Chapter 2
Modular Representations of Finite Groups

We are now ready to apply the results from the previous chapter to group rings of finite groups. If the order of the group is invertible in the base field, Maschke's Theorem 1.2.8 tells us that the group ring is semisimple, and semisimple rings are of less interest from the homological algebra point of view. Hence, we are mainly interested in base fields in which the group order is not invertible. Representations in this situation are of great interest, and there remain many unsolved questions. They will also provide the main application terrain and test ground for the theories we are going to develop in the sequel. In this chapter, we are dealing with groups, and the variable K will be used most often for subgroups. For this reason the base ring will usually be denoted by k.

2.1 Relatively Projective Modules

2.1.1 Relatively Projective Modules for Subalgebras

Let k be a field and let A be a k-algebra, then recall that an A-module P is projective if every k-split exact sequence

$$0 \longrightarrow M \longrightarrow N \xrightarrow{\pi} P \longrightarrow 0$$

is also split over A.

From this point of view it seems to be natural to define, for a subalgebra B of A, modules over A to be relatively projective with respect to B.

Definition 2.1.1 Let k be a commutative ring and let B be a k-subalgebra of the k-algebra A. An A-module Q is *relatively B-projective* if every exact sequence of A-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0$$

which is split as a sequence of B-modules, is also split as a sequence of A-modules.

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A relatively B-projective A-module has the nicest properties when A is a projective B-module. If A is projective as a B-module, then we may define the notion of being relatively projective with respect to a subalgebra by means of $Ext_A^1(Q, M)$ and $Ext_B^1(Q, M)$. Suppose A is projective as a B-module. Then any free A-module, considered as B-module by restriction, is again projective, and hence any projective A-module, considered as B-module by restriction, is a projective B-module. Hence the restriction of A-modules to B-modules maps projective modules to projective modules. Therefore, the restriction of a projective resolution

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q \longrightarrow 0$$

of Q as an A-module to B is a projective resolution of Q as a B-module. We may apply $Hom_A(-, N)$ or $Hom_B(-, N)$ to the same resolution. Since $Hom_A(-, N) \hookrightarrow Hom_B(-, N)$ we obtain a commutative diagram

$$\begin{array}{ccccccccc}
\ldots \leftarrow & Hom_A(P_2, N) & \leftarrow & Hom_A(P_1, N) & \leftarrow & Hom_A(P_0, N) & \leftarrow & Hom_A(Q, N) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots \leftarrow & Hom_A(P_2, N) & \leftarrow & Hom_B(P_1, N) & \leftarrow & Hom_B(P_0, N) & \leftarrow & Hom_B(Q, N)
\end{array}$$

which induces a mapping

$$Ext_A^i(Q, N) \xrightarrow{\rho} Ext_B^i(Q, N)$$

for all $i \geq 0$. This morphism is compatible with morphisms $N \to N'$ in the sense that the diagram

$$\begin{array}{ccc}
Ext_A^i(Q, N) & \xrightarrow{\rho} & Ext_B^i(Q, N) \\
\downarrow & & \downarrow \\
Ext_A^i(Q, N') & \xrightarrow{\rho} & Ext_B^i(Q, N')
\end{array}$$

obtained by the construction preceding Lemma 1.8.36 is commutative. The following lemma is therefore an immediate consequence of the definition.

Lemma 2.1.2 *Let k be a commutative ring and let B be a k-subalgebra of the k-algebra A. Suppose A is projective as a B-module. Then an A-module Q is relatively B-projective if the natural mapping $Ext_A^1(Q, N) \longrightarrow Ext_B^1(Q, N)$ is injective for all N.*

The most important case for the representation theory of groups will be the case of a kG-module M which is relatively projective with respect to a subgroup H of G, i.e. relatively projective with respect to the subalgebra kH.

Definition 2.1.3 Let G be a finite group, let k be a field and let H be a subgroup of G. A kG-module M is *H-projective* if M is relatively kH-projective.

Recall that for $g \in G$ and a kH-module M we denote by ${}^gM$ the $k(gHg^{-1})$-module given by the same k-module structure as M, and a gHg^{-1}-action on ${}^gM$ denoted by $\bullet$, with $ghg^{-1} \bullet m := h \cdot m$, and where $h \cdot m$ is understood to be the usual action of H on M.

A first observation is that if Q is H-projective, then Q is $g^{-1}Hg$-projective for all $g \in G$. Indeed, given a short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0$$

which splits as a sequence of $k(g^{-1}Hg)$-modules, then the sequence of kG-modules

$$0 \longrightarrow {}^gM \longrightarrow {}^gN \longrightarrow {}^gQ \longrightarrow 0$$

splits as a sequence of kH-modules. For every kG-module X there is a morphism of kG-modules $X \longrightarrow {}^gX$ given by $x \mapsto g \cdot x$. Hence there is an isomorphism of sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}^gM & \longrightarrow & {}^gN & \longrightarrow & {}^gQ & \longrightarrow & 0 \end{array}$$

which implies that Q is $g^{-1}Hg$-projective as well.

We remark that the same proof also works in the case of algebras.

Lemma 2.1.4 *If A is a k-algebra and Q is projective relative to the subalgebra B. Then Q is projective relative to the subalgebra uBu^{-1} for all units u of A.*

Proposition 2.1.5 *Let A be a k-algebra and let B be a subalgebra of A such that A is a projective B-module. Suppose that M is an A-module and suppose that the multiplication mapping $A \otimes_B M \longrightarrow M$ is split as a morphism between A-modules. Then M is relatively B-projective.*

Proof We need to show that the map

$$res : Ext^1_A(M, -) \longrightarrow Ext^1_B(M, -)$$

induced by restriction to B is injective. By hypothesis the multiplication map $A \otimes_B M \to M$ is split and so M is a direct factor of $A \otimes_B M$. Denote by $\mu : A \otimes_B M \longrightarrow M$ the multiplication and by $\sigma : M \longrightarrow A \otimes_B M$ the A-linear splitting. By Lemma 1.8.33 these maps produce a (split injective) map $Ext^1_A(M, -) \xrightarrow{\mu^*} Ext^1_A(A \otimes_B M, -)$. We obtain a diagram

$$\begin{array}{ccc} Ext^1_A(M, -) & \xrightarrow{res} & Ext^1_B(M, -) \\ \downarrow \mu^* & & \| \\ Ext^1_A(A \otimes_B M, -) & \overset{Proposition\ 1.8.31}{\simeq} & Ext^1_B(M, -) \end{array}$$

where σ^* and μ^* are the mappings induced by σ and μ on the Ext^1-groups. We need to show that this diagram is commutative. Indeed, let

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be the first terms of a projective resolution of M as an A-module. Then, by definition, $Ext^1_A(M, X)$ is a subquotient of $Hom_A(P_1, X)$ and analogously $Ext^1_A(A \otimes_B M, X)$ is a subquotient of $Hom_A(A \otimes_B P_1, X)$. Now

$$\begin{aligned} Hom_A(A \otimes_B P_1, X) &\overset{nat}{\simeq} Hom_B(P_1, X) \\ \varphi &\mapsto (p \mapsto \varphi(1 \otimes p)) \end{aligned}$$

and so the multiplication $A \otimes_B M \longrightarrow M$ induces the multiplication mapping $A \otimes_B P_1 \longrightarrow P_1$ which in turn induces

$$\begin{aligned} Hom_A(P_1, X) &\xrightarrow{\mu^*} Hom_A(A \otimes_B P_1, X) \\ \varphi &\mapsto (a \otimes p \mapsto \varphi(a \cdot p)) \end{aligned}$$

But this implies that $nat \circ \mu^*$ coincides with the restriction to B. Hence, *res* is injective if and only if μ^* is injective. But μ^* is split by σ^*, and therefore μ^* is injective. This proves the proposition. □

The converse of Proposition 2.1.5 is true as well, as is shown below.

Proposition 2.1.6 *Let k be a commutative ring and let A be a k-algebra containing a subalgebra B. Suppose that A is projective as a B-module. Then an A-module Q is relatively B-projective if and only if the multiplication mapping $A \otimes_B Q \longrightarrow Q$ is split as a morphism between A-modules.*

Proof In Proposition 2.1.5 we have seen that if $A \otimes_B Q \longrightarrow Q$ is split as a morphism between A-modules, then Q is relatively B-projective. In order to show the converse we observe that the epimorphism given by the multiplication mapping

$$A \otimes_B Q \xrightarrow{\mu} Q$$

is split as a mapping of B-modules by sending $q \in Q$ to $1 \otimes q \in A \otimes_B Q$. Indeed, call this mapping σ. Then

$$\sigma(bq) = 1 \otimes bg = b \otimes q = b \cdot (1 \otimes q)$$

for all $b \in B$ and $q \in Q$. Let $K := \ker(\mu)$. Then the exact sequence

$$0 \longrightarrow K \longrightarrow A \otimes_B Q \longrightarrow Q \longrightarrow 0$$

is split when considered as a sequence of B-modules. Since Q is relatively B-projective, the sequence is split as a sequence of A-modules, and therefore the multiplication mapping μ is split as a morphism between A-modules. □

Remark 2.1.7 In Lemma 2.1.4 we have seen that if Q is relatively B-projective for some k-subalgebra B of the k-algebra A, so that A is projective as a B-module, then Q is also relatively projective for any algebra conjugate to B. Moreover, trivially any A-module is relatively A-projective. Being relatively k-projective is exactly the same as being projective. If Q is relatively B-projective, then the definition implies immediately that Q is relatively C-projective for any subalgebra C of A containing B, and so that B is projective as a C-module.

A very special case, when Proposition 2.1.6 is always true, is the subject of the following statement.

Proposition 2.1.8 *Let k be a commutative ring, let B be a k-algebra and let A be a k-subalgebra. Then the multiplication map*

$$\begin{array}{ccc} B\otimes_A B & \stackrel{\mu}{\longrightarrow} & B \\ b_1\otimes b_2 & \mapsto & b_1 b_2 \end{array}$$

is split as a morphism of $B\otimes_k B^{op}$-modules if and only if there is an $\omega\in\mu^{-1}(1_B)$ such that $b\omega=\omega b$ for all $b\in B$.

If there is such an ω, then every B-module is A-projective and a B-module P is projective if P, considered as an A-module, is projective.

Proof Suppose that μ is split. Then there is a $\sigma: B\longrightarrow B\otimes_A B$ such that $\mu\circ\sigma=id_B$. Let $\omega:=\sigma(1_B)$. Then we get for all $b\in B$

$$b\cdot\omega=b\cdot\sigma(1)=\sigma(b)=\sigma(1)\cdot b=\omega\cdot b.$$

Suppose that ω exists. Then put $\sigma(b):=\omega\cdot b$ for all $b\in B$. Since

$$b_1\cdot\sigma(x)\cdot b_2=b_1\cdot\omega\cdot x\cdot b_2=\omega\cdot b_1\cdot x\cdot b_2=\sigma(b_1\cdot x\cdot b_2)$$

for all $x, b_1, b_2\in B$, this is a morphism of $B\otimes_k B^{op}$-modules. Moreover, for all $b\in B$ we get

$$\mu\circ\sigma(b)=\mu(\omega\cdot b)=\mu(\omega)\cdot b=1_B\cdot b=b$$

since $\omega\in\mu^{-1}(1_B)$.

Let M be a B-module. If μ is split by σ, then the multiplication map μ_M $B\otimes_A M\xrightarrow{\mu_M} M$ on M is split by $\sigma\otimes_B id_M$. Indeed, $\mu_M=\mu\otimes_B id_M$, where we identify

$$B\otimes_B M\simeq M \text{ and } B\otimes_A B\otimes_B M\simeq B\otimes_A M.$$

Hence M is a direct factor of $B\otimes_A M$. Now apply Proposition 1.8.31. This proves the statement. □

We shall now partially follow Külshammer [1].

Definition 2.1.9 Let k be a commutative ring, let B be a k-algebra and let A be a k-subalgebra. An *algebra extension* of A is a k-algebra C together with a k-algebra map $A \to C$. The extension $A \to B$ is *separable* if the multiplication

$$\begin{aligned} B \otimes_A B &\xrightarrow{\mu} B \\ b_1 \otimes b_2 &\mapsto b_1 b_2 \end{aligned}$$

is split as a morphism of B-B-bimodules.

We apply the concept to a Maschke-type statement (cf Theorem 1.2.8).

Proposition 2.1.10 *Let k be a commutative ring, let G be a group and let H be a subgroup of G. Suppose that the index $|G:H|$ is finite and invertible in k. Then the extension $kH \leq kG$ is separable.*

Proof Let

$$\omega := \left(\frac{1}{|G:H|}\sum_{gH\in G/H} g\otimes g^{-1}\right) \in kG \otimes_{kH} kG.$$

It is clear that ω is well-defined, in the sense that $g \otimes_H g^{-1}$ does not depend on the representative $g \in gH$. Moreover, for all $x \in G$ we get

$$\begin{aligned} x\cdot\omega = x\cdot\left(\frac{1}{|G:H|}\sum_{gH\in G/H} g\otimes g^{-1}\right) &= \left(\frac{1}{|G:H|}\sum_{gH\in G/H} xg\otimes g^{-1}\right) \\ &= \left(\frac{1}{|G:H|}\sum_{gH\in G/H} xg\otimes (xg)^{-1}x\right) \\ &= \left(\frac{1}{|G:H|}\sum_{xgH\in G/H} xg\otimes (xg)^{-1}\right)\cdot x = \omega\cdot x. \end{aligned}$$

Finally

$$\mu(\omega) = \mu\left(\frac{1}{|G:H|}\sum_{gH\in G/H} g\otimes g^{-1}\right) = \left(\frac{1}{|G:H|}\sum_{gH\in G/H} g\cdot g^{-1}\right) = 1_{kG}.$$

By Proposition 2.1.8 we obtain the statement. □

Lemma 2.1.11 *Let k be a commutative ring, let G be a group and let H be a subgroup of finite index. If the extension $kH \hookrightarrow kG$ is separable, then $|G:H|$ is invertible in k.*

Proof Suppose that the extension $kH \leq kG$ is separable. By Proposition 2.1.8 every kG-module M is kH-projective and by Proposition 2.1.6 we get that $\mu_M : kG \otimes_{kH} M \longrightarrow M$ is split. Choose $M = k$ the trivial module and let σ_k be a splitting. Then $\sigma_k(1)$ is in the trivial submodule of $kG \otimes_{kH} k \simeq kG/H$. However, the G-trivial submodule of kG/H is k-linearly generated by $\sum_{gH \in G/H} gH$. Hence

$$\sigma_k(1) = \lambda \cdot \sum_{gH \in G/H} gH$$

for some $\lambda \in k$. Since

$$1 = \mu_M \circ \sigma_k(1) = \lambda \cdot |G : H|$$

we get that $|G : H|$ is invertible in k. □

Lemma 2.1.12 *If the index of H in G is invertible in k, then any kG-module is projective relative to H.*

Proof By Proposition 2.1.10 we know that the extension $kH \leq kG$ is separable. Proposition 2.1.8 then shows that every kG-module is relatively kH-projective. □

2.1.2 Vertex and Source

Given an indecomposable A-module Q, is there some minimal subalgebra B_Q, uniquely defined up to conjugacy, such that Q is relatively B_Q-projective? We shall concentrate on the case $A = kG$ for a field k and subalgebras kH for subgroups H of G and mainly use Mackey's formula from Sect. 1.7.3 as a new tool available in this setting. Of course, if H is a subgroup of G, then kG is a free kH-module of rank $|G : H|$. Hence the results of Sect. 2.1.1 apply.

Definition 2.1.13 Let k be a field and let G be a finite group. Let M be an indecomposable kG-module. Then a subgroup D of G is called a *vertex* of M if M is relatively kD-projective but for any proper subgroup D' of D we have that M is not relatively kD'-projective.

We have seen that a vertex always exists, since any kG-module is relatively kG-projective by Remark 2.1.7. If k is of characteristic 0, then by Maschke's Theorem 1.2.8 any module is projective, and therefore the vertex is always the trivial group in this case.

Theorem 2.1.14 *Let G be a finite group and let k be a field of characteristic $p > 0$. Suppose M is an indecomposable kG-module. Then*

1. *any vertex of M is a p-subgroup of G,*
2. *any two vertices of M are conjugate in G.*

Proof The first statement is a direct consequence of Lemma 2.1.12.

We need to prove the second statement. Suppose M has vertices D_1 and D_2 and therefore M is relatively kD_1-projective and kD_2-projective for two p-subgroups D_1 and D_2 of G. We shall now use Proposition 2.1.6. Then M is a direct factor of $M\downarrow^G_{D_1}\uparrow^G_{D_1}$ and of $M\downarrow^G_{D_2}\uparrow^G_{D_2}$. But then

$$\begin{aligned} M \Big| (M\downarrow^G_{D_1}\uparrow^G_{D_1}) &\Rightarrow (M\downarrow^G_{D_2}) \Big| (M\downarrow^G_{D_1}\uparrow^G_{D_1}\downarrow^G_{D_2}) \\ &\Rightarrow (M\downarrow^G_{D_2}\uparrow^G_{D_2}) \Big| (M\downarrow^G_{D_1}\uparrow^G_{D_1}\downarrow^G_{D_2}\uparrow^G_{D_2}) \\ &\Rightarrow M \Big| (M\downarrow^G_{D_1}\uparrow^G_{D_1}\downarrow^G_{D_2}\uparrow^G_{D_2}) \end{aligned}$$

where the last implication holds since $M \Big| (M\downarrow^G_{D_2}\uparrow^G_{D_2})$. Hence

$$\left(M \Big| (M\downarrow^G_{D_1}\uparrow^G_{D_1})\right) \text{ and } \left(M \Big| (M\downarrow^G_{D_1}\uparrow^G_{D_1}\downarrow^G_{D_2}\uparrow^G_{D_2})\right).$$

But Mackey's Theorem 1.7.45 implies

$$M\downarrow^G_{D_1}\uparrow^G_{D_1}\downarrow^G_{D_2}\uparrow^G_{D_2} \simeq \bigoplus_{D_1gD_2 \in D_1\backslash G/D_2} {}^gM\downarrow^G_{D_1\cap {}^gD_2}\uparrow^G_{D_1\cap {}^gD_2}$$

and since M is indecomposable, there is a $g_0 \in G$ such that M is a direct factor of ${}^{g_0}M\downarrow^G_{D_1\cap {}^{g_0}D_2}\uparrow^G_{D_1\cap {}^{g_0}D_2}$. But D_1 is a vertex and any group $D_1 \cap {}^gD_2$ is a subgroup of D_1. Hence $D_1 \cap {}^{g_0}D_2 = D_1$ which is equivalent to $D_1 \subseteq {}^{g_0}D_2$. By the analogous argument interchanging D_1 and D_2 we get that D_2 is conjugate to a subgroup of D_1. This proves the second statement. □

In the case of group rings we will get a slightly more practical characterisation of relative projectivity than Proposition 2.1.6.

Proposition 2.1.15 *Let k be a field, let G be a finite group and let H be a subgroup of G. Then an indecomposable kG-module M is relatively H-projective if and only if M is a direct factor of $L\uparrow^G_H$ for some kH-module L. Here L can be taken to be $M\downarrow^G_H$.*

Proof We have already seen in Proposition 2.1.6 that M is relatively kH-projective if and only if the multiplication map $kG \otimes_{kH} M \longrightarrow M$ is split. But this shows that if M is relatively kH-projective, then M is a direct factor of $M\downarrow^G_H\uparrow^G_H$.

Clearly, if M is a direct factor of $M\downarrow^G_H\uparrow^G_H$ then M is a direct factor of $L\uparrow^G_H$ for some kH-module L.

Suppose now M is a direct factor of $L\uparrow^G_H$ and denote by $\iota : M \hookrightarrow L\uparrow^G_H$ the injection, and by $\pi : L\uparrow^G_H \twoheadrightarrow M$ the projection onto this direct factor. Then there is a kH-linear endomorphism

$$
\begin{aligned}
kG \otimes_{kH} L &\xrightarrow{\rho} kG \otimes_{kH} L \\
g \otimes x &\mapsto \begin{cases} g \otimes x & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

and put $\theta := \pi \circ \rho \circ \iota \in End_{kH}(M)$. For each endomorphism $\tau \in End_{kH}(M)$ we define a trace map

$$Tr_H^G(\tau) := \sum_{gH \in G/H} g\tau g^{-1},$$

that is $Tr_H^G(\tau)(m) := \sum_{gH \in G/H} g\tau(g^{-1}m)$ for all $m \in M$. We observe, just as in the proof of Maschke's theorem, that η is kG-linear. It is clear by definition that we have

$$Tr_H^G(\alpha \circ \tau \circ \beta) = \alpha \circ Tr_H^G(\tau) \circ \beta$$

for all kG-linear endomorphisms α and β of M. Moreover, we see that

$$Tr_H^G(\rho) = id_{L\uparrow_H^G}$$

and hence

$$Tr_H^G(\theta) = \pi \circ Tr_H^G(\rho) \circ \iota = \pi \circ \iota = id_M.$$

Let now

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\gamma} M \longrightarrow 0$$

be a short exact sequence of kG-modules. Suppose that γ has a splitting δ as a morphism of kH-modules, i.e. $\gamma \circ \delta = id_M$. Then put $\delta' := Tr_H^G(\delta \circ \theta)$ and observe

$$\gamma \circ \delta' = \gamma \circ Tr_H^G(\delta \circ \theta) = Tr_H^G(\gamma \circ \delta \circ \theta) = Tr_H^G(\theta) = id_M.$$

Since δ' is a kG-linear homomorphism, we get that γ is split as a morphism of kG-modules. This proves the proposition. □

Remark 2.1.16 We shall continue to study the properties of the map Tr_H^G in a slightly more general context and in more detail in Sect. 2.10.2.

We have seen in Proposition 2.1.15 that an indecomposable kG-module M has vertex D if and only if M is a direct factor of $M \downarrow_D^G \uparrow_D^G$ and D is minimal with respect to this property. Since M is indecomposable, M is a direct factor of $L \uparrow_D^G$ for an indecomposable direct factor L of $M \downarrow_D^G$. Indeed, if

$$M \downarrow_D^G = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

then

$$M \downarrow_D^G \uparrow_D^G = L_1 \uparrow_D^G \oplus \cdots \oplus L_n \uparrow_D^G$$

and by the Krull-Schmidt theorem there is an $i \in \{1, 2, \dots, n\}$ such that M is a direct factor of $L_i \uparrow_D^G$.

Definition 2.1.17 Let k be a field of characteristic $p > 0$ and let G be a finite group. Let M be an indecomposable kG-module with vertex D. An indecomposable kD-module L such that M is a direct factor of $L \uparrow_D^G$ is called a *source* of M.

A source is not unique in general. Indeed, let $N_G(D)$ be the normaliser of D in G. For an indecomposable kD-module L and $g \in N_G(D)$, since D is normal in $N_G(D)$ we get that ${}^gL$ is an indecomposable kD-module as well. Moreover,

$$\begin{aligned} {}^gL \uparrow_D^G &\xrightarrow{\psi} L \uparrow_D^G \\ x \otimes \ell &\mapsto x \cdot g^{-1} \otimes \ell \end{aligned}$$

is well-defined since for $d \in D$, $x \in kG$ and $\ell \in L$ we compute

$$\psi(xd \otimes \ell) = xdg^{-1} \otimes \ell = xg^{-1}gdg^{-1} \otimes \ell = xg^{-1} \otimes gdg^{-1}\ell = \psi(x \otimes d \cdot \ell).$$

Obviously ψ is bijective since an inverse is given by using g instead of g^{-1}. Finally ψ is kG-linear, since

$$\psi(h \cdot (x \otimes \ell)) = \psi(hx \otimes \ell) = hxg^{-1} \otimes \ell = h \cdot (xg^{-1} \otimes \ell) = h \cdot \psi(x \otimes \ell)$$

for all $x \in kG$, $\ell \in L$ and $h \in G$.

Proposition 2.1.18 *Let k be a field of characteristic $p > 0$, let G be a finite group, and let M be an indecomposable kG-module with vertex D. Then a source of M is unique up to conjugacy. More precisely, for any two sources L_1 and L_2, which are kD-modules, there is a $g \in N_G(D)$ such that ${}^gL_1 \simeq L_2$.*

Proof Any source L of M is a direct summand of $M \downarrow_D^G$. Hence L_2 is a direct factor of

$$L_1 \uparrow_D^G \downarrow_D^G \simeq \bigoplus_{DgD \in D\backslash G/D} {}^gL_1 \downarrow_{D \cap {}^gD}^D \uparrow^D$$

and since L_2 is indecomposable there is a $g \in G$ such that L_2 is a direct factor of ${}^gL_1 \downarrow_{D \cap {}^gD}^D \uparrow^D$. But since L_2 is a source of M, and hence M is a direct factor of $L_2 \uparrow_D^G$, we get that M is a direct factor of

$${}^gL_1 \downarrow_{D \cap {}^gD}^D \uparrow^D \uparrow_D^G = {}^gL_1 \downarrow_{D \cap {}^gD}^D \uparrow^G .$$

Now, D is a vertex of M, and hence M cannot be a direct factor of a module induced from a proper subgroup of D. Therefore $D \cap {}^gD = D$, which is equivalent to ${}^gD = D$, whence $g \in N_G(D)$. We obtain a $g \in N_G(D)$ such that L_2 is a direct factor of ${}^gL_1 \uparrow_{D \cap {}^gD}^D$. But $D = D \cap {}^gD$ and L_1 indecomposable implies that $L_2 \simeq {}^gL_1$. □

Lemma 2.1.19 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Then every vertex of the trivial kG-module is a Sylow p-subgroup of G.*

Proof Let D be a vertex of the trivial module k. Then k is a direct factor of kG/D. The trivial module is always a submodule with multiplicity 1 of the permutation module kG/D. Indeed, Frobenius reciprocity shows that $Hom_{kG}(k, k \uparrow_D^G) = Hom_{kD}(k, k)$, which is one-dimensional, generated by φ, say. Similarly, $Hom_{kG}(k \uparrow_D^G, k)$ is one-dimensional, generated by ψ, say. However $\psi \circ \varphi = |G : D| \cdot id_k$ and so, if k is a direct factor of kG/D, then D contains a Sylow p-subgroup. □

2.1.3 Green Correspondence

Let G be a finite group and let k be a field of characteristic $p > 0$.

We shall establish a correspondence between indecomposable modules over G with vertex D and indecomposable modules over H with vertex D as soon as $N_G(D) \leq H \leq G$. This correspondence is due to Green.

Let now D be a p-subgroup of G and let $H \leq G$ with $N_G(D) \leq H$. Then define

$$\mathcal{X}_{D,H} := \{X \leq G \mid \exists g \in G \setminus H : X \leq {}^gD \cap D\}$$
$$\mathcal{Y}_{D,H} := \{Y \leq G \mid \exists g \in G \setminus H : Y \leq {}^gD \cap H\}$$

and observe that

$$D \notin \mathcal{Y}_{D,H} \text{ and } \mathcal{X}_{D,H} \subseteq \mathcal{Y}_{D,H}.$$

Indeed, if $D \leq {}^gD \cap H$ for some $g \in G \setminus H$, then $D \leq {}^gD$, and since D is of finite order, we get $D = {}^gD$. Hence $g \in N_G(D)$ which was excluded. This contradiction shows that $D \notin \mathcal{Y}_{D,H}$. Since $D \leq N_G(D) \leq H$ by hypothesis, ${}^gD \cap D \leq {}^gD \cap H$ and so $\mathcal{X}_{D,H} \subseteq \mathcal{Y}_{D,H}$.

Lemma 2.1.20 *Let T be an indecomposable D-projective kH-module. Then $T \uparrow_H^G \downarrow_H^G \simeq T \oplus T_1$ for a kH-module T_1 so that all indecomposable direct factors of T_1 have vertex in $\mathcal{Y}_{D,H}$.*

Proof Since T is D-projective, there is an indecomposable kD-module S such that $T \mid S \uparrow_D^H$. Hence there is a kH-module T'' with $S \uparrow_D^H \simeq T \oplus T''$. Mackey's formula gives

$$T \uparrow_H^G \downarrow_H^G \simeq \bigoplus_{HgH \in H\backslash G/H} {}^gT \downarrow^{{}^gH}_{{}^gH \cap H} \uparrow^H = T \oplus T'$$
$$T'' \uparrow_H^G \downarrow_H^G \simeq \bigoplus_{HgH \in H\backslash G/H} {}^gT'' \downarrow^{{}^gH}_{{}^gH \cap H} \uparrow^H = T'' \oplus T'''$$

where the direct factor T (respectively T'') occurs for the class $HgH = H1H = H$ and

$$T' := \bigoplus_{HgH \in (H\backslash G/H)\backslash H} {}^gT \downarrow^{^gH}_{^gH\cap H} \uparrow^H .$$
$$T''' := \bigoplus_{HgH \in (H\backslash G/H)\backslash H} {}^gT'' \downarrow^{^gH}_{^gH\cap H} \uparrow^H .$$

But now

$$S \uparrow_D^G \downarrow_H^G \simeq \bigoplus_{DgH \in D\backslash G/H} {}^gS \downarrow^{^gD}_{^gD\cap H} \uparrow^H \simeq S \uparrow_D^H \oplus S'$$

where again the class $D1H = H$ gives the module $S \uparrow_D^H \simeq T \oplus T''$ and where the other direct summands give $S' \simeq T' \oplus T'''$. Moreover, this description give that all indecomposable direct summands of S' have vertex in $\mathcal{Y}_{D,H}$ and therefore all indecomposable direct factors of T' have vertex in $\mathcal{Y}_{D,H}$. But since the definition of T' is

$$T \uparrow_H^G \downarrow_H^G = T \oplus T'$$

we get the desired statement if we put $T_1 = T'$. □

Theorem 2.1.21 (Green correspondence) *Let k be a field of characteristic $p > 0$, let D be a p-subgroup of G and let $H \le G$ with $N_G(D) \le H$. Then*

1. *For every indecomposable kG-module M with vertex D there is a unique indecomposable kH-module $f(M)$ with vertex D and which is a direct summand of $M \downarrow_H^G$.*
2. *For every indecomposable kH-module N with vertex D there is a unique indecomposable kG-module $g(N)$ with vertex D and which is a direct summand of $N \uparrow_H^G$.*
3. *Each indecomposable direct summand of $M \downarrow_H^G /f(M)$ has vertex in $\mathcal{Y}_{D,H}$.*
4. *Each indecomposable direct summand of $N \uparrow_H^G /g(N)$ has vertex in $\mathcal{X}_{D,H}$.*
5. *$fg(M) \simeq M$ and $gf(N) \simeq N$.*

Proof Let M be an indecomposable kG-module with vertex D and source L. Then M is a direct summand of $L \uparrow_D^G$. Let $L \uparrow_D^H = N_0 \oplus N_0'$ for an indecomposable kH-module N_0 so that M is a direct summand of $N_0 \uparrow_H^G$. By Lemma 2.1.20 we know that $N_0 \uparrow_H^G \downarrow_H^G = N_0 \oplus N_1$ for some kH-module N_1 so that each indecomposable direct factor of N_1 has vertex in $\mathcal{Y}_{D,H}$. But $M \downarrow_H^G$ is a direct factor of $N_0 \uparrow_H^G \downarrow_H^G = N_0 \oplus N_1$. If N_0 is not a direct summand of $M \downarrow_H^G$, then each indecomposable direct summand of $M \downarrow_H^G$ has vertex in $\mathcal{Y}_{D,H}$. But this implies that M has vertex in $\mathcal{Y}_{D,H}$ since M is a direct summand of $M \downarrow_H^G \uparrow_H^G$. This contradiction proves 1 and 3 if we put $f(M) := N_0$.

On the other hand, let N be an indecomposable kH-module with vertex D. By Lemma 2.1.20 we know that N is a direct summand of $N \uparrow_H^G \downarrow_H^G$. Let M

be an indecomposable direct summand of $N \uparrow_H^G$ so that N is a direct summand of $M \downarrow_H^G$. Denote $N \uparrow_H^G = M \oplus M'$ for a kG-module M'. By Lemma 2.1.20 each indecomposable direct summand of $N \uparrow_H^G \downarrow_H^G$ different from N has vertex in $\mathcal{Y}_{D,H}$. Since N is a direct summand of $M \downarrow_H^G$, each indecomposable direct summand of $M' \downarrow_H^G$ has vertex in $\mathcal{Y}_{D,H}$. We have already shown 1 and 3 and see that therefore each indecomposable direct summand of $M \downarrow_H^G$ different from N has vertex in $\mathcal{Y}_{D,H}$. Putting $g(N) := M$ we have shown 2 in order to prove 4 we remark that if L denotes the source of N, then N is a direct factor of $L \uparrow_D^H$, and therefore $N \uparrow_H^G$ is a direct summand of $L \uparrow_D^H \uparrow_H^G = L \uparrow_D^G$. Therefore each indecomposable direct summand of $N \uparrow_H^G$ has vertex in $D \cap X$ for some $X \in \mathcal{Y}_{D,H}$. But X is a subgroup of ${}^gD \cap H$, and hence $D \cap X \leq D \cap {}^gD \cap H = D \cap {}^gD$. This describes exactly the groups in $\mathcal{X}_{D,H}$. We have proved 4.

The last point 5 is immediate by definition. □

Definition 2.1.22 Let k be a field of characteristic $p > 0$ and let G be a finite group. Let D be a p-subgroup of G and let H be a subgroup of G containing the normaliser $N_G(D)$ of D in G.

- Then for each indecomposable kG-module M with vertex D we define its *Green correspondent* to be the unique indecomposable direct summand $g(M)$ of $M \downarrow_H^G$ with vertex D.
- For each indecomposable kH-module N with vertex D we define its *Green correspondent* to be the unique indecomposable direct summand $f(N)$ of $N \uparrow_H^G$ with vertex D.

Green correspondence commutes with morphisms in a certain conceptual sense. We start with three indecomposable kG-modules M_1, M_2 and M_3 with vertices D_1, D_2 and D_3 and a subgroup H of G such that

$$N_G(D_1) \cdot N_G(D_2) \cdot N_G(D_3) \leq H.$$

Then the Green correspondents $g(M_1)$, $g(M_2)$ and $g(M_3)$ are defined; these are indecomposable kH-modules with vertex D_1, respectively D_2, respectively D_3 the unique direct summands of $M_1 \downarrow_H^G$, respectively $M_2 \downarrow_H^G$, respectively $M_3 \downarrow_H^G$ with vertex D_1, respectively D_2, respectively D_3. Let $\alpha \in Hom_{kG}(M_1, M_2)$ and $\beta \in Hom_{kG}(M_2, M_3)$. By restriction α and β induce $\alpha \in Hom_{kH}(M_1, M_2)$ and $\beta \in Hom_{kH}(M_2, M_3)$. Then by projection and inclusion of the direct summands $g(M_1)$, $g(M_2)$ and $g(M_3)$ into $M_1 \downarrow_H^G$, $M_2 \downarrow_H^G$ and $M_3 \downarrow_H^G$ we obtain $g(\alpha) \in Hom_{kH}(g(M_1), g(M_2))$ and $g(\beta) \in Hom_{kH}(g(M_2), g(M_3))$. In general we will not get $g(\beta \circ \alpha) = g(\beta) \circ g(\alpha)$. In order to see why this is, consider for all $i \in \{1, 2, 3\}$ the decomposition $M_i \downarrow_H^G = g(M_i) \oplus \check{M}_i$ for some kH-module $\check{M}_i$. Then

$$\alpha = \begin{pmatrix} g(\alpha) & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} g(\beta) & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{pmatrix}$$

for

$$\begin{array}{ll} \alpha_{1,2} \in Hom_{kH}(\check{M}_1, g(M_2)), & \beta_{1,2} \in Hom_{kH}(\check{M}_2, g(M_3)), \\ \alpha_{2,2} \in Hom_{kH}(\check{M}_1, \check{M}_2), & \beta_{2,2} \in Hom_{kH}(\check{M}_2, \check{M}_3), \\ \alpha_{2,1} \in Hom_{kH}(g(M_1), \check{M}_2), & \beta_{2,1} \in Hom_{kH}(g(M_2), \check{M}_3). \end{array}$$

Then

$$\begin{aligned} \beta \circ \alpha &= \begin{pmatrix} g(\beta) & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{pmatrix} \circ \begin{pmatrix} g(\alpha) & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} g(\beta) \circ g(\alpha) + \beta_{1,2} \circ \alpha_{2,1} & g(\beta) \circ \alpha_{1,2} + \beta_{1,2}\alpha_{2,2} \\ \beta_{2,1} g(\alpha) + \beta_{2,2}\alpha_{2,1} & \beta_{2,1}\alpha_{1,2} + \beta_{2,2}\alpha_{2,2} \end{pmatrix} \end{aligned}$$

so that by this naive definition

$$g(\beta \circ \alpha) = g(\beta) \circ g(\alpha) + \beta_{1,2} \circ \alpha_{2,1}.$$

We will now introduce a limited version of a concept which will be of crucial importance in Chap. 5. Our argument is taken from the much more general approach of Auslander-Kleiner [2].

