b\}.$$

Lemma 2.1 *For all a, b, $a \times b$ exists, i.e., $\forall a \forall b \exists c\ c = a \times b$.*

Proof $a \times b$ may be separated from $\mathscr{P}(\mathscr{P}(a \cup b))$. □

We also define $a_1 \times \cdots \times a_{n+1}$ to be $(a_1 \times \cdots \times a_n) \times a_{n+1}$ and

$$a^n = \underbrace{a \times \cdots \times a}_{n\text{-times}}.$$

An *n-ary relation r* is a subset of $a_1 \times \cdots \times a_n$ for some sets $a_1, \ldots, a_n$. The n-ary relation r is *on a* iff $r \subset a^n$. If r is a binary (i.e., 2-ary) relation, then we often write $x\ r\ y$ instead of $(x, y) \in r$, and we define the *domain of r* as

$$\mathrm{dom}(r) = \{x \colon \exists y\ x\ r\ y\}$$

and the *range of r* as

$$\mathrm{ran}(r) = \{y \colon \exists x\ x\ r\ y\}.$$

A relation $r \subset a \times b$ is a *function* iff

$$\forall x \in \mathrm{dom}(r) \exists y \forall y' (y' = y \leftrightarrow x\ r\ y').$$

If $f \subset a \times b$ is a function, and if $x \in \mathrm{dom}(f)$, then we write $f(x)$ for the unique $y \in \mathrm{ran}(f)$ with $(x, y) \in f$.

A function f is a *function from d to b* iff $d = \mathrm{dom}(f)$ and $\mathrm{ran}(f) \subset b$ (sic!), which we also express by writing

$$f \colon d \to b.$$

The set of all functions from d to b is denoted by ${}^d b$.

Lemma 2.2 *For all d, b, ${}^d b$ exists.*

Proof ${}^d b$ may be separated from $\mathscr{P}(d \times b)$. □

If $f: b \to d$ and $g: d \to e$, then we write $g \circ f$ for the function from b to e which sends $x \in b$ to $g(f(x)) \in e$.

If $f: d \to b$, then f is *surjective* iff $b = \text{ran}(f)$, and f is *injective* iff

$$\forall x \in d \forall x' \in d (f(x) = f(x') \to x = x').$$

f is *bijective* iff f is surjective and injective.

If $f: d \to b$ and $a \subset d$, then $f \restriction a$, the *restriction of f to a*, is that function $g: a \to b$ such that $g(x) = f(x)$ for every $x \in a$. We write $f''a$ for the *image of a under f*, i.e., for the set $\{y \in \text{ran}(f) : \exists x \in a\ y = f(x)\}$. Of course, $f''a = \text{ran}(f \restriction a)$.

If $f: d \to b$ is injective, and if $y \in \text{ran}(f)$, then we write $f^{-1}(y)$ for the unique $x \in \text{dom}(f)$ with $f(x) = y$. If $c \subset b$, then we write $f^{-1''}c$ for the set $\{x \in \text{dom}(f) : f(x) \in c\}$.

A binary relation $\le$ on a set a is called a *partial order on a* iff $\le$ is *reflexive* (i.e., $x \le x$ for all $x \in a$), $\le$ is *symmetric* (i.e., if $x, y \in a$, then $x \le y \wedge y \le x \to x = y$), and $\le$ is *transitive* (i.e., if $x, y, z \in a$ and $x \le y \wedge y \le z$, then $x \le z$). In this case we call $(a, \le)$ (or just a) a *partially ordered set*. If $\le$ is a partial order on a, then $\le$ is called *linear* (or *total*) iff for all $x \in a$ and $y \in a$, $x \le y$ or $y \le x$.

If $(a, \le)$ is a partially ordered set, then we also write $x < y$ iff $x \le y \wedge x \ne y$. Notice that $x \le y$ iff $x < y \vee x = y$. We shall also call $<$ a partial order.

