Chapter 2
Feedback Control Theory Continued

2.1 Introduction

In the previous chapter, the response characteristic of simple first and second order transfer functions were studied. It was shown that first order transfer function, sometimes called first order lag, has an overdamped response and the output lags input as it was shown in the frequency response. The second order transfer function as was shown in the previous chapter can have overdamped response for $\zeta > 1$ and can be oscillatory for $\zeta < 1$.

It becomes clear that transfer function of higher order may become unstable. Stability of control system is a major consideration and must be studied. If the system is stable, it must be studied how oscillatory the system is.

2.2 Routh–Hurwitz Stability Criteria

For a system with transfer function of

$$\frac{Y(s)}{X(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_0} \tag{2.1}$$

it is better to study stability in time domain. Converting Eq. (2.1) in differential form yields

$$a_n \cdot \frac{\mathrm{d}^n}{\mathrm{d}n^n} y + a_{n-1} \cdot \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} y + \cdots + a_0 \cdot y := x(t) \tag{2.2}$$

For this kind of differential equations, there are two solutions. One is the transient response and the other one is steady state solution. The steady state solution is obtained by assuming a solution in the form of the input $x(t)$ with unknown coefficients. Then the solution is substituted in the differential equations and the unknown coefficients are obtained by equating the coefficient of the same order in s. Usually

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the solutions in control are obtained for step, ramp, acceleration, and sinusoidal inputs.

More interesting is the transient response, which determines stability and how oscillatory the output is. To obtain the transient response the right hand side of Eq. (2.2) is equated to zero and a solution in the form of

$$y(t) = e^{st} \tag{2.3}$$

is assumed. Substituting Eq. (2.3) in Eq. (2.2) and after some algebraic manipulation yields

$$a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0 = 0 \tag{2.4}$$

Equation (2.4) is known as the characteristic equation and the roots of the characteristic equation determine the transient response. For stability all real parts of the roots must be negative. For real physical system, the complex roots appear in conjugate. This means that if Eq. (2.4) contains complex conjugate the response could be overdamped or oscillatory.

Routh–Hurwitz method is a quick way of establishing the stability of the system. Unfortunately, this method does not indicate how oscillatory the system is.

To determine stability an array in the following form is constructed

$$\begin{array}{lllll}
s^n & : a_n & a_{n-2} & a_{n-4} & \ldots\ldots\ldots\ldots 0 \\
s^{n-1} & : a_{n-1} & a_{n-3} & a_{n-5} & \ldots\ldots\ldots\ldots 0 \\
s^{n-2} & : b_1 & b_2 & b_3 & \ldots\ldots\ldots\ldots 0 \\
s^{n-3} & : c_1 & c_2 & c_3 & \ldots\ldots\ldots\ldots 0 \\
. & & & & \\
. & & & & \\
. & & & & \\
s^1 & : g_1 & 0 & & \\
s^0 & : h_1 & 0 & &
\end{array}$$

where

$$b_1 = \frac{1}{a_{n-1}}(a_{n-1}a_{n-2} - a_n a_{n-3})$$

$$b_2 = \frac{1}{a_{n-1}}(a_{n-1}a_{n-4} - a_n a_{n-5})$$

$$b_3 = \frac{1}{a_{n-1}}(a_{n-1}a_{n-6} - a_n a_{n-7})$$

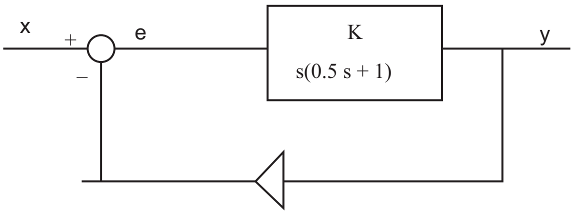

Fig. 2.1 Block diagram of a simple position control system

$$c_1 = \frac{1}{b_1}(b_1 a_{n-3} - b_2 a_{n-1})$$

$$c_2 = \frac{1}{b_1}(b_1 a_{n-5} - b_3 a_{n-1})$$

The calculation for the parameters in the above array may be written in determent form.