Suppose $D_1 = D_2 = D_3 =: D$. Then put

$$\mathcal{Y} := \{\, Y \leq G \mid Y \leq {}^{g}D \cap H \quad \text{for some } g \in G \setminus H\}$$

and for two indecomposable kG-modules M_1 and M_2 with vertices D_1 and D_2 let

$$Hom_{H,\mathcal{Y}}(g(M_1), g(M_2)) := Hom_{kH}(g(M_1), g(M_2))/Hom_{kH}^{\mathcal{Y}}(g(M_1), g(M_2))$$

and let $Hom_{kH}^{\mathcal{Y}}(g(M_1), g(M_2))$ be the set of those homomorphisms $\gamma : g(M_1) \longrightarrow g(M_2)$ such that there is a kH-module L which is a direct sum of indecomposable kH-modules L_i so that each L_i is relatively projective with respect to some $Y_i \in \mathcal{Y}$ and so that there are morphisms $\delta : g(M_1) \longrightarrow \bigoplus_i L_i$ and $\epsilon : \bigoplus_i L_i \longrightarrow g(M_2)$ with $\gamma = \epsilon \circ \delta$.

It is immediate to see that the composition of kH-linear morphisms induces a natural and well-defined composition of classes of morphisms in $Hom_{H,\mathcal{Y}}$.

In a completely analogous manner we define

$$\mathcal{X} := \{\, X \leq G \mid X \leq {}^{g}D \cap D \quad \text{for some } g \in G \setminus H\}$$

and for two indecomposable kH-modules N_1 and N_2 with vertices D_1 and D_2 let

$$Hom_{G,\mathcal{X}}(f(M_1), f(M_2)) := Hom_{kG}(f(N_1), f(N_2))/Hom_{kH}^{\mathcal{X}}(f(N_1), f(N_2))$$

and let $Hom_{kG}^{\mathcal{X}}(f(N_1), f(N_2))$ be the set of those homomorphisms $\gamma : f(N_1) \longrightarrow f(N_2)$ such that there is a kG-module L which is a direct sum of indecomposable

kG-modules L_i so that each L_i is relatively projective with respect to some $X_i \in \mathcal{X}$ and so that there are morphisms $\delta : f(N_1) \longrightarrow \bigoplus_i L_i$ and $\epsilon : \bigoplus_i L_i \longrightarrow f(N_2)$ with $\gamma = \epsilon \circ \delta$.

Again, the composition of morphisms induces a well-defined composition of classes of morphisms in $Hom_{G,\mathcal{X}}$.

Proposition 2.1.23 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let D be a p-subgroup of G and let H be a subgroup of G containing the normaliser $N_G(D)$ of D in G. Let M_1, M_2 and M_3 be indecomposable kG-modules with vertex D and let N_1, N_2 and N_3 be indecomposable kH-modules with vertex D.*

- *Then for all $\alpha \in Hom_{kG}(M_1, M_2)$ and $\beta \in Hom_{kG}(M_2, M_3)$ we have*

$$g(\beta \circ \alpha) = g(\beta) \circ g(\alpha) \in Hom_{H,\mathcal{Y}}(g(M_1), g(M_3)).$$

- *Moreover, for all $\rho \in Hom_{kH}(N_1, N_2)$ and $\sigma \in Hom_{kH}(N_2, N_3)$ we have*

$$f(\sigma \circ \rho) = f(\sigma) \circ f(\rho) \in Hom_{G,\mathcal{X}}(f(N_1), f(N_3)).$$

Proof The proof of the first part has already been done in the remarks before the proposition. The statement for f is completely analogous. □

2.2 Clifford Theory

We come to the question of whether it is possible to link the representation theory of a group G over a commutative ring R to the representation theory of some normal subgroup N of G over R. Of course in this generality the question cannot have a positive answer, since the trivial group with one element has a trivial representation theory and of course is a normal subgroup of every group. But we shall see that some information can be obtained.

2.2.1 Clifford's Main Theorem

In order to solve this problem one needs to study the concept of an inertia group. Given a group G and a normal subgroup N, let R be a commutative ring and let M be an RN-module. Then for all $g \in G$ conjugation with g gives an automorphism γ_g on N, since N is normal in G. Hence we may study the twisted KN-module ${}^{\gamma_g}M$ (cf Definition 1.7.39) and denote it again, slightly abusing the notation, by ${}^{g}M$. We have already used this construction in previous sections, such as for Mackey's formula and for Green correspondence.

Definition 2.2.1 Let G be a group and let N be a normal subgroup of G. Let R be a commutative ring and let M be an RN-module. Then

$$I_G(M) := \{g \in G|\ {}^gM \simeq M \text{ as } RN\text{-module}\}$$

is the *inertia group* of M in G.

Lemma 2.2.2 *For G a group, R a commutative ring, N a normal subgroup of G and M an RN-module, the inertia group $I_G(M)$ is a subgroup of G and N as well as the centraliser $C_G(N)$ of N in G is a subgroup of $I_G(M)$.*

Proof Indeed, $\gamma(gh) = \gamma(g)\gamma(h)$ and so

$${}^{\gamma(gh)}M \simeq {}^{\gamma(g)}\left({}^{\gamma(h)}M\right) \simeq {}^{\gamma(g)}M \simeq M$$

whenever $g, h \in I_G(M)$. A similar proof shows $g \in I_G(M) \Rightarrow g^{-1} \in I_G(M)$.

The fact that N is a subgroup of $I_G(M)$ is a consequence of Lemma 1.7.41. The fact that $C_G(N)$ is a subgroup of $I_G(M)$ follows from the definition of a twisted module. This proves the lemma. □

In order to illustrate Clifford's theorem we examine Mackey's formula in the case when $K = H$ is normal in G.

We get

$$\begin{aligned} M \uparrow_H^G \downarrow_H^G &\simeq \bigoplus_{HgH \in H\backslash G/H} \left({}^g\left(M \downarrow^H_{(H\cap gHg^{-1})}\right)\right) \uparrow^H_{(gHg^{-1}\cap H)} \\ &\simeq \bigoplus_{gH\in G/H} \left({}^g\left(M\downarrow_H^H\right)\right)\uparrow_H^H \simeq \bigoplus_{gH\in G/H} {}^gM. \end{aligned}$$

In particular we see that, as an RH-module, the induced module $M \uparrow_H^G$ is a direct sum of conjugates ${}^gM$ of M. Within the classes of G/H precisely the classes $I_G(M)/H$ lead to conjugates which are isomorphic to M.

Theorem 2.2.3 (Clifford's theorem) *Let G be a finite group and let N be a normal subgroup of G. Let R be a commutative ring such that the Krull-Schmidt theorem is valid for RS-modules, for all subgroups S of G. Let M be an indecomposable RN-module and suppose*

$$M \uparrow_N^{I_G(M)} \simeq M_1 \oplus M_2 \oplus \cdots \oplus M_r$$

for indecomposable $RI_G(M)$-modules $M_1, M_2, \ldots, M_r$.

Then for all $i, j \in \{1, 2, \ldots, r\}$ we have that $M_i \uparrow^G_{I_G(M)}$ is indecomposable and $M_i \uparrow^G_{I_G(M)} \simeq M_j \uparrow^G_{I_G(M)}$ implies $M_i \simeq M_j$.

Proof As in the remarks preceding the statement of the theorem we get

$$M\uparrow_N^G\downarrow_N^G\simeq\bigoplus_{gN\in G/N}{}^gM$$

and since for every $g_1, g_2\in G$ one has ${}^{g_1}M\simeq{}^{g_2}M$ whenever $g_1I_G(M)=g_2I_G(M)$, we get

$$M\uparrow_N^G\downarrow_N^G\simeq\bigoplus_{gI_G(M)\in G/I_G(M)}({}^gM)^{|I_G(M):N|}.$$

But, by Lemma 1.7.29 we have

$$M\uparrow_N^G\simeq\left(M\uparrow_N^{I_G(M)}\right)\uparrow_{I_G(M)}^G\simeq M_1\uparrow_{I_G(M)}^G\oplus M_2\uparrow_{I_G(M)}^G\oplus\cdots\oplus M_r\uparrow_{I_G(M)}^G.$$

Hence there are integers $n_1, n_2, \ldots, n_r$ with $\sum_{i=1}^r n_i=|I_G(M):N|$ such that

$$M_i\uparrow_{I_G(M)}^G\downarrow_N^G\simeq\bigoplus_{gI_G(M)\in G/I_G(M)}({}^gM)^{n_i}.$$

Since, as before, for every $g_1, g_2\in G$ one has ${}^{g_1}M\simeq{}^{g_2}M$ if and only if $g_1I_G(M)=g_2I_G(M)$, we get $M_i\downarrow_N^{I_G(M)}\simeq M^{n_i}$.

Corollary 1.7.46 implies that for each $i\in\{1,2,\ldots,r\}$ the $RI_G(M)$-module M_i is a direct factor of $M_i\uparrow_{I_G(M)}^G\downarrow_{I_G(M)}^G$. Hence for all $i\in\{1,2,\ldots,r\}$ there is an indecomposable RG-module F_i such that F_i is a direct factor of $M_i\uparrow_{I_G(M)}^G$ and such that M_i is a direct factor of $F_i\downarrow_{I_G(M)}^G$. Since $M_i\downarrow_N^{I_G(M)}\simeq M^{n_i}$ one has that M^{n_i} is a direct factor of $(F_i\downarrow_{I_G(M)}^G)\downarrow_N^{I_G(M)}=F_i\downarrow_N^G$. But for each $i\in\{1,2,\ldots,r\}$ one has that F_i is an RG-module, and so if M^{n_i} is a direct factor of $F_i\downarrow_N^G$, then ${}^gM^{n_i}$ is a direct factor of ${}^gF_i\downarrow_N^G\simeq F_i\downarrow_N^G$ for all $g\in G$. We use again that for every $g_1, g_2\in G$ one has ${}^{g_1}M\simeq{}^{g_2}M$ if and only if $g_1I_G(M)=g_2I_G(M)$, so that whenever we sum over cosets modulo $I_G(M)$ we get different isomorphism classes of modules, and so, $\bigoplus_{gI_G(M)\in G/I_G(M)}({}^gM)^{n_i}$ is a direct factor of $F_i\downarrow_N^G$. But

$$M_i\uparrow_{I_G(M)}^G\downarrow_N^G\simeq\bigoplus_{gI_G(M)\in G/I_G(M)}({}^gM)^{n_i}$$

and F_i is a direct factor of $M_i\uparrow_{I_G(M)}^G$ with

$$F_i\downarrow_N^G\simeq\bigoplus_{gI_G(M)\in G/I_G(M)}({}^gM)^{n_i}$$

as well. Therefore, $F_i \simeq M_i \uparrow_{I_G(M)}^G$ for all $i \in \{1, 2, \ldots, r\}$. This shows that $M_i \uparrow_{I_G(M)}^G$ is indecomposable for all $i \in \{1, 2, \ldots, r\}$.

We need to show that $M_i \uparrow_{I_G(M)}^G \simeq M_j \uparrow_{I_G(M)}^G$ implies $M_i \simeq M_j$.

Using Mackey's theorem and the fact that N is normal in G and contained in $I_G(M)$, we have

$$(\dagger): \quad M_i \uparrow_{I_G(M)}^G \downarrow_N^G \simeq \bigoplus_{gI_G(M) \in G/I_G(M)} {}^g\left(M_i \downarrow_N^{I_G(M)}\right).$$

Suppose M_j is a direct factor of $M_i \uparrow_{I_G(M)}^G \downarrow_{I_G(M)}^G$. Since M_i is a direct factor of $M_i \uparrow_{I_G(M)}^G \downarrow_{I_G(M)}^G$, and since its restriction to N corresponds to the coset $I_G(M)$ in $G/I_G(M)$ in the decomposition ($\dagger$), the restriction of M_j to N is a direct sum of copies of ${}^gM$ for $g \in G \setminus I_G(M)$. But since ${}^gM \not\simeq M$ if and only if $g \in G \setminus I_G(M)$ we get that $M_i \uparrow_{I_G(M)}^G \simeq M_j \uparrow_{I_G(M)}^G$ implies $M_i \simeq M_j$. □

Often $I_G(M)$ is smaller than G, but not always. We will see now what happens if $I_G(M) = G$ and give a criterion on M which implies $I_G(M) \neq G$.

Corollary 2.2.4 *Let R be a commutative ring and G be a finite group such that the Krull-Schmidt property holds for all RL-modules, for all subgroups L of G. Let H be a normal subgroup of G and let S be a simple RG-module. Let T be a simple submodule of $S \downarrow_H^G$. If $I_G(T) = G$, then $S \downarrow_H^G \simeq T^n$ for some integer n.*

Proof Indeed, $\sum_{g \in G} gT$ is a KG-submodule of S. Since S is simple, $\sum_{g \in G} gT = S$. But, $gT \simeq {}^gT$ as KH-module, and so gT is a simple KH-module as well. Since $I_G(T) = G$ we get $T \simeq {}^gT \simeq gT$ for all $g \in G$ and so $S \downarrow_A^G \simeq T^n$, for some integer n. □

Example 2.2.5 Suppose G is a group and N is a subgroup of the centre of G. Then for all commutative rings R and all RG-modules M one has $I_G(M) = G$. Indeed, Lemma 2.2.2 shows that $I_G(M)$ contains $C_G(N)$ but since $N \subseteq Z(G)$ one has

$$G \supseteq I_G(M) \supseteq C_G(N) \supseteq C_G(Z(G)) = G.$$

Definition 2.2.6 Let G be a group and let R be a commutative ring. An RG-module S is *faithful* if whenever for $g \in G$ one has $g \cdot s = s$ for all $s \in S$, then $g = 1$.

Proposition 2.2.7 (Blichfeldt) *Let G be a finite group and let k be a field.*

Let S be a simple faithful kG-module and let A be an abelian normal subgroup of G not contained in the centre of G. Suppose that k is a splitting field for A. Let T be a simple submodule of $S \downarrow_A^G$. Then $I_G(T) \neq G$.

Proof Suppose $I_G(T) = G$. We know that T is a direct summand of $S \downarrow_A^G$. By Corollary 2.2.4 all simple direct summands of $S \downarrow_A^G$ are isomorphic to T. Since k is a splitting field for A, each simple kA-module is one-dimensional and therefore $S \downarrow_A^G \simeq T^{\dim_k(S)}$.

Since each simple kA-module is one-dimensional, there is a homomorphism of groups $\alpha : A \longrightarrow k^\times$ such that for all $t \in T$ we have $a \cdot t = \alpha(a)t$. Since $S \downarrow_A^G \simeq T^{\dim_k(S)}$, one gets for all $a \in A$ and all $s \in S$ that $a \cdot s = \alpha(a)s$. Therefore $\sigma|_A = \alpha$, denoting by $\sigma : G \longrightarrow Gl_{\dim_k(S)}(k)$ the group homomorphism which defines the kG-module structure on S. But the image A' of α in $Gl_{\dim_k(S)}(k)$ are multiples of the identity matrix and so A' is central in $Gl_{\dim_k(S)}(k)$. The kG-module S is faithful if and only if σ is injective. Hence $G \simeq \sigma(G) \subseteq Gl_{\dim_k(S)}(k)$. Since

$$A' = \sigma(A) \subseteq Z(Gl_{\dim_k(S)}(k)) \subseteq Z(\sigma(G))$$

we get that A is contained in the centre of G. This contradiction proves the proposition. □

2.2.2 *Group Graded Rings and Green's Indecomposability Theorem*

Green correspondence studies the correspondence between specific indecomposable summands of modules induced from subgroups. Sometimes, these induced modules are decomposable, sometimes they are not. Green's indecomposability theorem provides a very useful criterion which tells us when the induced module is indecomposable. The main tools to obtain the result are Clifford's theorem and the notion of a grading of a ring by a group.

Definition 2.2.8 Let G be a group, let R be a commutative ring and let A be an R-algebra. We say that A is *graded by* G if for every $g \in G$ there is an R-module A_g such that

$$A \simeq \bigoplus_{g \in G} A_g$$

as an R-module, and such that

$$A_g \cdot A_h \subseteq A_{gh}$$

for all $g, h \in G$. We say that A is *strongly graded by* G if

$$A_g \cdot A_h = A_{gh}$$

for all $g, h \in G$.

If $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ are G-graded R-algebras, then we say that a *morphism* $f : A \longrightarrow B$ *is graded* if $f(A_g) \subseteq B_g$ for all $g \in G$. If $G = (\mathbb{Z}, +)$, then we say that f *is graded of degree* $n \in \mathbb{Z}$ if $f(A_m) \subseteq A_{n+m}$ for all $m \in \mathbb{Z}$.

Example 2.2.9 The following examples of graded algebras will be used later.

- Let k be a commutative ring and let G be a group with normal subgroup N. Then kG is strongly G/N-graded. Indeed,

$$kG = \bigoplus_{gN \in G/N} k(g \cdot N)$$

where $k(g \cdot N)$ is the free k-module generated by the coset $g \cdot N$. Moreover,

$$(g \cdot N) \cdot (h \cdot N) = g \cdot h \cdot (h^{-1}Nh) \cdot N = gh \cdot N$$

for all $g, h \in G$.
- We come to an example which will be examined in more detail in the sequel, and which motivates the whole subsection. Let k be a field, let G be a finite group, let N be a normal subgroup and let M be an indecomposable kN-module. Then $End_{kN}(M) =: B$ is a local algebra by Lemma 1.4.6. Consider $M \uparrow_N^G = kG \otimes_{kN} M$ and observe that

$$M \uparrow_N^G \downarrow_N^G = \bigoplus_{gN \in G/N} kgN \otimes_{kN} M = \bigoplus_{gN \in G/N} {}^gM.$$

Suppose now that the inertia group $I_G(M)$ of M is G. Then

$$\begin{aligned} End_{kG}(M \uparrow_N^G) &\simeq Hom_{kN}(M, M \uparrow_N^G \downarrow_N^G) \\ &\simeq \bigoplus_{gN \in G/N} Hom_{kN}(M, {}^gM) = \bigoplus_{gN \in G/N} B_g \end{aligned}$$

where $B_g := Hom_{kN}(M, {}^gM) = B$ since ${}^gM \simeq M$ for all $g \in G$. We see that $A := End_{kG}(M \uparrow_N^G)$ is a G/N-graded k-algebra with $A_{gN} = B_g$ for all $gN \in G/N$. Some properties of group graded algebras are immediate.

Lemma 2.2.10 *Let G be a group, let k be a commutative ring, and let A be a G-graded k-algebra. The neutral element of G is denoted by* 1.

Then A_1 is a subring of A and for each $g \in G$ we get that A_g is an A_1-A_1-bimodule. Moreover $A^\times \cap A_1 = A_1^\times$. In particular if A is local then A_1 is local.

Proof Since $1 \cdot 1 = 1$ in G we get $A_1 \cdot A_1 \subseteq A_1$. If $g \in G \setminus \{1\}$, then

$$A_g \cdot A_g \subseteq A_{g^2} \neq A_g$$

and hence every idempotent in A is in A_1. In particular the multiplicatively neutral element 1_A of A is in A_1. This shows that A_1 is a subring of A. Since $A_1 \cdot A_g \subseteq A_g \supseteq A_g \cdot A_1$ for all $g \in G$, we obtain that A_g is an $A_1 - A_1$-bimodule.

Let $N := A \setminus A^\times$ and $N_1 := A_1 \setminus A_1^\times$ be the set of non-units of A and of A_1. If A is local, N is an A-ideal. We show first that $N_1 = N \cap A_1$. Indeed, if $n_1 \in A_1$, and

$$a = \sum_{g \in G} a_g \in A = \bigoplus_{g \in G} A_g$$

so that $n_1 \cdot a = 1_A$, then $n_1 \cdot a = \sum_{g \in G} n_1 a_g$ where $n_1 a_g \in A_g$ for each $g \in G$. Hence $n_1 \cdot a = 1$ implies $n_1 \cdot a_1 = 1$ and therefore $n_1 \in A_1^\times$. Hence

$$A_1^\times \supseteq A^\times \cap A_1.$$

The other inclusion is a consequence of the fact that A_1 is a subring.

Let $n_1, n_2 \in A_1 \setminus A_1^\times$ and let $a_1 \in A_1$. Then

$$a_1 \cdot n_1 \in A_1 \cdot N_1 \subseteq A_1 \cap (A \cdot N) \subseteq A_1 \cap N = N_1.$$

Similarly, $n_1 \cdot a_1 \in N_1$. Since $n_1, n_2 \in A_1 \setminus A_1^\times$, and since $A_1^\times = A_1 \cap A^\times$, we get that $n_1, n_2 \notin A^\times$. Since A is local, $n_1 + n_2 \notin A^\times$ and moreover $n_1 + n_2 \in A_1$ since A_1 is a subring. Hence $n_1 + n_2 \in N_1$. This shows that N_1 is a two-sided ideal of A_1 and we obtain the statement. □

Lemma 2.2.11 *Let k be an algebraically closed field of characteristic $p > 0$, and let $N \trianglelefteq G$ be a normal subgroup of the finite group G. Suppose G/N is a p-group. Let M be an indecomposable kN-module and suppose that $I_G(M) = G$. Then $J := \mathrm{rad}(End_{kN}(M)) \cdot End_{kG}(M \uparrow_N^G)$ is a two-sided ideal of $End_{kG}(M \uparrow_N^G)$, $J \leq rad(End_{kG}(M \uparrow_N^G))$ and $End_{kG}(M \uparrow_N^G)/J \simeq kG/N$.*

Proof We recall the second part of Example 2.2.9. Let k be an algebraically closed field of characteristic $p > 0$, let G be a group and $N \trianglelefteq G$ and suppose that G/N is a p-group. Let M be an indecomposable kN-module with endomorphism ring B. We assume again that $I_G(M) = G$. Then $End_{kG}(M \uparrow_N^G)$ is a G/N-graded k-algebra $A = \bigoplus_{gN \in G/N} B_g$ with $B_g = Hom_{kN}(M, {}^gM)$ for all $g \in G$. Now, B_g is a B-module and we shall consider $\mathrm{rad}(B) \cdot B_g = B_g \cdot \mathrm{rad}(B) = \mathrm{rad}_B(B_g)$.

Since M is indecomposable, B is local. Since k is algebraically closed, $B/\mathrm{rad}(B) \simeq k$ and $\mathrm{rad}_B(B_g)$ is of codimension 1 in B_g.

Now, let $g, h \in G/N$ and let $b_g \in B_g$ and $b_h \in B_h$. Then $b_g \cdot b_h \in B_{gh}$. Moreover, for each $g \in G$ fix an element $i_g \in B_g \backslash \mathrm{rad}_B(B_g)$ in B_g. Since $B \cdot i_g = i_g \cdot B = B_g$, we get that $\mathrm{rad}(B) \cdot A$ is a two-sided ideal of A. Every element of B_g is of the form $b \cdot i_g$ for some $b \in B_1$. For all $g, h \in G$ there is an $f(g, h) \in k^\times$ such that

$$i_g \cdot i_h - f(g, h) \cdot i_{gh} \in \mathrm{rad}(B_g).$$

Now, the associativity of the multiplication in A gives that for $g_1, g_2, g_3 \in G/N$ we obtain in $A/\mathrm{rad}(B) \cdot A$

$$\begin{aligned} f(g_1, g_2) f(g_1 g_2, g_3) i_{g_1 g_2 g_3} &= (i_{g_1} \cdot i_{g_2}) \cdot i_{g_3} = i_{g_1} \cdot (i_{g_2} \cdot i_{g_3}) \\ &= f(g_1, g_2 g_3) f(g_2, g_3) i_{g_1 g_2 g_3} \end{aligned}$$

and hence

$$f(g_1, g_2)f(g_1g_2, g_3) = f(g_1, g_2g_3)f(g_2, g_3)$$

for all $g_1, g_2, g_3 \in G/N$. Definition 1.8.38 shows that f is a 2-cocycle with values in $k^\times$ with trivial G/N-action and Lemma 1.8.49 shows that f is actually a 2-coboundary. Therefore there is a $\sigma : G \longrightarrow k^\times$ such that

$$f(g_1, g_2) = \sigma(g_1) \cdot \sigma(g_2) \cdot \sigma(g_1g_2)^{-1}$$

for all $g_1, g_2 \in G/N$. But then replacing i_g by $j_g := i_g \cdot \sigma(g)^{-1}$ we obtain

$$\begin{aligned} j_gj_h &= i_g \cdot \sigma(g)^{-1} \cdot i_h \cdot \sigma(h)^{-1} \\ &= i_{gh} \cdot f(g, h) \cdot \sigma(g)^{-1} \cdot \sigma(h)^{-1} \\ &= i_{gh} \cdot \sigma(g) \cdot \sigma(h) \cdot \sigma(gh)^{-1} \cdot \sigma(g)^{-1} \cdot \sigma(h)^{-1} \\ &= i_{gh} \cdot \sigma(gh)^{-1} \\ &= j_{gh} \end{aligned}$$

for all $g, h \in G/N$, as an equation in $A/\mathrm{rad}(B) \cdot A$. Therefore $A = \bigoplus_{gN \in G/N} B \cdot j_g$, where $j_g \cdot j_h - j_{gh} \in \mathrm{rad}_B(B_{gh})$. Moreover, $j_g \cdot B = B \cdot j_g = B_g$ for all $g \in G$ and hence $\mathrm{rad}(B) \cdot A$ is a two-sided ideal of A. Further, $A/\mathrm{rad}(B)A = kG/N$. This proves the lemma. □

Theorem 2.2.12 (Green's indecomposability theorem) *Let k be an algebraically closed field of characteristic $p > 0$ and let G be a finite group. If N is a normal subgroup of G such that G/N is a p-group, then for every indecomposable kN-module M the module $M \uparrow_N^G$ is indecomposable again.*

Proof Clifford's Theorem 2.2.3 implies that we just need to show that $M \uparrow_N^{I_G(M)}$ is indecomposable. Hence, we may suppose that $G = I_G(M)$. Consider Lemma 2.2.11 and put $A := End_{kG}(M \uparrow_N^G)$ and $B := End_{kN}(M)$. We therefore obtain $A/(\mathrm{rad}(B) \cdot A) \simeq kG/N$. Using Proposition 1.6.22 we get $A/\mathrm{rad}(A) = k$ since G/N is a p-group and hence A is local. □

2.3 Brauer Correspondence

A special case of the Green correspondence is the so-called Brauer correspondence which establishes a bijection between direct factors of the group ring kG as a ring and those of certain subgroups $H \leq G$ containing specific p-subgroups.

Definition 2.3.1 Let k be a field of characteristic $p > 0$ and let G be a finite group. A *block* B of kG is an indecomposable direct factor of kG as a ring. In other words, $kG = B \times B_1$ where B and B_1 are k-algebras and B is indecomposable as a k-algebra.

We remark that if M is an indecomposable A-module for a k-algebra A, and if $A = A_1 \times A_2$ for two k-algebras A_1 and A_2, then M is either an A_1-module or an A_2-module, but not both. Indeed, let $e_1^2 = e_1 \in Z(A)$ so that $e_1A = A_1$ and $e_2 := 1 - e_1$. Then $M = e_1M \oplus e_2M$ as A-modules, and since M is indecomposable $e_1M = 0$ or $e_2M = 0$.

Definition 2.3.2 Let k be a field of characteristic $p > 0$ and let G be a finite group. Let M be an indecomposable kG-module. Let B be the unique block of kG such that M is a B-module. We say that M *belongs to the block* B. The trivial kG-module belongs to the *principal block* $B_0(kG)$ of kG.

We now link module and ring direct factors.

Lemma 2.3.3 *Let R be a commutative ring, let A be an R-algebra and let A_1 be a direct factor of A as a ring. Then A_1 is a direct factor as an $A \otimes_R A^{op}$-module of A. If A_2 is a direct factor as an $A \otimes_R A^{op}$-module of A, then A_2 is a direct factor of A as a ring.*

Proof We obtain $A_1 = e_1A$ for some idempotent $e_1 \in Z(A)$. Then for all $a, b, c \in A$ we get $a \otimes b \in A \otimes_R A^{op}$ and compute

$$(a \otimes b) \cdot ce_1 = a(ce_1)b = (acb)e_1$$

and therefore Ae_1 is an $A \otimes_R A^{op}$-submodule of A. Moreover, by the same argument $A(1 - e_1)$ is an $A \otimes_R A^{op}$-submodule of A as well and $A = Ae_1 \oplus A(1 - e_1)$ as $A \otimes_R A^{op}$-modules.

Conversely, let $A = A_2 \oplus A_2'$ as $A \otimes_R A^{op}$-modules. Then $Z(A) = End_{A\otimes_R A^{op}}(A)$ and there is an idempotent $e_2 \in Z(A) = End_{A\otimes_R A^{op}}(A)$ so that $e_2(A) = e_2 \cdot A = A_2$. Hence A_2 is an algebra and moreover $A = Ae_2 \times A(1 - e_2)$ as algebras. □

We obtain that direct factors of an algebra A as a ring are the same as direct factors of A as $A \otimes_R A^{op}$-modules. The notion of decomposability is the same, when regarded as module or as algebras.

Now, let G be a group and let k be a commutative ring. Then

$$\Delta : G \longrightarrow G \times G$$

is defined by $\Delta(g) := (g, g)$ for all $g \in G$. Observe that $k\Delta(G)$ is a subring of $k(G \times G)$ and

$$k(G \times G) \simeq kG \otimes_k (kG)^{op}$$

where the isomorphism is given by $g \otimes h \mapsto g \otimes h^{-1}$. In this way, Δ extends k-linearly to an algebra homomorphism $kG \longrightarrow kG \otimes_k (kG)^{op}$ which sends $g \in G$ to $g \otimes g^{-1}$. Therefore $kG \otimes_k (kG)^{op}$ becomes a $k\Delta(G)$-right module, where $g \in G$ acts by right multiplication with $\Delta(g)$.

Proposition 2.3.4 *Let k be a field of characteristic $p > 0$, let G be a finite group and let B be a block of kG. Then the $kG \otimes_k kG^{op}$-module B has a vertex in the set of subgroups of $\Delta(G)$.*

Proof We first recall that kG is a $kG \otimes_k (kG)^{op}$-module by the action $(g \otimes h) \cdot x := gxh$ for $g, h \in G$ and $x \in kG$. The algebra $kG \otimes_k (kG)^{op}$ is a $k\Delta(G)$-right module as seen above.

In order to prove the proposition, we use Proposition 2.1.15. Consider k as a trivial $\Delta(G)$-module. Then define

$$\begin{aligned} \psi : kG &\longrightarrow (kG \otimes_k (kG)^{op}) \otimes_{k\Delta(G)} k \\ \sum_{g \in G} r_g g &\mapsto \sum_{g \in G} r_g ((g \otimes 1) \otimes 1) \end{aligned}$$

The mapping ψ is certainly k-linear. Let $g, h_1, h_2 \in G$. Then

$$\begin{aligned} \psi((h_1 \otimes h_2) \cdot g) &= \psi(h_1 g h_2^{-1}) = (h_1 g h_2^{-1} \otimes 1) \otimes 1 = (h_1 g \otimes h_2) \otimes 1 \\ &= (h_1 \otimes h_2) \cdot ((g \otimes 1) \otimes 1) = (h_1 \otimes h_2) \cdot \psi(g) \end{aligned}$$

and so ψ is $kG \otimes_k kG^{op}$-linear. Moreover, the multiplication mapping

$$\begin{aligned} \mu : (kG \otimes_k (kG)^{op}) \otimes_{k\Delta(G)} k &\longrightarrow kG \\ (g \otimes h) \otimes x &\mapsto xgh^{-1} \end{aligned}$$

for $g, h \in G$ and $x \in k$ gives a morphism of $(kG \otimes_k (kG)^{op})$-modules. Finally, $\mu \circ \psi = id$.

Since kG is therefore $k\Delta(G)$-projective, the same holds for each direct summand. By Lemma 2.3.3 this shows that there are vertices of blocks which are subgroups of $\Delta(G)$. □

Note that the vertex is unique up to conjugacy only. Hence any subgroup conjugate in $G \times G$ to $\Delta(D)$ is a vertex as well. Such a conjugate will not be in $\Delta(G)$ in general.

Definition 2.3.5 Let k be a field of characteristic $p > 0$ and let G be a finite group. A *defect group* of a block B is a subgroup D of G such that $\Delta(D)$ is the vertex of B. The *defect of B* is the integer $\log_p(|D|)$.

Recall that a vertex, and hence also a defect group, is a p-group.

Proposition 2.3.6 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let M be an indecomposable kG-module that belongs to the block B. If B has defect group D, then there is a subgroup of D that is a vertex of M.*

Proof Let M be an indecomposable kG-module that belongs to the block B. Therefore M is a B-module, and we need to show that the multiplication mapping

$$B \otimes_{kD} M \longrightarrow M$$

is split. But we know that the multiplication mapping

$$(kG \otimes_k (kG)^{op}) \otimes_{k\Delta(D)} B \longrightarrow B$$

is split. Since all direct factors except B multiply to 0 on B we get a split morphism

$$(B \otimes_k B^{op}) \otimes_{k\Delta(D)} B \longrightarrow B.$$

Observe how the tensor product $k(G \times G) \otimes_{k\Delta(D)} B$ is formed. Indeed,

$$(g_1, g_2) \otimes db = (g_1 d, d^{-1} g_2) \otimes b$$

for all $g_1, g_2 \in G$, $d \in D$ and $b \in B$. The whole is a kG-right module by putting

$$((g_1, g_2) \otimes b) \cdot g = (g_1, g_2 g) \otimes b$$

and a kG-left module by putting

$$g \cdot ((g_1, g_2) \otimes b) = (gg_1, g_2) \otimes b.$$

We tensor this multiplication mapping with M over B, and so we get a split epimorphism

$$((B \otimes_k B^{op}) \otimes_{k\Delta(D)} B) \otimes_B M \longrightarrow B \otimes_B M.$$

Of course, $B \otimes_B M \simeq M$ by the multiplication map. Moreover, the B-right module structure of $((B \otimes_k B^{op}) \otimes_{k\Delta(D)} B)$ comes from the B-right module structure of the middle term B^{op}. Hence

$$\begin{array}{rcl} ((B \otimes_k B^{op}) \otimes_{k\Delta(D)} B) \otimes_B M & \simeq & (B \otimes_k M) \otimes_{k\Delta(D)} B \\ b_1 \otimes b_2 \otimes b_3 \otimes m & \mapsto & b_1 \otimes b_2 m \otimes b_3 \end{array}$$

and

$$\begin{array}{rcl} (B \otimes_k M) \otimes_{k\Delta(D)} B & \twoheadrightarrow & (B \otimes_k M) \\ b_1 \otimes m \otimes b_3 & \mapsto & b_1 b_3 \otimes m \end{array}$$

which induces a split epimorphism $B \otimes_D M \longrightarrow M$ as required. $\square$

Corollary 2.3.7 *The defect group of the principal block is a Sylow p-subgroup of G.*

Proof Indeed, by Lemma 2.1.19 the vertex of the trivial module is a Sylow p-subgroup of G. Hence the defect group of the principal block is at least a Sylow p-subgroup. The defect group is a p-group and therefore the defect group cannot be bigger than a Sylow p-subgroup. Proposition 2.3.6 then proves the statement. □

Corollary 2.3.8 *Let G be a p-group and k a field of characteristic p. Then the defect group of kG is G.*

Proof Indeed, kG is local by Proposition 1.6.22. □

Proposition 2.3.9 *If k is an algebraically closed field, then a block B is semisimple if and only if its defect group is the trivial group* (*i.e. the defect is* 0).

