Chapter 2
Physical Illustrations

2.1 Manifolds

2.1.1 The Configuration Space of a Mechanical System

The modern geometric viewpoint in Physics owes a great deal to Lagrange's Analytical Mechanics (Lagrange 1788). In Lagrange's view, a mechanical system is characterized by a finite number n of *degrees of freedom* to each of which a *generalized coordinate* is assigned. A *configuration* of the system is thus identified with an ordered n-tuple of real numbers. What are these numbers coordinates of? An answer to this question, which could not have been, and was in fact not, asked at the time, would have brought Lagrange close to the general concept of topological manifold. Riemann, who invented the term *manifold* in his epoch making inaugural lecture (Riemann 1854)[1] three quarters of a century later, associated it with the existence of a smooth metric.

As an elementary example of Lagrange's approach, consider a cart moving on a frictionless rail and carrying the point of suspension of a plane pendulum, as shown in Fig. 2.1. This mechanical system can be characterized by two independent degrees of freedom. If we were to adopt as generalized coordinates the horizontal displacements, x_1 and x_2, of the cart and the pendulum tip, respectively, measured from some vertical reference line, we would find that to an arbitrary combination of these two numbers, there may correspond up to two different configurations. If, to avoid this problem, we were to replace the second generalized coordinate by the angular deviations θ of the pendulum from the vertical, we would find that a given configuration can be characterized by an infinite number of values of the second coordinate, due to the additive freedom of 2π. If we attempt to solve this

[1]Riemann's work is translated and reproduced in its entirety in Spivak (1979), itself an invaluable source for the study of differential geometry.

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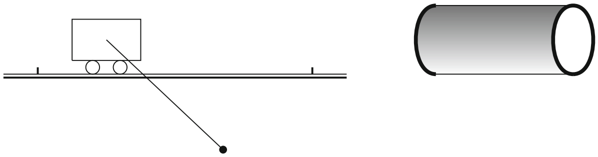

Fig. 2.1 A mechanical system and its configuration space

problem by limiting the range of θ to the interval $[0, 2\pi)$, we lose continuity of the representation (since two neighbouring configurations would correspond to very distant values of the coordinates).

Ironically, in the Preface to his masterpiece, Lagrange proudly boasts that the reader "will not find any figures in this work. The methods that I expose herein require neither geometrical nor mechanical constructions or reasonings, but only algebraic operations …" Let us, however, go against Lagrange's own belief and attempt to draw a mental picture of the topology of the situation. Since the cart is constrained to move along the (infinite) rail, we conclude that its configurations can be homeomorphically mapped onto $\mathbb{R}$. The pendulum tip can describe a circumference around any position of the cart. It is not difficult to conclude that the configuration space of the double pendulum is given by the surface of a cylinder. Now that this basic geometric (topological) question has been settled, we realize that an atlas of this cylinder must consist of at least two charts. But the central conceptual gain of the geometrical approach is that the configuration space of a mechanical system, whose configurations are defined with continuity in mind, can be faithfully represented by a unique topological manifold, up to a homeomorphism. Notice that if the cart were constrained to move between two fixed terminal points, the configuration space would become a manifold with boundary (the lateral surface of a finite cylinder including its two end circumferences, as shown in the figure).

2.1.2 The Configuration Space of a Deformable Body

A mechanical system, such as the one discussed in the previous example, is a finite collection of material points and/or rigid bodies. Topologically, therefore, this collection constitutes a 0-dimensional manifold. Its configuration space is an n-dimensional topological manifold. In the case of a deformable continuous medium, the body itself can be regarded as a 3-dimensional manifold (with or without boundary) having an infinite number of degrees of freedom. By analogy with Lagrange's approach, we are thus naturally led to an infinite-dimensional configuration space. Each element of this space, namely, each configuration of the body, is a one-to-one continuous map of the body into $\mathbb{R}^3$. The theory of infinite-dimensional manifolds in general and of manifolds of maps in particular is beyond our scope. The reader is referred to Lang (1972) and Binz et al. (1988).