For stability, all the coefficients of the characteristic equation must be positive and there must not be sign changes in the first column of the array. Any sign change indicates that there is a root with positive real part. If an element in the first column is zero, a small positive ε is assumed and the sign change is determined when ε tends to zero. If all elements in a row are zero, there is a root with positive real part or zero real part.

There are computer programs that calculate the roots of the characteristic equation. In this case, the roots can be plotted on a complex plane. This brings us to a powerful analysis of stability known as Root Locus method.

2.3 Root Locus Method

It was shown that the stability of control system could be studied by the roots of characteristic equation. In this section, the Root Locus will be studied for a second order system. For higher order system there, is an analytical approach that can be used to plot the Root Locus from the open loop transfer equation. The method is tedious and the loci are plotted from the zeros and poles of the open loop transfer function. It should be mentioned that the number of loci are equal of the order of the characteristic equation. The loci will end to the zeros or infinity as the gain of the system is increased.

The block diagram of a negative feedback of a simple servo position control is shown in Fig. 2.1. The integrator shows the fact that the position is obtained from velocity and the first order lag shows that because of the inertia there is a time lag.

The closed loop transfer function can be obtained by using the block diagram algebra. With some manipulation it can be shown that the closed loop transfer function becomes

$$\frac{y}{x} = \frac{k}{s(0.5s + 1) + k} \tag{2.5}$$

The characteristic equation is, therefore,

$$0.5 \cdot s^2 + s + k := 0 \tag{2.6}$$

The roots are

$$s_{1,2} = -1 \pm \sqrt{1 - 2k} \tag{2.7}$$

Now the root locus can be obtained by varying k from zero to infinity. Some important points are the roots when $k=0$, $k=0.5$ and $k>0.5$. For $k<0.5$, the roots are negative starting from the points 0 and -2. And there is a breakaway point at point $k=0.5$ where the roots becomes complex and as k is increased, the two complex roots move towards infinity parallel to the imaginary axis. By comparing the transfer function with the second order transfer function studied in previous chapter, it can be shown that

$$\omega_n = \sqrt{2k}$$

and

$$\zeta = \sqrt{\frac{1}{2k}}$$

And it can be shown that

$$\cos\theta = \zeta$$

The Root Locus for the above second order system is shown in Fig. 2.2. X and Y contain the real and imaginary part of the root for values of Gain (K). In this case, there are two loci, which end at infinity. The roots are shown by crosses.

Figure 2.3 shows a simple third order transfer function which could represent a position control system with DC motor.

This model contains the effect of inductance in the system. In the open loop transfer function, there is an integrator and two first order lags. Therefore, there are three poles at $s=0$, $s=-1$, $s=-2$. The loci start at these three poles and end to infinity as the gain K is increased. It should be noted that there is no zeros, which makes the numerator of the open loop transfer function equal to zero. The closed loop transfer function may be calculated as