Proof If B is semisimple, then B^{op} is semisimple, and $B \otimes_k B^{op}$ is also semisimple. Hence the $B \otimes_k B^{op}$-module B is projective, and therefore B is a direct factor of $B \otimes_k B^{op}$. Hence the defect group of B is the trivial group.

If B has the trivial group as defect group, then for every B-module M the multiplication map $B \otimes_k M \longrightarrow M$ is split. Hence M is a direct factor of $B^{\dim_k(M)}$, and therefore M is projective. This shows that B is semisimple. □

Theorem 2.3.10 (Brauer's first main theorem) *Let k be a field of characteristic* $p > 0$ *and let G be a finite group. Then there is a one to one correspondence between blocks B of kG with defect group D and blocks b of* $N_G(D)$ *with defect group D. This bijection is obtained in such a way that b is a direct factor of* $B \downarrow^{G\times G}_{N_G(D)\times N_G(D)}$, *and b is the unique direct factor with this property.*

Proof Let B be a block of kG with defect group D. Then B is an indecomposable $\Delta(D)$-projective $k(G \times G)$-module and we look at $N_{G\times G}(\Delta(D))$. Now, $(g_1, g_2) \in N_{G\times G}(\Delta(D))$ if and only if there is a $d' \in D$ such that

$$(g_1, g_2)(d, d)(g_1^{-1}, g_2^{-1}) = (g_1 d g_1^{-1}, g_2 d g_2^{-1}) = (d', d').$$

Hence

$$(g_1, g_2) \in N_{G\times G}(\Delta(D)) \Leftrightarrow g_2^{-1} g_1 \in C_G(D) \text{and} g_1, g_2 \in N_G(D).$$

Therefore

$$N_{G\times G}(\Delta(D)) = \Delta(N_G(D)) \cdot (C_G(D) \times C_G(D)).$$

Now, kG is a free $kN_G(D)$-module, and therefore $kG = kN_G(D)^{|G:N_G(D)|}$ as $kN_G(D)$-left modules. Since each of these factors is a $k(N_G(D) \times N_G(D))$-module we get that kG is a direct sum of $|G : N_G(D)|$ copies of the $k(N_G(D) \times N_G(D))$-module $kN_G(D)$. Hence the restriction of B to $k(N_G(D) \times N_G(D))$ is a direct sum of blocks of $kN_G(D)$.

We observe now that $C_G(D) \le N_G(D)$ and hence

$$\Delta(N_G(D)) \cdot (C_G(D) \times C_G(D)) \le N_G(D) \times N_G(D).$$

The Green correspondence implies that there is a unique indecomposable $k\,(N_G(D) \times N_G(D))$-module with vertex $\Delta(D)$ within these direct factors. By Lemma 2.3.3 the direct factors of $kN_G(D)$ as $k\,(N_G(D) \times N_G(D))$-modules are precisely the blocks of $kN_G(D)$. Hence there is a unique block b of $kN_G(D)$ with defect group D and which is a direct factor of $B^{G\times G}_{N_G(D)\times N_G(D)}$. □

Definition 2.3.11 Let k be a field of characteristic $p > 0$ and let G be a finite group. Let B be a block of kG with defect group D. The unique block b of $kN_G(D)$ with defect group D which occurs as a direct summand of the restriction of B as a $k(N_G(D) \times N_G(D))$-module is called the *Brauer correspondent* of B.

2.4 Properties of Defect Groups, Brauer and Green Correspondence

Defect groups are far from arbitrary. Only few possibilities can occur and these possibilities are governed by the intersection behaviour of the Sylow p-subgroups of the group. We introduce a result due to Green.

Proposition 2.4.1 (Green [3]) *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let B be a block of kG. Then every defect group of G is the intersection of two Sylow p-subgroups of G.*

Proof Let S be a Sylow p-subgroup of G. As in the proof of Proposition 2.3.4 we define a morphism μ of $kG \otimes_k (kG)^{op} \simeq k(G \times G)$-modules

$$\begin{array}{rcl} k(G \times G) \otimes_{k\Delta(G)} k & \longrightarrow & kG \\ (g, h) \otimes 1 & \mapsto & gh^{-1} \end{array}$$

for all $g, h \in G$. This is clearly well-defined and surjective. We showed there that μ is split by the mapping ψ defined as

$$\begin{array}{rcl} kG & \longrightarrow & k(G \times G) \otimes_{k\Delta(G)} k \\ g & \mapsto & (g, 1) \otimes 1 \end{array}$$

which is $k(G \times G)$ linear as well. Hence, kG is induced from a trivial $k\Delta(G)$-module. We see that the source of every block is the trivial module. In order to apply Mackey decomposition we shall need to consider representatives of double classes $\Delta(G)\backslash(G \times G)/(S \times S)$. We see first that

$$\Delta(G)\backslash(G \times G) = \bigcup_{g \in G} \Delta(G) \cdot (1, g).$$

Hence, double class representatives are elements $(1, g)$ for a set T of elements $g \in G$. Let B be a block of kG. Then by Green correspondence $B \downarrow^{G\times G}_{S\times S}$ has a unique direct summand $g(B)$ with vertex $\Delta(D)$ and source the trivial module. We compute

$$k \uparrow^{G\times G}_{\Delta(G)} \downarrow^{G\times G}_{S\times S} = \bigoplus_{h \in T} {}^{(1,h)}k \downarrow^{(1,h)\Delta(G)(1,h)^{-1}}_{(1,h)\Delta(G)(1,h)^{-1}\cap S\times S} \uparrow^{S\times S}$$

and observe that

$$(1, h)\Delta(G)(1, h)^{-1} \cap (S \times S) = \Delta(S \cap {}^{h}S)$$

and that the conjugate of the trivial module is still the trivial module. Hence

$$k \uparrow^{G\times G}_{\Delta(G)} \downarrow^{G\times G}_{S\times S} = \bigoplus_{h \in T} k \uparrow^{S\times S}_{\Delta(S\cap^{h}S)} .$$

Now, $S \times S$ is a p-group, its group ring is local and therefore each transitive permutation module (cf Definition 1.7.47), which is a quotient of the regular module and therefore has a simple head, is indecomposable. Hence each of the factors $k \uparrow^{S\times S}_{\Delta(S\cap^{h}S)} \simeq k(S \times S)/\Delta(S \cap {}^{h}S)$ is indecomposable and we get for the Green correspondent $g(B) \simeq k \uparrow^{S\times S}_{\Delta(S\cap^{h}S)}$ for some $h \in G$, which implies that the defect group of B is ${}^{h}S \cap S$. □

Corollary 2.4.2 *Let k be a field of characteristic $p > 0$ and let G be a finite group with a normal p-subgroup P. Then the defect group of every block of kG contains P.*

Proof Since any two Sylow p-subgroups of G are conjugate, we may choose one Sylow p-subgroup S freely. Let D be a defect group of G and let S be a Sylow p-subgroup of G containing D. Then any other Sylow p-subgroup T of G is conjugate to S, but since P is normal, P is fixed under conjugation and hence $P \leq T$. Since $D = S \cap T$, we obtain $P \leq D$. □

A certain converse to Proposition 2.3.6 is true as well.

Proposition 2.4.3 *If B is a block of kG with defect group D and if b is the Brauer correspondent of B in $kN_G(D)$, and if M is an indecomposable B-module with vertex D, then the Green correspondent of M in $kN_G(D)$ is a b-module. Conversely, if N is a b-module with vertex D, then the Green correspondent of N in kG is a B-module.*

Proof Since Green correspondence and Brauer correspondence are bijections we only need to prove one of the two statements. The other statement then follows by this bijection. Put $H := N_G(D)$.

Let N be an indecomposable kH-module with vertex D, suppose that N belongs to the block b of kH, and suppose that b has defect group D. The Green correspondent $g(N)$ of N has vertex D as well, by definition of the Green correspondence. Since $g(N)$ is indecomposable, there is a unique block $\hat{B}$ of kG such that $g(N)$ is a $\hat{B}$-module.

We will first show that $\hat{B}$ is a direct factor of $k(G \times G) \otimes_{k(H\times H)} b$. Since N is a b-module, the restriction of the standard isomorphism $kH \otimes_{kH} N \simeq N$ to the direct factor b of kH gives an isomorphism $b \otimes_{kH} N \simeq N$. Since N is a direct factor of the restriction of $g(N)$ to kH, we get that N is a direct factor of $b \otimes_{kH} g(N)$. But then

$$\begin{aligned}\big(k(G \times G) \otimes_{k(H\times H)} b\big) \otimes_{kG} g(N) &= kG \otimes_{kH} b \otimes_{kH} kG \otimes_{kG} g(N)\\ &= kG \otimes_{kH} b \otimes_{kH} g(N)\end{aligned}$$

has a direct factor $kG \otimes_{kH} N$. In particular, $g(N)$ is a module over a direct factor of $k(G \times G) \otimes_{k(H\times H)} b$ and therefore $\hat{B}$ is a direct factor of $b \uparrow_{H\times H}^{G\times G}$.

By Proposition 2.3.6 the defect group $\hat{D}$ of $\hat{B}$ contains D. Brauer correspondence was introduced as a special case of Green correspondence and this shows that the Brauer correspondent B of b is the unique indecomposable direct factor of $k(G \times G) \otimes_{k(H\times H)} b$ with vertex D. All the other indecomposable direct factors of $k(G \times G) \otimes_{k(H\times H)} b$ have vertex in

$$\mathcal{X}_{\Delta(D),H\times H} = \{X \le G \times G \mid \exists g \in (G \times G) \setminus (H \times H) : X \le \Delta(D) \cap {}^{g}\Delta(D)\}.$$

In particular, let $\tilde{B}$ be an indecomposable direct factor of $k(G\times G)\otimes_{k(H\times H)} b$ different from B. Then the vertex of $\tilde{B}$ is a proper subgroup of $\Delta(D)$, using that $H = N_G(D)$. Since $D \le \hat{D}$ we get that $B = \hat{B}$. □

Corollary 2.4.4 (Brauer's third main theorem) *The Brauer correspondent of the principal block is the principal block.*

Proof We have seen by Proposition 2.4.3 that the trivial module corresponds to Brauer correspondents. Since the Green correspondent of the trivial module is the trivial module we obtain the statement. □

In a block with defect group D there is always a module with vertex D. In order to see this we first need a lemma.

Lemma 2.4.5 *Let k be a field of characteristic $p > 0$, let G be a finite group, let D be a p-subgroup of G, suppose H is a normal subgroup of G and suppose $D \le H \trianglelefteq G$. If N is an indecomposable kH-module with vertex D, then $N \uparrow_H^G$ is a direct sum of indecomposable kG-modules $N_1, N_2, \dots, N_s$ so that the vertex of N_i is D for all i.*

Proof $N \uparrow_H^G = N_1 \oplus \cdots \oplus N_m$ for indecomposable kG-modules N_i; for all $i \in \{1, 2, \dots, m\}$. By definition, each N_i has a vertex included in D. Moreover

$$N_1 \downarrow_H^G \oplus \cdots \oplus N_m \downarrow_H^G = N \uparrow_H^G \downarrow_H^G = \bigoplus_{gH \in G/H} {}^g N$$

by Mackey decomposition and the decomposition above. Since H is normal in G, we see that ${}^gN$ has vertex ${}^gD$ as a kH-module since conjugation by g is an automorphism of the algebra kH. Hence $N_i \downarrow_H^G$ is isomorphic to a direct sum of modules ${}^gN$ for certain g, and each of these modules has vertex ${}^gD$. Therefore N_i has vertex ${}^gD$ for some g, which is the same as having vertex D. □

Proposition 2.4.6 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let B be a block of kG with defect group D. Then there is an indecomposable B-module with vertex D.*

Proof Let $H := N_G(D)$ and observe that $\mathrm{rad}(kD) \cdot kH$ is a nilpotent ideal since $D \trianglelefteq H$ and since $\mathrm{rad}(kD)$ is nilpotent. Hence $\mathrm{rad}(kD) \cdot kH \le \mathrm{rad}(kH)$. The vertex of the trivial kD-module k is D. Hence $k \uparrow_D^H = T_1 \oplus \cdots \oplus T_n$ for indecomposable kH-modules $T_1, T_2, \ldots, T_n$. By Lemma 2.4.5 all modules T_i for $i \in \{1, 2, \ldots, n\}$ have vertex D. Since D is a p-group, we get $k = kD/\mathrm{rad}(kD)$ and therefore

$$k \uparrow_D^H = kD/\mathrm{rad}(kD) \uparrow_D^H = kH \otimes_{kD} (kD/\mathrm{rad}(kD)) = kH/(\mathrm{rad}(kD)kH)$$

where we use for the second isomorphism that induction transforms exact sequences into exact sequences. Since $\mathrm{rad}(kD) \cdot kH \le \mathrm{rad}(kH)$ all isomorphism types of simple kH-modules appear as direct factors of the head of the modules $T_1, T_2, \ldots, T_n$. Let $\hat{e}$ be the central idempotent of kG such that $\hat{e} \cdot kG = B$ and let e be the central idempotent of kH such that $e \cdot kH$ is the Brauer correspondent b of B. Then there is an i such that $e \cdot T_i \neq 0$. Indeed, $e \cdot S = S$ holds for all simple kH-modules S which belong to b. Let S be a simple b-module which is a direct factor of $T_i/\mathrm{rad}(T_i)$. Then

$$T_i \longrightarrow\!\!\!\!\!\rightarrow T_i/\mathrm{rad}(T_i)$$

gives that e does not act as 0 on T_i, since it does not act as 0 on the quotient $T_i/\mathrm{rad}(T_i)$. Since T_i is indecomposable, T_i is a b-module and e acts as 1 on T_i. By Proposition 1.10.26 the Green correspondent $g(T_i)$ of T_i in kG is a B-module with vertex D. This proves the proposition. □

Let A be a symmetric k-algebra. Then there is a non-degenerate symmetric associative bilinear form

$$\langle\ ,\ \rangle : A \times A \longrightarrow k.$$

Let $e^2 = e$ be an idempotent in A. Restricting to $eAe \times eAe$ gives that $\langle\ ,\ \rangle$ induces a symmetric associative bilinear form

$$\langle\ ,\ \rangle : eAe \times eAe \longrightarrow k.$$

Moreover, since $\langle\ ,\ \rangle : A \times A \longrightarrow k$ is non-degenerate, for all $x \in eAe$ there is a $y \in A$ such that $\langle x, y\rangle \neq 0$. Then for all $x \in eAe$ and $y \in A$ we get

$$\langle x, y\rangle = \langle exe, y\rangle = \langle exe, ey\rangle = \langle ey, exe\rangle = \langle eye, exe\rangle = \langle exe, eye\rangle$$

and so we may take $y \in eAe$. Therefore for a symmetric algebra A and an idempotent $e \in A$ the algebra eAe is also symmetric. Recall from Proposition 1.10.26 that kG is a symmetric k-algebra. Hence each block of kG is a symmetric k-algebra.

We have proved the following lemma.

Lemma 2.4.7 *Let k be a field, let A be a symmetric k-algebra, let $e^2 = e \neq 0$ be an idempotent of A and let G be a finite group. Then eAe is also a symmetric k-algebra. In particular each block B of kG is a symmetric k-algebra.* □

2.5 Orders and Lattices

So far we have mainly looked at algebras over fields and their modules. It will be important to look at modules over more general local commutative rings R and to link modules over R-algebras Λ to modules over $R/\mathrm{rad}(R) \otimes_R \Lambda$. This works particularly well in a case we will describe now.

2.5.1 Discrete Valuation Rings

We recall some well-known elementary facts about commutative rings that are going to be used in the sequel. We will only recall the facts that are absolutely necessary for the sequel and keep this section to a strict minimum. The subject of commutative rings is vast and not considered here as the main focus of the book. For an opposite point of view one might consult Curtis-Reiner [4, 5]. For interested readers we recommend Cassels-Fröhlich [6] or Serre [7]. We follow the beginning of the presentation in Cassels-Fröhlich [6].

Definition 2.5.1 Let K be a field with unit group K^*. A map

$$v : K \longrightarrow \mathbb{Z} \cup \{\infty\}$$

is a discrete valuation of K if

- $v(K^*) = \mathbb{Z}$ and $v(ab) = v(a) + v(b)$ for all $a, b \in K^*$,
- $v(0) = \infty$,
- $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$.

Here we define $\infty > a$ for all $a \in \mathbb{Z}$.

Lemma 2.5.2 *Let v be a discrete valuation of K. If $\upsilon(x) \neq \upsilon(y)$ then $\upsilon(x+y) = \min\{\upsilon(x), \upsilon(y)\}$. The set $R_\upsilon := \{x \in K \mid \upsilon(x) \geq 0\}$ is a local subring of K with field of fractions K and the principal ideal $\wp_\upsilon := R_\upsilon \setminus \upsilon^{-1}(0)$ is its unique maximal ideal.*

Proof The first axiom implies that R_υ is multiplicatively and additively closed. Since $1 \in K$ is an idempotent, we see that $\upsilon(1)$ is an additive idempotent of $\mathbb{Z}$, and therefore $\upsilon(1) = 0$. We claim that $\upsilon^{-1}(0) = R_\upsilon^\times$ are the units of R_υ. Indeed, given a with $\upsilon(a) = 0$, we consider $a^{-1} \in K$ and get

$$0 = \upsilon(1) = \upsilon(a \cdot a^{-1}) = \upsilon(a) + \upsilon(a^{-1}).$$

This implies $\upsilon^{-1}(0) = R_\upsilon^\times$. Then the third axiom implies that $(R_\upsilon, +)$ is a group, and hence that R_υ is a subring of K. Since υ is surjective, 1 is in the image of υ and there is a $\pi \in R_\upsilon$ such that $\upsilon(\pi) = 1$. The first axiom implies that $\upsilon(\pi^k) = k$ for all $k \in \mathbb{Z}$. Given $a \in R_\upsilon$ with $\upsilon(a) = k$, then $\upsilon(a \cdot \pi^{-k}) = \upsilon(a) - k = 0$, and so $u_a := a \cdot \pi^{-k} \in R_\upsilon^\times$. This shows that $\wp_\upsilon$ is the unique maximal ideal of R_υ. If $x, y \in K^\times$, then $x = \pi^{\upsilon(x)} \cdot u_x$ and $y = \pi^{\upsilon(y)} \cdot u_y$ for units u_x and u_y of R_υ. Hence R_υ is a principal ideal domain and any ideal is of the form $\pi^n R_\upsilon$. Suppose without loss of generality that $\upsilon(x) < \upsilon(y)$. Hence

$$\upsilon(x+y) = \pi^{\upsilon(x)} \cdot u_x + \pi^{\upsilon(y)} \cdot u_y = \pi^{\upsilon(x)}(u_x + \pi^{\upsilon(y)-\upsilon(x)} u_y)$$

and u_x a unit, whereas $\pi^{\upsilon(y)-\upsilon(x)} u_y \in \wp_\upsilon$. Therefore $u_x + \pi^{\upsilon(y)-\upsilon(x)} u_y \in R_\upsilon^\times$ since we have already shown that R_υ is local. Hence $\upsilon(x+y) = \upsilon(x)$ as claimed. □

Definition 2.5.3 A generator π of the maximal ideal $\wp_\upsilon$ of R_υ is called a *uniformiser* of R_υ.

A discrete valuation gives rise to a metric $d(x, y) := 2^{-\upsilon(x-y)}$ for all $x, y \in K$. Indeed, $d(x, y) = 0 \Leftrightarrow \upsilon(x - y) = \infty \Leftrightarrow x = y$ and $d(x, y) = d(y, x)$ by the proof of Lemma 2.5.2, and we have $d(x, y) \leq d(x, z) + d(z, y)$ since $\upsilon(x + y) \geq \min(\upsilon(x), \upsilon(y))$.

As in real analysis, we can define Cauchy sequences and convergence of sequences.

Definition 2.5.4 A sequence $(a_n)_{n\in\mathbb{N}}$ of elements in K is a *Cauchy sequence* if and only if

$$\forall \epsilon > 0\ \exists N(\epsilon) \in \mathbb{N}\ \forall n, m > N(\epsilon) :\ d(a_n, a_m) < \epsilon.$$

A sequence $(a_n)_{n\in\mathbb{N}}$ *converges to* $a \in K$ if and only if

$$\forall \epsilon > 0\ \exists N(\epsilon) \in \mathbb{N}\ \forall n > N(\epsilon) :\ d(a_n, a) < \epsilon.$$

It is immediate that convergent sequences are Cauchy sequences. The converse is not always true, as we know from real analysis, for example.

Definition 2.5.5 A field K is *complete with respect to the valuation* υ on K if every Cauchy sequence converges. A ring R is a *discrete valuation ring* if there is a field K with discrete valuation υ such that $R = R_\upsilon$. A discrete valuation ring R is *complete* if its field of fractions is complete.

We can complete a given discrete valuation.

Proposition 2.5.6 *Given a discrete valuation υ on K, there is a field $\hat{K}_\upsilon$ and a complete discrete valuation $\hat{\upsilon}$ on $\hat{K}_\upsilon$ such that K is a dense subfield of $\hat{K}_\upsilon$ and the restriction of $\hat{\upsilon}$ to K is υ. Moreover if* $\mathrm{rad}(R_\upsilon) = \pi R_\upsilon$ *then* $\mathrm{rad}(\hat{R}_\upsilon) = \pi \hat{R}_\upsilon$. *We call $\hat{R}_\upsilon$ the completion of R at υ.*

Proof Let K be a field with discrete valuation υ, then let $\mathcal{C}_\upsilon$ be the set of all Cauchy sequences in K. Two Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are called equivalent if the sequence $(a_n - b_n)_{n\in\mathbb{N}}$ converges to 0. Let $\hat{K}_\upsilon$ be the set of equivalence classes of Cauchy sequences.

This set carries a natural ring structure given by $(a_n) + (b_n) := (a_n + b_n)$ and $(a_n)\cdot(b_n) := (a_n b_n)$ for all $(a_n), (b_n) \in \mathcal{C}_\upsilon$. This gives the ring the structure of a field. Indeed, if (a_n) does not converge to 0, then there is a $\delta > 0$ such that $d(a_n, 0) \geq \delta$ for all $n \geq N_0$. In particular $a_n \neq 0$ for all $n \geq N_0$. Put $\check{a}_n := a_n^{-1}$ for $n \geq N_0$ and $\check{a}_n = 1$ for $n < N_0$. The sequence $(\check{a}_n)$ is again a Cauchy sequence. Moreover, $(a_n) \cdot (\check{a}_n)$ converges to the constant sequence 1. Hence $\hat{K}_\upsilon$ is a field.

We may embed K into $\hat{K}_\upsilon$ by sending $a \in K$ to the constant sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = a$ for all $n \in \mathbb{N}$.

The valuation υ induces a valuation $\hat{\upsilon}$ on $\hat{K}_\upsilon$ by putting $\hat{\upsilon}\,((a_n)) := \lim_{n\to\infty} (\upsilon(a_n))$. We need to show that this is well-defined. Indeed, let (a_n) be a Cauchy sequence. Suppose (a_n) does not converge to 0. Hence there is an $N_0 \in \mathbb{N}$ such that $d(a_n, 0) > \delta$ for all $n \geq N_0$. Since $d(a_n, a_m) < \epsilon$ for $n, m > N(\epsilon)$ and since $d(a_n, a_m) = 2^{-\upsilon(a_n - a_m)} = 2^{-\min\{\upsilon(a_n),\upsilon(a_m)\}}$ we get that the sequence $(\upsilon(a_n))_{n\in\mathbb{N}}$ is ultimately constant. Hence $\hat{\upsilon}$ is again a discrete valuation and the restriction of $\hat{\upsilon}$ to constant sequences, i.e. the image of K in $\hat{K}_v$, is v. The field K is dense in $\hat{K}_\upsilon$. Indeed, if $\epsilon > 0$ and $x \in \hat{K}_\upsilon$, let x be represented by a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$. Then for all $n > N_0$ we get that $2^{-\hat{\upsilon}(x - x_n)} < \varepsilon$. This shows that for all $n > N_0$ the elements $x_n \in K$ are in an ϵ-disk around x, and hence K is dense in $\hat{K}_\upsilon$. □

Example 2.5.7 The set $\mathbb{Q}$ carries a valuation υ_p for every prime integer p. For any integer $m \in \mathbb{Z} \setminus \{0\}$ let $\upsilon_p(m) := s$ if $p^s \mid m$ but $p^{s+1} \nmid m$. Let $q = \frac{a}{b}$ for coprime integers $a > 0$ and b. Then put $\upsilon_p(q) := \upsilon_p(a) - \upsilon_p(b)$. This is a discrete valuation. The valuation ring with respect to υ_p is

$$\mathbb{Z}_p := \left\{ \frac{a}{b} \mid b > 0;\ a \text{ and } b \text{ are coprime}, p \nmid b \right\}.$$

The completion with respect to υ_p is $\hat{\mathbb{Q}}_p$, which is the field of fractions of formally infinite p-adic expansions, i.e.

$$\hat{\mathbb{Z}}_p := \left\{ \sum_{n=0}^{\infty} a_n p^n \mid \forall n \in \mathbb{N} : \; a_n \in \{0, 1, \ldots, p-1\} \right\}.$$

Another way of looking at the ring $\hat{\mathbb{Z}}_p$ is to consider the sequence of quotients

$$\longrightarrow \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\psi_m} \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\psi_{m-1}} \mathbb{Z}/p^{m-1}\mathbb{Z} \longrightarrow$$

and the unique ring R equipped with ring epimorphisms $\varphi_m : R \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$ such that $\varphi_m = \psi_m \circ \varphi_{m+1}$ for all m and such that whenever there is another ring S equipped with ring homomorphisms χ_m satisfying $\chi_m = \psi_m \circ \chi_{m+1}$, then there is a unique ring homomorphism $\chi : S \longrightarrow R$ with $\chi_m = \varphi_m \circ \chi$ for all m. It is an easy exercise to see that $\hat{\mathbb{Z}}_p$ has this property with respect to the sequence

$$\longrightarrow \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\psi_m} \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\psi_{m-1}} \mathbb{Z}/p^{m-1}\mathbb{Z} \longrightarrow .$$

We shall see later that this is an instance of an inverse limit of rings. We call the ring $\hat{\mathbb{Z}}_p$ the ring of p-adic integers and the field $\hat{\mathbb{Q}}_p$ the field of p-adic numbers.

Example 2.5.8 Not all complete discrete valuation rings are of characteristic 0, however we will mainly use the case of characteristic 0 in the sequel.

Let k be any field and let $k(X)$ be the field of rational functions with coefficients in k. Define for any polynomial $p(X) \in k[X]$ the number $v_X(p(X)) := s$ if $X^s \mid p(X)$ but $X^{s+1} \nmid p(X)$. Then, for two coprime polynomials $p(X) \neq 0$ and $q(X)$, define

$$v_X\left(\frac{p(X)}{q(X)}\right) := v_X(p(X)) - v_X(q(X)).$$

We observe that v_X is a discrete valuation and the valuation ring contains $k[X]$. The completion of $k(X)$ with respect to v_X is field of Laurent power series with coefficients in k.

Proposition 2.5.9 *Let k be a perfect field of characteristic $p > 0$, then there is a complete discrete valuation ring R of characteristic* 0 *with $R/\mathrm{rad}(R) \simeq k$.*

A proof can be found, for example, in [7, Chap. II, Sect. 6, Thèorème 5].

Remark 2.5.10 If R is a discrete valuation ring, then we may also define Cauchy sequences in a free R-module R^n. If R is complete, then each Cauchy sequence in R^n converges.

Remark 2.5.11 Let R be a discrete valuation ring and let M be a finitely generated R-module. Recall that the modification of Gauss' algorithm to obtain the classification theorem for finitely generated abelian groups via elementary divisors of integral matrices can also be formulated for matrices over principal ideal domains. This yields a classification of modules over principal ideal domains, namely M is isomorphic to a direct product of a free module and modules of the form $R/\pi^m R$ for some $m \in \mathbb{N}$.

The observation in Example 2.5.7 can be used to define completions in a more general setting. We shall use this generalisation only in two cases, namely the classical Noether-Deuring theorem, and its generalisation to derived categories.

Lemma 2.5.12 *Let R be a commutative ring and let I be a proper ideal of R. Then we can form quotients of R modulo powers of I so that we obtain a sequence* $\ldots \twoheadrightarrow R/I^n \xrightarrow{\psi_{n-1}} R/I^{n-1} \twoheadrightarrow \ldots \twoheadrightarrow R/I$. *There is a unique ring $\hat{R}_I$ and morphisms $\hat{R}_I \xrightarrow{\varphi_n} R/I^n$ such that $\psi_n \circ \varphi_{n+1} = \varphi_n$ for all n, and such that whenever there is a commutative ring S with mappings $S \xrightarrow{\chi_n} R/I^n$ satisfying $\psi_n \circ \chi_{n+1} = \chi_n$ for all n, then there is a unique ring homomorphism $S \xrightarrow{\chi} \hat{R}_I$ such that $\chi_n = \varphi_n \circ \chi$ for all n.*

Idea of Proof The construction of $\hat{R}_I$ is analogous to the explicit description of the p-adic integers in Example 2.5.7, namely $\hat{R}_I = \{(r_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} R/I^n \mid \chi_n(r_{n+1}) = r_n \ \forall n \geq 1\}$. □

Definition 2.5.13 The ring $\hat{R}_I$ is the *I-adic completion* of R.

We shall see later that $\hat{R}_I$ is actually a special case of a projective limit.

An especially important case is given when R is commutative and semilocal, i.e. R has only finitely many maximal ideals (cf [8]).

Proposition 2.5.14 (Eisenbud [9, Corollary 7.6]) *Let R be a semilocal commutative ring and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be its maximal ideals. Then for $I := \mathrm{rad}(R)$ we get for the I-adic completion $\hat{R}_I \simeq \hat{R}_{\mathfrak{m}_1} \times \cdots \times \hat{R}_{\mathfrak{m}_1}$.*

Proof (sketch): In this case $I = \bigcap_{i=1}^r \mathfrak{m}_i = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_r$. Then $I^n = \mathfrak{m}_1^n \cdot \ldots \cdot \mathfrak{m}_r^n$ and since $\mathfrak{m}_i^n + \mathfrak{m}_j^n = R$ for all $i \neq j$, we get by the Chinese Reminder Theorem that $R/I^n = R/\mathfrak{m}_1^n \times \cdots \times R/\mathfrak{m}_r^n$. Since $R/\mathfrak{m}_i^n$ is local we have $R/\mathfrak{m}_i^n = (R/\mathfrak{m}_i^n)_{\mathfrak{m}_i} = R_{\mathfrak{m}_i}/\mathfrak{m}_i^n R_{\mathfrak{m}_i}$, where we denote by $R_{\mathfrak{m}_i}$ the localisation at $\mathfrak{m}_i$. The construction in Lemma 2.5.12 then proves the statement. □

2.5.2 *Classical Orders and Their Lattices*

Throughout this section let R be a commutative ring without zero-divisors and field of fractions K. Then we want to formalise R-algebras which are "big subrings" of finite dimensional semisimple K-algebras. The notion for being "big" corresponds to the concept of containing a K-basis.

Definition 2.5.15 An R-algebra Λ is an *R-order* (or equivalently a classical R-order) in the semisimple K-algebra A if Λ is projective as an R-module, and if $K \otimes_R \Lambda \simeq A$ as K-algebras.

Example 2.5.16 Let us give some examples for properties of R-orders.

1. For every integer $n \in \mathbb{N}$ we have that $Mat_{n\times n}(R)$ is an R-order in $Mat_{n\times n}(K)$.
2. Let $\wp$ be a projective ideal of the ring R. Then the subring $\Lambda := \begin{pmatrix} R & R \\ \wp & R \end{pmatrix}$ of $Mat_{n\times n}(R)$ given by the set of matrices with lower left coefficients in $\wp$ is an R-order in $Mat_{n\times n}(K)$. Observe that Λ is a proper subring of $Mat_{n\times n}(R)$ of index $|R/\wp|$.
3. Fix a projective ideal $\wp$ of R and let $R - R := \{(x, y) \in R^2 \mid x - y \in \wp\}$. Then $R - R$ is an R-order in $K \times K$. Hence an R-order may be indecomposable in a decomposable K-algebra.
4. If Λ has R-torsion (the torsion submodule of Λ is the set of $\lambda \in \Lambda$ such that the kernel of $R \ni r \mapsto r \cdot \lambda \in \Lambda$ is not zero), then Λ is not an R-order since in this case Λ is not R-projective.
5. Let G be a finite group and let R be a discrete valuation ring of characteristic 0 and with field of fractions K. Then RG is an R-order in KG.

Let R be a discrete valuation ring with residue field k, let Λ be an R-order, let $A := k \otimes_R \Lambda$ and let M be a Λ-module, then $\overline{M} := k \otimes_R M$ is an A-module. If M is projective, then $\overline{M}$ is projective as an A-module. The main interest in using complete discrete valuation rings in this setting is the following proposition.

Proposition 2.5.17 *Let R be a Noetherian complete discrete valuation ring with residue field k and with field of fractions K. Let Λ be an R-order and let $A = k \otimes_R \Lambda$. Let $\pi : \Lambda \longrightarrow A$ be the residue mapping. Then for every idempotent $\overline{e}^2 = \overline{e} \in A$ there is an idempotent $e \in \Lambda$ such that $\pi(e) = \overline{e}$. In particular for every projective indecomposable A-module $\overline{P}$ there is a projective A-module P such that $\overline{P} \simeq k \otimes_R P$.*

Proof Recall that R is a complete discrete valuation ring and hence any projective R-module is free. Let π be a uniformiser of R. Since Λ is a finitely generated projective R-module, it is a free R-module of finite rank, we shall need to produce a Cauchy sequence of elements converging to an idempotent. If we have a sequence of elements $e_n \in \Lambda$ so that $e_n^2 - e_n \in \pi^n\Lambda$ and $e_n - e_{n-1} \in \pi^{n-1}\Lambda$, then the sequence e_n converges to an element $e \in \Lambda$ with $e^2 - e = 0$.

We proceed by induction. Let e_1 be an element such that $\pi(e_1) = \overline{e}$, let $n \geq 2$ and suppose we have constructed e_m for $m < n$ so that $e_m^2 - e_m \in \pi^m\Lambda$ and so that $e_m - e_{m-1} \in \pi^{m-1}\Lambda$. Put $e_n := 3e_{n-1}^2 - 2e_{n-1}^3$. Then

$$\begin{aligned} e_n^2 - e_n &= (3e_{n-1}^2 - 2e_{n-1}^3)^2 - (3e_{n-1}^2 - 2e_{n-1}^3) \\ &= (3e_{n-1}^2 - 2e_{n-1}^3)(3e_{n-1}^2 - 2e_{n-1}^3 - 1) \\ &= e_{n-1}^2(3 - 2e_{n-1})(3e_{n-1}^2 - 2e_{n-1}^3 - 1) \\ &= -(3 - 2e_{n-1})(1 + 2e_{n-1})(e_{n-1}^2 - e_{n-1})^2 \end{aligned}$$

where the last equation is verified by an elementary multiplication. But since $e_{n-1}^2 - e_{n-1} \in \pi^{n-1}\Lambda$, we obtain

$$e_n^2 - e_n \in \pi^{2(n-1)}\Lambda \subseteq \pi^n\Lambda.$$

If we consider e_n in $\Lambda/\pi^{n-1}\Lambda$, we get that $e_{n-1}^2 = e_{n-1}$ in $\Lambda/\pi^{n-1}\Lambda$ and so we compute in $\Lambda/\pi^{n-1}\Lambda$

$$e_n = 3e_{n-1}^2 - 2e_{n-1}^3 = 3e_{n-1} - 2e_{n-1} = e_{n-1} \in \Lambda/\pi^{n-1}\Lambda.$$

Hence

$$e_n - e_{n-1} \in \pi^{n-1}\Lambda.$$

This proves that (e_n) is a Cauchy sequence converging to $e \in \Lambda$ in the sense of Remark 2.5.10. The construction implies that

$$e^2 - e \in \bigcap_{n\in\mathbb{N}} \pi^n \Lambda = 0$$

and we have proved the result. □

Corollary 2.5.18 *Let G be a finite group and let R be a complete discrete valuation ring with residue field k. Then for every block B of RG the algebra $k\otimes_R B$ is a block of kG and for every block b of kG there is a unique block B of RG such that $b = k\otimes_R B$.*

Proof We know that the centre of RG is R-linearly generated by the conjugacy class sums of G, and the centre of kG is k-linearly generated by the conjugacy class sums of G. Hence the ring epimorphism $RG \longrightarrow\!\!\!\!\rightarrow kG$ induces a ring epimorphism $Z(RG) \longrightarrow\!\!\!\!\rightarrow Z(kG)$. Proposition 2.5.17 then proves the statement. □

Example 2.5.19 If R is not complete then it is not true that idempotents lift as in Proposition 2.5.17. We consider $\mathbb{Z}_pC_q$ for two different primes p and q. This is an order in $\mathbb{Q}C_q$ where we obtain

$$\mathbb{Q}C_q \simeq \mathbb{Q} \times \mathbb{Q}(\zeta_q)$$

for a primitive q-th root of unity ζ_q in $\mathbb{C}$ so that there are exactly two idempotents corresponding to the projection onto each of the direct factors. Since $\mathbb{Z}_pC_q$ is a subring of $\mathbb{Q}C_q$ there are at most two idempotents. $\mathbb{F}_pC_q$ is an $\mathbb{F}_p$-algebra. If $\mathbb{F}_p$ contains a primitive q-th root of unity, then $\mathbb{F}_pC_q = (\mathbb{F}_p)^q$ is a direct product of q direct factors $\mathbb{F}_p$. This happens if q divides $p-1$ since the multiplicative group of $\mathbb{F}_p$ is cyclic. Such pairs of primes exist, for example $p = 11$ and $q = 5$. Other occurrences of this kind of phenomenon appear for different reasons. It is easily seen that the Frobenius automorphism on $\mathbb{F}_2C_7$ has order 3. Indeed, $\mathbb{F}_2C_7 \simeq \mathbb{F}_2[X]/(X^7 - 1)$ and the Frobenius automorphism sends X to X^2, X^2 to X^4 and X^4 to X. Also the element X^3 lies in an orbit of length 3 under the Frobenius automorphism. Hence

$$\mathbb{F}_2C_7 \simeq \mathbb{F}_2 \times \mathbb{F}_8 \times \mathbb{F}_8$$

and there are three primitive idempotents in this algebra.