2.2 Groups

2.2.1 Local Symmetries of Constitutive Laws

Think of a material point as a small (infinitesimal) die in $\mathbb{R}^3$ that can be deformed into small arbitrary parallelepipeds by means of regular linear maps. These maps are, therefore, represented by non-singular matrices $\mathbf{F}$. Consider now a scalar function of state, or *constitutive function* ψ, such as a stored elastic energy, that depends exclusively on $\mathbf{F}$ via a *constitutive equation* or *constitutive law*

$$\psi = \psi(\mathbf{F}). \tag{2.1}$$

The *general linear group* $GL(3;\mathbb{R})$, that is, the (topological) group of all non-singular 3×3 real matrices $\mathbf{G}$, acts to the right on the collection $\mathcal{F}$ of all possible constitutive equations of the form (2.1) according to the following prescription:

$$R_{\mathbf{G}}\psi(\mathbf{F}) = \psi(\mathbf{FG}). \tag{2.2}$$

An element $\mathbf{G} \in GL(3;\mathbb{R})$ is a *material symmetry* of the constitutive law ψ if

$$R_{\mathbf{G}}\psi = \psi, \tag{2.3}$$

or, more explicitly, if

$$\psi(\mathbf{FG}) = \psi(\mathbf{F}) \tag{2.4}$$

identically for all $\mathbf{F} \in GL(3;\mathbb{R})$. We also say that the constitutive equation $\psi = \psi(\mathbf{F})$ is *invariant* under the right action of the element $\mathbf{G}$ of the general linear group $GL(3;\mathbb{R})$. It is not difficult to show that the collection of symmetries of a given constitutive law ψ is a subgroup $\mathcal{G}_\psi$ of $GL(3;\mathbb{R})$, called the *material symmetry group* of ψ.

Clearly, the unit element of $GL(3;\mathbb{R})$, namely the unit matrix $\mathbf{I}$, is a trivial symmetry of all constitutive laws. On the other hand, it is not difficult to construct examples of constitutive laws that have non-trivial symmetries. For example, any function of the determinant of $\mathbf{F}$ is invariant under the action of any matrix $\mathbf{G}$ with unit determinant.

The general linear group $GL(3;\mathbb{R})$ has also a natural *left action* on constitutive laws according to the formula

$$L_{\mathbf{G}}\psi(\mathbf{F}) = \psi(\mathbf{GF}). \tag{2.5}$$

Of particular interest is the left action of the orthogonal group $O(3,\mathbb{R}) \subset GL(3;\mathbb{R})$. Physically, it represents the influence of the choice of Galilean frame (or observer) on the constitutive law. According to the *principle of material frame indifference* all constitutive laws are invariant under a this action. More precisely,

$$\psi(\mathbf{QF}) = \psi(\mathbf{F}) \qquad \forall \mathbf{Q} \in O(3, \mathbb{R}). \tag{2.6}$$

A restricted version of this principle postulates the invariance under the proper orthogonal group only. While the validity of either of these principles may be a matter of dispute, the adoption of the first leads to the trivial conclusion that all constitutive laws are invariant under multiplication by the element $-\mathbf{I}$. Since this particular multiplication is commutative, we conclude that all constitutive laws abiding by this principle are materially symmetric under an inversion about the origin.

We ask now whether the right action of $GL(3; \mathbb{R})$ on $\mathcal{F}$ is transitive. The answer is clearly negative, since otherwise all constitutive equations would be identical to each other. Is the action free? Again, the answer is negative since, as shown above, there exist constitutive equations with non-trivial symmetries. Finally, one might have expected that the action be at least effective. That would have meant that if an element $\mathbf{G} \in GL(3; \mathbb{R})$ leaves all constitutive laws invariant then it must necessarily be the group unit $\mathbf{I}$. If the set $\mathcal{F}$ is assumed to abide by the general principle of material frame indifference, all constitutive laws are invariant under the action of $-\mathbf{I}$, in which case the action is not even effective.

2.3 Fibre Bundles

2.3.1 Space-Time

2.3.1.1 Aristotelian Space-Time

Starting from our everyday perception of events, we reason, like Aristotle, that time and space are two separate entities with absolute meaning, independent of each other and of the presence of observers. This naive conception leads us immediately to the consideration of space-time as a Cartesian product of two sets. Moreover, as repeatedly pointed out by ancient thinkers, these two entities seem to leave no gaps either between instants of time or locations in space. Finally, only one parameter is needed for the determination of time, while three are needed for location. We arrive thus at a space-time complex consisting of a product bundle, as described in Sect. 1.4.1. The base is a 1-dimensional topological manifold $\mathcal{Z}$ (the time line) and the fibre is a 3-dimensional topological manifold manifold $\mathcal{E}$. We call the product topological manifold