Let $(a, \le)$ be a partially ordered set, and let $b \subset a$. We say that x is a *maximal element of b* iff $x \in b \wedge \neg\exists y \in b\ x < y$. We say that x is *the maximum of b*, $x = \max(b)$, iff $x \in b \wedge \forall y \in b\ y \le x$. We say that x is a *minimal element* of b iff $x \in b \wedge \neg\exists y \in b\ y < x$, and we say that x is *the minimum of b*, $x = \min(b)$, iff $x \in b \wedge \forall y \in b\ x \le y$. Of course, if $x = \max(b)$, then x is a maximal element of b, and if $x = \min(b)$, then x is a minimal element of b. We say that x is an *upper bound of b* iff $y \le x$ for each $y \in b$, and we say that x is a *strict upper bound of b* iff $y < x$ for each $y \in b$; x is the *supremum of b*, $x = \sup(b)$, iff x is the minimum of the set of all upper bounds of b, i.e., if x is an upper bound and

$$\forall y \in a (\forall y' \in b\ y' \le y \to x \le y).$$

If $x = \max(b)$, then $x = \sup(b)$. We say that x is a *lower bound of b* iff $x \le y$ for each $y \in b$, and we say that x is a *strict lower bound of b* iff $x < y$ for all $y \in b$; x is the *infimum of b*, $x = \inf(b)$, iff x is the maximum of the set of all lower bounds of b, i.e., if x is a lower bound and

$$\forall y \in a (\forall y' \in b\ y \le y' \to y \le x).$$

If $x = \min(b)$, then $x = \inf(b)$. If $\le$ is not clear from the context, then we also say "$\le$-maximal element," "$\le$-supremum," "$\le$-upper bound," etc.

Let $(a, \le_a)$, $(b, \le_b)$ be partially ordered sets. A function $f: a \to b$ is called *order-preserving* iff for all $x, y \in a$,

$$x \le_a y \Longleftrightarrow f(x) \le_b f(y).$$

If $f: a \to b$ is order-preserving and f is bijective, then f is called an *isomorphism*, also written

$$(a, \le_a) \overset{f}{\cong} (b, \le_b).$$

$(a, \le_a)$ and $(b, \le_b)$ are called *isomorphic* iff there is an isomorphism $f: a \to b$, written

$$(a, \le_a) \cong (b, \le_b).$$

The following concept plays a key role in set theory.

Definition 2.3 Let $(a, \le)$ be a partial order. Then $(a, \le)$ is called a well-ordering iff for every $b \subset a$ with $b \ne \emptyset$, $\min(b)$ exists.

The natural ordering on $\mathbb{N}$ is a well-ordering, but there are many other well-orderings on $\mathbb{N}$ (cf. Problem 2.7).

Lemma 2.4 *Let $(a, \le)$ be a well-ordering. Then $\le$ is total.*

Proof If $x, y \in a$, then $\min(\{x, y\}) \le x$ and $\min(\{x, y\}) \le y$. Hence if $\min(\{x, y\}) = x$, then $x \le y$, and if $\min(\{x, y\}) = y$, then $y \le x$. □

Lemma 2.5 *Let $(a, \le)$ be a well-ordering, and let $f: a \to a$ be order-preserving. Then $f(x) \ge x$ for all $x \in a$.*

Proof If $\{x \in a : f(x) < x\} \ne \emptyset$, set

$$x_0 = \min(\{x \in a : f(x) < x\}).$$

Then $y_0 = f(x_0) < x_0$ and so $f(y_0) < f(x_0) = y_0$, as f is order-preserving. But this contradicts the choice of x_0. □

Lemma 2.6 *If $(a, \le)$ is a well-ordering, and if $(a, \le) \overset{f}{\cong} (a, \le)$, then f is the identity.*

Proof By the previous lemma applied to f as well as to f^{-1}, we must have $f(x) \ge x$ as well as $f^{-1}(x) \ge x$, i.e., $f(x) = x$, for every $x \in a$. □

Lemma 2.7 *Suppose that $(a, \le_a)$ and $(b, \le_b)$ are both well-orderings such that $(a, \le_a) \cong (b, \le_b)$. Then there is a unique f with $(a, \le_a) \overset{f}{\cong} (b, \le_b)$.*