Let R be a discrete valuation ring with residue field $k = R/\mathrm{rad}(R)$ and let Λ be an R-order. Of course, if M is a $k \otimes_R \Lambda$-module, then M is also a Λ-module since there is a surjective ring homomorphism $\Lambda \longrightarrow\!\!\!\!\to k \otimes_R \Lambda$. However, dealing with orders we are really mainly interested in R-projective Λ-modules.

Definition 2.5.20 Let R be a commutative ring without zero-divisors and let Λ be an R-order. Then a finitely generated Λ-module L is called a *Λ-lattice* if L is projective as an R-module.

We have seen in Example 2.5.10 that for a discrete valuation ring R with residue field $k = R/\mathrm{rad}(R)$ and an R-order Λ there may be $k \otimes_R \Lambda$-modules that are not of the form $k \otimes_R L$ for a Λ-lattice L.

A very important property of completions is given in the following lemma.

Lemma 2.5.21 *Let R be a discrete valuation ring and let $A \overset{\iota}{\hookrightarrow} B$ be an inclusion of finitely generated R-modules. Then $\hat{R} \otimes_R A \xrightarrow{id_{\hat{R}} \otimes \iota} \hat{R} \otimes_R B$ is an inclusion of $\hat{R}$-modules, and $\hat{R} \otimes_R A = 0 \Rightarrow A = 0$.*

Moreover, for a uniformiser π of R we have $\hat{R}/\pi^\ell \hat{R} \simeq R/\pi^\ell R$ for all $\ell > 0$.

Proof Let π be a uniformiser of R. Since R is local any finitely generated indecomposable R-module is either isomorphic to R or to $R/\pi^n R$ for some $n > 0$. Now, trivially $\hat{R} \otimes_R R \simeq \hat{R}$.

We shall show $\hat{R}/\pi^\ell \hat{R} \simeq R/\pi^\ell R$. Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in R. Then $2^{-\nu(a_n - a_m)} < \epsilon$ for $n, m \geq N_\epsilon$ and so $a_n - a_m \in \pi^\ell R$ if $\epsilon < 2^{-\ell}$ and $n, m > N_\epsilon$. Hence, modulo π^ℓ, all Cauchy sequences are ultimately constant, and the Cauchy sequence gives, modulo $\pi^\ell \hat{R}$, the same element as the constant sequence with constant generic term a_{N_ϵ}. Hence

$$\hat{R}/\pi^\ell \hat{R} \simeq R/\pi^\ell R$$

as claimed. This shows at once that

$$\hat{R} \otimes_R A = 0 \Rightarrow A = 0.$$

We still need to show that $A \leq B$ implies $\hat{R} \otimes_R A \leq \hat{R} \otimes_R B$. For this we use that R is a subring of $\hat{R}$, and hence is an R-module, by considering constant Cauchy sequences. Now, by Remark 2.5.11 each finitely generated R-module is either free or isomorphic to $R/\pi^\ell R$ for some ℓ. We claim that $R/\pi^\ell R$ is not a submodule of $\hat{R}$. Indeed, let the equivalence class of the Cauchy sequence $(a_n)_{n \in \mathbb{N}}$ be a generator of this module. Then $\pi^\ell \cdot (a_n)_{n \in \mathbb{N}} = (\pi^\ell a_n)_{n \in \mathbb{N}}$ converges to 0. Hence for all $\epsilon > 0$ there is an N_ϵ such that

$$2^{-\upsilon(\pi^\ell a_n)} = 2^{-(\ell + \nu(a_n))} < \epsilon \quad \forall n > N_\epsilon$$

and hence $(a_n)_{n \in \mathbb{N}}$ converges to 0. Therefore the Cauchy sequence $(a_n)_{n \in \mathbb{N}}$ represents the same element as 0. Therefore each finitely generated submodule of $\hat{R}$ is a free R-module.

Now, for each index set I and each family A_i of R-submodules of the R-module A and each R-module C we have the following equality

$$\left(\sum_{i\in I} A_i\right)\otimes_R C = \sum_{i\in I}(A_i\otimes_R C)\,.$$

Indeed, for finite sets I this is clear. If I is an infinite set, then let x be an element of the module on the left of the equation. Then there are only finitely many elements of I involved, and so we get the inclusion from left to right. Conversely let y be an element of the module on the right. Then again only a finite number of elements of I are involved and we get that y is also in the module on the left.

Now, each module is the sum of its finitely generated submodules. This proves the statement. □

Definition 2.5.22 We call an R-module S *flat* if for every monomorphism $\alpha : A \hookrightarrow B$ of R-modules, $id_S\otimes_R\alpha : S\otimes_R A \to S\otimes_R B$ is also a monomorphism. An R-module S is *faithfully flat* if S is flat and if $S\otimes_R A = 0$ implies $A = 0$.

Of course free modules are faithfully flat, but completions are also faithfully flat as we have seen in Lemma 2.5.21. Projective modules are flat but are not always faithfully flat. We shall study this property in more detail in Sect. 3.8.

Let R be a complete discrete valuation ring, let Λ be an R-order and let L be a Λ-lattice. For each $f \in End_\Lambda(L)$ define

$$\ker(f)^\infty := \{x\in L \mid \forall n\in\mathbb{N}\ \exists m\geq 0 :\ f^m(x)\in \mathrm{rad}^n(R)L\}$$

and

$$\mathrm{im}(f)^\infty := \bigcap_{n\in\mathbb{N}} \mathrm{im}(f^n).$$

Lemma 2.5.23 *Let R be a complete discrete valuation ring with uniformiser π. Then Fitting's lemma holds for endomorphisms of lattices L over R-orders Λ in the sense that there for every $f\in End_\Lambda(L)$ there is a decomposition $L = \ker(f)^\infty\oplus \mathrm{im}(f)^\infty$ so that f restricted to $\mathrm{im}(f)^\infty$ is an automorphism and f restricted to $\ker(f)^\infty$ is nilpotent modulo π^m for all m.*

Proof For all $n\in\mathbb{N}$ we see that L/π^nL is a $\Lambda/\pi^n\Lambda$-module. Since $\Lambda/\pi^n\Lambda$ is artinian and Noetherian, we get by Fitting's lemma

$$L/\pi^n = \mathrm{im}(f)^\infty/\pi^n \oplus \ker(f)^\infty/\pi^n$$

for all $n\in\mathbb{N}$. Moreover $\mathrm{im}(f)^\infty/\pi^{n+1} \longrightarrow\!\!\!\!\rightarrow \mathrm{im}(f)^\infty/\pi^n$ for all $n\in\mathbb{N}$ and likewise for $\ker(f)^\infty$. □

Corollary 2.5.24 *Let R be a complete discrete valuation ring. Then the Krull-Schmidt theorem holds for lattices over R-orders.*

Proof We review the proof of the Krull-Schmidt theorem in our setting. Fitting's lemma in the version of Lemma 2.5.23 again tells us that the endomorphism ring of an indecomposable lattice is local. Lemma 1.4.6 reads then that every endomorphism of an indecomposable lattice is either bijective or nilpotent modulo π^m for all m for a uniformiser π of R. In particular the series of endomorphisms used at the end of the proof of Lemma 1.4.6 converges in the topology induced by the valuation of R. Lemma 1.4.7 holds without any change, and the reader is invited to check that the proof of the Krull-Schmidt theorem holds word for word as well in this setting, without change. □

Lemma 2.5.25 *Let R be a discrete valuation ring and let $\hat{R}$ be its completion. Let Λ be an R-algebra and let $\hat{\Lambda} := \hat{R} \otimes_R \Lambda$. Then for all Λ-modules M and N we define $\hat{M} := \hat{R} \otimes_R M$ and $\hat{N} := \hat{R} \otimes_R N$. Suppose M is finitely generated. Then we have*

$$\hat{R} \otimes_R Ext^i_\Lambda(M, N) \simeq Ext^i_{\hat{\Lambda}}(\hat{M}, \hat{N})$$

for all $i \geq 0$.

Proof We shall prove that the canonical homomorphism

$$\begin{aligned} Hom_\Lambda(M, N) &\longrightarrow Hom_{\hat{\Lambda}}(\hat{M}, \hat{N}) \\ \varphi &\mapsto id_{\hat{R}} \otimes_R \varphi \end{aligned}$$

induces an isomorphism

$$\hat{R} \otimes_R Hom_\Lambda(M, N) \simeq Hom_{\hat{\Lambda}}(\hat{M}, \hat{N}).$$

For this we need that if $A \longrightarrow B$ is a monomorphism of R-modules, then $\hat{R} \otimes_R A \longrightarrow \hat{R} \otimes_R B$ is a monomorphism of $\hat{R}$-modules by Lemma 2.5.21.

Then we see that for every $\hat{\Lambda}$-module X we have

$$Hom_{\hat{R}\otimes_R\Lambda}(\hat{R} \otimes_R M, X) = Hom_\Lambda(M, Hom_{\hat{R}}(\hat{R}, X)) = Hom_\Lambda(M, X)$$

and for every free Λ-module Λ^n we obtain

$$Hom_{\hat{\Lambda}}(\hat{\Lambda}^n, \hat{R} \otimes_R N) = \hat{R} \otimes_R N^n = \hat{R} \otimes_R Hom_\Lambda(\Lambda^n, N).$$

Hence, taking a free resolution of M as a Λ-module

$$\ldots \longrightarrow \Lambda^{n_1} \longrightarrow \Lambda^{n_0} \longrightarrow M \longrightarrow 0$$

we may tensor first by $\hat{R}$ over R, and then take $Hom_{\hat{\Lambda}}(-, \hat{N})$ or first take $Hom_\Lambda(-, N)$ and tensor with $\hat{R}$ over R afterwards. To shorten the notation we use the abbreviation

$$\hat{R} \otimes_R Hom_\Lambda(X, Y) = \widehat{Hom_\Lambda(X, Y)}.$$

The result on free modules is the same, and since tensor products with $\hat{R}$ over R turn exact sequences into exact sequences, we obtain a diagram with commutative squares:

$$\begin{array}{ccccccc} \widehat{Hom_\Lambda(M, N)} & \hookrightarrow & \widehat{Hom_\Lambda(\Lambda^{n_0}, N)} & \to & \widehat{Hom_\Lambda(\Lambda^{n_1}, N)} & \to & \widehat{Hom_\Lambda(\Lambda^{n_2}, N)} \to \\ \| & & \| & & \| & & \| \\ Hom_{\hat{\Lambda}}(\hat{M}, \hat{N}) & \hookrightarrow & Hom_{\hat{\Lambda}}(\hat{\Lambda}^{n_0}, \hat{N}) & \to & Hom_{\hat{\Lambda}}(\hat{\Lambda}^{n_1}, \hat{N}) & \to & Hom_{\hat{\Lambda}}(\hat{\Lambda}^{n_2}, \hat{N}) \to \end{array}$$

For the first row we obtain $Ext^n_{\hat{\Lambda}}(\hat{M}, \hat{N})$ as the quotient of the kernel modulo the image of the mappings in the n-th term, and for the second row we obtain this way $\hat{R} \otimes_R Ext^n_{\hat{\Lambda}}(M, N)$ since $\hat{R}$ is faithfully flat as an R-module, and hence $\hat{R} \otimes_R -$ turns exact sequences into exact sequences. □

A very nice application of completions is the following generalisation of a result of Noether and Deuring, due to Roggenkamp. This is formulated in the more general context of I-adic completions, rather than discrete valuation rings. We shall briefly indicate what will be needed.

Remark 2.5.26 The statement of Lemma 2.5.25 remains true in the more general context of $\mathrm{rad}(R)$-adic completions of commutative Noetherian local rings R. Furthermore the lifting of idempotents result Proposition 2.5.17 still holds true in this more general context (cf Eisenbud [9, Corollary 7.5]).

Similar to the Hilbert basis theorem we have the following result.

Proposition 2.5.27 *Let R be a commutative Noetherian ring and let $\mathfrak{m}$ be a maximal ideal of R. Then the $\mathfrak{m}$-adic completion $\hat{R}_\mathfrak{m}$ is Noetherian as well. Moreover, $R/\mathfrak{m} = \hat{R}_\mathfrak{m}/\mathfrak{m}\hat{R}_\mathfrak{m}$ and $\hat{R}_\mathfrak{m}$ is a faithfully flat R-module.*

A proof can be found e.g. in Eisenbud [9, Theorem 7.1 and 7.2] or in Zariski-Samuel [10, Chap. VIII, Sect. 3, Corollary 5].

Theorem 2.5.28 (Noether-Deuring for fields, Roggenkamp [11]) *Let R be a local commutative Noetherian ring without zero-divisors and let Λ be a Noetherian R-algebra. Suppose S is either a commutative R-algebra which is a faithful finitely generated projective R-module or the* $\mathrm{rad}(R)$*-adic completion of R. Suppose that M and N are finitely generated Λ-modules so that $End_\Lambda(M)$ and $End_\Lambda(N)$ are finitely generated R-modules. Then M is a direct summand of N as Λ-modules if and only if $S \otimes_R M$ is a direct summand of $S \otimes_R N$ as $S \otimes_R \Lambda$-modules.*

Proof If M is a direct summand of N as Λ-module, then trivially $S \otimes_R M$ is a direct summand of $S \otimes_R N$ as $S \otimes_R \Lambda$-modules.

We need to show the converse, and so let $\hat{\sigma}$ be a split monomorphism $S \otimes_R M \overset{\hat{\sigma}}{\hookrightarrow} S \otimes_R N$. Suppose in the first step that $S = \hat{R}$ is the completion of R at the unique prime of R. We have seen from Lemma 2.5.25 (or Remark 2.5.26 for rad(R)-adic completions) that

$$\hat{R} \otimes_R Ext^n_\Lambda(M, N) = Ext^n_{\hat{R}\otimes_R \Lambda}(\hat{R} \otimes_R M, \hat{R} \otimes_R N)$$

for all $n \geq 0$. For $n = 0$ we get that $\hat{\sigma} = \sum_{i=1}^n \hat{r}_i \otimes_R \sigma_i$ for homomorphisms $\sigma_i \in Hom_\Lambda(M, N)$ and $\hat{r}_i \in \hat{R}$. Since $R/\mathrm{rad}(R) \simeq \hat{R}/\mathrm{rad}(\hat{R})$ by Proposition 2.5.6 (or Lemma 2.5.26 in the context of rad(R)-adic completions) for all $\hat{r}_i \in \hat{R}$ there is an r_i in R such that $r_i - \hat{r}_i \in \mathrm{rad}(\hat{R})$. Let

$$\sigma := \sum_{j=1}^{n} r_i \sigma_i \in Hom_\Lambda(M, N).$$

We claim that σ is a split monomorphism. To prove this it is sufficient to show that $id_{\hat{R}} \otimes_R \sigma$ is a split monomorphism in $Hom_{\hat{R}\otimes_R \Lambda}(\hat{R} \otimes_R M, \hat{R} \otimes_R N)$ since then the exact sequence

$$0 \longrightarrow M \xrightarrow{\sigma} N \longrightarrow N/\mathrm{im}(\sigma) \longrightarrow 0$$

is mapped to the zero element in

$$\hat{R} \otimes_R Ext^1_\Lambda(N/\mathrm{im}(\sigma), M) = Ext^1_{\hat{R}\otimes_R \Lambda}(\hat{R} \otimes_R N/\mathrm{im}(\sigma), \hat{R} \otimes_R M).$$

Since $\hat{\sigma}$ is a split injection by hypothesis we get a homomorphism

$$\hat{\tau} \in Hom_{\hat{R}\otimes_R \Lambda}(\hat{R} \otimes_R N, \hat{R} \otimes_R M)$$

so that

$$\hat{\tau} \circ \hat{\sigma} = id_{\hat{R}\otimes_R M}.$$

We have

$$\begin{aligned}
\hat{\tau} \circ (id_{\hat{R}} \otimes_R \sigma) - id_{\hat{R}\otimes_R M} &= \hat{\tau} \circ ((id_{\hat{R}} \otimes_R \sigma) - \hat{\sigma}) \\
&= \hat{\tau} \circ \left(\sum_{i=1}^{n} (r_i - \hat{r}_i) \otimes \sigma_i \right) \in \mathrm{rad}(\hat{R}) \cdot End_{\hat{\Lambda}}(\hat{M})
\end{aligned}$$

since by the choice of r_i we get for all i that $r_i - \hat{r}_i \in \mathrm{rad}(\hat{R})$. Since $End_\Lambda(M)$ is finitely generated as R-module, $End_{\hat{\Lambda}}(\hat{M})$ is finitely generated as an $\hat{R}$-module. Therefore

$$\hat{\tau} \circ (id_{\hat{R}} \otimes_R \sigma) \in id_{\hat{R}\otimes_R M} + \mathrm{rad}(\hat{R}) \cdot End_{\hat{\Lambda}}(\hat{M})$$

is a unit in $End_{\hat{\Lambda}}(\hat{M})$, using Nakayama's Lemma 1.6.5. Therefore $id_{\hat{R}} \otimes_R \sigma$ is a split monomorphism and hence σ is a split monomorphism.

In order to prove the case when S is a faithful finitely generated projective R-module we observe that the completion $\hat{R} \otimes_R S$ of S is a projective and hence free $\hat{R}$-module, whence $\hat{R} \otimes_R S = \hat{R}^n$. Hence

$$S \otimes_R M \,|S \otimes_R N \Rightarrow \hat{R} \otimes_R S \otimes_R M \,\Big|\hat{R} \otimes_R S \otimes_R N \Rightarrow \hat{R}^n \otimes_R M \,\Big|\hat{R}^n \otimes_R N \;.$$

By the Krull-Schmidt theorem for lattices over $\hat{R}$-orders (Corollary 2.5.24) we get

$$\hat{R} \otimes_R M \,\Big|\hat{R} \otimes_R N$$

and this implies $M\,|N$ by the first step of the proof. □

Remark 2.5.29 We remark that Theorem 2.5.28 includes the case of a field R and a finite extension field S of K.

Under the hypothesis of Theorem 2.5.28 we get that $S \otimes_R M \simeq S \otimes_R N$ as $S \otimes_R \Lambda$-modules implies $M \simeq N$ as Λ-modules.

Example 2.5.30 We mention the original goal of Noether and Deuring. Let K be a field and let L be a Galois extension of K with $G := Gal(L/K)$. Then L is a KG-module, since G acts on L as K-linear automorphisms. We get that $L \otimes_K L \simeq \sum_{g \in G} L_g$, where $L_g = L$ for each $g \in G$, since the various K-linear embeddings of L into L describe precisely the Galois group G. Moreover, G permutes the summands L_g i.e. $h \cdot L_g = L_{hg}$ for $g, h \in G$. Hence we get the regular module

$$L \otimes_K L \simeq LG \simeq L \otimes_K KG$$

and by the Noether-Deuring theorem $L \simeq KG$ as KG-modules.

This is the easiest case of the so-called Galois module structure of a Galois extension. Most interesting is the case when K and L are algebraic number fields and we consider the algebraic integers $\mathcal{O}_L$ in L as an $\mathcal{O}_K G$-module. This is a highly complicated and difficult problem. We refer to Fröhlich's monograph [12] for further reading.

2.6 The Cartan-Brauer Triangle

Sometimes it is useful to consider only the composition factors of a module, with multiplicity, or the number of copies of a special isomorphism class of a projective module occurring in a particular projective module. The abstract tool for this is the Grothendieck group, which we shall introduce now. The Cartan mapping associates to each projective indecomposable module its composition factors. Many interesting properties of algebras are encoded in the Cartan mapping.

2.6.1 Grothendieck Groups

Definition 2.6.1 Let S be a commutative Noetherian ring and let Γ be a Noetherian S-algebra. The *Grothendieck group* $G_0(\Gamma - mod)$ of $\Gamma - mod$ is the quotient of the free abelian group on the set of isomorphism classes $\{M\}$ of finitely generated Γ-modules M modulo the subgroup generated by the relations

$$\{M\} - \{N\} - \{L\}$$

whenever

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of A-modules. We denote by $[M]$ the image of an isomorphism class $\{M\}$ in $G_0(\Gamma - mod)$.

Proposition 2.6.2 *Let S be a commutative ring and let Γ be an artinian S-algebra. Then $G_0(\Gamma - mod)$ is a free abelian group with basis*

$$\{[L] \mid L \textit{ simple } \Gamma\textit{-module}\}.$$

Proof Since Γ is artinian, every finitely generated Γ-module has a composition series using Proposition 1.6.20 and Theorem 1.6.26. By Lemma 1.2.27 every indecomposable Γ-module M has a simple submodule L. Hence

$$0 \longrightarrow L \longrightarrow M \longrightarrow M/L \longrightarrow 0$$

is exact and by induction on the length of a composition series we get that

$$[M] = \sum_{L \in cf(M)} [L]$$

where $cf(M)$ is the set of composition factors of M. Hence

$$\mathcal{S}_\Gamma := \{[L] \mid L \text{ simple } \Gamma\text{-module}\}$$

is a generating set of $G_0(\Gamma - mod)$.

We need to show that $G_0(\Gamma - mod)$ is free over $\mathcal{S}_\Gamma$ as an abelian group. Let $\mathcal{S}S(\Gamma)$ be the free abelian group generated by the isomorphism classes of semisimple Γ-modules $\{T\}$ modulo the relations

$$\{T_1 \oplus T_2\} - \{T_1\} - \{T_2\}.$$

Denote by $[T]$ the image of the element $\{T\}$ in $\mathcal{S}S(\Gamma)$. It is clear by definition that $\mathcal{S}S(\Gamma)$ is a free abelian group with basis $\{[L] \mid L \text{ simple } \Gamma\text{-module}\}$.

We get a homomorphism of abelian groups

$$\mathcal{S}S(\Gamma) \xrightarrow{\sigma} G_0(\Gamma - mod)$$

via $\sigma([T]) := [T]$. Conversely define a homomorphism of abelian groups

$$G_0(\Gamma - mod) \xrightarrow{\rho} \mathcal{S}S(\Gamma)$$

by $\rho([M]) := \sum_{L \in cf(M)} [L]$ where $cf(M)$ is the set of composition factors of M. Then it is clear that this mapping is well-defined since composition series of a submodule U of M and the quotient module M/U give composition series of M. Moreover, σ is surjective since we have seen that $\mathcal{S}_\Gamma$ is in the image of σ and that this set is generating. Finally $\rho \circ \sigma = id_{\mathcal{S}S(\Gamma)}$ since the composition series of a semisimple module is readily established. Hence σ is injective as well. Therefore $G_0(\Gamma - mod)$ is free over $\mathcal{S}_\Gamma$ as an abelian group. □

Definition 2.6.3 Let S be a commutative ring and let Γ be a Noetherian S-algebra. Then $K_0(\Gamma)$ is the quotient of the free abelian group on the isomorphism classes $\{P\}$ of finitely generated projective Γ-modules P by the subgroup generated by the expressions $\{P_1 \oplus P_2\} - \{P_1\} - \{P_2\}$. Denote by $[P]$ the image of $\{P\}$ in $K_0(\Gamma)$.

2.6.2 Cartan and Decomposition Maps

There is an obvious homomorphism of abelian groups

$$\begin{aligned} K_0(\Gamma) &\xrightarrow{c} G_0(\Gamma - mod) \\ [P] &\mapsto [P] \end{aligned}$$

This map is well-defined since a direct sum of two projective modules gives rise to a short exact sequence.

Since Γ is artinian, by Proposition 1.9.6 each simple Γ-module S has a projective cover P_S and this projective cover is unique up to isomorphism. We may choose the set

$$\{[L] \mid L \text{ simple } \Gamma\text{-module}\}$$

as a $\mathbb{Z}$-basis for $G_0(\Gamma - mod)$ and the set

$$\{[P_L] \mid L \text{ simple } \Gamma\text{-module}\}$$

as a $\mathbb{Z}$-basis of $K_0(\Gamma)$. With respect to this basis the group homomorphism c is given by a matrix C_Γ with coefficients in $\mathbb{N}$.

Definition 2.6.4 Let S be a commutative ring and let Γ be an artinian S-algebra. Then the natural homomorphism of abelian groups

$$\begin{aligned} K_0(\Gamma) &\longrightarrow G_0(\Gamma - mod) \\ [P] &\mapsto [P] \end{aligned}$$

is the *Cartan mapping* c of Γ. The matrix C_Γ of c with respect to the basis formed by the isomorphism classes of simple modules and their projective covers respectively is called the *Cartan matrix*.

Let R be a complete discrete valuation ring, let $k = R/\mathrm{rad}(R)$ be its residue field and let $K = \mathrm{frac}(R)$ be its field of fractions. Let Λ be an R-order and let $A := k \otimes_R \Lambda$ be the finite dimensional residue algebra of Λ.

By Proposition 2.5.17 we know that every projective indecomposable A-module $\overline{P}$ is isomorphic to $k \otimes_R P$ for some projective indecomposable Λ-module P. Hence, in this case we get that $K_0(A) \simeq K_0(\Lambda)$ and the Cartan mapping is actually a mapping

$$K_0(\Lambda) \xrightarrow{c} G_0(k \otimes_R \Lambda - mod).$$

However, we get another homomorphism e of abelian groups

$$\begin{aligned} K_0(\Lambda) &\xrightarrow{e} K_0(K \otimes_R \Lambda) \\ [P] &\mapsto [K \otimes_R P] \end{aligned}$$

which is well-defined since

$$K \otimes_R (P \oplus Q) \simeq (K \otimes_R P) \oplus (K \otimes_R Q)$$

for any Λ-modules P and Q. We will finally define a group homomorphism

$$K_0(K \otimes_R \Lambda) \xrightarrow{d} G_0(k \otimes_R \Lambda - mod)$$

called the *decomposition map* in the following way:

For a finitely generated $K \otimes_R \Lambda$-module $\hat{M}$ choose an arbitrary K-basis $\{m_1, m_2, \ldots, m_t\}$ of $\hat{M}$. Then $M := \sum_{i=1}^{s} \Lambda \cdot m_i$ is a Λ-submodule of the $K \otimes_R \Lambda$-module $\hat{M}$, and therefore a Λ-lattice. Moreover, since $\{m_1, m_2, \ldots, m_t\}$ is a K-basis of $\hat{M}$, we get $K \otimes_R M = \hat{M}$. Of course $\hat{M}$ may contain many different such Λ-modules M.

Lemma 2.6.5 *Suppose R is a complete discrete valuation ring with residue field k, uniformiser π and field of fractions K. Suppose k is a splitting field for $k \otimes_R \Lambda$. Let $\hat{M}$ be a finitely generated $K \otimes_R \Lambda$-module and let M be a Λ-lattice with $K \otimes_R M = \hat{M}$. Then $[k \otimes_R M] \in G_0(k \otimes_R \Lambda - mod)$ does not depend on the choice of M in $\hat{M}$.*

Proof Put $\overline{M} := k \otimes_R M$ and $\overline{\Lambda} := k \otimes_R \Lambda$ to simplify the notation. Let $L_1, \ldots, L_n$ be representatives of the isomorphism classes of simple $\overline{\Lambda}$-modules.

We need to compute the multiplicity of a simple module L_i as a composition factor of $k \otimes_R M$. Let $\overline{P}_i$ be the projective cover of L_i as a $k \otimes_R \Lambda$-module, and let P_i be the projective cover of L_i as a Λ-module. Observe that Proposition 2.5.17 implies that the projective cover P_i exists and that $k \otimes_R P_i \simeq \overline{P}_i$.

If L_i is a composition factor of $\overline{M}$, then there is a submodule U of $\overline{M}$ such that L_i is in the socle of $\overline{M}/U$. Hence, there is a non-zero $\overline{\Lambda}$-module homomorphism $\overline{P}_i \longrightarrow \overline{M}/U$ and since $\overline{P}_i$ is projective, there is a non-zero $\overline{\Lambda}$-module homomorphism $\overline{P}_i \longrightarrow \overline{M}$. Conversely, any $\overline{\Lambda}$-linear homomorphism $\overline{P}_i \longrightarrow \overline{M}$ is induced by a composition factor of $\overline{M}$ isomorphic to L_i. The number of composition factors of $\overline{M}$ isomorphic to L_i is $\dim_k(Hom_{\overline{\Lambda}}(\overline{P}_i, \overline{M}))$ since k is a splitting field of $\overline{\Lambda}$. Hence

$$[\overline{M}] = \sum_{i=1}^{n} \left(\dim_k(Hom_{\overline{\Lambda}}(\overline{P}_i, \overline{M}))\right) \cdot [L_i].$$

But now,

$$\dim_k(Hom_{\overline{\Lambda}}(\overline{P}_i, \overline{M})) = \dim_k(Hom_{\Lambda}(P_i, \overline{M}))$$

since any homomorphism $P_i \longrightarrow \overline{M}$ factors through $P_i \longrightarrow \overline{P}_i$, the module $\overline{M}$ being annihilated by π. But, $M \longrightarrow \overline{M}$ is surjective and P_i is projective. Therefore

$$\dim_k(Hom_{\Lambda}(P_i, \overline{M})) = \mathrm{rank}_R(Hom_{\Lambda}(P_i, M))$$

since the projectivity of P_i implies that every homomorphism $P_i \longrightarrow \overline{M}$ factors through $M \longrightarrow \overline{M}$ and since M is a lattice, $Hom_{\Lambda}(P_i, M)$ is free as an R-module. But now,

$$\mathrm{rank}_R(Hom_{\Lambda}(P_i, M)) = \dim_K(Hom_{\Lambda}(K \otimes_R P_i, K \otimes_R M)).$$

Indeed, the tensor product $K \otimes_R -$ gives that the left-hand side is at most as big as the right-hand side, and given a morphism $\alpha \in Hom_{\Lambda}(K \otimes_R P_i, K \otimes_R M)$, using that M is finitely generated as an R-module, there is an integer $u(\alpha) \in \mathbb{Z}$ such that $\alpha(x) \in \pi^{u(\alpha)} M$ for all $x \in P_i$. Hence we get equality in the above equation. Therefore

$$[\overline{M}] = \sum_{i=1}^{n} \left(\dim_K(Hom_{K \otimes_R \Lambda}(K \otimes_R P_i, K \otimes_R M))\right) \cdot [L_i].$$

This expression is independent of the choice of M inside $K \otimes_R M$. □

Definition 2.6.6 Let R be a complete discrete valuation ring, let $k = R/\mathrm{rad}(R)$ be its residue field, let K be its field of fractions, and let Λ be an R-order. Suppose that k is a splitting field for $k \otimes_R \Lambda$. Let $\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_s$ be representatives of the isomorphism classes of the simple $K \otimes_R \Lambda$-modules and choose representatives $P_1, P_2, \ldots, P_n$ of the isomorphism classes of the indecomposable projective $k \otimes_R \Lambda$-modules. The numbers

$$d_{i,j} := \dim_K(Hom_{K \otimes_R \Lambda}(K \otimes_R P_i, \hat{V}_j))$$

are the *decomposition numbers* of Λ. The matrix $D := (d_{i,j})_{1\le i\le s;1\le j\le n}$ is the *decomposition matrix* of Λ.

2.6.3 The Cartan-Brauer Triangle

As in the previous section, R is a complete discrete valuation ring with field of fractions K and residue field k, Λ is an R-order and $A = k\otimes_R \Lambda$ is a finite dimensional k-algebra. Suppose that k is a splitting field for A. Recall that $e : K_0(\Lambda) \longrightarrow G_0(K\otimes_R \Lambda - mod)$ is given by tensoring with K over R. Now, if K is a splitting field for $\hat{\Lambda} := K \otimes_R \Lambda$, we obtain the following formula. For every $\hat{\Lambda}$-module $\hat{M}$ the multiplicity of the simple $\hat{\Lambda}$-module $\hat{V}_i$ as a direct summand of $\hat{M}$ is $\dim_K(Hom_{\hat{\Lambda}}(\hat{M}, \hat{V}_i))$. Hence

$$[\hat{M}] = \sum_{\ell=1}^{s} \left(\dim_K(Hom_{\hat{\Lambda}}(\hat{M}, \hat{V}_\ell)\right) [\hat{V}_\ell].$$

In particular, for $\hat{M} = K \otimes_R P_i$ one obtains the coefficients of the matrix associated to the $\mathbb{Z}$-linear mapping e with respect to the basis $\{[P_i] \mid i \in \{1, \ldots, n\}\}$ of $K_0(\Lambda)$ and $\{[\hat{V}_j] \mid j \in \{1, \ldots, s\}\}$ of $G_0(\hat{\Lambda} - mod)$. These are precisely the transposes of the coefficients of d. Therefore e is the transpose of d, i.e. $e = d^{tr}$.

Proposition 2.6.7 *Let R be a complete discrete valuation ring and let k be its residue field, and let K be its field of fractions. Let Λ be an R-order and suppose that k is a splitting field for $k \otimes_R \Lambda =: A$ and that K is a splitting field for $K \otimes_R \Lambda$. Then $C = D \cdot D^{tr}$, or more precisely the coefficients of the Cartan matrix of A, are $c_{i,j} = \sum_{\ell=1}^{s} d_{i,\ell} d_{\ell,j}$.*

Proof We shall show that c factorises as $c = d \circ e$. Consider

$$c : K_0(\Lambda) \longrightarrow G_0(\overline{\Lambda} - mod)$$

which maps $[P]$ to $[k \otimes_R P]$ inside $G_0(\overline{\Lambda} - mod)$. First, take

$$e : K_0(\Lambda) \longrightarrow G_0(\hat{\Lambda} - mod)$$

which is just given by the tensor product $[P] \mapsto [K \otimes_R P]$. Then d is given by taking a Λ-lattice L inside $K \otimes_R P$ and then considering $k \otimes_R L$ inside $G_0(\overline{\Lambda} - mod)$. Hence we can take $L = P$, since by Lemma 2.6.5 the result does not depend on this choice. This shows that the composition $d \circ e$ maps $[P]$ to $[k \otimes_R P]$ and this is precisely $c([P])$. $\square$

Graphically, Proposition 2.6.7 says that for an R-order Λ with K the field of fractions and k the residue field of the complete discrete valuation ring R the diagram

$$\begin{tikzcd} & G_0(K \otimes_R \Lambda - mod) \arrow[dr, "d"] & \\ K_0(\Lambda) \arrow[ur, "e"] \arrow[rr, "c"] & & G_0(k \otimes_R \Lambda - mod) \end{tikzcd}$$

is commutative if K and k are splitting fields for $K \otimes_R \Lambda$ and for $k \otimes_R \Lambda$ respectively.

