Preface

This book gives an introduction to the representation theory of finite groups and finite dimensional algebras via homological algebra. In particular, it emphasises common techniques and unifying themes between the two subjects.

Until the 1990s, the representation theory of finite groups and the representation theory of finite dimensional algebras had developed quite independently of one another, with few connections between them. Then, starting at about 1990 a unifying theme was discovered in form of methods from homological algebra. In the representation theory of algebras, homological approaches had already been accepted and used for quite a long time. This was certainly a consequence of the most influential work of Auslander and Reiten who preferred this abstract approach. The techniques of triangulated categories were brought to the representation theory of algebras by Happel in [1]. However, before this time, methods from homological algebra were not yet fully accepted in the representation theory of groups. The relevance of stable categories was known, but no systematic approach was really undertaken. The situation was different for the representation theory of Lie algebras and of algebraic groups. Since that part of representation theory is very close to algebraic geometry, homological algebra methods and in particular derived categories had long been established as an important tool.

The bridge from the representation theory of algebras to the representation theory of groups via homological algebra was fully established in 1989, when Rickard proved a Morita theory for derived categories and Broué pronounced his most famous abelian defect conjecture. The most appealing abelian defect conjecture uses very deep knowledge of the structure of algebras and its homological algebra. It cannot be approached by dealing only with finite groups, progress can be achieved only by taking into consideration further algebraic structures and their homological algebra. Most interestingly, the conjecture and the methods to approach it offer a bridge to questions close to algebraic topology, and is at once a very challenging field of applications for the representation theory of algebras. Now, more than two decades after the abelian defect group conjecture was stated, categories are everywhere in the representation theory of groups and homological algebra methods as well as methods from the representation theory of algebras are omnipresent. In the reverse direction, motivations and methods from the representation theory of finite groups have become central and important in the representation theory of finite dimensional algebras.

While Broué's abelian defect conjecture was in a certain sense, the crystallisation point for the use of common techniques in the representation theory of groups and of algebras, it is now commonly admitted that these modern techniques are useful when studying group representations, even if one is not really interested in Broué's abelian defect conjecture. This is the point of view we will present. We shall present the homological algebra methods that are useful for all kinds of questions in the representation theory of algebras and of groups, and thus provide a solid background for further studies. We will not, or better to say almost not, consider the abelian defect conjecture, but the reader will acquire knowledge of the necessary techniques which will enable him to pursue developments in the direction of this and related conjectures.

At the present time, no textbook is available which gives an introduction to representations of groups and algebras at the same time, focussing on methods from homological algebra. In my opinion, a student learning the representation theory of either groups or algebras should master techniques from both sides and the unifying homological algebra needed. This book is meant to give an introduction with this goal in mind.

More explicitly, the general principle is the following. Classical representation theory deals with modules over an algebra A or over a group algebra $A = KG$. We emphasise here the category $\mathcal{C} = A\text{-}mod$ of these modules. It soon becomes clear that the module category of these algebras or groups encodes the relevant phenomena, but these cannot easily be obtained from the latter. To circumvent this problem, another category $\mathcal{D}_{\mathcal{C}}$ obtained from the category $\mathcal{C}$ is introduced. The choice of the category $\mathcal{D}_{\mathcal{C}}$ depends on the kind of question one wants to study. The main kind of problem is then to characterise in terms of $\mathcal{C}_1$ and $\mathcal{C}_2$ when $\mathcal{D}_{\mathcal{C}_1}$ is equivalent to $\mathcal{D}_{\mathcal{C}_2}$. Moreover, it is obviously interesting to find properties of $\mathcal{C}_1$ that are invariant under an equivalence between $\mathcal{D}_{\mathcal{C}_1}$ and $\mathcal{D}_{\mathcal{C}_2}$, so that $\mathcal{C}_2$ also has this property whenever there is such an equivalence and $\mathcal{C}_1$ has the requested property.

In our case we first study the module category $\mathcal{C} = A\text{-}mod$, and choose $\mathcal{D}_{\mathcal{C}} = \mathcal{C}$. The relevant equivalence then is called Morita equivalence. Then, in a second attempt, we choose $\mathcal{D}_{\mathcal{C}}$ to be the stable category, to be introduced later. A rich structure is available for this choice of $\mathcal{D}_{\mathcal{C}}$, at least when A is self-injective. There are two possible equivalences in this case, in order to strengthen possible implications if necessary. Third, we choose $\mathcal{D}_{\mathcal{C}}$ to be the derived category of complexes of A-modules. In this case, many classical results are known, and in some sense the structure is aesthetically the most appealing. Recently, an equivalence weaker than equivalences between derived categories, but more general than stable equivalences, coming from so-called singular categories has been examined intensively, and we shall present some of the known results. In addition to this, structure results from very classical theory also heavily depend on equivalence between these subsequent categories. For example, the structure theorem of blocks with cyclic defect group is intimately linked with questions about equivalences in stable categories. We shall present the details in the relevant Chap. 5. Of course, for all this, some preparations are necessary. Not all of the necessary mathematics is part of the standard material in algebra classes, so we need to close this gap. In the introductory chapters, all the necessary background is presented in a uniform way.