Proof If $(a, \le_a) \overset{f}{\cong} (b, \le_b)$ and $(a, \le_a) \overset{g}{\cong} (b, \le_b)$, then $(a, \le_a) \overset{g^{-1} \circ f}{\cong} (a, \le_a)$, so $g^{-1} \circ f$ is the identity, so $f = g$. □

If $(a, \le)$ is a partially ordered set, and if $x \in a$, then we write $(a, \le) \restriction x$ for the partially ordered set

$$(\{y \in a\colon y < x\}, \le \cap \{y \in a\colon y < x\}^2),$$

i.e., for the restriction of $(a, \le)$ to the predecessors of x.

Lemma 2.8 *If $(a, \le)$ is a well-ordering, and if $x \in a$, then $(a, \le) \ncong (a, \le) \restriction x$.*

Proof If $(a, \le) \overset{f}{\cong} (a, \le) \restriction x$, then $f\colon a \to a$ is order-preserving with $f(x) < x$. This contradicts Lemma 2.5. □

Theorem 2.9 *Let $(a, \le_a)$, $(b, \le_b)$ be well-orderings. Then exactly one of the following statements holds true.*

(1) $(a, \le_a) \cong (b, \le_b)$
(2) $\exists x \in b\ (a, \le_a) \cong (b, \le_b) \restriction x$
(3) $\exists x \in a\ (a, \le_a) \restriction x \cong (b, \le_b)$.

Proof Let us define $r \subset a \times b$ by

$$(x, y) \in r \iff (a, \le_a) \restriction x \cong (b, \le_b) \restriction y.$$

By the previous lemma, for each $x \in a$ there is at most one $y \in b$ such that $(x, y) \in r$ and vice versa. Therefore, r is an injective function from a subset of a to b. We have that r is order-preserving, because, if $x <_a x'$ and

$$(a, \le_a) \restriction x' \overset{f}{\cong} (b, \le_b) \restriction y,$$

then

$$(a, \le_a) \restriction x \overset{f \restriction \{y \in a: y < x\}}{\cong} (b, \le_b) \restriction f(x),$$

so that $r(x) = f(x) < y = r(x')$.

If both $a \setminus \mathrm{dom}(r)$ as well as $b \setminus \mathrm{ran}(r)$ were nonempty, say $x = \min(a \setminus \mathrm{dom}(r))$ and $y = \min(b \setminus \mathrm{dom}(r))$, then

$$(a, \le_a) \restriction x \overset{r}{\cong} (b, \le_b) \restriction y,$$

so that $(x, y) \in r$ after all. Contradiction! □

The following Theorem is usually called ZORN*'s Lemma*. The reader will gladly verify that its proof is performed in the theory ZC.

Theorem 2.10 (Zorn) Let $(a, \le)$ be a partial ordering, $a \ne \emptyset$, such that for all $b \subset a, b \ne \emptyset$, if $\forall x \in b \forall y \in b (x \le y \vee y \le x)$, then b has an upper bound. Then a has a maximal element.

Proof Fix $(a, \leq)$ as in the hypothesis. Let

$$A = \{\{(b, x) : x \in b\} : b \subset a, b \neq \emptyset\}.$$

Notice that A exists, as it can be separated from $\mathscr{P}(\mathscr{P}(a) \times \bigcup \mathscr{P}(a))$. (AC), the axiom of choice, gives us some set f such that for all $y \in A$ there is some z with $y \cap f = \{z\}$, which means that for all $b \subset a$, $b \neq \emptyset$, there is some unique $x \in b$ such that $(b, x) \in f$. Therefore, f is a function from $\mathscr{P}(a) \setminus \{\emptyset\}$ to a such that $f(b) \in b$ for every $b \in \mathscr{P}(a) \setminus \{\emptyset\}$.

Let us now define a binary relation $\leq^*$ on a as follows.

We let W denote the set of all well-orderings $\leq'$ of subsets b of a such that for all $u, v \in b$, if $u \leq' v$, then $u \leq v$, and for all $u \in b$, writing

$$B_u^{\leq'} = \{w \in a : w \text{ is a } \leq\text{–upper bound of } \{v \in b : v <' u\}\},$$

$B_u^{\leq'} \neq \emptyset$ and $u = f(B_u^{\leq'})$. Notice that W may be separated from $\mathscr{P}(a^2)$.