Corollary 2.6.8 *Let R be a complete discrete valuation ring with field of fractions K and let k be its residue field. Let A be a finite-dimensional k-algebra, so that k is a splitting field for A. If there is an R-order Λ with $k \otimes_R \Lambda \simeq A$, and such that K is a splitting field of $K \otimes_R \Lambda$, then the Cartan matrix of A is symmetric.*

Proof This is immediate using Proposition 2.6.7 and the fact that

$$c^{tr} = \left(d \circ d^{tr}\right) = \left(d^{tr}\right)^{tr} \circ d^{tr} = d \circ d^{tr} = c. \qquad \square$$

Example 2.6.9 Let

$$A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$$

be the algebra of upper triangular matrices over a field k. The Cartan matrix of A is

$$C_A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

whence there is no order reducing to A.

More generally, for a finite dimensional quiver algebra $k\Gamma$ the Cartan matrix can be read off from the quiver directly. The coefficient (i, j) is the number of paths from i to j. Hence the Cartan matrix of $k\Gamma$ is never symmetric when $k\Gamma$ is finite dimensional.

Remark 2.6.10 Let Γ be an artinian K-algebra over a field K. Suppose K is a splitting field for Γ then we may define a $\mathbb{Z}$-bilinear mapping

$$\langle\ ,\ \rangle : K_0(\Gamma) \times G_0(\Gamma - mod) \longrightarrow \mathbb{Z}$$

by defining

$$\langle [P], [M] \rangle := \dim_K(Hom_\Gamma(P, M))$$

for each projective Γ-module P and each Γ-module M and extending this to all of $K_0(\Gamma)$ and $G_0(\Gamma - mod)$ by putting

$$\langle [P] - [Q], [M] - [N] \rangle := \langle [P], [M] \rangle - \langle [P], [N] \rangle - \langle [Q], [M] \rangle + \langle [Q], [N] \rangle$$

for all projective Γ-modules P and Q and all Γ-modules M and N. We need to see that this is actually well-defined. Indeed, since

$$Hom_\Gamma(P_1 \oplus P_2, M) \simeq Hom_\Gamma(P_1, M) \oplus Hom_\Gamma(P_2, M)$$

the form is well-defined on the first variable. For projective Γ-modules P we have for each exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of Γ-modules an exact sequence of K-modules

$$0 \longrightarrow Hom_\Gamma(P, L) \longrightarrow Hom_\Gamma(P, M) \longrightarrow Hom_\Gamma(P, N) \longrightarrow 0.$$

Therefore the bilinear form is well-defined on the second variable as well.

Now, we have a basis for $K_0(\Gamma)$

$$\{[P_L] \mid L \text{ simple } \Gamma\text{-module}\}$$

and a basis for $G_0(\Gamma - mod)$

$$\{[L] \mid L \text{ simple } \Gamma\text{-module}\}.$$

Since $Hom_\Gamma(P_L, L) = K$ and $Hom_\Gamma(P_L, L') = 0$ if L' is a simple module not isomorphic to L, we see that the two bases are dual to each other with respect to the bilinear form $\langle\ ,\ \rangle$. The form is non-degenerate.

2.7 Defining Blocks by Non-vanishing Ext^1 Between Simple Modules

In this section we shall give a characterisation of blocks, and more generally of indecomposable direct factors of finite dimensional k-algebras over a field k. This characterisation is useful in certain technical considerations and gives another illuminating view on the indecomposable factors of an algebra.

Let A be a finite dimensional k-algebra and let

$$A = A_1 \times \cdots \times A_n$$

be a decomposition of A into indecomposable direct factors. Equivalently we decompose $1 \in Z(A)$ into primitive central idempotents

$$1 = e_1 + \cdots + e_n$$

and we know by Proposition 1.9.14 that this decomposition is unique.

Analogously to the case of a block of a group ring, given in Definition 2.3.2, we say that an indecomposable A-module belongs to the block A_i if M is an A_i-module. It is clear that there is a unique A_i such that M belongs to A_i. Indeed,

$$M = e_1 M \oplus \cdots \oplus e_n M$$

as A-modules since $e_i \in Z(A)$ for all $i \in \{1, \ldots, e_n\}$ and therefore, since M is indecomposable, there is a unique e_i with $M = e_i M$. Hence $A_j \cdot M = 0$ for all $j \neq i$, where $A_i := A \cdot e_i$.

Lemma 2.7.1 *Let M and N be two indecomposable A-modules and suppose $Ext^1_A(M, N) \neq 0$. Then M and N belong to the same block A_i.*

Proof Let

$$0 \longrightarrow N \xrightarrow{\nu} X \xrightarrow{\mu} M \longrightarrow 0$$

be a non-split short exact sequence. Suppose M belongs to A_i and suppose N belongs to A_j where $i \neq j$. Then

$$0 \longrightarrow e_i N \longrightarrow e_i X \longrightarrow e_i M \longrightarrow 0$$

and

$$0 \longrightarrow e_j N \longrightarrow e_j X \longrightarrow e_j M \longrightarrow 0$$

are short exact sequences of A-modules. Moreover

$$\mu(e_i x) = e_i \mu(x) = \mu(x)$$

for all $x \in X$ and likewise

$$\nu(y) = \nu(e_j y) = e_j \nu(y)$$

for all $y \in N$. Hence, since $X = e_i X \oplus (1 - e_i) X$, we see that ν is an isomorphism $N \simeq (1 - e_i) X$ and μ is an isomorphism $e_i X \simeq M$. Therefore the sequence

$$0 \longrightarrow N \xrightarrow{\nu} X \xrightarrow{\mu} M \longrightarrow 0$$

splits, contrary to our assumption. $\square$

Definition 2.7.2 Let k be a field and let A be a finite dimensional k-algebra. Two simple A-modules S and T are in the same *block-class* if there is a sequence of simple A-

modules $S_1, S_2, \dots, S_n$ such that $S_1 = S$ and $S_n = T$ and such that $Ext^1_A(S_i, S_{i+1}) \neq 0$ or $Ext^1_A(S_{i+1}, S_i) \neq 0$ for all $i \in \{1, 2, \dots, n-1\}$.

Remark 2.7.3 Curtis-Reiner [13] use the notion of a "linkage class" for this concept. Lemma 2.7.1 shows that if two simple modules S and T belong to the same block class then S and T belong to the same block.

We come to the main result of this section.

Proposition 2.7.4 *Let k be a field and let A be a finite dimensional k-algebra. Then two simple A-modules S and T belong to the same block of A if and only if they belong to the same block class of A.*

Proof Remark 2.7.3 and Proposition 2.7.1 show that two simple modules in the same block class belong to the same block.

Suppose S and T belong to the same block. We need to show that there is a sequence $S = S_1, S_2, \dots, S_n = T$, such that $Ext^1_A(S_i, S_{i+1}) \neq 0$ or $Ext^1_A(S_{i+1}, S_i) \neq 0$ for all $i \in \{1, 2, \dots, n-1\}$. For every simple A-module L we choose its projective cover P_L and obtain $L \simeq P_L/\mathrm{rad}(P_L)$. From the exact sequence

$$0 \longrightarrow \mathrm{rad}(P_L) \longrightarrow P_L \longrightarrow L \longrightarrow 0$$

we obtain an exact sequence

$$Hom_A(L, M) \hookrightarrow Hom_A(P_L, M) \to Hom_A(\mathrm{rad}(P_L), M) \twoheadrightarrow Ext^1_A(L, M)$$

using Definition 1.8.19 and Lemma 1.8.8 since P_L is projective. If M is a simple module non-isomorphic to L, then $Hom_A(P_L, M) = 0$ and we get

$$Hom_A(\mathrm{rad}(P_L), M) \simeq Ext^1_A(L, M).$$

Since M was assumed to be simple,

$$Hom_A(\mathrm{rad}(P_L), M) \simeq Hom_A(\mathrm{rad}(P_L)/\mathrm{rad}^2(P_L), M).$$

But, $\mathrm{rad}(P_L)/\mathrm{rad}^2(P_L)$ is semisimple and we get that $Ext^1_A(L, M) \neq 0$ if and only if M is a direct summand of $\mathrm{rad}(P_L)/\mathrm{rad}^2(P_L)$.

Definition 2.7.5 We associate a quiver to a finite dimensional algebra A, called the *Ext-quiver*, as follows. The vertices are the isomorphism classes $\{L\}$ of simple A-modules L. For every simple A-module L and projective cover P_L decompose

$$\mathrm{rad}(P_L)/\mathrm{rad}^2(P_L) = L_1 \oplus \cdots \oplus L_n$$

and draw arrows $\{L\} \longrightarrow \{L_1\}, \{L\} \longrightarrow \{L_2\}, \dots, \{L\} \longrightarrow \{L_n\}$.

The statement of Proposition 2.7.4 is that S and T belong to the same block if and only if the vertices $\{S\}$ and $\{T\}$ belong to the same connected component of the Ext-quiver Γ_A associated to A. Let Γ_S be the connected component of Γ_A containing $\{S\}$ and let Γ_T be the connected component of Γ_A containing $\{T\}$.

Let

$$P_S := \bigoplus_{L\in\Gamma_S} P_L^{n_L}$$

and

$$P_{\not S} := \bigoplus_{L\notin\Gamma_S} P_L^{n_L}$$

where $P_L^{n_L}$ is a direct factor of the regular A-module, but $P_L^{n_L+1}$ is not a direct factor of the regular module. Suppose

$$\Gamma_T \subseteq \Gamma_{\not S}.$$

Then

$$P_S \oplus P_{\not S} \simeq A$$

and

$$\begin{pmatrix} End_A(P_S) & Hom_A(P_{\not S}, P_S) \\ Hom_A(P_S, P_{\not S}) & End_A(P_{\not S}) \end{pmatrix} = End_A(P_S \oplus P_{\not S}) = End_A(A) = A^{op}.$$

We shall show that the hypothesis implies

$$Hom_A(P_{\not S}, P_S) = 0 = Hom_A(P_S, P_{\not S}).$$

Suppose for a moment we have shown this. Then

$$A = End_A(P_S)^{op} \times End_A(P_{\not S})^{op}$$

is decomposable and S belongs to $End_A(P_S)$ whereas T belongs to $End_A(P_{\not S})$.

Suppose $Hom_A(P_S, P_{\not S}) \neq 0$. Then there is a projective indecomposable P_L, being a direct summand of P_S, and a projective indecomposable P_M, being a direct summand of $P_{\not S}$ such that $Hom_A(P_M, P_L) \neq 0$. This means that M is a direct factor of $\mathrm{rad}^i(P_L)/\mathrm{rad}^{i+1}(P_L)$ for some $i \geq 1$. We may choose L and M so that i is minimal possible. We claim that in this case $i = 1$. Indeed

$$\mathrm{rad}^i(P_L)/\mathrm{rad}^{i+1}(P_L) \hookrightarrow \mathrm{rad}^{i-1}(P_L)/\mathrm{rad}^{i+1}(P_L) \twoheadrightarrow \mathrm{rad}^{i-1}(P_L)/\mathrm{rad}^i(P_L)$$

is an exact non split sequence and M is a direct factor of $\mathrm{rad}^i(P_L)/\mathrm{rad}^{i+1}(P_L)$. Hence $M \oplus U = \mathrm{rad}^i(P_L)/\mathrm{rad}^{i+1}(P_L)$. We get a short exact sequence

$$0 \to M \to \mathrm{rad}^{i-1}(P_L)/(\mathrm{rad}^{i+1}(P_L) + U) \to \mathrm{rad}^{i-1}(P_L)/\mathrm{rad}^i(P_L) \to 0$$

and claim that this sequence is non-split. If not, M would be direct factor of the quotient $\mathrm{rad}^{i-1}(P_L)/(\mathrm{rad}^{i+1}(P_L)+U)$. But then M would be a direct factor of $\mathrm{rad}^{i-1}(P_L)/\mathrm{rad}^{i}(P_L)$ since M is simple and since the radical is the intersection of the kernels of all homomorphisms to simple modules. This contradicts the minimality of i. Hence

$$0 \to M \to \mathrm{rad}^{i-1}(P_L)/(\mathrm{rad}^{i+1}(P_L)+U) \to \mathrm{rad}^{i-1}(P_L)/\mathrm{rad}^{i}(P_L) \to 0$$

is non-split. We decompose

$$\mathrm{rad}^{i-1}(P_L)/\mathrm{rad}^{i}(P_L) = T_1 \oplus \cdots \oplus T_m$$

into a direct sum of simple modules. Then taking pre-images of T_j inside $\mathrm{rad}^{i-1}(P_L)$ $(\mathrm{rad}^{i+1}(P_L)+U)$ gives exact sequences

$$0 \to M \to V_j \to T_j \to 0$$

for all $j \in \{1, \ldots, m\}$. If all these sequences split by morphisms

$$\sigma_j : T_j \longrightarrow V_j \subseteq \mathrm{rad}^{i-1}(P_L)/(\mathrm{rad}^{i+1}(P_L)+U),$$

then the sequence

$$0 \to M \to \mathrm{rad}^{i-1}(P_L)/(\mathrm{rad}^{i+1}(P_L)+U) \to \mathrm{rad}^{i-1}(P_L)/\mathrm{rad}^{i}(P_L) \to 0$$

splits by the morphism $\sum_{j=1}^{m} \sigma_j$. Hence there is a $j_0 \in \{1, \ldots, m\}$ such that

$$0 \to M \to V_j \to T_{j_0} \to 0$$

is non-split. However $\{T_{j_0}\}$ is in Γ_S by minimality of i and $\{M\}$ is in $\Gamma_{\mathcal{S}}$.

The case $Hom_A(P_{\mathcal{S}}, P_S) \neq 0$ gives a non-split exact sequence as above by symmetry. This completes the proof of Proposition 2.7.4. $\square$

Example 2.7.6 In the proof of Proposition 2.7.4 we used a very important concept, the concept of an *Ext*-quiver of an algebra. We shall give some examples.

1. Let $A = kG$ where k is a field of characteristic $p > 0$ and G is a finite p-group. Then the *Ext*-quiver of A is a single vertex with n loops, where n is the p-rank of the Frattini quotient $G/\Phi(G)$ of G. The Frattini quotient of G is the largest elementary abelian quotient of the p-group G. Equivalently, the preimages of the generators of the Frattini quotient form a minimal generating set of G.
2. The *Ext*-quiver of the algebra of upper triangular $n \times n$ matrices over k is

$$\{1\} \longrightarrow \{2\} \longrightarrow \ldots \longrightarrow \{n\}.$$

We see that in the second case the path algebra given by this *Ext*-quiver is isomorphic to the algebra of upper triangular matrices. This is not an accident, as we shall see later. For a precise statement we need the concept of a Morita equivalence given in Chap. 4.

2.8 The Structure of Serial Symmetric Algebras

There are numerous applications of the results of Sect. 2.7. We shall give one application here, which will be of some importance later, but which is also interesting in its own right.

Recall from Remark 1.6.31 that an A-module is uniserial if it has only one composition series and that a ring is serial if each indecomposable projective A-module is uniserial.

We shall study the special case of a symmetric serial algebra. This case will be useful later, but it also gives an example of the power of the structure we have already obtain at this stage. We shall follow here Linckelmann's thesis [14].

Lemma 2.8.1 *Let A be a finite dimensional k-algebra. Then every projective indecomposable A-module is uniserial if and only if every indecomposable module is uniserial. Moreover, each indecomposable A-module M is uniserial if and only if for each indecomposable A-module M we have* $\mathrm{rad}^i(M)/\mathrm{rad}^{i+1}(M)$ *is either simple or* 0*, for all i.*

Proof Let M be an indecomposable A-module. We claim that each of the modules $\mathrm{rad}^i(M)/\mathrm{rad}^{i+1}(M)$ is simple or 0 for each i. Indeed, let P_M be its projective cover. If P_M is indecomposable, then we are done. Indeed, since M is a quotient of P_M, if M admits two different decomposition series, P_M also admits two different decomposition series. But, if $\mathrm{rad}^i(M)/\mathrm{rad}^{i+1}(M) = S \oplus T$ for a simple S and a non-zero semisimple module T, and i is minimal with this property, then

$$M \supseteq \mathrm{rad}(M) \supseteq \cdots \supseteq \mathrm{rad}^i(M) \supseteq \mathrm{rad}^{i+1}(M) + S \supseteq \mathrm{rad}^{i+1}(M) \supseteq \ldots$$

and

$$M \supseteq \mathrm{rad}(M) \supseteq \cdots \supseteq \mathrm{rad}^i(M) \supseteq \mathrm{rad}^{i+1}(M) + T \supseteq \mathrm{rad}^{i+1}(M) \supseteq \ldots$$

can be completed to two different composition series, refining T into a direct sum of simple modules. But this shows that the radical series of M is the only composition series of M.

So, assuming P_M is decomposable, we proceed by induction on the number of indecomposable direct summands. Let $P_M = P_0 \oplus P_1$ with P_0 indecomposable. Let

$$\begin{array}{ccc} P_M & \stackrel{\pi_M}{\longrightarrow} & M \\ \downarrow \alpha & & \downarrow \beta \\ P_0 & \stackrel{\pi_0}{\longrightarrow} & M_0 \end{array}$$

be a pushout diagram, where α is the natural projection. Since α and π_M are epimorphisms, β and π_0 are also epimorphisms (cf Lemma 1.8.27). Since P_0 is projective, π_0 lifts to a morphism $\sigma : P_0 \longrightarrow M$ such that $\beta \circ \sigma = \pi_0$. Since π_M is an epimorphism there is a $\tau : P_0 \longrightarrow P_M$ with $\pi_M \circ \tau = \sigma$.

But P_0 is uniserial and therefore $\ker(\pi_0) = \mathrm{rad}^{n_0}(P_0)$ for some $n_0 \in \mathbb{N}$. We may choose P_0 so that n_0 is maximal. Hence τ induces a homomorphism $M_0 \stackrel{\check{\gamma}}{\longrightarrow} P_M/\mathrm{rad}^{n_0}P_M$. Since n_0 is maximal there is an epimorphism $P_M/\mathrm{rad}^{n_0}P_M \longrightarrow M$ so that τ induces $\gamma : M_0 \longrightarrow M$ with $\gamma \circ \pi_0 = \pi_M \circ \tau$.

But this shows

$$\beta \circ \gamma \circ \pi_0 = \beta \circ \pi_M \circ \tau = \beta \circ \sigma = \pi_0.$$

Now, π_0 is an epimorphism, and therefore $\beta \circ \gamma = id_{M_0}$. Therefore $M \simeq M_0 \oplus M'$ where $M' = \ker \beta$. We have proved the statement since by the induction hypothesis, M' is a direct sum of uniserial modules. □

Lemma 2.8.2 *Let k be a field and let A be a finite dimensional, indecomposable, self-injective k-algebra. Let $S_1, \ldots, S_n$ be representatives of the isomorphism classes of simple A-modules, and let P_i be the projective cover of S_i for each i. Then the following are equivalent.*

1. *A is serial.*
2. *Each indecomposable A-module is uniserial.*
3. *For each indecomposable A-module M the module* $\mathrm{rad}^i(M)/\mathrm{rad}^{i+1}(M)$ *is either simple or* 0, *for all i.*
4. *There is an element σ of the symmetric group $\mathfrak{S}_n$ such that*

$$\mathrm{rad}(P_i)/\mathrm{rad}^2(P_i) \simeq S_{\sigma(i)}.$$

In this case, and if A is symmetric, σ is a transitive cycle of length n; i.e. we can renumber the simple modules so that $\sigma = (1\ 2\ \ldots\ n)$. Moreover, the Loewy structure of the projective indecomposable A-modules is the Loewy structure of the Nakayama algebra $N_n^{e\cdot n+1}$ over K with n simple modules and

$$\mathrm{rad}^{e\cdot n+1}(A) = 0 \neq \mathrm{rad}^{e\cdot n}(A).$$

Proof The equivalence of the first three items has already been shown in Lemma 2.8.1. The fact that 4 implies 3 is trivial. Suppose now 3. Then $\mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)$ is simple and we define σ by the property

$$\mathrm{rad}(P_i)/\mathrm{rad}^2(P_i) \simeq S_{\sigma(i)}.$$

Then $P_{\sigma(i)}$ is the projective cover of $\mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)$, and therefore also of $\mathrm{rad}(P_i)$. But this implies that $\mathrm{rad}(P_{\sigma(i)})$ maps surjectively to $\mathrm{rad}(\mathrm{rad}(P_i)) = \mathrm{rad}^2(P_i)$ and therefore $P_{\sigma^2(i)}$ is the projective cover of $\mathrm{rad}^2(P_i)$. Recursively, $P_{\sigma^n(i)}$ is the projective cover of $\mathrm{rad}^n(P_i)$. Since A is self-injective, using Remark 1.10.32, there is an n such that $\mathrm{rad}^n(P_i) = \mathrm{soc}(P_i) = S_{\nu(i)}$, where ν is the Nakayama permutation. Since $\nu \in \mathfrak{S}_n$, also $\sigma \in \mathfrak{S}_n$.

Suppose that A is symmetric. Hence $\nu = id$. The composition factors of P_i are therefore formed by the orbit of i under σ. Furthermore, the composition factors of $P_{\sigma^j(i)}$ are in the same σ-orbit of i. Now, Proposition 2.7.4 implies that A is indecomposable if and only if σ is a cycle of length n in $\mathfrak{S}$.

We can now see that the Loewy structure of A and of the projective indecomposable A-modules is actually the Loewy structure of the symmetric Nakayama algebra $N_n^{e \cdot n+1}$ and its projective indecomposable modules. The only remaining possibility is that the radical length of P_i is not $e \cdot n+1$ where e is independent from i. Assuming this is the case, then suppose the radical length of P_i is $n \cdot f + 1$ and the radical length of $P_{\sigma^{-1}(i)}$ is $n \cdot e+1$ for $e > f$. Then P_i is a submodule of $P_{\sigma^{-1}(i)}/\mathrm{soc}(P_{\sigma^{-1}(i)})$. Since A is self-injective, this would imply that $P_{\sigma^{-1}(i)}$ is not indecomposable, a contradiction. □

Remark 2.8.3 We have seen that all indecomposable modules are quotients of projective indecomposable modules. Moreover, this property also holds for N_n^{en+1}. Hence, we get a bijection between the indecomposable modules of N_n^{en+1} and the indecomposable A-modules. This bijection preserves homomorphism spaces in the sense that $Hom_A(M_1, M_2) \simeq Hom_{N_n^{en+1}}(\beta M_1, \beta M_2)$ and the isomorphism is compatible with composition of morphisms. This is a particular case of a correspondence studied in Chap. 4, namely a Morita equivalence. In other words, A is Morita equivalent to N_n^{en+1} in the notation which will be introduced there.

Theorem 2.8.4 (cf e.g. Linckelmann [14, Théorème 2.1]) *Let k be a field and let A be a finite dimensional symmetric k-algebra. Then the following statements are equivalent.*

1. *There is a $t \in A$ such that* $\mathrm{rad}(A) = A \cdot t$ *or* $\mathrm{rad}(A) = t \cdot A$.
2. *The A-modules* $A/\mathrm{rad}(A)$ *and* $\mathrm{rad}(A)/\mathrm{rad}^2(A)$ *are isomorphic.*
3. *A is serial and all indecomposable projective A-modules appear with the same multiplicity as direct factors of A.*

If one of the conditions in 1 or 2 or 3 is satisfied, we obtain

$$t \cdot A = \mathrm{rad}(A) = A \cdot t.$$

Proof Let $\langle\ ,\ \rangle : A \times A \longrightarrow k$ be a symmetrising form on A, and let $M^{\perp} := \{a \in A \mid \langle m, a\rangle = 0\ \forall m \in M\}$ for each subset M of A. Suppose that $\mathrm{rad}(A) = A \cdot t$ and fix $a \in A$. If $t \cdot a = 0$, then

$$0 = A \cdot t \cdot a = \mathrm{rad}(A) \cdot a$$

and since $\langle m, a\rangle = \langle ma, 1\rangle$ we get $a \in \mathrm{rad}(A)^\perp$. Since A is symmetric, $\langle a, b\rangle = \langle b, a\rangle$ for all $b \in \mathrm{rad}(A)$. Moreover, $\mathrm{rad}(A)$ is a two-sided ideal and hence $t \cdot A \subseteq \mathrm{rad}(A) = A \cdot t$. Hence

$$\begin{aligned}\langle m, a\rangle = 0\ \forall m \in \mathrm{rad}(A) \Leftrightarrow 0 &= \langle a, m\rangle\ \forall m \in \mathrm{rad}(A)\\ \Rightarrow 0 &= \langle a, tb\rangle = \langle at, b\rangle\ \forall b \in A,\end{aligned}$$

and this is possible only if $a \cdot t = 0$ since $\langle\ ,\ \rangle$ is non-degenerate. We obtain that the map

$$\begin{aligned} A \cdot t &\longrightarrow t \cdot A\\ a \cdot t &\mapsto t \cdot a\end{aligned}$$

is well-defined and injective. This gives $\dim_k(t \cdot A) \geq \dim_k(A \cdot t)$. Since $t \cdot A \subseteq \mathrm{rad}(A) = A \cdot t$ we get

$$t \cdot A = A \cdot t = \mathrm{rad}(A).$$

This immediately shows that

$$\mathrm{rad}^s(A) = A \cdot t^s = t^s \cdot A.$$

Hence condition 1 implies that $\mathrm{rad}(A) = A \cdot t = t \cdot A$.

We shall show that condition 1 implies that multiplication by t from the right gives an isomorphism $A/\mathrm{rad}(A) \simeq \mathrm{rad}(A)/\mathrm{rad}^2(A)$. Indeed, since $A \cdot t^2 = \mathrm{rad}^2(A)$, the mapping

$$\begin{aligned} A/\mathrm{rad}(A) &\xrightarrow{\tau} \mathrm{rad}(A)/\mathrm{rad}^2(A)\\ a + \mathrm{rad}(A) &\mapsto a \cdot t + \mathrm{rad}^2(A)\end{aligned}$$

is a well-defined A-module homomorphism. Moreover, $a \cdot t \in \mathrm{rad}^2(A) = A \cdot t^2$ implies $a \in A \cdot t = \mathrm{rad}(A)$, whence τ is injective. Since $\mathrm{rad}(A) = A \cdot t$, it is clear that τ is surjective as well. This proves condition 2.

Assume that there is an A-module isomorphism

$$A/\mathrm{rad}(A) \xrightarrow{\tau} \mathrm{rad}(A)/\mathrm{rad}^2(A).$$

Let $S_1, S_2, \ldots, S_n$ be representatives of the isomorphism classes of the simple A-modules, and let P_i be the projective cover of S_i for each i. Iterating τ shows that τ^n gives an isomorphism of A-modules

$$A/\mathrm{rad}(A) \xrightarrow{\tau^n} \mathrm{rad}^n(A)/\mathrm{rad}^{1+n}(A).$$

Composing τ with the natural projection $A \longrightarrow A/\mathrm{rad}(A)$ we get a morphism $A \longrightarrow \mathrm{rad}(A)/\mathrm{rad}^2(A) \subseteq A/\mathrm{rad}^2(A)$. Since A maps surjectively to $A/\mathrm{rad}^2(A)$, the fact that A is projective gives the existence of a morphism $\hat{\tau} : A \longrightarrow A$ fitting into a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\hat{\tau}} & A \\ \downarrow & & \downarrow \\ A/\mathrm{rad}(A) & \xrightarrow{\tau} & A/\mathrm{rad}^2(A) \end{array} .$$

But, since τ has image $\mathrm{rad}(A)/\mathrm{rad}^2(A)$, we have $\hat{\tau}(A) = \mathrm{rad}(A)$. Therefore we obtain a commutative diagram

$$\begin{array}{ccc} A & \overset{\hat{\tau}}{\twoheadrightarrow} & \mathrm{rad}(A) \\ \downarrow & & \downarrow \\ A/\mathrm{rad}(A) & \overset{\tau}{\twoheadrightarrow} & \mathrm{rad}(A)/\mathrm{rad}^2(A) \end{array} .$$

Since $End_A(A) = A^{op}$, we obtain that τ is multiplication by t on the right. We restrict to an indecomposable projective direct summand $P_i = Ae_i$ of A, for some $e_i^2 = e_i \in A$. Then

$$e_i^2 \cdot t = e_i \cdot t = t \cdot f = t \cdot f^2$$

for some $f \in A$. Hence $f - f^2$ is annihilated by t. By the lifting of idempotents theorem Proposition 1.9.17 we may choose f so that $f^2 = f$. Moreover, using right modules instead, the symmetry of the situation implies that f is a primitive idempotent $e_{\sigma(i)}$. We obtain that each simple A-module S_j occurs as a direct factor of some $P_{\sigma^{-1}(j)}/\mathrm{rad}(P_{\sigma^{-1}(j)})$ with multiplicity 1. Hence there is a permutation $\sigma \in \mathfrak{S}_n$ so that

$$P_i/\mathrm{rad}(P_i) \simeq S_{\sigma(i)}.$$

Lemma 2.8.2 shows that A is serial. Now, let

$$A/\mathrm{rad}(A) = \bigoplus_{i=1}^{n} S_i^{n_i}.$$

Then multiplication by t maps S_i to $S_{\sigma(i)}$ and hence

$$\mathrm{rad}(A)/\mathrm{rad}^2(A) = \bigoplus_{i=1}^{n} S_{\sigma(i)}^{n_i} \simeq \bigoplus_{i=1}^{n} S_i^{n_i} \simeq A/\mathrm{rad}(A).$$

Since σ is a cycle of length n, $n_i = n_j$ for all i, j and therefore all simple A-modules occur with the same multiplicity in $A/\mathrm{rad}(A)$. We have shown that condition 2 implies condition 3.

Now assume condition 3 Lemma 2.8.2 shows that each simple A-module occurs with the same multiplicity in $A/\mathrm{rad}(A)$ as in $\mathrm{rad}(A)/\mathrm{rad}^2(A)$. This gives an isomorphism $A/\mathrm{rad}(A) \simeq \mathrm{rad}(A)/\mathrm{rad}^2(A)$. By the proof of 2 implies 3 this shows that $\mathrm{rad}(A) = A \cdot t$ and we obtain condition 1. This proves the theorem. □

2.9 Külshammer Ideals

Let G be a finite group and let k be a field of characteristic $p > 0$. Theorem 1.5.4 is a simple formula for the number of simple KG modules when K is a splitting field for G of characteristic 0. A similar formula is desirable for the case when the field k is of characteristic $p > 0$ and is a splitting field for G.

2.9.1 Definitions of Külshammer Spaces

The route we take follows the original considerations of Brauer and the refinement proposed by Külshammer.

We start with the following simple observation. In Sect. 2.9.1 and occasionally in the subsequent sections we denote our base field by K, since we need k for our index sets. Index considerations occur frequently in this section.

Definition 2.9.1 For any field K and any K-algebra A we define the *commutator subspace* $[A, A]$ to be the K-subspace generated by the elements $\{ab - ba \mid a, b \in A\}$. We define $HH_0(A) := A/[A, A]$, the commutator quotient.

We mention that the set of commutators $\{ab - ba \mid a, b \in A\}$ is not in general a vector space. Moreover, a very important remark is that $[A, A]$ is *not* an ideal of A. Hence $A/[A, A]$ is not a ring, and is *not* an A-module.

Lemma 2.9.2 *Let K be a field.*

- *Let $A = Mat_{n\times n}(K)$ be the matrix ring over K. Then $[A, A]$ is of codimension 1 in A.*
- *For all K-algebras A_1 and A_2 we get*

$$[A_1 \times A_2, A_1 \times A_2] = [A_1, A_1] \times [A_2, A_2]$$

and hence

$$HH_0(A_1 \times A_2) = HH_0(A_1) \times HH_0(A_2).$$

Proof For the first statement we use the usual matrix trace $trace : Mat_{n\times n}(K) \to K$ which is easily seen to be K-linear and observe that

$$\begin{aligned} trace(MN - NM) &= trace(MN) - trace(NM) \\ &= trace(MN) - trace(MN) = 0. \end{aligned}$$

Hence

$$[A, A] \subseteq \{M \in Mat_{n\times n}(K) \mid trace(M) = 0\} = \ker(trace)$$

which is of codimension 1 in A. Now, let $E_{i,j} = (e_{k,\ell})_{1\le k,\ell\le n}$ be the matrix with $e_{k,\ell} = \delta_{i,k}\delta_{j,\ell}$; where as usual we denote by $\delta_{x,y}$ the Kronecker delta which is 1 if $x = y$ and 0 otherwise. Then if $i \neq j$ we have that $E_{i,j}E_{j,i} - E_{j,i}E_{i,j}$ is the matrix with diagonal entry 1 in position (i, i), diagonal entry -1 in position (j, j) and 0 elsewhere. Moreover $E_{k,j}E_{j,i} - E_{j,i}E_{k,j}$ for $k \neq i$ is the matrix with coefficient 1 in position (k, i) and 0 elsewhere. It is obvious that these matrices span the set of matrices with trace 0. Since all these matrices are commutators we have proved the first statement.

For the second statement we just observe that elements in A_1 commute with elements in A_2. This proves the lemma. □

Recall from Definition 1.9.17 the notion of a free algebra.

Lemma 2.9.3 *Let K be a field of characteristic $p > 0$ and let A be a K-algebra. Then the $\mathbb{F}_p$-linear mapping*

$$\begin{aligned} A &\xrightarrow{\mu_p} A \\ a &\mapsto a^p \end{aligned}$$

extends to an $\mathbb{F}_p$-linear mapping

$$\begin{aligned} HH_0(A) &\xrightarrow{\mu_p} HH_0(A) \\ a + [A, A] &\mapsto a^p + [A, A]. \end{aligned}$$

Proof We need to show that if $a \in A$ and $b \in [A, A]$, then

$$(a + b)^p - a^p \in [A, A].$$

In particular we need to show that $b^p \in [A, A]$. We consider the element $(x + y)^p$ in the free K-algebra F_2 in two variables x and y. The cyclic group $C_p = \langle c\rangle$ acts on the set of products of p factors by cyclic permutation of the factors. Now, since p is prime, orbits are of length 1 or of length p. Moreover, $z - cz \in [F_2, F_2]$ since if $z = z_1z_2 \dots z_p$, then $cz = z_2 \dots z_pz_1$ and

$$z - cz = z_1z_2 \dots z_p - z_2 \dots z_pz_1 = [z_1, z_2 \dots z_p] \in [F_2, F_2].$$

Since the only orbits of length 1 are those of x^p or y^p the above implies that

$$(x+y)^p - x^p - y^p \in [F_2, F_2].$$

Therefore for all $a, b \in A$ we get

$$(ab - ba)^p - (ab)^p - (ba)^p \in [A, A]$$

and hence $\mu_p([A, A]) \subseteq [A, A]$. Moreover if $a \in A$ and $b \in [A, A]$, then

$$(a+b)^p - a^p - b^p \in [A, A]$$

and since $b^p \in [A, A]$ we obtain the statement. □

Definition 2.9.4 Let K be a field of characteristic $p > 0$ and let A be a K-algebra. Then we define the *Külshammer spaces*

$$T_n(A) := \{a \in A \mid a^{p^n} \in [A, A]\}$$

and

$$T(A) := \bigcup_{n=1}^{\infty} T_n(A).$$

By Lemma 2.9.3 we get that

$$[A, A] \subseteq T_n(A).$$

Observe that $T_n(A)$ is a K-subspace of A. Of course $T_0(A) = [A, A]$ and $T_n(A) \subseteq T_{n+1}(A)$ for all $n \in \mathbb{N}$. Moreover we recall from Proposition 1.6.18 that for artinian algebras the Jacobson radical is the largest nilpotent ideal of A and hence we get for an artinian algebra A that

$$[A, A] + \mathrm{rad}(A) \subseteq T(A).$$

Proposition 2.9.5 *Let K be a field of characteristic $p > 0$ and let A be an artinian K-algebra. Then $T_n(A)$ is a K-subspace of A and*

$$[A, A] = T_0(A) \subseteq T_1(A) \subseteq T_2(A) \subseteq \ldots T(A) = [A, A] + \mathrm{rad}(A)$$

is an increasing sequence of K-subspaces of A. In particular

$$T(A/\mathrm{rad}(A)) = [A/\mathrm{rad}(A), A/\mathrm{rad}(A)].$$

Proof Almost all of the statements have been proved in the discussion preceding the proposition. The only missing part is the statement that $T(A) = [A, A] + \mathrm{rad}(A)$. In order to prove this, we first observe that

$$T(A/\mathrm{rad}(A)) = T(A)/\mathrm{rad}(A).$$

Indeed, the ring homomorphism

$$\rho : A \longrightarrow A/\mathrm{rad}(A)$$

is compatible with the p-power mapping μ_p. Hence

$$\mu_p(a + \mathrm{rad}(A)) = \mu_p(a) + \mathrm{rad}(A).$$

Moreover ρ is a ring homomorphism and hence

$$\rho([A, A]) = [A/\mathrm{rad}(A), A/\mathrm{rad}(A)].$$

This gives

$$a^{p^n} \in [A, A] \Rightarrow (a + [A, A])^{p^n} \in [A/\mathrm{rad}(A), A/\mathrm{rad}(A)].$$

Therefore $\rho(T_n(A)) \subseteq T_n(A/\mathrm{rad}(A))$ for all $n \geq 0$, which implies

$$T(A)/\mathrm{rad}(A) \subseteq T(A/\mathrm{rad}(A))$$

since $\mathrm{rad}(A) \subseteq T(A)$. Given $\rho(a) \in T(A/\mathrm{rad}(A))$, there is an $n \in \mathbb{N}$ such that

$$\rho(a^{p^n}) = \rho(a)^{p^n} \in \rho([A, A]) = [A/\mathrm{rad}(A), A/\mathrm{rad}(A)]$$

and therefore

$$a^{p^n} \in [A, A] + \ker(\rho) = [A, A] + \mathrm{rad}(A) \subseteq T(A).$$

Since $\mathrm{rad}(A) \subseteq T(A)$, we have $a^{p^n} \in T(A)$, which implies $a \in T(A)$. □

2.9.2 The Number of Simple Modules

Proposition 2.9.6 *Let K be a field and let A be an artinian K-algebra such that K is a splitting field for A. Then* $\dim_K(A/T(A))$ *is the number of isomorphism classes of simple A-modules.*

Proof Wedderburn's theorem shows that $A/\mathrm{rad}(A)$ is isomorphic to a direct product of s matrix algebras over K. The number of simple A-modules equals s. Proposition 2.9.5 and Lemma 2.9.2 show that

$$\dim_K((A/\mathrm{rad}(A))/(T(A/\mathrm{rad}(A))) = s.$$

Moreover,

$$A/T(A) = A/([A, A] + \mathrm{rad}(A)) = (A/\mathrm{rad}(A))/(T(A/\mathrm{rad}(A))).$$

This proves the statement. □

Theorem 2.9.7 (Brauer) *Let k be a splitting field for the group G and suppose that the characteristic of k is $p > 0$. Then the number of isomorphism classes of simple kG-modules is equal to the number of conjugacy classes of G of elements $g \in G$ such that the order of g is not divisible by p.*

Proof We need to show that $kG/T(kG)$ has a k-basis given by representatives of conjugacy classes of elements of G which have order relatively prime to p.