$$\mathcal{S}_A = \mathcal{Z} \times \mathcal{E} \tag{2.7}$$

the *Aristotelian space-time continuum*. Recall that the *Cartesian product* of two sets is the set formed by all ordered pairs such that the first element of the pair belongs to the first set and the second element belongs to the second set. Thus, the elements s of $\mathcal{S}_A$, namely the events, are ordered pairs of the form (t, x), where $t \in \mathcal{Z}$ and $x \in \mathcal{E}$. In other words, for any given $s \in \mathcal{S}_A$, we can determine independently its corresponding temporal and spatial components. In mathematical terms, we say that the 4-dimensional (product) manifold $\mathcal{S}_A$ is endowed with two *projection maps*,

$$\pi_1 : \mathcal{S}_A \longrightarrow \mathcal{Z}, \tag{2.8}$$

and

$$\pi_2 : \mathcal{S}_A \longrightarrow \mathcal{E}, \tag{2.9}$$

defined, respectively, by:

$$\pi_1(s) = \pi_1(t, x) = t, \tag{2.10}$$

and

$$\pi_2(s) = \pi_2(t, x) = x. \tag{2.11}$$

2.3.1.2 Proto-Galilean Space-Time

If, not following the historical development, we would start our analysis of space-time from an arbitrary 4-dimensional topological manifold $\mathcal{S}$, a natural exercise could consist of imposing upon this event manifold a variety of geometric conditions and determining the physical ramifications arising from the resulting geometric structures. We have indeed followed this policy in the previous section when we assumed that $\mathcal{S}$ is a product bundle. The physical consequences of this assumption are the absolute character of simultaneity and of spatial location.

We consider now the case in which space-time is a topological fibre bundle $(\mathcal{C}, \pi, \mathcal{Z}, \mathcal{E}, \mathcal{G})$ with projection π, base $\mathcal{Z}$, typical fibre $\mathcal{E}$ and structure group $\mathcal{G}$. Following Segev and Epstein (1980), we call the resulting entity *proto-Galilean space-time*. The physical meaning of this structure is that, while time retains its absolute character, the determination of event locations is dependent on the local trivialization chosen to represent the bundle. The physical interpretation of any such trivialization cannot be other than that of a (temporary) observer. Two different observers agree, therefore, on the issue of simultaneity. They can tell unequivocally, for instance, whether or not two light flashes occurred at the same time and, if not, which preceded which and by how much. Nevertheless, in the case of two non-simultaneous events, they will in general disagree on the issue of position. For example, an observer carrying a pulsating flashlight, will interpret successive flashes as happening always 'here', while an observer receding uniformly from the first will reckon the successive flashes as happening farther and farther away as time goes on. For a given observer, and for some interval of time, $\mathcal{S}$ looks like $\mathcal{S}_A$.

Moreover, the role of the structure group $\mathcal{G}$ is that of monitoring the allowable observer transformations. A *world line* is a cross section $\gamma : \mathcal{Z} \to \mathcal{S}$ of the space-time bundle.

Classical (Newtonian-Galilean) space time is a particular case of the proto-Galilean variety. The following extra assumptions have to be made to obtain this particular structure: (i) $\mathcal{S}$ is a trivializable bundle; (ii) the typical fibre $\mathcal{E}$ is a 3-dimensional Euclidean affine space; the base manifold $\mathcal{Z}$ is a 1-dimensional Euclidean affine space (essentially $\mathbb{R}$); (iii) the structure group is the group of Galilean transformations (translations, rotations and reflections, that is, *isometries*, or transformations that preserve the Euclidean metric in the typical fibre).

An added feature of classical space-time, arising from dynamical considerations, is that, among all possible trivializations, it is possible to distinguish some that not only preserve the Euclidean structure but also represent changes of observers that travel with respect to each other at a fixed inclination (i.e., without angular velocity) and at a constant velocity of relative translation. Observers related in this way are said to be *inertially related*. It is possible, accordingly, to divide the collection of all observers into equivalence classes of inertially related observers. Of all these inertial classes, Newton declared one to be privileged above all others. This is the class of *inertial observers*, for which the laws of Physics acquire a particularly simple form.

2.3.1.3 Relativistic Space-Time

The revolution brought about by the theory of Relativity (both in its special and general varieties) can be said to have destroyed the bundle structure altogether. In doing so, it in fact simplified the geometry of space-time, which becomes just a 4-dimensional manifold $\mathcal{S}_R$. On the other hand, instead of having two separate metric structures, one for space and one for time, Relativity assumes the existence of a space-time metric structure that involves both types of variables into a single construct. This type of metric structure is what Riemann had already considered in his pioneering work on the subject, except that Relativity (so as to be consistent with the Lorentz transformations) required a metric structure that could lead both to positive and to negative squared distances between events, according to whether or not they are reachable by a ray of light. In other words, the metric structure of Relativity is not positive definite. By removing the bundle structure of space time, Relativity was able to formulate a geometrically simpler picture of space time, although the notion of simplicity is in the eyes of the beholder.