Let us show that if $\leq', \leq'' \in W$, then $\leq' \subset \leq''$ or else $\leq'' \subset \leq'$. Let $\leq' \in W$ be a well-ordering of $b \subset a$, and let $\leq'' \in W$ be a well-ordering of $c \subset a$.

By Theorem 2.9, we may assume by symmetry that either $(b, \leq') \cong (c, \leq'')$ or else there is some $v \in c$ such that $(b, \leq') \cong (c, \leq'') \upharpoonright v$. Let $g : b \to c$ be such that

$$(b, \leq') \overset{g}{\cong} (c, \leq'') \text{ or } (b, \leq') \overset{g}{\cong} (c, \leq'') \upharpoonright v.$$

We aim to see that g is the identity on b.

Suppose not, and let $u_0 \in b$ be $\leq'$-minimal in

$$\{w \in b : g(w) \neq w\}.$$

Writing $\bar{g} = g \upharpoonright \{w \in b : w <' u_0\}$,

$$(b, \leq') \upharpoonright u_0 \overset{\bar{g}}{\cong} (c, \leq'') \upharpoonright g(u_0),$$

and $\bar{g}$ is in fact the identity on $\{w \in b : w <' u_0\}$, so that

$$\{w \in b : w <' u_0\} = \{w \in c : w <'' g(u_0)\}.$$

But then $B_{u_0}^{\leq'} = B_{g(u_0)}^{\leq''} \neq \emptyset$ and thus

$$u_0 = f(B_{u_0}^{\leq'}) = f(B_{g(u_0)}^{\leq''}) = g(u_0).$$

Contradiction!

We have shown that if $\leq', \leq'' \in W$, then $\leq' \subset \leq''$ or $\leq'' \subset \leq'$.

But now $\bigcup W$, call it $\leq^*$, is easily seen to be a well-ordering of a subset b of a. Setting

$$B = \{w \in a : w \text{ is a } \leq \text{–upper bound of } b\},$$

our hypothesis on $\leq$ gives us that $B \neq \emptyset$. Suppose that b does have a maximum with respect to $\leq$. We must then have $B \cap b = \emptyset$, and if we set

$$u_0 = f(B)$$

and $\leq^{**} = \leq^* \cup \{(u, u_0) : u \in b\} \cup \{(u_0, u_0)\}$, then $B = B^{\leq^{**}}_{u_0}$. It is thus easy to see that $\leq^{**} \in W$. This gives $u_0 \in b$, a contradiction!

Thus b has a maximum with respect to $\leq$. ZORN's Lemma is shown. □

The following is a special case of ZORN's lemma (cf. Problem 3.10).

Corollary 2.11 (Hausdorff Maximality Principle) *Let $a \neq \emptyset$, and let $A \subset \mathscr{P}(a)$ be such that for all $B \subset A$, if $x \subset y \vee y \subset x$ for all $x, y \in B$, then there is some $z \in A$ such that $x \subset z$ for all $x \in B$. Then A contains an $\subset$-maximal element.*

In the next chapter, we shall use the HAUSDORFF Maximality Principle to show that every set can be well-ordered (cf. Theorem 3.23).

It is not hard to show that in the theory ZF, (AC) is in fact equivalent with ZORN's Lemma, with the HAUSDORFF Maximality Principle, as well as with the assertion that every set can be well-ordered, i.e., that for every set x there is some well-order $<$ on x (cf. Problem 3.10).