Of course, G is a generating set of $kG/T(kG)$ as a vector space since it generates kG. If $g, h \in G$ then $hgh^{-1} - g = [h, gh^{-1}]$ so that two conjugate elements in G give the same element in $kG/T(kG)$. Moreover, let $g = g_p \cdot g_q \in G$ for g_p of prime power order p^m and g_q relatively prime to p. Further g_p and g_q commute. Indeed, the subgroup $\langle g \rangle$ of G generated by g is cyclic, hence abelian, and therefore the classification of finitely generated abelian groups gives the statement. By Lemma 2.9.3

$$\begin{aligned}(g - g_q)^{p^m} - g^{p^m} + g_q^{p^m} &= (g - g_q)^{p^m} - g_p^{p^m} g_q^{p^m} + g_q^{p^m} \\ &= (g - g_q)^{p^m} - g_q^{p^m} + g_q^{p^m} \\ &= (g - g_q)^{p^m} \in [kG, kG]\end{aligned}$$

and so

$$g \equiv g_q \mod T(kG).$$

Therefore, we get that a set of representatives of conjugacy classes of elements of G which have order relatively prime to p is a generating set of $kG/T(kG)$.

We need to prove that this set is linearly independent. Let C be the set of conjugacy classes g^G of G and let

$$Q := \{g^G \in C \mid \text{the order of } g \text{ is not divisible by } p\}$$

and let Q_e be a set of representatives of elements $g \in G$ with $g^G \in Q$, i.e. for all $g_1, g_2 \in Q_e$ we have $g_1^G = g_2^G \Rightarrow g_1 = g_2$ and $\{g^G \mid g \in Q_e\} = Q$.

Let $\alpha = \sum_{g \in Q_e} \alpha_g g \in T(kG)$ for coefficients $\alpha_g \in k$. Since the order of each $g \in Q_e$ is not divisible by p, there is an $m \in \mathbb{N}$ such that $g^{p^m} = g$ for all $g \in Q_e$. We may even choose m so big that $\alpha^{p^m} \in [kG, kG]$. But

$$\alpha^{p^m} = \left(\sum_{g \in Q_e} \alpha_g g\right)^{p^m} \equiv \left(\sum_{g \in Q_e} (\alpha_g)^{p^m} g\right) \mod [kG, kG].$$

Moreover $[kG, kG]$ is contained in the set of elements $\sum_{g \in G} \beta_g g$ such that $\beta_g = \beta_h$ whenever g is conjugate to h in G. Since we were picking one conjugate in each conjugacy class in Q_e we get that $\alpha_g = 0$ for all $g \in Q_e$. This proves the statement. □

Corollary 2.9.8 *Let k be a field of characteristic $p > 0$ and let G be a finite p-group. Then the trivial module is the only simple kG-module, kG is a local algebra and*

$$I(kG) := \ker(kG \longrightarrow k) = \langle g - 1 \mid g \in G\rangle_{k\text{- vector space}} = \mathrm{rad}(kG).$$

Proof Let $\overline{k}$ be an algebraic closure of k. Then by Theorem 2.9.7 we know that the trivial module is the only simple $\overline{k}G$-module, using that all elements of G except the neutral element have order divisible by p. But then k is actually already a splitting field since the endomorphism ring of the trivial module is k. Hence there is only one simple kG-module, the trivial module, which is of multiplicity 1 since its dimension is 1. Since the k-vector space generated by the elements $g - 1$, $g \in G$, is of codimension 1, and since these elements are obviously in the kernel of $kG \longrightarrow k$, we get the generating set of $I(kG)$ as claimed and so we obtain the statement. □

Remark 2.9.9 We have seen this fact already by more elementary methods in Proposition 1.6.22.

2.9.3 Külshammer Ideals of Symmetric Algebras

In this subsection we again follow Külshammer [15–18].

Let K be a field of characteristic $p > 0$ and let A be a symmetric K-algebra. Then there is a non-degenerate symmetric associative bilinear form

$$\langle\ ,\ \rangle : A \times A \longrightarrow K.$$

We shall consider for all subsets U of A orthogonal spaces $U^\perp$ with respect to this symmetrising form, i.e.

$$U^\perp := \{a \in A \mid \langle a, u\rangle = 0\ \forall u \in U\}.$$

Lemma 2.9.10 *Let K be a field of characteristic $p > 0$ and let A be a finite dimensional symmetric K-algebra. Denote by $Z(A)$ the centre of A and by $[A, A]$ the commutator space of A. Then*

$$[A, A]^\perp = Z(A) \textit{ and } \mathrm{rad}(A)^\perp = \mathrm{soc}(A).$$

Therefore the symmetrising form $\langle\ ,\ \rangle : A \times A \longrightarrow K$ induces a non-degenerate bilinear form

$$\langle\ ,\ \rangle : Z(A) \times A/[A, A] \longrightarrow K.$$

Proof Let $a, b, c \in A$. Then

$$\begin{aligned}\langle a, bc - cb\rangle &= \langle a, bc\rangle - \langle a, cb\rangle = \langle ab, c\rangle - \langle cb, a\rangle \\ &= \langle c, ab\rangle - \langle c, ba\rangle = \langle c, ab - ba\rangle\end{aligned}$$

and therefore

$$\begin{aligned}a \in [A, A]^{\perp} &\Leftrightarrow \forall b, c \in A : \langle a, bc - cb\rangle = 0 \Leftrightarrow \forall b \in A : (ba - ab) \in A^{\perp} = 0 \\ &\Leftrightarrow a \in Z(A).\end{aligned}$$

Let I be a left ideal of A. We claim that $I^{\perp}$ is a right ideal. Let $a \in A$ and $b \in I^{\perp}$. Then, since I is an ideal,

$$\langle ba, I\rangle = \langle b, aI\rangle \subseteq \langle b, I\rangle = 0$$

and $ba \in I^{\perp}$. Likewise, since $\langle\ ,\ \rangle$ is symmetric, if I is a right ideal, then $I^{\perp}$ is a left ideal. Let J be a subset of A. Then

$$\langle bJ, \mathrm{rad}(A)\rangle = \langle b, J \cdot \mathrm{rad}(A)\rangle$$

for all $b \in A$ and so $\mathrm{rad}(A)^{\perp} \cdot \mathrm{rad}(A) \subseteq A^{\perp} = 0$ since $\mathrm{rad}(A)^{\perp}$ is a two-sided ideal of A. This shows that $\mathrm{rad}(A)^{\perp}$ consists of the elements in A which annihilate $\mathrm{rad}(A)$. Now, the elements of A which annihilate $\mathrm{rad}(A)$ are precisely the elements in the socle of A. This proves the statement. □

Proposition 2.9.11 *Let K be a field of characteristic $p > 0$ and let A be a finite dimensional symmetric K-algebra. Then $T_n(A)^{\perp}$ is an ideal of the centre $Z(A)$ of A and $T(A)^{\perp} = \mathrm{soc}(A) \cap Z(A)$.*

Proof Since by Proposition 2.9.5 we have $T(A) = [A, A] + \mathrm{rad}(A)$, we get

$$T(A)^{\perp} = ([A, A] + \mathrm{rad}(A))^{\perp} = [A, A]^{\perp} \cap \mathrm{rad}(A)^{\perp} = Z(A) \cap \mathrm{soc}(A).$$

For the next statement we observe that $T_n(A)$ is a $Z(A)$-module. Indeed, for all $z \in Z(A)$ and for all $a, b \in A$ we compute

$$z \cdot (ab - ba) = zab - zba = (za)b - b(za) \in [A, A]$$

and hence $[A, A]$ is a $Z(A)$-module. Moreover, using Lemma 2.9.10, we get

$$x \in T_n(A) \Rightarrow x^{p^n} \in [A, A] \Rightarrow (zx)^{p^n} = z^{p^n} x^{p^n} \in z^{p^n}[A, A] \subseteq [A, A]$$

and hence $T_n(A)$ is a $Z(A)$-module. Moreover, if $z \in Z(A)$ and $b \in T_n(A)^{\perp}$, we compute

$$\langle bz, T_n(A)\rangle = \langle b, zT_n(A)\rangle = 0 \text{ since } \langle b, T_n(A)\rangle = 0$$

since $T_n(A)$ is a $Z(A)$-module. Hence $T_n(A)^{\perp}$ is an ideal in $Z(A)$. □

Reynolds defined an ideal of group algebras via a property similar to the one used in Brauer's Theorem 1.5.4. He obtained that the so-defined ideal equals $\mathrm{soc}(A) \cap Z(A)$ where $A = kG$ is a group algebra of a finite group G over an algebraically closed field k of characteristic $p > 0$.

Definition 2.9.12 Let A be a finite dimensional K-algebra over a field K. Then $R(A) := Z(A) \cap \mathrm{soc}(A)$ is the *Reynolds ideal* of A. If K has characteristic $p > 0$ then $T_n(A)^\perp$ is the n-th *Külshammer ideal* of A.

Recall that the statement of Proposition 2.9.11 means that for finite dimensional symmetric algebras A over a perfect field K of characteristic $p > 0$ we get a sequence of ideals

$$R(A) = T(A)^\perp = \bigcap_{n=0}^{\infty} T_n(A)^\perp \subseteq \cdots \subseteq T_2(A)^\perp \subseteq T_1(A)^\perp \subseteq T_0(A)^\perp = Z(A)$$

of the centre of A. The centre hence carries a very nice additional ideal structure coming from the fact that $Z(A)$ is the centre of an algebra A. Observe that the same commutative algebra may be the centre of various different algebras.

Example 2.9.13 Let K be a field of characteristic 2, fix $c \in K$ and let $A(c)$ be the algebra given by the quiver

$$\begin{tikzcd} \underset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \underset{1}{\bullet} \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

modulo the relations

$$\gamma\beta = 0, \alpha\beta\gamma = \beta\gamma\alpha, \alpha^2 = c \cdot \alpha\beta\gamma.$$

This algebra occurs in Erdmann's classification of blocks with dihedral defect groups and is called $\mathcal{D}(2A)^1(c)$ in the notation used there. The algebra actually occurs as a principal block for a certain group ring.

If $c \neq 0$, we may replace β by $c \cdot \beta$ and observe that $A(c) \simeq A(1)$ in this case. Hence we may consider the two cases $c \in \{0, 1\}$. The two projective indecomposable modules have radical length 4 independently of the value of c. We get

$$\begin{array}{rlrl} P_0/\mathrm{rad}(P_0) &= S_0; & P_1/\mathrm{rad}(P_1) &= S_1; \\ \mathrm{rad}(P_0)/\mathrm{rad}^2(P_0) &= S_0 \oplus S_1; & \mathrm{rad}(P_1)/\mathrm{rad}^2(P_1) &= S_0; \\ \mathrm{rad}^2(P_0)/\mathrm{rad}^3(P_0) &= S_0 \oplus S_1; & \mathrm{rad}^2(P_1)/\mathrm{rad}^3(P_1) &= S_0; \\ \mathrm{rad}^3(P_0) &= S_0; & \mathrm{rad}^3(P_1) &= S_1. \end{array}$$

The algebra has dimension 10. Every element in the centre is a linear combination of closed paths since we may multiply by the idempotents e_0 and e_1 from the left and the right. More precisely if z is in the centre $e_i z = z e_i$ and this implies that z is a linear combination of paths, and the paths starting at i also end at i. The closed paths of A are

$$e_0, \alpha, \beta\gamma, \alpha\beta\gamma, e_1, \gamma\alpha\beta.$$

Every element in the socle is in the centre since multiplying with any arrow gives 0, and multiplying by idempotents from the left and from the right gives the same result since the socle has a basis given by the closed paths $\alpha\beta\gamma$ and $\gamma\alpha\beta$. Hence we need to consider central elements as a linear combination of

$$e_0, \alpha, \beta\gamma, e_1.$$

Since a central element must commute with β we get that the coefficient of e_0 and of e_1 has to be equal. On the other hand, $e_0 + e_1 = 1$ is central. Can we find coefficients $d_0, d_1 \in K$ so that

$$d_0 \cdot \alpha + d_1 \cdot \beta\gamma \in Z(A(c))?$$

We have that

$$\gamma \cdot (d_0 \cdot \alpha + d_1 \cdot \beta\gamma) = d_0\gamma\alpha \stackrel{!}{=} (d_0 \cdot \alpha + d_1 \cdot \beta\gamma) \cdot \gamma = 0$$

implies that $d_0 = 0$. However, $\beta\gamma \in Z(A(c))$ since it commutes with α, with e_0 and multiplication with any other arrow gives 0. We have proved

$$\begin{aligned} Z(A(c)) &= K \cdot 1 + K \cdot \alpha\beta\gamma + K \cdot \gamma\alpha\beta + K \cdot \beta\gamma \\ &\simeq K[X, Y, Z]/(XY, XZ, YZ, X^2, Y^2, Z^2) \end{aligned}$$

independently of c. The algebra $A(c)$ is symmetric. Indeed, choose the basis

$$B := \{e_0, e_1, \alpha, \beta, \gamma, \alpha\beta, \beta\gamma, \gamma\alpha, \alpha\beta\gamma, \gamma\alpha\beta\}$$

and define a linear form $\psi : A(c) \longrightarrow K$ by $\psi(\alpha\beta\gamma) = 1 = \psi(\gamma\alpha\beta)$ and $\psi(b) = 0$ for all $b \in B \setminus \{\alpha\beta\gamma, \gamma\alpha\beta\}$. Proposition 1.10.18 shows that in this way we get a non-degenerate associative bilinear form $\langle x, y\rangle := \psi(xy)$ which is symmetric in this specific case, as is readily verified.

Since $[A(c), A(c)]^\perp = Z(A(c))$ by Lemma 2.9.10 the commutator space $[A(c), A(c)]$ is of dimension $10 - 4 = 6$. Of course, all non-closed paths are commutators with an idempotent, and so

$$\beta, \gamma, \alpha\beta, \gamma\alpha \in [A(c), A(c)].$$

Moreover,

$$\alpha\beta\gamma - \gamma\alpha\beta = [\alpha\beta, \gamma] \in [A(c), A(c)].$$

Finally

$$[\beta, \gamma] = \beta\gamma - \gamma\beta = \beta\gamma \in [A(c), A(c)].$$

We shall compute $T_1(A(c))$. As we know

$$[A(c), A(c)] \subseteq T_1(A(c)) \subseteq \mathrm{rad}(A(c))$$

we need only decide whether a linear combination

$$s_1\alpha + s_2\alpha\beta\gamma$$

is in $T_1(A(c))$. Since $(\alpha\beta\gamma)^2 = 0$ we get that $\alpha\beta\gamma \in T_1(A(c))$ for all c. But $\alpha^2 = c \cdot \alpha\beta\gamma$ and so

$$\alpha \in T_1(A(0))$$

whereas

$$\alpha \notin T_1(A(1)).$$

Hence

$$T_1(A(c)) = \begin{cases} [A(c), A(c)] + K \cdot \alpha\beta\gamma & \text{if } c = 1, \\ [A(c), A(c)] + K \cdot \alpha\beta\gamma + K \cdot \alpha & \text{if } c = 0. \end{cases}$$

Of course $T_2(A(c)) = \mathrm{rad}(A(c))$ since the radical of $A(c)$ has nilpotence degree 4. Since $\beta\gamma$ is not orthogonal to α we get

$$T_1(A(c))^\perp = \begin{cases} \mathrm{soc}(A(c)) & \text{if } c = 1, \\ \mathrm{rad}(Z(A(c))) & \text{if } c = 0. \end{cases}$$

Under the identification $Z(A(c)) \simeq K[X, Y, Z]/(XY, XZ, YZ, X^2, Y^2, Z^2)$ we get that

$$T_1(A(1))^\perp = < X, Y > \text{ and } T_1(A(0))^\perp = < X, Y, Z > .$$

As a whole we see that the commutative ring $Z = Z(A(c))$, which is independent of c, gets an additional ideal structure from the fact that it is the centre of $A(c)$ and the ideal structure of Külshammer ideals depends on c.

We note that Holm and Zimmermann give another closely related example in [20] where the centres of two algebras are isomorphic and where the first Külshammer ideal is of the same codimension for both algebras, but where the quotients of the centre modulo the first Külshammer ideal gives two non-isomorphic rings.

Let A be a symmetric K-algebra for a field K of characteristic $p > 0$. Let $\langle\ ,\ \rangle$ be a symmetrising form for A. By Lemma 2.9.10 this symmetrising form induces a non-degenerate bilinear form

$$\langle\ ,\ \rangle : Z(A) \times A/[A, A] \longrightarrow K.$$

We have seen that the p-power map $\mu_p : a \longrightarrow a^p$ is a semilinear map

$$A/[A, A] \longrightarrow A/[A, A],$$

i.e. $\mu_p(a + b + [A, A]) = \mu_p(a) + \mu_p(b) + [A, A]$ and $\mu_p(\lambda \cdot a + [A, A]) = \lambda^p \cdot \mu_p(a) + [A, A]$ for all $a, b \in A$ and $\lambda \in K$.

Hence there is a left adjoint $\zeta_p : Z(A) \longrightarrow Z(A)$ to μ_p with respect to $\langle\ ,\ \rangle$. In other words there is a semilinear map $\zeta_p : Z(A) \longrightarrow Z(A)$ so that for all $a \in A$ and $z \in Z(A)$ we get

Lemma 2.9.14 $\langle z, \mu_p(a + [A, A])\rangle = \langle\zeta_p(z), a + [A, A]\rangle^p$.

Of course, an iteration of this relation gives

$$\langle z, \mu_p^n(a + [A, A])\rangle = \langle\zeta_p^n(z), a + [A, A]\rangle^{p^n}.$$

Lemma 2.9.15 *Let A be a symmetric K-algebra for a field K of characteristic $p > 0$ and let ζ_p be the left adjoint to the p-power map $\mu_p : A/[A, A] \longrightarrow A/[A, A]$ with respect to a fixed symmetrising form. Then $T_n(A)^\perp = \mathrm{im}(\zeta_p^n)$ for all n, where the orthogonal space is taken with respect to this same fixed symmetrising form.*

Proof Since $T_n(A) = \ker(\mu_p^n)$, dualising this equality using the symmetrising form we get $T_n(A)^\perp = \mathrm{im}(\zeta_p^n)$. □

Remark 2.9.16 We should alert the reader that ζ_p depends heavily on the symmetrising form. Different symmetrising forms lead to different mappings ζ_p. However the image does not depend on the choice of the form.

2.9.4 Further Properties of Külshammer Ideals of Group Algebras

In this section we follow Külshammer [19].

Let k be a field of characteristic $p > 0$ and let G be a finite group. We can describe $T_n(kG)$ in terms of the basis G of kG. Indeed, recall first that $[kG, kG] \leq T_n(kG)$. Then for all $g, h \in G$ we get

$$hgh^{-1} - g = [h, gh^{-1}] \in [kG, kG] \leq T_n(kG).$$

Hence, denoting by $C_g := \{hgh^{-1} \mid h \in G\}$ the conjugacy class of $g \in G$ and by $Cl(G) := \{C_g \mid g \in G\}$ the conjugacy classes of G, we obtain that $\sum_{g\in G} \alpha_g g \in T_n(kG)$ if and only if

$$
\begin{aligned}
0 \equiv \left(\sum_{g\in G} \alpha_g g\right)^{p^n} &\equiv \sum_{g\in G} (\alpha_g g)^{p^n} \\
&\equiv \sum_{C_g\in Cl(G)} \left(\sum_{h\in C_g} \alpha_h^{p^n}\right) g^{p^n} \mod [kG, kG]).
\end{aligned}
$$

We have proved the following lemma.

Lemma 2.9.17 *Let k be a field of characteristic $p > 0$, G be a finite group and, for all $C \in Cl(G)$, let $C^{p^{-n}} := \{g \in G |\ g^{p^n} \in C\}$. Then*

$$T_n(kG) = \left\{ \sum_{g\in G} \alpha_g g \mid \forall C \in Cl(G) : \sum_{g\in C^{p^{-n}}} \alpha_g = 0 \right\}.$$

For a finite group H we denote its exponent by $\exp(H)$. The exponent of a finite group is the least common multiple of the orders of its elements.

Lemma 2.9.18 *Let k be a field of characteristic $p > 0$, G be a finite group and let S_p be a Sylow p-subgroup of G. Then*

$$\exp(S_p) = \min\{p^n \mid T_n(kG) = T(kG)\}.$$

Proof Uniquely decomposing each $g \in G$ into an element g_p of order p^m for some p and g'_p of order relatively prime to p, we get $g^{p^n} = g_{p'}^{p^n} \cdot g_p^{p^n}$. Then by the proof of Theorem 2.9.7 we see that $T_n(kG) = T(kG)$ if and only if the order of g^{p^n} is not divisible by p for all $g \in G$. This happens if and only if $p^n \geq \exp(S_p)$ since $g_p \in hS_ph^{-1}$ for some $h \in G$ by Sylow's theorem. □

For an integer m we define $m_p := \max\{p^n |\ p^n \text{ divides } m\}$, the biggest power of p that divides m. We obtain immediately from Lemma 2.9.18 the following Corollary.

Corollary 2.9.19 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Then G has a cyclic Sylow p-subgroup if and only if*

$$\min\{p^n \mid T_n(kG) = T(kG)\} = |G|_p.$$

We shall examine the consequences of Lemma 2.9.17 and obtain a k-basis for $T_n(kG)^\perp$ as a corollary.

Lemma 2.9.20 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Then $T_n(kG)^\perp$ has a k-basis*

$$\left\{\sum_{g\in C^{p^{-n}}} g|\ C \in Cl(G) \text{ and } C^{p^{-n}} \neq \emptyset\right\}$$

where $C^{p^{-n}} := \{h \in G|\ h^{p^n} \in C\}$.

Proof It is clear that the elements in $\{\sum_{g\in C^{p^{-n}}} g|\ C \in Cl(G)\}$ are linearly independent since G is linearly independent in kG. Moreover,

$$a \in T_n(kG)^{\perp} \Leftrightarrow \forall t \in T_n(kG):\ \langle a, t\rangle = 0.$$

Lemma 2.9.17 shows that

$$T_n(kG) = \left\{\sum_{g\in G} \alpha_g g \mid \forall C \in Cl(G):\ \sum_{g\in C^{p^{-n}}} \alpha_g = 0\right\}.$$

But recall the definition of $\langle\ ,\ \rangle$ for kG from Proposition 1.10.26:

$$\langle g, h\rangle = \begin{cases} 1 & \text{if } gh = 1 \\ 0 & \text{if } gh \neq 1 \end{cases}$$

and then extend bilinearly to a bilinear form $kG \times kG \longrightarrow k$. The following will be used later:

Corollary 2.9.21 *For any subset $S \subseteq G$ we have*

$$\left\langle \sum_{s\in S} s, \sum_{g\in G} \alpha_g g\right\rangle = \sum_{s\in S}\left\langle s, \sum_{g\in G} \alpha_g g\right\rangle = \sum_{s\in S} \alpha_{s^{-1}}.$$

Proof Indeed, this is just a corollary of the definition of the standard symmetrising form of kG. □

We continue with the proof of Lemma 2.9.20. Therefore

$$\begin{aligned} T_n(kG) &= \left\{x \in kG \mid \forall C \in Cl(G):\ \langle x, \sum_{g\in C^{p^{-n}}} g\rangle = 0\right\} \\ &= \bigcap_{C\in Cl(G)} \left(\sum_{g\in C^{p^{-n}}} g\right)^{\perp} \end{aligned}$$

and by consequence

$$T_n(kG)^\perp = \sum_{C\in Cl(G)} k\cdot\left(\sum_{g\in C^{p^{-n}}} g\right).$$

In other words, $T_n(kG)^\perp$ has a basis consisting of the elements $\sum_{g\in C^{p^{-n}}} g$ for all $C \in Cl(G)$. This finishes the proof of Lemma 2.9.20. □

Recall again that for every $g \in G$ one has a unique decomposition $g = g_p \cdot g_{p'} = g_{p'} \cdot g_p$ where g_p is an element of p-power order and $g_{p'}$ has order not divisible by p. This follows from the classification of finitely generated abelian groups applied to the cyclic group generated by g.

Corollary 2.9.22 (Reynolds) *Let k be a field of characteristic $p > 0$ and let G be a finite group. Then for all $g \in G$ let S_g be the set of those elements h of G such that $h_{p'}$ is conjugate to $g_{p'}$. Then the k-vector space generated by $\sum_{h\in S_g} h$ for all $g \in G$ is equal to* $\mathrm{soc}(kG)\cap Z(kG)$.

Proof This follows from the above together with the fact that $g_{p'}$ is conjugate to $h_{p'}$ if and only if there is an integer n such that g^{p^n} is conjugate to h^{p^n}. Then for $n \geq \exp(S_p)$ for some Sylow p subgroup S_p of G we obtain the statement from the fact that

$$T_n(kG)^\perp = T(kG)^\perp = \mathrm{soc}(kG)\cap Z(kG)$$

which was shown above. □

Recall the definition of ζ_p from Lemma 2.9.14. It is easy to compute the element $\zeta_p^n(\sum_{g\in C} g)$ for each $C \in Cl(G)$.

Lemma 2.9.23 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Then for all $C \in Cl(G)$ we have $\zeta_p^n(\sum_{g\in C} g) = \sum_{g\in G;\, g^{p^n}\in C} g$.*

Proof We use the symmetrising form

$$\langle\ ,\ \rangle : Z(kG)\times kG/[kG,kG] \longrightarrow k,$$

Lemma 2.9.14 and Corollary 2.9.21 to get

$$\begin{aligned}\left\langle \zeta_p^n\left(\sum_{g\in C} g\right), \sum_{h\in G}\alpha_h h\right\rangle^{p^n} &= \left\langle \sum_{g\in C} g, \left(\sum_{h\in G}\alpha_h h\right)^{p^n}\right\rangle = \left\langle \sum_{g\in C} g, \sum_{h\in G}\alpha_h^{p^n} h^{p^n}\right\rangle\\ &= \sum_{h\in G;\, h^{p^n}\in C}\alpha_{h^{-1}}^{p^n} = \left(\sum_{h\in G;\, h^{p^n}\in C}\alpha_{h^{-1}}\right)^{p^n}\\ &= \left\langle \sum_{g\in C^{p^{-n}}} g, \sum_{h\in G}\alpha_h h\right\rangle^{p^n}\end{aligned}$$

and this shows

$$\sum_{g \in C^{p^{-n}}} g = \zeta_p^n (\sum_{g \in C} g)$$

as claimed. □

2.10 Brauer Constructions and p-Subgroups

We shall introduce a very useful construction, due to Brauer, which links the representation theory of a group G over a field of characteristic p to the representation theory of centralisers of p-subgroups. This construction is more technical compared to what we did before. However, it is crucial for more subtle properties of Brauer and Green correspondences.

2.10.1 The Brauer Homomorphism and Osima's Theorem

Recall that $C_G(Q)$ is the centraliser of Q in G.

Definition 2.10.1 Given a field k of characteristic $p > 0$ and a finite group G we define for any p-subgroup Q of G the *Brauer homomorphism* by

$$\beta_Q\left(\sum_{g \in G} \alpha_g g\right) := \sum_{g \in C_G(Q)} \alpha_g g$$

and obtain a mapping $\beta_Q : kG \longrightarrow kC_G(Q)$.

Lemma 2.10.2 *Let k be a field k of characteristic $p > 0$ and let G be a finite group with p-subgroup Q. Then the restriction of the Brauer homomorphism β_Q to $Z(kG)$ induces an algebra homomorphism $Z(kG) \longrightarrow Z(kC_G(Q))$.*

Proof It is clear that β_Q is k-linear. Moreover, given a conjugacy class $C \in Cl(G)$, if $C \cap C_G(Q) \neq \emptyset$, then $C \cap C_G(Q)$ is a union of conjugacy classes of $C_G(Q)$. Hence $\beta_Q(C) \in Z(kC_G(Q))$.

We need to show that β_Q is multiplicative. Let $C_1, C_2 \in Cl(G)$ be two conjugacy classes of G and put $x_1 := \sum_{g \in C_1} g$ and $x_2 := \sum_{g \in C_2} g$. Since by Lemma 1.5.3 the conjugacy class sums form a basis for the centre of kG, we only need to show $\beta_Q(x_1)\beta_Q(x_2) = \beta_Q(x_1 x_2)$. But

$$x_1 x_2 = \sum_{g \in G} \gamma_g g \text{ and } \beta_Q(x_1)\beta_Q(x_2) = \sum_{h \in C_G(Q)} \delta_h h$$

for

$$\gamma_g = |\{(u, v) \in C_1 \times C_2 \mid uv = g\}|$$

and

$$\delta_h = |\{(u, v) \in (C_1 \times C_2) \cap (C_G(Q) \times C_G(Q)) \mid uv = h\}|\,.$$

We observe that Q acts on C_1 and on C_2 by conjugation and $C_1 \cap C_G(Q)$ respectively $C_2 \cap C_G(Q)$ are the fixpoints of this action. Hence Q acts on $C_1 \times C_2$ by conjugation diagonally on each factor and the fixpoints are $(C_1 \times C_2) \cap (C_G(Q) \times C_G(Q))$. Now, if $(u, v) \in C_1 \times C_2$ and $q \in Q$, then

$$(quq^{-1})(qvq^{-1}) = quvq^{-1} = qgq^{-1}$$

if $uv = g$. Therefore the action of Q on $C_1 \times C_2$ restricts to an action on

$$\{(u, v) \in C_1 \times C_2 \mid uv = g\}$$

whenever $g \in C_G(Q)$. The set

$$\{(u, v) \in (C_1 \times C_2) \cap (C_G(Q) \times C_G(Q)) \mid uv = g\}$$

corresponds to the fixpoints, i.e. orbits of length 1, of this action. All other orbits are of length a non-trivial power of p, since Q is a p-group. Hence

$$\forall g \in C_G(Q) : \quad \gamma_g = \delta_g \in k$$

which proves the lemma. □

Corollary 2.10.3 *Given a field k of characteristic $p > 0$ and a finite group G with p-subgroup Q, then the kernel of $\beta_Q : Z(kG) \longrightarrow Z(C_G(Q))$ has a k-basis given by*

$$\left\{ \sum_{g \in C} g \mid C \in Cl(G) \text{ and } C \cap C_G(Q) = \emptyset \right\}.$$

Proof Indeed, this follows immediately from the definition of β_Q. □

Remark 2.10.4 Now, consider the relation $C \cap C_G(Q) = \emptyset$ for some $C \in Cl(G)$. This means that no element in C centralises all of Q. This is equivalent to the statement that for all $g \in C$ there is a $q \in Q$ such that $qg \neq gq$. This in turn is equivalent to the fact that for all $g \in C$ the group Q is not totally contained in $C_G(g)$. Since Q is a p-group, using Sylow's theorems this last statement is equivalent to the statement that the group Q is not contained in any Sylow p-subgroup of $C_G(g)$ of any $g \in C$. Denote by $Syl_p(H)$ the set of Sylow p-subgroups of H. Then we get

$$[C \cap C_G(Q) = \emptyset] \Leftrightarrow \left[\forall\, S \in \bigcup_{g \in C} Syl_p(C_G(g)) :\; Q \not\leq S \right].$$

The following consequence of Lemma 2.10.2 is a theorem of Osima with a proof due to Donald Passman.

Proposition 2.10.5 (Osima) *Let k be a field of characteristic $p > 0$, let G be a finite group and let $e^2 = e \in Z(kG)$ be a central idempotent of kG. If $e = \sum_{g \in G} \epsilon_g g$, then $\epsilon_g = 0$ if p divides the order of g.*

Proof Let $g \in G$, suppose $\epsilon_g \neq 0$ and let $g = g_p \cdot g_{p'}$ be the decomposition of g into a product of an element g_p of a power of p and an element $g_{p'}$ of order r not divisible by p. Let Q be the group generated by g_p and suppose that $Q \neq 1$. Since $g_{p'}$ commutes with g_p we get that $g \in C_G(Q)$. Since $e^2 = e$, using Lemma 2.10.2, $\beta_Q(e)$ is a central idempotent of $kC_G(Q)$. Hence we may assume that $G = C_G(Q)$ and $g_p \in Z(G)$. Now, p is a unit in $\mathbb{Z}/r\mathbb{Z}$ and so it is of finite order m' in $(\mathbb{Z}/r\mathbb{Z})^\times$. Let $|G| = p^s t$ for some integer $s \in \mathbb{N}$ and some integer $t \in \mathbb{N}$ such that p does not divide t. Increasing m' if necessary, there is an integer $m \geq s$ such that r divides $p^m - 1$ and therefore

$$\left(g_{p'}^{-1} e\right)^{p^m} = \left(g_{p'}^{-1}\right)^{p^m} e^{p^m} = g_{p'}^{-1} e.$$

Moreover, if $g_{p'}^{-1} e = \sum_{h \in G} \phi_h h$, then $\phi_{g_p} \neq 0$ since we assumed that $\epsilon_g \neq 0$. Now

$$\left(g_{p'}^{-1} e\right)^{p^m} = \left(\sum_{h \in G} \phi_h h\right)^{p^m} \equiv \sum_{h \in G} \phi_h^{p^m} h^{p^m} \quad \text{mod } [kG, kG]$$

and since we assumed $m \geq s$, the order of h^{p^m} is not divisible by p for all $h \in G$. Since $\phi_{g_p} \neq 0$, and since h^{p^m} is of order relatively prime to p, we obtain for the difference

$$c := g_{p'}^{-1} e - \sum_{h \in G} \phi_h^{p^m} h^{p^m} = \left(g_{p'}^{-1} e\right)^{p^m} - \left(\sum_{h \in G} \phi_h h\right)^{p^m} \in [kG, kG].$$

We define the coefficients $\gamma_h \in k$ by

$$c = \sum_{h \in G} \gamma_h h$$

and we get that $\gamma_{g_p} \neq 0$. Since $[kG, kG]$ is k-linearly generated by elements $xy - yx$ for $x, y \in G$, there are $x, y \in G$ such that $xy \neq yx$ and such that $g_p = xy$. Since $g_p \in Z(G)$ we obtain

$$yx = x^{-1}(xy)x = x^{-1}g_p x = g_p = xy$$

which contradicts the above established fact that $xy - yx \neq 0$. □

2.10.2 Higman's Criterion, Traces and the Brauer Quotient

We come to a classical object of study which we have already used in the proof of Proposition 2.1.15. For a group Γ and a $k\Gamma$-module M denote by

$$M^\Gamma := \{m \in M \mid \gamma m = m \ \forall \gamma \in \Gamma\}$$

the space of Γ-fixed points.