Where is it appropriate to start and where to end? In my opinion, the classical representation theory of finite groups and semisimple algebras is still a very good starting point. The subject is explicit, provides plenty of nice examples, and is appealing for most people entering the subject. Moreover, when introducing the basic objects it is natural to pave the way in a manner that the methods needed to understand the bridge between representations of algebras and of groups become familiar naturally. The student I have in mind has taken a basic linear algebra course and knows some basics on rings, fields and groups, as I believe is done nowadays at all universities around the world. Normally, the student is at the end of his fourth year of university and wants to learn representation theory. In this sense, the book is self-contained. The very few results that are not proved concern number theoretic statements on completions or set theoretical details, where a proof would have led to distant foreign grounds. In this case references are given, so that the reader willing to explore these points can find accessible sources easily. I also believe that a mathematician wanting to see what this theory is about and trying to learn the main results should be able to profit from the book. This is the goal I tried to achieve.

The book is meant to give the material for lectures starting at the level of the end of the first year of a master's degree in the Bologna treaty scheme of the European Union from 1999, then specializing in the first half of the second year and finally choosing a more advanced topic in the second half of the last year. The first chapter has a level of difficulty appropriate to the first year of the master's degree, the second chapter is an introduction to group representations, and should be adapted to the first semester of the second year. The following chapters can be studied in parts only, and offer many different ways to approach more specialised topics. Chapter 1 is compulsory for the rest of the book, in the sense that the basic methods and main motivations from ring and module theory are provided there. Of course, the advised reader can skip this chapter. Chapter 2 gives an introduction to the representation theory of groups, which frequently provides examples and applications for more general concepts throughout the book. Again, most of the content can also be obtained from other text books, but some concepts are original here and cannot easily be found elsewhere.

The second part of the book deals with more advanced subjects. The homological algebra used in representation theory is developed and the main focus is given to equivalences between categories of representations. Chapter 3 is the entrance gate to the second part. Chapters 4–6 give specialisations in different directions, and depending on how the reader wants to continue, different parts of Chap. 3 are necessary. Sections 3.1–3.3 are necessary for each of the remaining chapters, whereas Chap. 4 does not need any further information. Section 3.4 is the technical framework for all that is studied in Chaps. 5 and 6. Section 3.5 is crucial and it prepares Sects. 3.6 and 3.7. In particular, Sects. 3.5 and 3.7 are both used in an essential way in Chaps. 5 and 6. Section 3.9 is technical and is used for a few details in proofs appearing in Chap. 6.

Chapter 4 presents the classical Morita theory from the 1950s in such a way that the more recent developments of the subsequent chapters become natural. As main applications, we give Külshammer's proof of Puig's structure theorem for nilpotent blocks in Sect. 4.4.2 and Gabriel's theorem on algebras presented by quivers and relations in Sect. 4.5.1.

Chapter 5 describes stable categories and the equivalences between them. As an application the structure of blocks with cyclic defect groups as Brauer tree algebras is proved in Sect. 5.10. Section 5.11 gives a very nice and not widely known result due to Reiten, and the method of proof is interesting in its own right. As far as I know, this is the first documentation of this result in book form.

Chapter 6 then introduces the reader to derived equivalences, and at least from Sect. 6.9 onwards parts of Chap. 5 are also used. For Rickard's Morita theory I choose a proof which combines results from Rickard and from Keller. The proof of the basic fact may be slightly more involved than in Rickard's original approach, but it has the advantage that it answers questions left open in Rickard's original approach concerning the actual constructibility of the objects. Readers interested in getting the most direct proof for Rickard's Morita theory for derived categories may like to consult the treatment in [2]. However, only an existence proof is given there and no construction of the basic object is shown, except in a very specific situation.

All chapters arose from lectures I gave at the Université de Picardie over the years, and most chapters served as class notes for lectures I gave at the appropriate level. I owe much to the students and their questions and remarks during the lectures, alerting me when the presentation of preliminary versions had to be improved.

Representation theory is vast, and even the topics covered in the book can be developed much further. In order to keep the book to a reasonable size, I decided to leave out many important topics, such as Rickard's result on splendid equivalences. Initially I planned to include much more of the ordinary representation theory as well so that it could be used as manuscript for an entire one-semester course. However, this project would have increased the book further to an unreasonable size, and since the subsequent chapters do not use ordinary representation theory, I had to refrain from developing the first chapter further in this direction. The representation theory of algebras is in some sense unthinkable without an introduction to Auslander-Reiten quivers and almost split sequences. I have not given an introduction to this most important theory here. Many textbooks, such as [3] or [4] have appeared recently and cover Auslander-Reiten theory in a very nice and an accessible way. Almost split sequences have not been used throughout the book, although many properties in Chap. 5 in particular can be shown with only a little use of this technique. The book [4] gives some examples, and [5] also uses them without hesitation and with breathtaking speed. Another omission is the rapidly developing theory of cluster categories and cluster algebras. I feel that the theory is not yet settled enough to be given a definitive treatment. A very nice and useful topic which I omitted, is geometric representation theory, in the sense of methods from algebraic geometry used in the representation theory of algebras. It would have made a great deal of sense to include this powerful tool, however, it needs quite a few prerequisites in algebraic geometry at various levels. To introduce the algebraic geometry needed for even the most elementary purposes would have increased the size of the book considerably.

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Amiens, August 2013 Alexander Zimmermann

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