2.2 Gödel–Bernays Class Theory

There is another axiomatization of set theory, BGC, which is often more convenient to use. Its language is the same one as $\mathscr{L}_\in$, except that in addition there is a second type of variables. The variables $x, y, z, \ldots, a, b, \ldots$ of $\mathscr{L}_\in$ are supposed to range over *sets*, whereas the new variables, $X, Y, Z, \ldots, A, B, \ldots$ are supposed to range over *classes*. Each set is a class, and a given class is a set iff it is a member of some class (equivalently, of some set). Classes which are not sets are called *proper classes*. Functions may now be proper classes. The axioms of the BERNAYS–GÖDEL *class theory* BG are (Ext), (Fund), (Pair), (Union), (Pow), (Inf) exactly as before together with the following ones:

$$\forall X \forall Y \forall x ((x \in X \leftrightarrow x \in Y) \rightarrow X = Y) \tag{2.1}$$

$$\forall x \exists X \; x = X \tag{2.2}$$

$$\forall X \, (\exists Y \; X \in Y \leftrightarrow \exists x \; x = X) \tag{2.3}$$

If F is a (class) function, then $F''a$ is a set for each set a, (Rep*) and for all φ such that φ is a formula of the language of BG, which contains exactly $x, X_1, \ldots, X_k$ (but not Y) as its free variables and which does not have quantifiers ranging over classes (in other words, φ results from a formula φ' of the language of ZF by replacing free occurences of set variables by class variables), then

$$\forall X_1 \ldots X_k \exists Y \forall x (x \in Y \leftrightarrow \varphi). \qquad (\mathrm{Comp}_\varphi)$$

(Comp_φ) is called the *comprehension axiom* for φ, and the collection of all (Comp_φ) is called the *comprehension schema*. The BERNAYS–GÖDEL *class theory with choice*, BGC, is the theory BG plus the following version of the axiom of choice:

$$\text{There is a (class) function } F \text{ such that } \forall x (x \neq \emptyset \rightarrow F(x) \in x). \qquad (\mathrm{AC})$$

It can be shown that ZFC and BGC prove the same theorems in their common language $\mathscr{L}_\in$ (i.e., BGC is conservative over ZFC).

If φ is a formula as in (Comp_φ), then we shall write $\{x : \varphi\}$ for the class given by (Comp_φ). (Rep^*) says that for all class functions F and for all sets a, $F''a = \{y : \exists x\, (x, y) \in F\}$ is a set.

We shall write V for the *universe of all sets*, i.e., for $\{x : x = x\}$. V cannot be a set, because otherwise

$$R = \{x \in V : x \notin x\}$$

would be a set, and then $R \in R$ iff $R \notin R$. This is another instantiation of RUSSELL's antinomy.

If A is a class, then we write

$$\bigcup A = \{x : \exists y \in A\, x \in y\}$$

and

$$\bigcap A = \{x : \forall y \in A\, x \in y\}.$$

$\bigcup A$ and $\bigcap A$ always exist, and $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = V$.

It may be shown that in contrast to ZFC, BGC can be *finitely* axiomatized. BGC will be the theory used in this book.

The books [15, 18, 23] present introductions to axiomatic set theory.

2.3 Problems

2.1. Let $k \in \mathbb{N}$. Show that there cannot be sets $x_1, x_2, \ldots, x_k$ such that $x_1 \in x_2 \in \ldots \in x_k \in x_1$.

2.2. Show that for all x, y, (x, y) exists. Show that if $(x, y) = (x', y')$, then $x = x'$ and $y = y'$. Show that for all a, b, $a \times b$ exists (cf. Lemma 2.1). Show that for all d, b, ${}^d b$ exists (cf. Lemma 2.2). Which axioms of ZF do you need in each case? Show that (Pair) may be derived from the rest of the axioms of ZF (from which ones?).

2.3. Show that neither in (Aus_φ) nor in (Rep_φ), as formulated on p. 11, we could have allowed b to occur freely in φ. Show that the separation schema (Aus) can be derived from the rest of the axioms of ZF augmented by the statement $\exists x\ x = \emptyset$.

2.4. Show that the following "version" of (AC) is simply false:

$$\forall x(\forall y \in x\ y \neq \emptyset) \to \exists z \forall y \in x \exists u\ z \cap y = \{u\}).$$

2.5. Show tht every partial order can be extended to a linear order. More precisely: Let a be any set. Show that for any partial order $\leq$ on a there is a linear order $\leq'$ on a with $\leq \subset \leq'$.