$$\frac{y}{x} := \frac{K}{0.5 \cdot s^3 + 1.5 \cdot s^2 + s + k} \tag{2.8}$$

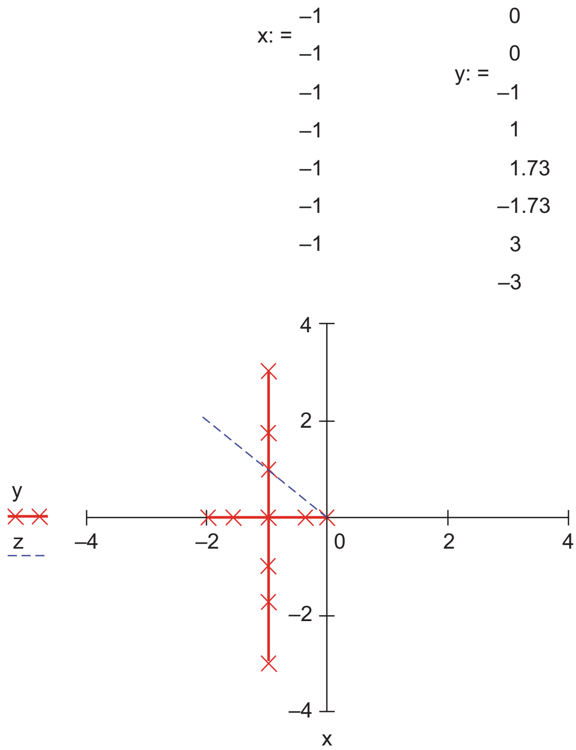

Fig. 2.2 Root Locus for a second order system

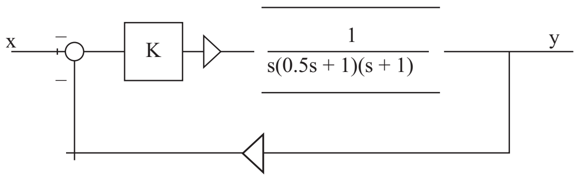

Fig. 2.3 Block diagram of a third order system with gain *K*

With the characteristic equation of

$$0.5 \cdot s^3 + 1.5s^2 + s + k := 0 \tag{2.9}$$

The MathCAD Polyroots expression can be used to calculate the roots of characteristic equation for various *K*. The root locus for this system is shown in Fig. 2.4. At $K=0$, there are three negative real roots. As *K* is increased the two real roots move towards each other and the third real root moves towards infinity. The two real roots break away from the real axis and become complex. As *k* is increased, the real part of complex roots becomes positive showing that the system becomes unstable. The root locus in this case enter the right-hand side of the *s*-plane.

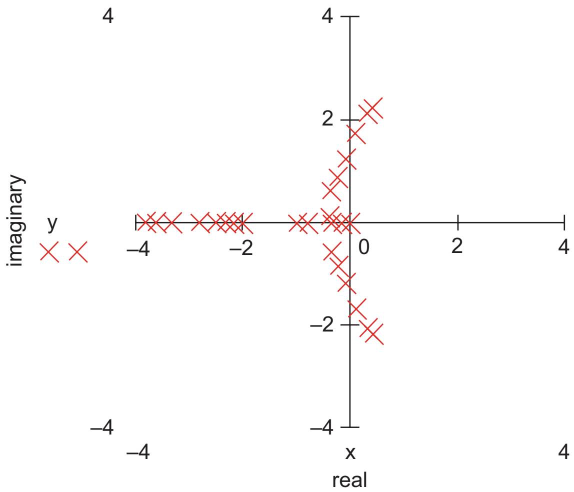

Fig. 2.4 A typical root locus for a third order characteristic equation

The root locus for this system is shown in Fig. 2.4. Higher order systems have more roots and the root locus becomes more complicated. There are also computer programs, which presents the root locus from the open loop transfer function removing the need to calculate the closed loop transfer function. In this book, the MathCAD computer program is used throughout to plot the root locus. There are also methods of calculating the gain for each location of the roots. With MathCAD polyroots facility, the roots can be calculated for each gain or parameter of interest separately. The correct gain or parameter of interest can be obtained by inspection of the table of roots and there is no need to go into details of the graphical methods.

There are computer programs that can calculate the roots of characteristic equation of any order. MatLab and MathCAD are two of the most commonly used computer programs. The Root Locus can then be plotted. The damping ratio indicated by the real part and the frequency of oscillation is determined by the imaginary part. For most control systems, a damping ratio of 0.7–1 is preferred. The damping ratio of 0.7 will result in small overshoot and the damping ratio of 1 results in no overshoot. This means that all roots must lie between the ±45° lines on the left side of the s-plane.

For stability all roots must lie in the left hand side of the s-plane and the imaginary part show in fact the frequencies of oscillation. The further away from the origin the faster the response to a step input. For a good response to step input all roots must lie in the 45° and the negative real axis. For complicated system which have many roots the root nearest to the imaginary axis dominates the step input response.