2.3.2 Microstructure

2.3.2.1 Shells

One of the many different ways to describe a *shell* in structural engineering is to regard it as the product bundle of a two-dimensional manifold $\mathcal{B}$ times the open (or sometimes closed) segment $\mathcal{F} = (-1, 1) \in \mathbb{R}$. The base manifold is known as the *middle surface* while the fibre conveys the idea of thickness, eventually responsible for the bending stiffness of the shell. The fact that this is a product bundle means that one can in a natural way identify corresponding locations throughout the thickness at different points of the middle surface. Thus, two points of the shell standing on different points of the middle surface can be said to correspond to each other if they have the same value of the second projection. This fact can be interpreted as being on the same side of the middle surface and at the same fraction of the respective thicknesses.

2.3.2.2 General Microstructure

In a more general context, we can consider three-dimensional bodies for which the usual kinematic degrees of freedom are supplemented with extra (internal) degrees of freedom intended to describe a physically meaningful counterpart. This idea, going at least as far back as the pioneering work of the Cosserat brothers (Cosserat and Cosserat 1909), applies to diverse materials, such as liquid crystals and granular media. The base manifold represents the matrix, or *macromedium*, while the fibres represent the *micromedium* (the elongated molecules or the grains, as the case may be). An example of this situation is provided by an everyday material such as concrete, which is formed by embedding in a cement matrix an aggregate consisting of stones whose size is relatively large when compared with the grains of cement. Each of these stones can then be considered as a *micromedium*. In a continuum model we expect to have these micromedia continuously assigned to each point of the matrix, thus generating a fibre bundle, whose typical fibre is the micromedium. In contradistinction with the case of the shell, there is no canonical correspondence between points belonging to micromedia attached at different points of the macromedium.

2.4 Groupoids

2.4.1 Material Uniformity

2.4.1.1 An Imprecisely Defined Material Body

In Sect. 2.2.1 we introduced the intuitive idea of a material point as a small die of material that can be subjected to linear deformations and whose constitutive response is governed by one or more constitutive equations. Following this imprecise intuitive line of thought, we can consider a sort of “continuous collection” of such material points[2] and regard the resulting entity as a material body. Each of the constituent material points is endowed with its own constitutive law and, if we cavalierly denote by $\mathbf{X}$ a running three-dimensional variable indicating the location of the body $\mathcal{B}$, we obtain the constitutive law of the body as some function

$$\psi = \psi(\mathbf{F}, \mathbf{X}). \tag{2.12}$$

2.4.1.2 Distant Versus Local Symmetry

The collection of material points just introduced bears a resemblance to the collection of tiles making up the bathroom floor of Box 1.8. Recall that each tile has a symmetry group consisting of certain rotations and reflections. But it is also intuitively recognized that the floor as a whole has a repetitive pattern and, therefore, some extra symmetry. In the case of a material body, each material point may have a constitutive law with local symmetries, as discussed in Sect. 2.2.1. Moreover, the fact that two distant points may be made of the same material should be understood as an extra degree of symmetry that the body may possess, just as in the case of the bathroom floor, where distant tiles happen to have the same shape. This analogy should not be pushed too far, but it serves to trigger a useful picture and to understand the unifying role that the concept of groupoid plays in terms of encompassing all types of symmetries.

2.4.1.3 Material Isomorphisms

We want to formalize the answer to the question: Are two material points $\mathbf{X}_1$ and $\mathbf{X}_2$ made of the same material? We reason that for this to be so, the only possible difference between the local constitutive equations $\psi(\mathbf{F}, \mathbf{X}_1)$ and $\psi(\mathbf{F}, \mathbf{X}_2)$ must be a fixed *transplant* represented by some matrix $\mathbf{P}_{12}$ such that

$$\psi(\mathbf{F}, \mathbf{X}_2) = \psi(\mathbf{F}\mathbf{P}_{12}, \mathbf{X}_1), \tag{2.13}$$

identically for all deformations $\mathbf{F}$. Indeed, in this case we would agree that the responses of the two points are exactly the same except for the fact that the die at point $\mathbf{X}_2$ is a rotated or otherwise distorted version of the die at point $\mathbf{X}_1$. In the standard Continuum Mechanics terminology, such a material transplant is known as a *material isomorphism*. It is not difficult to verify that material isomorphism is an equivalence relation. A body is said to be *materially uniform* if all its points are mutually materially isomorphic.