2.6. Show that in the theory ZF, the following statements are equivalent.

(i) (AC).
(ii) For every x such that $y \neq \emptyset$ for every $y \in x$ there is a *choice function*, i.e., some $f: x \to \bigcup x$ such that $f(y) \in y$ for all $y \in x$.

2.7. (a) Let $\leq$ denote the natural ordering on $\mathbb{N}$, and let $m \in \mathbb{N}$, $m \geq 2$. Let the ordering $\leq^m$ on $\mathbb{N}$ be defined as follows. $n \leq^m n'$ iff either $n \equiv n' (\mathrm{mod}\ m)$ and $n \leq n'$, or else if $k < m$, $k \in \mathbb{N}$, is least such that $n \equiv k (\mathrm{mod}\ m)$ and $k' < m$, $k' \in \mathbb{N}$, is least such that $n' \equiv k' (\mathrm{mod}\ m)$, then $k < k'$. Show that $\leq^m$ is a well-ordering on $\mathbb{N}$.
(b) Let, for $m \in \mathbb{N}$, $\leq^m$ be *any* well-ordering of $\mathbb{N}$, and let $\varphi: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection. Let us define $\leq'$ on $\mathbb{N}$ by $n \leq' n'$ iff, letting $(m, q) = \varphi(n)$ and $(m', q') = \varphi(n')$, $m < m'$ or else $m = m'$ and $q \leq^m q'$. Show that $\leq'$ is a well-ordering of $\mathbb{N}$.

2.8. (**Cantor**) Let $(a, <)$ be a linear order. $(a, <)$ is called *dense* iff for all $x, y \in a$ with $x < y$ there is some $z \in a$ with $x < z < y$. Show that if $(a, <)$ is dense (and a has more than one element), then $<$ is not a well-ordering on a. $(a, <)$ is said to have *no endpoints* iff for all $x \in a$ there are $y, z \in a$ with $y < x < z$. Let $(a, <_a)$ and $(b, <_b)$ be two dense linear orders with no endpoints such that both a and b are countable. Show that $(a, <_a)$ is isomorphic to $(b, <_b)$. [Hint. Write $a = \{x_n : n \in \mathbb{N}\}$ and $b = \{y_n : n \in \mathbb{N}\}$, and construct $f: a \to b$ by recursively choosing $f(x_0)$, $f^{-1}(y_0)$, $f(x_1)$, $f^{-1}(y_1)$, etc.]

2.9. Show that there is a set A of pairwise non-isomorphic linear orders on $\mathbb{N}$ such that $A \sim \mathbb{R}$.

2.10. Show that every axiom of ZFC is provable in BGC.
Let us introduce ACKERMANN'*s set theory*, AST. The language of AST arises from $\mathscr{L}_\in$ by adding a single constant, say $\dot{v}$. The axioms of AST are (Ext), (Fund), (Aus), as well as (Str) and (Refl) which are formulated as follows.

$$\forall x \in \dot{v}\, \forall y\, ((y \in x \vee y \subset x) \to y \in \dot{v}). \tag{Str}$$

Let φ be any formula of $\mathscr{L}_\in$ in which exactly $v_1, \ldots, v_k$ occur freely. Then $\varphi^{\dot{v}}$ results fromφ by replacing every occurence of $\forall x$ by $\forall x \in \dot{v}$ and every occurence of $\exists x$ by $\exists x \in \dot{v}$. Then

$$\forall v_1 \in \dot{v} \ldots \forall v_k \in \dot{v}\, (\varphi^{\dot{v}} \longleftrightarrow \varphi). \tag{Refl$_\varphi$}$$

(Refl) is the schema of all (Refl$_\varphi$), where φ is a formula of $\mathscr{L}_\in$ (in which $\dot{v}$ does not occur). (Str) states that $\dot{v}$ is "supertransitive," and (Refl) states (as a schema) that $\dot{v}$ is a fully elementary submodel of V, the universe of all sets.

2.11. **(W. Reinhardt)** Show that every axiom of ZF is provable in AST.
AST is also conservative over ZF, cf. Problem 5.15.

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