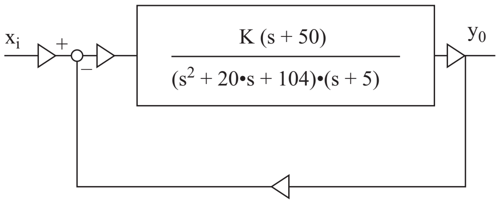

Fig. 2.5 A simple unity feedback system

2.4 Important Features of Root Locus

Usually the transfer function of elements of a control system is in form of first or second order lag form and the closed loop system is in the form of several cascaded of these transfer functions. This can be used to sketch the root locus manually from the open loop transfer functions. There are important features on the Root Locus that the loci can be sketched manually and they are useful to know. It can be used for simple system and it is very complicated for complex systems.

Without loss of generality consider the simple system shown in Fig. 2.5.

Where x_i is the input variable and (y_o) is the output variable to be controlled. The second order term in the denominator could be as a mass-spring-damper transfer equation. The other term in the numerator and in the denominator may represent a lead lag compensation network. The K is the parameter of interest that should be adjusted to obtain a stable and fast responding system.

The closed loop transfer function becomes,

$$\frac{y_o}{x_i} := \frac{[K \cdot (s+50)]}{\left(s^2+20s+104\right) \cdot (s+5) + K \cdot (s+50)}$$

Therefore the characteristic equation becomes,

$$\left(s^2+20s+104\right) \cdot (s+5) + K \cdot (s+50) := 0$$

Dividing both side of the above equation gives the characteristic equation in standard for as,

$$\frac{\left(s^2+20s+104\right) \cdot (s+5)}{K \cdot (s+50)} := -1 \tag{2.10}$$

The angle Law states that the angle of left side of Eq. 2.10, which is a complex number, should be ±180° measured from the real axis counter clockwise.

The magnitude law states that the magnitude of the left side of Eq. (2.10) should be −1.

These two laws and some feature of the loci help us to draw the root locus manually. The order of the denominator is $n=3$ and the order the numerator is $m=1$ for the open loop transfer function. From the characteristic equation is clear that there

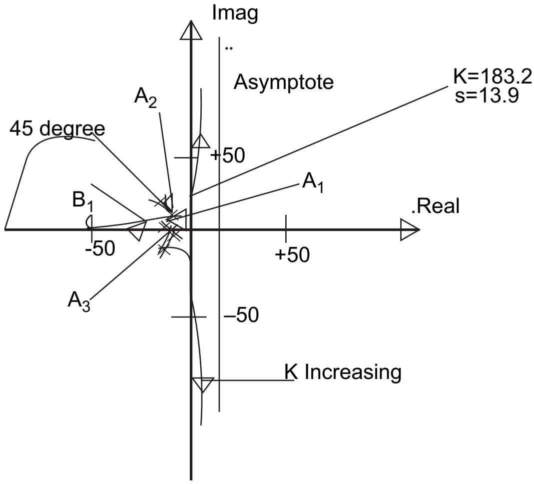

Fig. 2.6 Sketch of the root locus for the example considered in this section

are three loci. When $K=0$ the loci start at the pole of the transfer equation and end to the zero of the numerator and the remaining poles end to infinity. For this example the pole when $K=0$ are.

$$\left(s^2+20\cdot s+104\right)\cdot(s+5)+K\cdot(s+50):=0$$
$$s_1:=-5$$
$$F:=\begin{pmatrix}104\\20\\1\end{pmatrix} \tag{2.11}$$
$$s:=\text{polyroots}(F)$$
$$s_{2,3}:=\begin{pmatrix}-10-2i\\-10+2i\end{pmatrix}$$

It can be seen that even for simple second order equation the Mathcad software can be used to determine the roots. The poles are shown on the graph paper by crosses and zeros are shown by a circle. In this example there is only one zero and it is,

$$Z=-50$$

The poles and zeros are shown on Fig. 2.6.