Lemma 2.10.6 *Let k be a field and let G be a finite group. For every subgroup H of G and every kG-module M we get that for all $m \in M^H$ we have $\sum_{gH \in G/H} gm \subseteq M^G$.*

Proof If $m \in M^H$ consider $\sum_{gH \in G/H} gm$. Now multiplication by $g' \in G$ permutes the classes $gH \in G/H$ and the image is in M^G. □

Definition 2.10.7 Let k be a field of characteristic $p > 0$ and let G be a finite group. For every subgroup H of G and every kG-module M define the *trace from H to G* as

$$\begin{aligned} Tr_H^G : M^H &\longrightarrow M^G \\ m &\mapsto \sum_{gH \in G/H} gm \end{aligned}$$

We come to a useful observation linking traces and vertices, called Higman's criterion. Recall that if M and N are kG-modules, then $Hom_k(M, N)$ is a kG-module by $(gf)(m) := gf(g^{-1}m)$ for all $g \in G, f \in Hom_k(M, N)$ and $m \in M$. Now by Lemma 2.10.6 we get

$$Tr_H^G : Hom_{kH}(M \downarrow_H^G, N \downarrow_H^G) \longrightarrow Hom_{kG}(M, N).$$

Observe that $Hom_{kH}(M \downarrow_H^G, N \downarrow_H^G)$ is an $End_{kG}(N)$-$End_{kG}(M)$ bi-module by composition of mappings. Indeed, the composition $f \circ h$ of two maps $f \in Hom_{kH}(M \downarrow_H^G, N \downarrow_H^G)$ and $h \in Hom_{kG}(M, M)$ is in $Hom_{kH}(M \downarrow_H^G, N \downarrow_H^G)$, and likewise for endomorphisms of N.

Lemma 2.10.8 *Let G be a finite group, let k be a field and let H be a subgroup of G. Then for every kG-module M we get that $Tr_H^G : End_{kH}(M \downarrow_H^G) \longrightarrow End_{kG}(M)$ is a morphism of $End_{kG}(M)$-$End_{kG}(M)$ bimodules.*

Proof We need to show

$$Tr_H^G(h_1 \circ f \circ h_2) = h_1 \circ Tr_H^G(f) \circ h_2$$

for all $f \in Hom_{kH}(M \downarrow_H^G, M \downarrow_H^G)$ and $h_1, h_2 \in Hom_{kG}(M, M)$. To prove this we compute

$$\begin{aligned}
\left(h_1 \circ Tr_H^G(f) \circ h_2\right)(m) &= \sum_{gH \in G/H} h_1((gf)(h_2 m)) \\
&= \sum_{gH \in G/H} h_1(g(f(g^{-1}(h_2 m)))) \\
&= \sum_{gH \in G/H} g(h_1(f(h_2(g^{-1}m)))) \\
&= \sum_{gH \in G/H} g\left((h_1 \circ f \circ h_2)(g^{-1}m)\right) \\
&= \left(Tr_H^G(h_1 \circ f \circ h_2)\right)(m)
\end{aligned}$$

and obtain the result. □

Proposition 2.10.9 (D. Higman) *Let k be a field of characteristic $p > 0$, let G be a finite group and let $H \leq G$. Then a kG-module M is relatively H-projective if and only if $id_M \in Tr_H^G(Hom_{kH}(M \downarrow_H^G, M \downarrow_H^G))$.*

Proof Suppose $id_M \in Tr_H^G(Hom_{kH}(M \downarrow_H^G, M \downarrow_H^G))$. By Lemma 2.10.8 we obtain that $Tr_H^G(Hom_{kH}(M \downarrow_H^G, M \downarrow_H^G))$ contains $id_M \circ End_{kG}(M) = End_{kG}(M)$. Hence Tr_H^G is surjective. By definition the natural epimorphism of kG-modules

$$kG \otimes_{kH} M \longrightarrow kG \otimes_{kG} M \simeq M$$

is split if and only if the identity id_M is in the image of the induced morphism

$$Hom_{kG}(M, M \downarrow_H^G \uparrow_H^G) \xrightarrow{\mu} Hom_{kG}(M, M).$$

Recall the Frobenius reciprocity isomorphism

$$\begin{aligned}
Hom_{kH}(M \downarrow_H^G, M \downarrow_H^G) &\xrightarrow{\phi} Hom_{kG}(M, M \downarrow_H^G \uparrow_H^G) \\
\varphi &\mapsto \left(m \mapsto \sum_{gH \in G/H} g \otimes \varphi(g^{-1}m)\right)
\end{aligned}$$

from Remark 1.7.34. Hence $Tr_H^G = \mu \circ \phi$. Since id_M is in the image of Tr_H^G, the identity id_M is also in the image of μ.

Suppose conversely M is relatively H-projective. Proposition 2.1.6 shows that $M \downarrow_H^G \uparrow_H^G \longrightarrow M$ is split and hence the identity is in the image of μ and therefore in the image of Tr_H^G. □

Higman's criterion gives an interpretation of the defect group of a block in term of traces.

Lemma 2.10.10 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let $B = kG \cdot e$ be a block of kG with defect group D. Then $e \in Tr_{\Delta(H)}^{\Delta(G)}(kG^{\Delta(H)})$ if and only if H contains a conjugate of D.*

Proof As we have seen, B is projective relative to $\Delta(G)$. Hence H contains a defect group if and only if there is an $\alpha \in End_{k\Delta(H)}(kG)$ such that $id_B = Tr_{\Delta(H)}^{\Delta(G)}(\alpha)$. But then

$$\begin{aligned} e = Tr_{\Delta(H)}^{\Delta(G)}(\alpha)(e) &= \sum_{(g,g)\Delta(H)\in\Delta(G)/\Delta(H)} (g,g)\cdot\alpha((g^{-1},g^{-1})\cdot e) \\ &= \sum_{gH\in G/H} g\alpha(e)g^{-1} = Tr_H^G(\alpha(e)) \end{aligned}$$

which shows that $e \in Tr_{\Delta(H)}^{\Delta(G)}((kG)^{\Delta(H)})$. If conversely $e = Tr_{\Delta(H)}^{\Delta(G)}(a)$ for some $a \in (kG)^{\Delta(H)}$, then $\alpha(x) := x \cdot a$ for all $x \in kG^{\Delta(H)}$ defines a $k\Delta(H)$-linear endomorphism of kG. Then for all $x \in B$

$$\begin{aligned} Tr_{\Delta(H)}^{\Delta(G)}(\alpha)(x) &= \sum_{(g,g)\Delta(H)\in\Delta(G)/\Delta(H)} (g,g)\cdot\alpha((g^{-1},g^{-1})\cdot x) \\ &= \sum_{g\in G/H} g\cdot((g^{-1}\cdot x\cdot g)\cdot a)\cdot g^{-1} = x\cdot Tr_{\Delta(H)}^{\Delta(G)}(a) = x\cdot e = x \end{aligned}$$

and so $Tr_{\Delta(H)}^{\Delta(G)}(\alpha) = id_B$, which implies that B is $\Delta(H)$-projective by Higman's criterion. □

The following construction is useful mainly for p-subgroups. We shall explain below the reason for this restriction. Given a subgroup P of G, we may consider the submodule M^P of fixed-points of P. The group $N_G(P)$ acts on M^P since if $g \in N_G(P)$ and $m \in M^P$, then for all $h \in P$ we obtain

$$h\cdot(gm) = (hg)m = (g(g^{-1}hg))m = g((g^{-1}hg)m) = gm$$

since $g \in N_G(P)$ and therefore $(g^{-1}hg) \in P$ and hence $gm \in M^P$. Moreover, for all $Q \leq P$ we obtain trace functions

$$Tr_Q^P : M^Q \longrightarrow M^P$$

so that we may form $M(P) := M^P / \sum_{Q<P} Tr_Q^P(M^Q)$. Now, if P is not a p-group, then there exists a proper subgroup Q of P such that p does not divide $|P : Q|$. Maschke's argument then shows that $Tr_Q^P(M^Q) = M^P$. Hence the quotient construction $M(P)$ is reasonable only for p-groups P. Indeed, if $Q < P$, then $M^Q \geq M^P$ and a k-basis of M^P may be completed to a k-basis of M^Q. Let $\iota : M^P \longrightarrow M^Q$ be the natural embedding and $\pi : M^Q \longrightarrow M^P$ be the projection on M^P. There is no reason why π should be P-linear. However

$$\hat{\pi}(m) := \frac{1}{|P : Q|} \sum_{gP \in P/Q} g\pi(g^{-1}m)$$

defines a kP-linear projection which splits ι.

Definition 2.10.11 Let G be a finite group and let P be a p-subgroup of G. For every kG-module M we define the *Brauer quotient*

$$M(P) := M^P / \sum_{Q<P; Q \neq P} Tr_Q^P(M^Q)$$

which is naturally a $k(N_G(P)/P)$-module.

2.10.3 Brauer Pairs

We shall discuss an application of the Brauer quotient from Definition 2.10.11 due to Alperin, Broué and Puig. We shall follow the most elegant approach given by Külshammer [1].

Let k be a field of characteristic $p > 0$ and let G be a finite group.

Definition 2.10.12 A *Brauer pair* is a pair (P, e) where $P \leq G$ is a p-subgroup of G and e is a primitive central idempotent of $Z(kC_G(P))$.

In this section we shall prove Sylow-like theorems for Brauer pairs. Surprisingly the main tool to achieve this is the Brauer quotient.

Recall from Definition 2.10.1 the Brauer map

$$\beta_Q : kG \longrightarrow kC_G(Q)$$

for every p-subgroup Q of G.

Lemma 2.10.13 *The restriction of the Brauer map β_Q to*

$$\beta_Q : (kG)^Q \longrightarrow kC_G(Q)$$

is a surjective homomorphism of k-algebras.

Proof Indeed, a basis for the Q fixed points in kG is obtained by taking the sums of the orbits of the elements of G under the conjugation action. The orbit of an element $x \in G$ is isomorphic to $Q/C_Q(x)$ as a set with Q-action. If the orbit of the element x is of cardinality different from 1, then the sum of the elements of the orbit is $Tr_{C_Q(x)}^Q(x)$, and therefore belongs to $\sum_{S<Q} Tr_S^Q(kS)$. If the orbit of x is of length 1, then Q centralises x, and hence $x \in kC_G(Q)$. Therefore

$$(kG)^Q = kC_G(Q) \oplus \sum_{S<Q} Tr_S^Q(kS)$$

as k-vector spaces and the mapping β_Q is just the projection onto the first component. Moreover, $\sum_{S<Q} Tr_S^Q(kS)$ is a two-sided ideal of $(kG)^Q$. □

Corollary 2.10.14 *The kernel of the restriction of the map β_Q to $(kG)^Q$ equals $\sum_{S<Q} Tr_S^Q(kS)$.* □

We shall use the following observation in the proof of the main Theorem 2.10.16 below.

Lemma 2.10.15 *Let A and B be artinian k-algebras and let $\varphi : A \longrightarrow B$ be a homomorphism of algebras. Let $e \neq 0$ be an idempotent in B. Then there is a primitive idempotent f of A such that $\varphi(f) \cdot e \neq 0$. If e is primitive and B is commutative, then $\varphi(f) \cdot e = e$ and f is unique with this property.*

Proof Since A is artinian, we can find primitive idempotents $f_1, \dots, f_n$ of A such that $1 = \sum_{i=1}^n f_i$. Then

$$e = 1 \cdot e = \varphi(1) \cdot e = \varphi\left(\sum_{i=1}^n f_i\right) \cdot e = \sum_{i=1}^n \varphi(f_i) \cdot e.$$

This proves the existence of f. If B is commutative and e is primitive, then $\varphi(f) \cdot e$ and $(1 - \varphi(f)) \cdot e$ are both idempotents of B. Moreover,

$$e = e \cdot \varphi(f) + e \cdot (1 - \varphi(f))$$

and since e is primitive, and since $e \cdot \varphi(f) \neq 0$, we get $e = e \cdot \varphi(f)$. This also proves that $e \cdot \varphi(1 - f) = 0$, and hence unicity. □

Theorem 2.10.16 (Broué-Puig) *Let G be a finite group and let k be a field of characteristic $p > 0$. Let (P, e) be a Brauer pair of kG and let $Q \leq P$. Then there exists a unique primitive idempotent $f \in Z(kC_G(Q))$ such that for each primitive idempotent $e_P \in (kG)^P$ with $\beta_P(e_P)e \neq 0$ one has*

$$\beta_Q(e_P) \cdot f = \beta_Q(e_P).$$

We write in this case $(Q, f) \leq (P, e)$.

Proof (Külshammer [1]) We first prove unicity.

Let f_1 and f_2 be two primitive idempotents in $Z(kC_G(Q))$ such that for each primitive idempotent $e_P \in (kG)^P$ with $\beta_P(e_P) \neq 0$ one has

$$\beta_Q(e_P) \cdot f_1 = \beta_Q(e_P) = \beta_Q(e_P) \cdot f_2.$$

We have seen in Lemma 2.10.13 that β_Q is a ring homomorphism. By Lemma 2.10.15 there is a primitive idempotent e'_{i_0} of $(kG)^P$ with $\beta_P(e'_{i_0}) \cdot e \neq 0$. But since $C_G(Q) \geq C_G(P)$ we get $0 \neq \beta_Q(e'_{i_0})$ and therefore by the hypothesis

$$0 \neq \beta_Q(e'_{i_0}) = \beta_Q(e'_{i_0})f_2 = \beta_Q(e'_{i_0})f_1f_2.$$

This shows that $f_1f_2 \neq 0$ and since f_1 and f_2 are primitive central idempotents, $f_1 = f_2$.

We shall now prove the existence by induction on $|P : Q|$.

If $P = Q$, then we obtain that $\beta_P(e'_{i_0}) \cdot e \neq 0$. But $\beta_P(e'_{i_0})$ is a primitive idempotent in $kC_G(P)$. Indeed, this follows from the fact that

$$(kG)^P = kC_G(P) \oplus \sum_{S<P} Tr_S^P(kS)$$

and that $\sum_{S<P} Tr_S^P((kG)^S)$ is an ideal of $(kG)^P$. Therefore,

$$\beta_P(e'_{i_0}) \cdot e = \beta_P(e'_{i_0})$$

as claimed.

Suppose that $P > Q$. Then, since P is a p-group, $Q \neq N_P(Q) =: R$ (cf e.g. [21, III Hauptsatz 2.3]), and $Q < N_P(Q)$. Hence, by the induction hypothesis, there is a primitive idempotent $g \in Z(C_G(R))$ with $\beta_R(e_P) \cdot g = \beta_R(e_P)$ for every primitive idempotent $e_P \in (kG)^P$ with $\beta_P(e_P)e \neq 0$.

Now, Lemma 2.10.13 shows that β_R is a surjective ring homomorphism $(kG)^R \longrightarrow kC_G(R)$ and the restriction of β_R to $Z((kC_G(Q))^R)$ is a surjective ring homomorphism with image in $Z(kC_G(R))$. Indeed, any surjective ring homomorphism induces a (not necessarily surjective) homomorphism between the centres of the rings.

As in the first step there is a decomposition of $1 \in Z(kC_G(Q))^R$ into primitive idempotents. Since $\beta_Q : Z(kC_G(Q)^R) \to Z(kC_G(Q))$ is a ring homomorphism, by Lemma 2.10.15 there is a primitive idempotent $f \in Z((kC_G(Q))^R)$ such that $\beta_R(f) \cdot g = g$. Let f' be a primitive idempotent of $Z(kC_G(Q))$ so that $f \cdot f' = f'$, and let

$$I_{f'} := \{h \in N_G(Q) \mid h \cdot f' \cdot h^{-1} = f'\}.$$

It is clear that then $Tr_{I_{f'}}^R(f') = f$. Since $\ker \beta_R = \sum_{S<R} \mathrm{im}(Tr_S^R)$, the hypothesis $R \neq I_{f'}$ implies

$$\beta_R(f) = \beta_R(Tr_{I_{f'}}^R(f')) = 0,$$

a contradiction. Hence $R = I_{f'}$, and $f = f'$, which implies that f is actually a primitive idempotent in $Z(kC_G(Q))$.

Let e_P be a primitive idempotent of $(kG)^P$. We shall show that $\beta_Q(e_P)\cdot f = \beta_Q(e_P)$. Let S be a subgroup of P such that

$$Q < S \leq N_P(Q) = R.$$

By Lemma 2.10.15 there is a primitive idempotent e_R of $(kG)^R$ such that

$$e_R \cdot e_P = e_P \cdot e_R = e_R$$

and $\beta_R(e_R) \neq 0$. Then

$$\begin{aligned}\beta_R(e_R)g &= \beta_R(e_R \cdot e_P)g = \beta_R(e_R) \cdot \beta_R(e_P) \cdot g \\ &= \beta_R(e_R) \cdot \beta_R(e_P) = \beta_R(e_R \cdot e_P) = \beta_R(e_R) \neq 0.\end{aligned}$$

The induction hypothesis shows that there is a primitive idempotent h of $Z(kC_G(S))$ such that

$$\beta_S(e_R)h = \beta_S(e_R) \neq 0.$$

Hence

$$0 \neq \beta_S(e_R \cdot e_P)h = \beta_S(e_R) \cdot \beta_S(e_P)h$$

and therefore $\beta_S(e_P)h \neq 0$.

Again applying the induction hypothesis there is a primitive idempotent $h' \in Z(kC_G(S))$ with $\beta_S(e_P)h' = \beta_S(e_P) \neq 0$. Hence $\beta_S(e_R \cdot e_P)h' \cdot h \neq 0$ which implies $h'h \neq 0$. Since h' and h are both primitive and central, $h = h'$ and this implies $\beta_S(e_P)h = \beta_S(e_P)$.

We obtain

$$0 \neq \beta_R(e_R)g = \beta_R(e_R)\beta_R(f)g = \beta_R(\beta_Q(e_R)f)g.$$

Since $\beta_Q(e_R)$ is a primitive idempotent in $\beta_Q((kG)^R) = kC_G(Q)^R$, we get

$$\beta_Q(e_R)f = \beta_Q(e_R).$$

This shows

$$\beta_S(e_R)\beta_S(f)h = \beta_S(e_R)h \neq 0$$

and therefore

$$\beta_S(f)h \neq 0.$$

Since $\beta_S(f) \in Z(kC_G(S))$, we get $\beta_S(f)h = h$ and hence

$$\beta_S(\beta_Q(e_P)(1-f)) = \beta_S(e_P)(1-\beta_S(f)) = \beta_S(e_P) \cdot h \cdot (1-\beta_S(f)) = 0.$$

But this shows that

$$\beta_Q(e_P)(1-f) \in \bigcap_{Q<S\leq R} \ker(\beta_S|_{kC_G(Q)^S}) = Tr_Q^R(kC_G(Q)) = \beta_Q(\mathrm{im}(Tr_Q^P)).$$

Therefore

$$\beta_Q(e_P)(1-f) = (1-f)\beta_Q(e_P) \in \beta_Q(e_P \cdot \mathrm{im}(Tr_Q^P) \cdot e_P).$$

Since e_P is primitive, $e_P \cdot (kG)^P \cdot e_P$ is a local algebra, and since $e_P \cdot \mathrm{im}(Tr_Q^P) \cdot e_P$ is a proper ideal of this local algebra, it is a nilpotent ideal. Since $\beta_Q(e_P)(1-f)$ is an idempotent in this nilpotent ideal, the idempotent $\beta_Q(e_P)(1-f)$ has to be 0. Hence $\beta_Q(e_P) = \beta_Q(e_P) \cdot f$. This proves the statement. □

Corollary 2.10.17 *Let (P, e), (Q, f) and (R, g) be Brauer pairs of kG. Then*

1. *$(Q, f) \leq (P, e)$ if and only if $Q \leq P$ and there is a primitive idempotent e_P in $(kG)^P$ with $\beta_Q(e_P)e \neq 0 \neq \beta_Q(e_P)f$.*
2. *$(Q, f) \leq (R, g)$ and $(R, g) \leq (P, e)$ implies $(Q, f) \leq (P, e)$.*

Indeed, the first statement is a direct consequence of Theorem 2.10.16 and the second statement uses the first statement. □

Definition 2.10.18 Suppose (P, e) is a Brauer pair of kG. Then there is a unique block $B = kG \cdot b$ of kG such that $(1, b) \leq (P, e)$. We say that (P, e) *belongs to the block* $B = kG \cdot b$ if $(1, b) \leq (P, e)$.

Note that (P, e) belongs to the block kGb if and only if $\beta_P(b)e \neq 0$. If $(Q, f) \leq (P, e)$ and if (P, e) belongs to the block b, then Corollary 2.1017 shows that (Q, f) also belongs to B. We get a version of Sylow's theorem for Brauer pairs.

Proposition 2.10.19 *Let G be a finite group and let k be a field of characteristic $p > 0$. Let B be a block of kG with defect group P. Then there exists a Brauer pair (P, e) that belongs to B. For every Brauer pair (Q, f) that belongs to B there exists an $x \in G$ such that $(Q, f) \leq (xPx^{-1}, xex^{-1})$.*

Proof Let b be the central idempotent of kG so that $B = kGb$. By Lemma 2.10.10 there is an $a \in (kG)^P$ with $Tr_P^G(a) = b$. Then $b \in Z(kG) \subseteq (kG)^P$, and we may decompose $b = \sum_{i=1}^n e_i$ where $e_1, \ldots, e_n$ are primitive idempotents of $(kG)^P$. Now

$$\begin{aligned} b = b^2 = b \cdot Tr_P^G(a) &= \sum_{i=1}^n e_i Tr_P^G(a) \\ &= \sum_{i=1}^n Tr_P^G(e_i a) \in \sum_{i=1}^n Tr_P^G((kG)^P e_i (kG)^P). \end{aligned}$$

Since $Tr_P^G((kG)^P e_i (kG)^P)$ is an ideal of $Z(kG)$, Rosenberg's Lemma 1.9.16 shows that there is an $i_0 \in \{1, \dots, n\}$ with $b \in Tr_P^G((kG)^P e_{i_0}(kG)^P)$. If $e_{i_0} \in Tr_Q^P((kG)^Q)$ for some $Q < P$, then $b \in Tr_Q^P((kG)^Q)$ and hence $\beta_P(b) = 0$. But $\beta_P(b)e \neq 0$ since $(1, b) \leq (P, e)$. This contradiction shows that $e_{i_0} \notin Tr_Q^P((kG)^Q)$ for all $Q < P$. Again Rosenberg's Lemma 2.10.16 implies

$$e_{i_0} \notin \sum_{Q<P} Tr_Q^P((kG)^Q) = \ker \beta_P.$$

By Lemma 2.10.15 there is a primitive idempotent e of $Z(kC_G(P))$ such that $\beta_P(e_{i_0})e = \beta_P(e_{i_0}) \neq 0$. By construction (P, e) is a Brauer pair that belongs to B.

Let (Q, f) be a Brauer pair that belongs to $B = kGb$. By Corollary 2.10.17 this shows that $\beta_Q(b)f \neq 0$. We may apply Mackey's formula to get

$$b \in Tr_P^G((kG)^P e_{i_0}(kG)^P) \subseteq \sum_{QxP\in\, Q\backslash G/P} Tr^Q_{Q\cap P^x}((kG)^{Q\cap P^x} e^x_{i_0}(kG)^{Q\cap P^x})$$

and again by Rosenberg's Lemma 1.9.16 we get that there is an $x \in G$ such that $Q \leq P^x$ and $\beta_Q((kG)^Q e^x_{i_0}(kG)^Q \cdot f) \neq 0$. This implies $\beta_Q(e^x_{i_0}) \cdot f \neq 0$ and hence $(Q, f) \leq (xPx^{-1}, xex^{-1})$. □

Corollary 2.10.20 *Let k be an algebraically closed field of characteristic $p > 0$, let G be a finite group and let B be a block of kG with defect group D and central primitive idempotent b. Then there is a primitive idempotent f of $A := (kG)^D$ and a Brauer pair (D, e) that belongs to B such that*

$$b \in Tr_P^G(AfA) \subseteq AfA$$

and $\beta_D(f)e \neq 0$.

Proof This is precisely the construction of e_{i_0} in the proof of Proposition 2.10.19, and it is sufficient to take $f := e_{i_0}$. □

Remark 2.10.21 We shall use the results of this section, and in particular Theorem 2.10.16 will be used in Sect. 4.4.2.

There is another application of the preceding results concerning the relation between Green and Brauer correspondence.

Proposition 2.10.22 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let B be a block of kG with central primitive idempotent e and let D be a p-subgroup of G. Suppose H is a subgroup of G so that $C_G(D) \leq H \leq N_G(D)$. Then for every B-module M we get $M \downarrow_H^G = (\beta_D(e) \cdot M \downarrow_H^G) \oplus N$ and each indecomposable direct summand of N is relatively Q-projective for some $Q \leq H$ and $D \not\leq Q$.*

Proof $M = \beta_D(e) \cdot M \oplus (1 - \beta_D(e)) \cdot M$ as kH-modules. We define $N := (1 - \beta_D(e)) \cdot M$ and need to show that the vertex K of each indecomposable direct factor of N satisfies $D \not\leq K \leq H$. Since $eM = M$ we get that multiplication by $e - \beta_D(e)$ acts as the identity on N. By Lemma 2.10.13 and its proof we get that $kG^D = kC_G(D) \oplus \sum_{Q<D} Tr_Q^D((kG)^Q)$ and taking H-fixed points in this equation, we divide the orbits into those with trivial D-action, and those with non-trivial D-action. Therefore we obtain

$$kG^H = kC_G(D)^H \oplus \sum_{Q \leq H; D \not\leq Q} Tr_Q^H((kG)^Q).$$

Hence, $e - \beta_D(e) \in \sum_{Q \leq H; D \not\leq Q} Tr_Q^H((kG)^Q)$ and therefore

$$id_N \in \sum_{Q \leq H, D \not\leq Q} Tr_Q^H(End_{kQ}(N \downarrow_Q^H)).$$

Rosenberg's lemma and Higman's criterion show that for each indecomposable direct summand N' of N there is a $Q \leq H$ with $D \not\leq Q$ so that N' is projective relative Q. □

The following corollary generalises Proposition 2.4.3.

Corollary 2.10.23 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let D be a p-subgroup of G and let e be a central primitive idempotent of kG. Let M be an indecomposable kG-module and let $f(M)$ be its Green correspondent in $kN_G(D)$. Then $e \cdot M = M \Leftrightarrow \beta_D(e) \cdot f(M) = f(M)$.*

Proof This follows from Proposition 2.10.22 and the description of the Green correspondent. □

2.11 Blocks, Subgroups and Clifford Theory of Blocks

In this section we shall study normal defect groups and how blocks relate to this property.

2.11.1 Brauer Correspondence for Bigger Groups

Lemma 2.10.13 and its corollary has an interesting consequence for idempotents.

Proposition 2.11.1 *Let k be a field of characteristic $p > 0$ and let G be a finite group. If D is a normal p-subgroup of G, then every central idempotent of kG belongs to $kC_G(D)$.*

Proof Let e be a central idempotent of kG. Then by Lemma 2.10.13 we get

$$e \in Z(kG) = (kG)^G \subseteq (kG)^D = kC_G(D) \oplus J,$$

where we denote by $(kG)^G$ the G-fixed points of the conjugation action of G on kG. Hence $e = e_0 + r$ where $e_0 \in kC_G(D)$ and $r \in J$. Now, $J \subseteq \mathrm{rad}(kG)$ and hence J is a nilpotent ideal of $(kG)^D$. Then

$$e_0 + r = e = e^2 = e_0^2 + e_0 r + r e_0 + r^2$$

and therefore $e_0^2 = e_0$ since $e_0 r + r e_0 + r^2 \in J$, J being a two-sided ideal. Moreover,

$$e_0 + e_0 r = e_0 e = e e_0 = e_0 + r e_0$$

implies that e_0 commutes with r. Therefore, using that J is nilpotent and that $\binom{p^s}{\ell}$ is divisible by p as soon as $\ell \notin \{0, p^s\}$,

$$e_0 + r = e = e^{p^s} = e_0 + \sum_{\ell=1}^{p^s} \binom{p^s}{\ell} e_0 r^\ell = e_0$$

if $J^{p^s} = 0$. This proves the lemma. □

Remark 2.11.2 Let H be a subgroup of G such that $N_G(D) \le H \le G$. Then by Brauer's first main theorem for a block B of kG with defect group D, there is a unique block b of $kN_G(D)$ with defect group D so that b is a direct factor of $B \downarrow^{G\times G}_{N_G(D)\times N_G(D)}$. Observe that

$$B \downarrow^{G\times G}_{N_G(D)\times N_G(D)} = \left(B \downarrow^{G\times G}_{H\times H}\right) \downarrow^{H\times H}_{N_G(D)\times N_G(D)}$$

Hence if we decompose

$$B \downarrow^{G\times G}_{H\times H} = \beta_1 \oplus \cdots \oplus \beta_s$$

into blocks of kH, then there is a unique $i \in \{1, \ldots, s\}$ such that b is a direct factor of a $(\beta_i) \downarrow^{H\times H}_{N_G(D)\times N_G(D)}$. Hence there is also a unique block β of kH with defect group D such that β is a direct factor of $B \downarrow^{G\times G}_{H\times H}$. We may therefore extend Brauer correspondence to subgroups H containing $N_G(D)$.

Remark 2.11.3 Let k be a field of characteristic $p > 0$ and let G be a finite group. Let D be a p-subgroup of G and let H be a subgroup of G with $DC_G(D) \le H \le N_G(D)$. Let $e \neq 0$ be a central primitive idempotent of kH, and let $b = kH \cdot e$. Then by Proposition 2.11.1 we get that $e \in kC_G(D)$. Recall the Brauer homomorphism β_D from Definition 2.10.1. By Lemma 2.10.15 there is a unique primitive idempotent e_{i_0} of $Z(kG)$ with $e = e\beta_D(e_{i_0}) \neq 0$. For all the other primitive central idempotents $e_i \neq e_{i_0}$ in $Z(kG)$ we get $e \cdot \beta_D(e_i) = 0$. We say that B_{i_0} is the *Brauer correspondent* to b and write $B_{i_0} = b^G$. In general the Brauer correspondence for such general subgroups H is not bijective. In the case $H = N_G(D)$ this newly defined Brauer correspondence is exactly the bijective Brauer correspondence Theorem 2.3.10 we have seen earlier as a special case of Green correspondence. This is precisely the statement of Proposition 2.11.4 below.

The treatment of the next statement follows Benson [22, Sect. 6.2].

Proposition 2.11.4 *Let k be a field of characteristic $p > 0$ and let D be a p-subgroup of G. Let B be a block of kG with defect group D and let b be a block of $kN_G(D)$ with defect group D. Then b is the Brauer correspondent of B if and only if $B = b^G$.*

Proof Put $H := N_G(D)$ and let e be the central primitive idempotent such that $B = kG \cdot e$. Lemma 2.10.10 implies that if $B = kGe$ has defect group D, then H contains a conjugate of D if and only if $e \in Tr_{\Delta(H)}^{\Delta(G)}((kG)^{\Delta(H)})$. Rosenberg's Lemma 1.9.16 implies that B has defect group D if and only if $e \notin \sum_{Q<D} Tr_{\Delta(Q)}^{\Delta(G)}((kG)^{\Delta(Q)})$ but $e \in Tr_{\Delta(D)}^{\Delta(G)}((kG)^{\Delta(D)})$. We have seen in Lemma 2.10.13 and its corollary that the Brauer map $\beta_D : (kG)^D \longrightarrow kC_G(D)$ has a kernel which is generated by $\sum_{Q<D} Tr_Q^D(kG)$.

We obtained as an intermediate step that the block generated by e has defect group D if and only if $\beta_D(e) \neq 0$ and $e \in Tr_{\Delta(D)}^{\Delta(G)}((kG)^{\Delta(D)})$.

Next we get for each $x \in (kG)^D$, using Mackey's formula, that

$$\beta_D(Tr_D^G(x)) = \beta_D\left(\sum_{N_G(D)gD \in N_G(D)\backslash G/D} Tr_{N_G(D)\cap {}^gD}^{N_G(D)}(gx)\right) = \beta_D(Tr_D^{N_G(D)}(x))$$

since in the sum the terms for $N_G(D)gD \neq N_G(D)$ all disappear under the Brauer homomorphism. Now, $\beta_D(Tr_D^{N_G(D)}(x)) = Tr_D^{N_G(D)}(\beta_D(x))$. Indeed, the space $(kG)^{N_G(D)}$ is spanned by $N_G(D)$-orbits. Those orbits on which D acts trivially span precisely $(kC_G(D))^{N_G(D)}$ and the other orbits span $\sum_{D<Q\le N_G(D)} Tr_Q^{N_G(D)}(kG)$. Now, β_D projects onto $(kC_G(D))^{N_G(D)}$. This argument also shows that β_D induces a surjective map $Tr_D^G((kG)^D) \xrightarrow{\beta_D} Tr_D^{N_G(D)}((kC_G(D))^D)$. By Proposition 2.11.1 every central primitive idempotent of $kN_G(D)$ is an element in $kC_{N_G(D)}(D)$. If such a block has defect group D then the primitive central idempotent of this block belongs to $Tr_{\Delta(D)}^{\Delta(N_G(D))}((kC_G(D))^{\Delta(D)})$. Hence for a given block $B = kGe$ with defect group D there is a unique block b of $kN_G(D)$ with defect group D so that $B = b^G$. The Brauer correspondent of $B = kGe$ is the unique direct factor b of $B\downarrow_{N_G(D)\times N_G(D)}^{G\times G}$ with defect group D. Hence $\beta_D(e) \cdot b \neq 0$, and therefore b is the Brauer correspondent of B. $\square$

Remark 2.11.5 Let G be a finite group, let k be an algebraically closed field of characteristic $p > 0$, and let D be a p-subgroup of G. Suppose $DC_G(D) \le H \le G$ and let b be a block of kH with defect group D. Then by the Brauer correspondence and Proposition 2.11.4 there is a unique block b' of $kN_H(D)$ with defect group D so that $b = (b')^H$. But $N_H(D) \le N_G(D)$ and so $DC_G(D) \le N_H(D) \le N_G(D) \le G$. Hence we are in the same setting as Remark 2.11.3 and we know that there is a unique block B of kG such that $B = (b')^G$.