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[2]This notion will be made more precise when we define a differentiable manifold and its tangent bundle.

Notice that a material point is trivially isomorphic to itself (via the identity map), but it may also be non-trivially so (via a non-trivial material symmetry). We thus see that a material isomorphism is a generalization of the notion of local material symmetry to encompass what we may call distant material symmetries of a material body.

2.4.1.4 The Material Groupoid

Given a material body, whether uniform or not, we can imagine an arrow drawn for every material isomorphism between two points, including the material symmetries, namely, the cases whereby the source and target points coincide. In this way, without much further ado, we conclude that every material body with a specified constitutive law gives rise to a groupoid, which we shall call the *material groupoid* of the body. In case the body is materially uniform, we obtain a transitive groupoid. If, in addition, the constitutive equation is continuous in $\mathbf{X}$, we obtain a transitive topological groupoid.[3]

Consider now a non-uniform body. The material groupoid is still properly definable, except that it loses its transitivity. It may still preserve its continuity (namely, it may still be a topological groupoid). A good example of this last situation is provided by the so-called *functionally-graded materials*, which have continuously varying material properties tailored to specific applications.[4]

2.4.1.5 Material Principal Bundles

A *material principal bundle* of a materially uniform body $\mathcal{B}$ is any one of the equivalent principal bundles that can be obtained from the material groupoid. Physically speaking, a material principal bundle is obtained by arbitrarily singling out a material point $\mathbf{X}_0$, called the *material archetype* and replacing the material transplants between arbitrary pairs of points by material *implants* $\mathbf{P}(\mathbf{X})$ from the archetype to each and every point $\mathbf{X}$, as shown in Fig. 2.2. The constitutive equation of a uniform body thus conceived is given by:

$$\psi(\mathbf{F}, \mathbf{X}) = \bar{\psi}(\mathbf{F}\mathbf{P}(\mathbf{X})), \tag{2.14}$$

where we have indicate by $\bar{\psi}$ the constitutive law of the archetype.

[3]For the use of groupoids in the theory of material uniformity see Epstein and de León (1998).

[4]Under certain circumstances, however, the transitivity of the material groupoid of functionally graded materials can be restored by modifying the definition of material isomorphism (Epstein and de León 2000; Epstein and Elżanowski 2007).

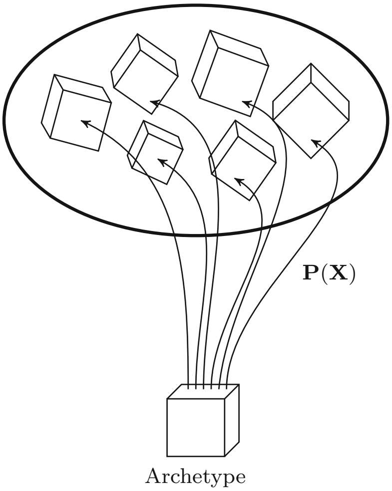

Fig. 2.2 A material principal bundle

We observe, however, that whereas the material groupoid always exists (whether or not the body is uniform), the material principal bundles can only be defined when the body is uniform. Then, and only then, we have a transitive topological groupoid to work with. In conclusion, although both geometrical objects are suitable for the description of the material structure of a body, the groupoid representation is the more faithful one, since it is unique and universal.

The structure group of a material principal bundle is, according to the previous construction, nothing but the material symmetry group of the archetype. As expected, it controls the degree of freedom available in terms of implanting this archetype at the points of the body.

A material principal bundle may, or may not, admit (global) cross sections. If it does, the body is said to be *globally uniform*. This term is slightly misleading, since uniformity already implies that *all* the points of the body are materially isomorphic. Nevertheless the term conveys the sense that the material isomorphisms can be prescribed smoothly in a single global chart of the body (which, by definition, always exists). Put in other terms, the existence of a global section implies (in a principal bundle, as we know) that the principal bundle is trivializable. A cross section of a principal bundle establishes, through the right action of the structure group, a global isomorphism between the fibres, also called a *distant parallelism*. In our context, this property will be called a *material parallelism*. If the structure group is discrete, the material parallelism is unique. Moreover, if the material symmetry group consists of just the identity, a uniform body must be globally uniform.

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