Those loci which go to infinity have asymptotes determined by the following equation,

$$\theta_1:=\frac{[(2\cdot 1+1)\cdot 180]}{n-m}$$
$$\theta_2:=\frac{-[(2\cdot 1+1)\cdot 180]}{n-m}$$

1 is any integer number. For real systems the complex roots appears in conjugate so for this complex conjugate there is one positive and one negative asymptote. Selecting one for the integer $l=1$ the angle of asymptotes becomes,

$$\theta_1 = 90$$
$$\theta_2 = -90$$

All asymptotes intersect the real axis at a single point and is given by,

$$d = \frac{\text{The sum of all open loop poles} - \text{the sum of open loop zeros}}{n - m}$$

Where n is the number of open loop poles and m is the number of open loop zeros.

It should be noted because all the complex roots appear in conjugate the distance d will be a real number. Therefore the distance d can be calculated as,

$$d := \frac{(-10 - 2i - 10 + 2i - 5 + 50)}{2}$$
$$d := 12.5$$

It is shown in Fig. 2.6.

The loci depart from complex open loop poles and arrive to the complex open loop zeros at angle that satisfy the angle law. For the example in hand the departure angle considering a point very near to the complex root poles becomes,

$$180 - 170 - 180 + \alpha = \pm(2l + 1)180$$

So $\alpha=\pm10°$. This is shown in Fig. 2.6.

Another important point is the intersection of the loci with Imaginary axes. The Routh–Hurwitz array is constructed for the system constructed above. The characteristic equation is given as below,

$$s^3 + 25 \cdot s^2 + (K + 204) \cdot s + 520 + 50k := 0$$

Constructing the Routh–Hurwitz array gives,

$$\begin{array}{llll} s^3 & 1 & 204 + K & 0 \\ s^2 & 25 & 520 + 50 \cdot K & 0 \\ s^1 & 183.2 - K & 0 & 0 \\ s^0 & \frac{1}{183.2 - K} \cdot (9626.4 + 8640 \cdot K - 50 \cdot K^2) & 0 & 0 \end{array}$$

The value of k must be as such that the first column must be all positive for the system to remain stable. Doing this the value of K where the loci crosses the imaginary axis can be found. The first column of third row indicates that the value of K must be less than 183.2. The first column of forth row is more complicated because there is the K to power of 2 and the denominator has also the coefficient K. It is clear that for $k = 183.2$ the first column fourth row for the above K is positive. To find the intersections points on the imaginary axis the auxiliary equation is formed as,

$$K := 183.2$$

$$25 \cdot s^2 + 520 + 50 \cdot K := 0$$

so,

$$s := \sqrt{\frac{-(520 + 50 \cdot K)}{50}}$$

$$s = 13.914i$$

$$s_1 := 13.91i$$

$$s_2 := -13.91i$$

These points are shown on the imaginary axes by crosses. The reader is encourage to do the Routh–Hurwitz algebra themselves to show the data presented here is correct. For design purpose the value of K must be selected to have a damping ratio 0.7. This damping ratio is achievable by drawing a 45° line from the origin and its intersection with the loci gives a damping ratio of 0.7. The following procedures remain as before only the magnitude changes. This is shown by point A on the loci. Then the value of gain is obtained by magnitude law as,

$$K := \frac{(A_1 \cdot A_{2\ldots})}{B_1 \cdot B_{2\ldots}}$$

These measurement is shown in Fig. 2.6, so,

$$K := \frac{(1.5 \cdot 2.5 \cdot 1.3)}{6}$$

$$K := 0.813$$

By remembering these important points the root locus can be sketched manually on a graph paper as shown in Fig. 2.6. This is just an approximate of the root locus and for complicated system, the procedures becomes complex. It is useful to know these points and when the root locus is plotted using a software like Mathcad these point

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