Lemma 2.11.6 (cf Benson [22, Lemma 6.2.7]) *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let D be a p-group and suppose $D \cdot C_G(D) \le H \le G$ for some subgroup H of G. Let b be a block of kH with defect group D. Then b^G is the unique block B of kG such that b is a direct factor of $B\downarrow_{H\times H}$.*

Proof Since by the proof of Proposition 2.3.4

$$kG = k((G\times G)/\Delta(G)) = k(G\times G)\otimes_{k\Delta(G)} k = k\uparrow_{\Delta(G)}^{G\times G}$$

we may apply Mackey's theorem and obtain

$$\begin{aligned} kG\downarrow_{H\times H}^{G\times G} &= k\uparrow_{\Delta(G)}^{G\times G}\downarrow_{H\times H} \\ &= \bigoplus_{(H\times H)(1,g)\Delta(G)\in(H\times H)\backslash(G\times G)/\Delta(G)} k\uparrow_{(H\times H)\cap\ {}^{(1,g)}\Delta(G)}^{H\times H} \\ &= \bigoplus_{(H\times H)(1,g)\Delta(G)\in(H\times H)\backslash(G\times G)/\Delta(G)} k\uparrow_{{}^{(1,g)}\Delta(H\cap\ {}^{g^{-1}}H)}^{H\times H}. \end{aligned}$$

If

$${}^{(1,g)}\Delta(H\cap\ {}^{g^{-1}}H) = \{(h, ghg^{-1}) \mid h\in H\cap\ {}^{g^{-1}}H\}$$

contains a subgroup of the form

$${}^{(h_1,h_2)}\Delta(D) = \{(h_1dh_1^{-1}, h_2dh_2^{-1}) \mid d\in D\}$$

for some $(h_1, h_2)\in H\times H$ then

$$h = h_1dh_1^{-1} \text{ and } ghg^{-1} = h_2dh_2^{-1}$$

which implies

$$gh_1dh_1^{-1}g^{-1} = h_2dh_2^{-1}$$

and hence

$$h_2^{-1}gh_1 \in C_G(D) \le H.$$

This shows $g\in H$. Therefore the only summands in

$$kG\downarrow_{H\times H}^{G\times G} = \bigoplus_{(H\times H)(1,g)\Delta(G)\in(H\times H)\backslash(G\times G)/\Delta(G)} k\uparrow_{{}^{(1,g)}\Delta(H\cap\ {}^{g^{-1}}H)}^{H\times H}$$

which can have defect group D are those with $g \in H$, or what is the same, the class given by $(1, 1)$. This term is obtained by putting $g = 1$ in the above direct sum, and is hence
$$k \uparrow_{\Delta(H)}^{H \times H} = kH.$$

This shows that $kG \downarrow_{H \times H}^{G \times G}$ has only one direct factor with vertex $\Delta(D)$, and therefore only one direct factor isomorphic to b. This implies that there is only one block $\check{B}$ of kG such that $\check{B} \downarrow_{H \times H}$ has a direct factor isomorphic to b.

We need to show that $\check{B} = b^G$. Suppose first that $H \leq N_G(D)$. Then let B be a block of kG such that b is not isomorphic to a direct factor of $B \downarrow_{H \times H}$. Let e and e' be the central primitive idempotents such that $b = e \cdot kH$ and $B = e' \cdot kG$. Consider kH as a direct factor of kG, and observe that the projection of $e \cdot e'$ onto this direct factor is 0. Hence $e\beta_D(e') = \beta_D(ee') = 0$ and therefore $B \neq b^G$.

If H is not contained in $N_G(D)$, then we may first pass to $N_H(D) \leq N_G(D)$, and then to H as is indicated in Remark 2.11.5. We have proved the lemma. □

2.11.2 Blocks and Normal Subgroups

This section closely follows the treatment in Benson [22].

The case of a normal p-subgroup D of G is particularly important. In this connection we prove the following general statement.

Lemma 2.11.7 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let $D \trianglelefteq G$ and D be a p-subgroup. Then the natural projection $G \longrightarrow G/D$ induces a ring epimorphism $kG \longrightarrow kG/D$ with nilpotent kernel $I(kD)G$.*

Proof The fact that $G \longrightarrow G/D$ induces a ring epimorphism $\pi_D : kG \longrightarrow kG/D$ follows from Lemma 1.2.3. The kernel $I(kD)G$ of this ring epimorphism is a two-sided ideal generated by the elements $(d-1)$ for $d \in D$. The ideal $I(kD)$ is a nilpotent ideal of kD by Proposition 1.6.22. Now $I(kD)G$ is generated as a k-vector space by the elements $(d-1)g$ for all $d \in D$ and $g \in G$, and hence for $d, d' \in D$ and $g, g' \in G$ we get
$$(d-1)g \cdot (d'-1)g' = (d-1)((gd'g^{-1}) - 1)gg'$$

where we observe that $gd'g^{-1} \in D$ since D is normal in G. This shows that $I(kD)G$ is nilpotent, and hence is in the radical of kG. □

Remark 2.11.8 The combination of Lemma 2.11.7 and Lemma 1.9.17 shows that if D is a normal p-subgroup of G, then we may lift idempotents of kG/D to idempotents of kG. However, a central idempotent of kG/D need not be lifted to a central idempotent of kG. Nevertheless, Proposition 2.11.1 gives the relevant statement for central idempotents. Lifting is not the correct concept there.

Definition 2.11.9 Let k be a field and let G be a finite group. Let e be an idempotent of kG. Denote by

$$I_G(b) := I_G(e) := \{g \in G \mid geg^{-1} = e\}$$

the *inertia group* of e, respectively of b.

It is clear that if e is an idempotent of kG, then geg^{-1} is again an idempotent of kG and that $I_G(e)$ is a subgroup of G. A particularly interesting use of this fact is the following. Let N be a normal subgroup of G and let b be a block of kN. Given $g \in G$, then $g \cdot b \cdot g^{-1}$ is a direct factor of

$$g \cdot kN \cdot g^{-1} = k(gNg^{-1}) = kN,$$

and hence is again a block of kN. If $b = e \cdot kN$, then

$$g \cdot b \cdot g^{-1} = b \Leftrightarrow g \cdot e \cdot g^{-1} = e \Leftrightarrow g \in I_G(e)$$

and moreover $g \cdot e \cdot g^{-1}$ is again a primitive central idempotent of kN. Hence,

$$(*) \quad b \cap (g \cdot b \cdot g^{-1}) = 0 \text{ if } g \in G \setminus I_G(e).$$

Therefore we may consider

$$\begin{aligned}\hat{b} &:= \bigoplus_{gI_G(b) \in G/I_G(b)} g \cdot b \cdot g^{-1} = \bigoplus_{gI_G(b) \in G/I_G(b)} g \cdot kNe \cdot g^{-1} \\ &= \bigoplus_{gI_G(b) \in G/I_G(b)} kN \cdot (g \cdot e \cdot g^{-1}),\end{aligned}$$

where the last equality follows since $N \trianglelefteq G$. Let

$$\sum_{gI_G(b) \in G/I_G(b)} g \cdot e \cdot g^{-1} =: \hat{e}.$$

Then

$$kN \cdot \hat{e} \subseteq \hat{b} \subseteq kG \cdot \hat{e}$$

by definition. Moreover,

$$B := kG \cdot \hat{e}$$

is a direct sum of blocks of kG. Indeed, $\hat{e}$ is an idempotent of kG since

$$(g \cdot e \cdot g^{-1}) \cdot (h \cdot e \cdot h^{-1}) = g \cdot \left(e \cdot \left((g^{-1}h) \cdot e \cdot (g^{-1}h)^{-1}\right)\right) \cdot g^{-1} = 0$$

if $g^{-1}h \notin I_G(b)$ by $(*)$. Then $\hat{e}$ is central since e is central in kN and conjugation by any $g \in G$ fixes $\hat{e}$ by construction.

However, in general, $\hat{e}$ is not primitive. If f is a central primitive idempotent of kG, then either $f\hat{e} = 0$, or $f\hat{e} = f$. We obtain $f\hat{e} = f$ if and only if $fe = e$.

Hence, for every block b of kN there are (possibly many) blocks $B = kGf$ of kG such that $B \cdot b = b$. In this case we say that B *covers* b and write $B = b^G$. We have proved the following.

Proposition 2.11.10 *Let k be a field of characteristic $p > 0$, let G be a finite group and let N be a normal subgroup of G. Let $b = kN \cdot e$ be a block of kN with block idempotent $e^2 = e \in Z(kN)$.*

- *Then for $\sum_{gI_G(b) \in G/I_G(b)} g \cdot e \cdot g^{-1} =: \hat{e}$ we get that $\hat{e}^2 = \hat{e} \in Z(kG)$ and $kG \cdot \hat{e}$ is a direct sum of blocks $B = kGf$ of kG. These are characterised by the equation $B \cdot \hat{e} = B$ or $f \cdot e = e$. We say that B covers b and write $B = b^G$.*
- *If b has defect group D and if $C_G(D) \leq N$, then there is a unique block B covering b.*
- *Moreover, precisely the $|G/I_G(b)|$ blocks $g \cdot b \cdot g^{-1}$ for $gI_G(b) \in G/I_G(b)$ are covered by B.*

Proof The first part was shown above. The third part is part of the definition of the idempotent $\hat{e}$. The only statement left to prove is that B is the unique block covering b if $C_G(D) \leq N$. But this statement is actually a consequence of Lemma 2.11.6. □

We even get an explicit description of $kG \cdot \hat{e}$ in terms of $kI_G(e) \cdot e$.

Lemma 2.11.11 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let $N \trianglelefteq G$ and let $b = kN \cdot e$ be a block of kN with $e^2 = e \in Z(kN)$. Suppose B is a block of kG covering b. Put $\hat{e} := \sum_{gI_G(b) \in G/I_G(b)} geg^{-1}$. Then there is a defect group D of B contained in $I_G(b)$. Moreover*

$$kG\hat{e} \simeq Mat_{|G/I_G(b)|}(kI_G(b) \cdot e).$$

Proof We need to restrict $kG \cdot \hat{e}$ to $N \times N$. First,

$$kG \simeq \bigoplus_{gN \in G/N} g \cdot kN$$

as $k(N \times N)$-modules, using the fact that N is normal in G. Further, the definition of $I_G(b)$ implies that b is actually a $\Delta(I_G(b)) \cdot (N \times N)$-module. Then we get the following isomorphisms as $k(N \times N)$-modules

$$\begin{aligned}
\hat{e} \cdot kG &= \hat{e} \cdot \left(\bigoplus_{gN \in G/N} g \cdot kN \right) \\
&= \bigoplus_{gN \in G/N} \hat{e} \cdot (g \cdot kN) \\
&= \bigoplus_{gN \in G/N} \bigoplus_{hI_G(b) \in G/I_G(b)} \left(h \cdot e \cdot h^{-1} \cdot g \cdot kN \right) \\
&= \bigoplus_{gN \in G/N} \bigoplus_{hI_G(b) \in G/I_G(b)} \left(h \cdot e \cdot (h^{-1} \cdot g) \cdot kN \cdot (g^{-1} \cdot h) \cdot (h^{-1} \cdot g) \right) \\
&= \bigoplus_{gN \in G/N} \bigoplus_{hI_G(b) \in G/I_G(b)} \left(h \cdot e \cdot kN \cdot (h^{-1} \cdot g) \right) \\
&= \bigoplus_{(h_1, h_2) \in (G \times G)/(\Delta(I_G(b)) \cdot (N \times N))} \left(h_1 \cdot e \cdot kN \cdot h_2^{-1} \right) \\
&= k(G \times G) \otimes_{k(\Delta(I_G(b)) \cdot (N \times N))} b
\end{aligned}$$

where we put $h_1 = h$ and $h_2 = g^{-1}h$ in the second last isomorphism. Now, the defect group is the smallest group D such that B is $\Delta(D)$-projective. Hence

$$\Delta(D) \leq \Delta(I_G(b)) \cdot (N \times N).$$

Since $N \leq I_G(b)$, since a defect group D of B is a subgroup of $\Delta(G)$ by Proposition 2.3.4, and since

$$\Delta(G) \cap (\Delta(I_G(b)) \cdot (N \times N)) = \Delta(I_G(b))$$

we get that $D \leq I_G(b)$.

We can consider $e \cdot kG$ as a $kI_G(b)$-left module and observe that

$$e \cdot kG = \bigoplus_{gI_G(b) \in G/I_G(b)} (e \cdot kI_G(b)) \cdot g^{-1}$$

so that $e \cdot kG$ is free of rank $|G/I_G(b)|$ as an $e \cdot kI_G(b)$-left module. Hence there is a ring isomorphism

$$End_{e \cdot kI_G(b)}(e \cdot kG)^{op} \simeq Mat_{|G/I_G(b)|}(e \cdot kI_G(b)).$$

Now, we observe $\hat{e}$ is central in kG and hence we may define a mapping

$$\begin{array}{rcl}
\hat{e} \cdot kG & \longrightarrow & End_{e \cdot kI_G(b)}(e \cdot kG)^{op} \\
\hat{e} \cdot x & \mapsto & (ey \mapsto ey\hat{e}x = e\hat{e}yx = eyx)
\end{array}$$

We see immediately that this map is a ring homomorphism. Consider the kernel of this ring homomorphism:

If $ey \mapsto eyx = 0$ for all $y \in kG$, then

$$\begin{aligned}\hat{e}x = (\hat{e}\hat{e})x = \hat{e}(\hat{e}x) &= \sum_{gI_G(b)\in G/I_G(b)} geg^{-1}(\hat{e}x) \\ &= \sum_{gI_G(b)\in G/I_G(b)} ge((g^{-1}\hat{e})x) = \sum_{gI_G(b)\in G/I_G(b)} g\cdot 0 = 0\end{aligned}$$

since $(g^{-1}\hat{e})$ is a possible y. Hence the ring homomorphism is injective.

We claim that the ring homomorphism is surjective as well. For this have

$$\begin{aligned}\hat{e}\cdot kG &= \bigoplus_{(h_1,h_2)\in(G\times G)/(\Delta(I_G(b))\cdot(N\times N))} \left(h_1\cdot e\cdot kN\cdot h_2^{-1}\right) \\ &= \bigoplus_{(h_1,h_2)\in(G\times G)/(I_G(b)\times I_G(b))} \left(h_1\cdot e\cdot kI_G(b)\cdot h_2^{-1}\right)\end{aligned}$$

as $k(N\times N)$-modules. Moreover, the lines and columns of $Mat_{|G/I_G(b)|}(e\cdot kI_G(b))$ are parameterised by classes $gI_G(b)$. The element $x\in ekI_G(b)$ in row $g_1I_G(b)$ and column $g_2I_G(b)$, and 0 elsewhere, is realised by right multiplication by $g_1xg_2^{-1}$. This proves the statement. □

Proposition 2.11.12 *Let k be a field of characteristic $p>0$ and let G be a finite group with normal subgroup N. If b is a block of kN generated by the central idempotent e, let B be a block of kG covering b. Then there is a defect group D of B such that $D\cap N$ is a defect group of b.*

Proof Let D be a defect group of B. Since by the proof of Proposition 2.3.4

$$kG = k(G\times G)\otimes_{\Delta(G)} k,$$

the $k(G\times G)$-module B is a direct factor of $k\uparrow_{\Delta(D)}^{G\times G}$ and $B\downarrow_{\Delta(D)}^{G\times G}$ has the trivial module k as a direct factor. This shows that $B\downarrow_{(\Delta(D)\cap(N\times N)}^{G\times G}$ also has the trivial module as a direct factor. Hence, some direct factor of the restriction $B\downarrow_{N\times N}^{G\times G}$ has vertex in $\Delta(D)\cap(N\times N)=\Delta(D\cap N)$.

We shall consider now $B\downarrow_{N\times N}^{G\times G}$. As in the proof of Lemma 2.11.11, we get that $B\downarrow_{N\times N}^{G\times G}$ is a direct sum of $k(N\times N)$-modules of the form $g_1bg_2^{-1}$ for $g_1,g_2\in G$. Denote the defect group of b by D_b and recall that then $\Delta(D_b)$ is the vertex of b as a $k(N\times N)$-module. Then the vertex of each of the $k(N\times N)$-modules $g_1bg_2^{-1}$ are $G\times G$-conjugates of $\Delta(D_b)$ and hence some G-conjugate of $D\cap N$ is in D_b.

Conversely, since b is a direct factor of $B\downarrow_{N\times N}^{G\times G}$, b is relatively projective to some $\Delta(G)$-conjugate of $\Delta(D\cap N)$. Hence we get equality and the result is proven. □

Although in the previous results of this subsection we did not need the base field to be algebraically closed, in the next lemma we do need this hypothesis.

Lemma 2.11.13 *Let k be an algebraically closed field of characteristic $p > 0$ and let G be a finite group. Let N be a normal subgroup of G and let $b = kN \cdot e$ be a block of kN where $e^2 = e \in Z(kN)$, and suppose that b is covered by the block B of kG with defect group D chosen so that $D \cap N$ is a defect group of b. If $C_G(D \cap N) \leq N$ then $|I_G(b) : D \cdot N|$ is not divisible by p.*

Proof By Lemma 2.11.6 B is the only block of kG such that b is a direct factor of $B^{G\times G}_{N\times N}$, and hence B is the unique block covering b. As in the proof of Lemma 2.11.11, by Definition of $I_G(b)$ we see that b is a $\Delta(I_G(b)) \cdot (N \times N)$-module. Since b is a block of kN, the restriction of b as a $\Delta(I_G(b)) \cdot (N \times N)$-module to $N \times N$ is still indecomposable.

Let $S \in Syl_p(I_G(b))$. Then the restriction of b to $\Delta(S) \cdot (N \times N)$ is still indecomposable. Since by hypothesis $D \cap N$ is a defect group of b, the $\Delta(S) \cdot (N \times N)$-module b is projective relative to $\Delta(D) \cdot (N \times N)$. Green's indecomposability Theorem 2.2.12 shows that if L is a $\Delta(D) \cdot (N \times N)$-module such that b is a direct factor of $L \uparrow^{\Delta(S)\cdot(N\times N)}_{\Delta(D)\cdot(N\times N)}$, then this induced module is indecomposable. Hence

$$b \simeq L \uparrow^{\Delta(S)\cdot(N\times N)}_{\Delta(D)\cdot(N\times N)} .$$

But $b \downarrow^{\Delta(S)\cdot(N\times N)}_{\Delta(D)\cdot(N\times N)}$ is indecomposable, as we have just seen. Therefore, by Mackey's formula,

$$\begin{aligned} b \downarrow^{\Delta(S)\cdot(N\times N)}_{\Delta(D)\cdot(N\times N)} &= L \uparrow^{\Delta(S)\cdot(N\times N)}_{\Delta(D)\cdot(N\times N)} \downarrow^{\Delta(S)\cdot(N\times N)}_{\Delta(D)\cdot(N\times N)} \\ &= \bigoplus_{DNsDN \in D\backslash S/D} {}^{s}L \downarrow_{\Delta(DN \cap {}^{s}DN)\cdot(N\times N)} \uparrow^{\Delta({}^{s}D)\cdot(N\times N)} . \end{aligned}$$

Since the left-hand side is indecomposable, the right-hand side is indecomposable as well and there is only one double-class. Therefore $S \cdot N = D \cdot N$. Hence $|I_G(b) : DN|$ is not divisible by p. $\square$

Proposition 2.11.14 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let Q be a p-group of G and suppose $G = Q \cdot C_G(Q)$. Then the natural map $G \longrightarrow G/Q$ induces a bijective correspondence between blocks of kG with defect group D and blocks of kG/Q with defect group D/Q.*

Proof Since $Q \trianglelefteq (Q \cdot C_G(Q)) = G$ we may apply Lemma 2.4.2 to get $Q \leq D$. Lemma 2.11.1 shows that each primitive central idempotent of kG belongs to $kC_G(Q)$. But $G = Q \cdot C_G(Q)$ implies

$$Z(G) = Z(Q \cdot C_G(Q)) \geq Z(C_G(Q))$$

and so the primitive central idempotents of kG are precisely the central primitive idempotents of $kC_G(Q)$. Finally, the kernel of $kG \longrightarrow kG/Q$ is generated as an ideal by the elements $q-1$, for $q \in Q$, and is therefore nilpotent. Moreover,

$$kG/Q = k((Q \cdot C_G(Q))/Q) = k(C_G(Q)/(Q \cap C_G(Q))) = k(C_G(Q)/Z(Q))$$

and therefore the commutative diagram

$$\begin{array}{ccc} Z(kC_G(Q)) & \hookrightarrow & kG \\ \downarrow & & \downarrow \\ Z(k(G/Q)) & \hookrightarrow & kG/Q \end{array}$$

has surjective vertical morphisms. Hence every central primitive idempotent of kG/Q lifts to a central primitive idempotent of kG, and of course every central primitive idempotent of kG maps to a central primitive idempotent of kG/Q. This proves the statement. □

We summarise what we proved so far.

Proposition 2.11.15 (cf Benson [22, Theorem 6.4.3]) *Let k be an algebraically closed field of characteristic $p > 0$, let G be a finite group. Then there is a bijection between each of the following sets.*

1. *Blocks of kG with defect group D.*
2. *Blocks of $kN_G(D)$ with defect group D.*
3. *$N_G(D)$-conjugacy classes of blocks b of $kDC_G(D)$ with defect group D such that p does not divide $|I_{N_G(D)}(b) : DC_G(D)|$.*
4. *$N_G(D)$-conjugacy classes of blocks b of $kDC_G(D)/D$ of defect 0 such that p does not divide $|I_{N_G(D)}(b) : DC_G(D)|$.*

Proof The bijection between (1) and (2) is the Brauer correspondence. The bijection between (3) and (4) is Proposition 2.11.14. The bijection between (2) and (3) is obtained as follows. Since $C_G(D) = C_{N_G(D)}(D)$, we may assume that $G = N_G(D)$ and we may therefore assume that $D \trianglelefteq G$. Let $N := DC_G(D)$, let B be a block of kG with defect group D and let b be a block of kN that is covered by B. Lemma 2.11.13 shows that p does not divide $|I(b) : DC_G(D)|$ and by Proposition 2.11.12 b also has defect group D. Proposition 2.11.10 shows that the blocks covered by B form a conjugacy class under the $N_G(D)$-action.

Conversely if b is a block of kN, then by Proposition 2.11.10 there is a unique block B of kG covering b. By Lemma 2.11.11 the defect group D of B can be chosen so that $D \leq I(b)$. But p does not divide $|I_{N_G(D)}(b) : DC_G(D)|$, so that $I_{N_G(D)}(b) \cap N = D$. □

2.12 Representation Type of Blocks, Cyclic Defect

Since $D \trianglelefteq N_G(D)$ we see that it is important in view of the Brauer and Green correspondence to study the blocks of a group ring with normal defect group. The case of a normal cyclic defect group is particularly simple.

2.12.1 The Structure of Blocks with Normal Cyclic Defect Group

The results of this section are partially a special case of the results of Sect. 2.8. However, the abstract case with which we dealt there is made explicit here for group rings and the structure constants which we obtained there get a group theoretical interpretation here.

Proposition 2.12.1 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let B be a block of G with normal cyclic defect group $D \trianglelefteq G$. Then* $\mathrm{rad}(B) = (d-1)B$ *where d is a generator of D. Moreover each indecomposable projective B-module is uniserial of Loewy length $|D|$. If m is the number of isomorphism classes of simple B-modules, then there are exactly $m \cdot |D|$ isomorphism classes of indecomposable B-modules.*

Proof Let $kG \cdot e = B$ for an idempotent $e \in Z(kG)$. Since D is normal, $\mathrm{rad}(kD) \cdot B = \mathrm{rad}(kD) \cdot kG \cdot e$ is an ideal in $\mathrm{rad}(B)$ by Lemma 2.11.7, i.e. $\mathrm{rad}(kD) \cdot B \subseteq \mathrm{rad}(B)$. Moreover, since D is cyclic, $\mathrm{rad}(kD) = (d-1) \cdot kD$ and since D is normal in G, we get

$$(d-1) \cdot kG = kG \cdot (d-1)$$

and therefore

$$(d-1) \cdot B = B \cdot (d-1) \subseteq \mathrm{rad}(B).$$

We shall show that $B/(d-1)B$ is semisimple. Indeed, since B is a direct factor of kG, we get that $B/(d-1)B$ is a direct factor of $kG/(d-1)kG = k(G/D)$. Let

$$0 \longrightarrow S \longrightarrow U \longrightarrow T \longrightarrow 0$$

be an exact sequence of $B/(d-1)B$-modules. Hence D acts trivially on each of these modules. Since k is a field, and all short exact sequences of k-modules are split as a sequence of k-modules, this shows that the sequence is actually split as a sequence of kD-modules. Since S, T and U are B-modules, they have their vertex in D, hence are relatively D-projective by Proposition 2.3.6. This shows that the sequence

$$0 \longrightarrow S \longrightarrow U \longrightarrow T \longrightarrow 0$$

is actually split as a sequence of kG-modules. Hence we obtain the result. Theorem 2.8.4 shows that B is serial. Lemma 2.4.7 shows that B is actually a symmetric algebra. Lemma 2.8.2 shows that each indecomposable B-module is uniserial. All projective indecomposable B-modules have the same Loewy length, and this Loewy length is obviously the nilpotency degree of $\mathrm{rad}(kD)$, whence $|D|$. Moreover, each indecomposable B-module is uniserial, hence isomorphic to some $P/\mathrm{rad}^n P$ for some integer n and a projective indecomposable B-module P. There are precisely $|D| \cdot m$ such modules, if there are m isomorphism classes of simple B-modules. □

As a corollary we mention a result due to Michler.

Corollary 2.12.2 (Michler) *Let B be a block with normal cyclic defect group $D = \langle d | d^{p^m} \rangle$, and let M be an indecomposable B-module. Then there is an integer s and a primitive idempotent e of B such that $M \simeq B \cdot (d-1)^s \cdot e$. In particular there are only finitely many isomorphism classes of indecomposable B-modules.* □

Corollary 2.12.3 *Let B be a block with normal cyclic defect group, then each projective indecomposable B-module is uniserial.* □

Proposition 2.12.4 *Let k be an algebraically closed field and let G be a finite group. Let B be a block of kG with normal cyclic defect group D. Then there are e isomorphism classes of simple B-modules represented by $S_1, S_2, \dots, S_n$. The simple modules can be numbered in such a way that for the projective covers P_i of S_i one gets* $\mathrm{rad}^i P_j / \mathrm{rad}^{i+1} P_j \simeq S_{i+j}$ *for all $j \in \{1, \dots, n\}$ and for all $i \in \{0, 1, \dots, |D|\}$. Moreover, n divides $|D| - 1$.*

Proof The only new statement is the fact that n divides $|D|-1$. But this follows from the fact that B is symmetric, and that the Loewy structure of B is the same as for the Nakayama algebra N_n^{en+1}. Since $en + 1 = |D|$, we see that n divides $|D| - 1$. The rest is just a reformulation of Theorem 2.8.4, Lemma 2.4.7 and Lemma 2.8.2. □

Remark 2.12.5 We will come back to the theory of blocks with cyclic defect group D in Sect. 5.10. Blocks with cyclic defect group are actually very well understood and the structure described in Proposition 2.12.4 is its simplest model. Indeed, the attentive reader will observe that the composition series of the blocks with cyclic normal defect group are the same as for the Nakayama algebra $N_n^{|D|+1}$. In Example 4.4.3 we will see that this is actually a very close correspondence, a Morita equivalence, which will be introduced in Chap. 4.

2.12.2 Blocks of Finite Representation Type

We are ready to deal with the number of isomorphism classes of indecomposable modules. Cyclic defect groups play a prominent role here.

Definition 2.12.6 Let k be a commutative ring and let A be a k-algebra satisfying the Krull-Schmidt theorem. If there are only a finite number of isomorphism classes of finitely generated indecomposable A-modules, then we say that A is of *finite representation type.*

Algebras satisfying the Krull-Schmidt theorem and which are not of finite representation type are of *infinite representation type.*

Remark 2.12.7 We shall develop this theme further in Sect. 5.10. The notion of representation type is crucial for the representation theory of finite dimensional algebras.

We have seen in Corollary 2.12.2 that blocks with cyclic defect groups are of finite representation type. As a first result in this section we shall prove the converse: blocks of finite representation type have cyclic defect group.

We shall start with an example.

Example 2.12.8 Let p be a prime number, let k be an algebraically closed field of characteristic $p > 0$ and let

$$G = C_p \times C_p = \langle x, y \mid x^p, y^p, xyx^{-1}y^{-1} \rangle$$

be the direct product of two cyclic groups of order p. We claim that there are infinitely many isomorphism classes of indecomposable kG-modules.

We see that

$$\begin{aligned} kG &\simeq k[X, Y]/(X^p - 1, Y^p - 1) \simeq k[X, Y]/((X - 1)^p, (Y - 1)^p) \\ &\simeq k[X, Y]/(X^p, Y^p) \end{aligned}$$

just as we did for the group ring of a cyclic group of prime power order. We may write this algebra as a quiver algebra with one vertex, two loops ξ and η and relations $\xi^p, \eta^p, \xi\eta - \eta\xi$.

A representation of dimension d of this algebra is therefore a vector space V of dimension d together with two commuting endomorphisms x and y of V which are nilpotent of degree at most p. Isomorphism classes are realised by simultaneous conjugation of these endomorphisms. Hence we may suppose that one of these endomorphisms, say x, is in Jordan normal form.

Let $d = p$. Suppose moreover now that x has only one Jordan block, that is x is nilpotent of degree p, and not of lower degree. Hence we can choose a basis of V so that x is represented by the matrix

$$N := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}.$$

Then an elementary matrix multiplication shows that y commutes with x if and only if

$$y = \sum_{j=0}^{p-1} \alpha_j N^j$$

for $\alpha_j \in k$. We claim that if $\alpha_j = 0$ for all $j \neq 1$ we obtain an indecomposable representation $R(\alpha_1)$, and two of these representations $R(\alpha_1)$ and $R(\beta_1)$ are isomorphic if and only if $\alpha_1 = \beta_1$.

First, $R(\alpha_1)$ is indecomposable since its restriction to the action of x is the regular $k[X]/X^p$-module, and hence is already indecomposable.

Second, if $R(\alpha_1) \simeq R(\beta_1)$, then there is an invertible endomorphism σ of V commuting with x and conjugating $\alpha_1 x$ to $\beta_1 x$. But since σ commutes with x, it commutes with $\alpha_1 x$ as well.

Since k is algebraically closed, and hence infinite, we have proved the statement.

What is the vertex of the module $R(\alpha)$? Since $R(\alpha)$ is of dimension p and since kG is a local algebra of dimension p^2, the module $R(\alpha)$ is not a projective module. Hence, the vertex of $R(\alpha)$ is at least isomorphic to C_p. There are p different subgroups isomorphic to C_p in C_p^2. Since there are only finitely many isomorphism classes of indecomposable modules of kC_p, only finitely many isomorphism classes of modules can have vertex C_p for each of the finitely many copies of $C_p \leq C_p^2$. Since there are infinitely many indecomposable modules of the form $R(\alpha)$ we know that infinitely many of these have vertex $G = C_p^2$.

We can now formulate the converse to Corollary 2.12.2.

Proposition 2.12.9 *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let B be a block with defect group D. If B is of finite representation type, then D is a cyclic group. In particular a block of a finite group over a field k of characteristic $p > 0$ is of finite representation type if and only if its defect group is cyclic.*

Proof Suppose to the contrary that D is not cyclic. Then, denoting by D' the commutator subgroup, D/D' is abelian not cyclic, and hence there is a quotient $D \longrightarrow\!\!\!\!\rightarrow C_p \times C_p$. Since by Example 2.12.8 there are infinitely many indecomposable $k(C_p \times C_p)$-modules with vertex $C_p \times C_p$, using the morphism $D \longrightarrow\!\!\!\!\rightarrow C_p \times C_p$ there are infinitely many indecomposable kD-modules. If M is a $k(C_p \times C_p)$-module with vertex $C_p \times C_p$, then M, regarded as a kD-module, has vertex D. Indeed, if M is a direct factor of $M \downarrow_S \uparrow^D$ for a proper subgroup S of D, since the kernel of the projection $D \longrightarrow\!\!\!\!\rightarrow C_p \times C_p$ acts trivially on M, the group S is actually the preimage in D of a subgroup $\overline{S}$ of $C_p \times C_p$. Since the vertex of M is $C_p \times C_p$, we get that $\overline{S} = C_p \times C_p$ and hence $S = D$.

We form $M \uparrow_D^{N_G(D)}$. Since D is a p-group, D acts trivially on any simple $kN_G(D)$-module, and hence any simple $kN_G(D)$-module is actually a $kN_G(D)/D$-module. However, the socle of M is a trivial D-module k^s. Hence $kN_G(D) \otimes_{kD} k^s \simeq (kN_G(D)/D)^s$ is a submodule of $M \uparrow_D^{N_G(D)}$. But the socle of $kN_G(D)$ is the same as the socle of $kN_G(D)/D$ and hence any simple $kN_G(D)$-module is a submodule of $M \uparrow_D^{N_G(D)}$. Therefore, for every block b' of $kN_G(D)$ we get $b' \cdot M \uparrow_D^{N_G(D)} \neq 0$.

Let b be the Brauer correspondent of B in $kN_G(D)$. Then $b \cdot M \uparrow_D^{N_G(D)} \neq 0$ by the above. Let M' be an indecomposable direct factor of $b \cdot M \uparrow_D^{N_G(D)}$. Now $M \uparrow_D^{N_G(D)} \downarrow_D$ is a direct sum of $N_G(D)$-conjugates of M, the module M is a source of M'. This implies that two non-isomorphic, non-$N_G(D)$-conjugate $k(C_p \times C_p)$-modules M_1 and M_2 as constructed above yield two non-isomorphic b-modules M_1' and M_2'. Since for a given M there are only finitely many $N_G(D)$-conjugates, we obtain the statement. □

Example 2.12.10 The above construction is actually an instance of what we have seen already. We return to Example 1.6.23. Recall that the Kronecker quiver Q is given by

$$Q: \quad \begin{tikzcd} \bullet \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \bullet \end{tikzcd}$$

and we have seen in Example 1.6.23 that for every field K there are infinitely many isomorphism classes of indecomposable KQ-modules of dimension 2, and that these modules are parameterised by a projective line. Now, in identifying the two idempotents of KQ we observe that the algebra KQ has a quotient $A := K\overline{Q}/I$ where $\overline{Q}$ is the quiver

$$\overline{Q}: \quad \begin{tikzcd} \bullet \arrow[loop left, "\overline{\alpha}"] \arrow[loop right, "\overline{\beta}"] \end{tikzcd}$$

and where I is the ideal

$$I =< (\overline{\alpha})^2, (\overline{\beta})^2, \overline{\alpha}\overline{\beta}, \overline{\beta}\overline{\alpha} > .$$

Graphically this is most evident since by identifying the two vertices, the two arrows become loops. The relations arise since we cannot compose any arrows in Q, and so any composition in $\overline{Q}$ has to be in the ideal. The two-dimensional indecomposable KQ-modules from Example 1.6.23 can again be found in A. What are the two-dimensional A-modules? A has to be a nilpotent 2×2 matrix. Hence, by conjugating properly, $\overline{\alpha}$ may be represented by the matrix $\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ for $\lambda \in \{0, 1\}$. If $\lambda = 1$, then $\overline{\beta}$ also needs to be represented by such a matrix, annihilating the representing matrix of $\overline{\alpha}$ from the left and from the right. This shows that $\overline{\beta}$ is also represented by a matrix $\begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$ for $\mu \in K$. If $\lambda = 1$ and $\mu \neq 0$, then each of these representations B^μ is indecomposable and $B^\mu \simeq B^{\mu'}$ if and only if $\mu = \mu'$. Indecomposability is verified by looking at the endomorphism ring. An endomorphism is a 2×2-matrix, commuting with the action of $\overline{\alpha}$ and $\overline{\beta}$. Only scalar multiples of the identity matrix satisfy this condition. An isomorphism is given by a regular 2×2-matrix, commuting with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and conjugating $\begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & \mu' \\ 0 & 0 \end{pmatrix}$. There is no such matrix if $\mu \cdot \mu' \neq 0$ and $\mu \neq \mu'$.

This shows that A is representation infinite. Trivially, if B is another K-algebra which admits a quotient A, then B is representation infinite as well. Indeed, any representation of A becomes a representation of B via the ring homomorphism $B \longrightarrow\!\!\!\rightarrow A$, indecomposability is preserved by the surjectivity of $B \longrightarrow\!\!\!\rightarrow A$ and by the same argument two of these B-modules are isomorphic if they are over A.

An example is $K(C_p \times C_p)$ where K is an algebraically closed field of characteristic $p > 0$. Since

$$K(C_p \times C_p) \simeq K[X, Y]/(X^p, Y^p) \longrightarrow\!\!\!\rightarrow K[X, Y]/(X^2, Y^2, XY) \simeq A$$

we obtain that $K(C_p \times C_p)$ is representation infinite. This is another way to approach parts of Proposition 2.12.9.

Remark 2.12.11 The complete structure of blocks with cyclic defect group will be determined in Sect. 5.10, namely Theorem 5.10.37. This quite involved theory can be shown using more elaborate methods, namely equivalences between stable categories, which we are going to study in Chap. 5, and the theory of nilpotent blocks, which will be developed in Sect. 4.4.2.

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