Chapter 2
Optimal Control Problems with Singleton Turnpikes

In this chapter we study the structure of solutions of a discrete-time control system with a compact metric space of states X which arises in economic dynamics. This control system is described by a nonempty closed set $\Omega \subset X \times X$ which determines a class of admissible trajectories (programs) and by a bounded upper semicontinuous objective function $v : X \times X \to R^1$ which determines an optimality criterion. We show the stability of the turnpike phenomenon under small perturbations of the objective function v and the set Ω.

2.1 Preliminaries and Stability Results

Let (X, ρ) be a compact metric space. For each $x \in X$ and each nonempty set $C \subset X$ set

$$\rho(x, C) = \inf\{\rho(x, y) : \ y \in C\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \leq r\}.$$

We equip the space $X \times X$ with the metric ρ_1 defined by

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2), \ x_1, x_2, y_1, y_2 \in X.$$

For each $(x_1, x_2) \in X \times X$ and each nonempty set $C \subset X \times X$ set

$$\rho_1((x_1, x_2), C) = \inf\{\rho_1((x_1, x_2), (y_1, y_2)) : \ (y_1, y_2) \in C\}.$$

Denote by $\mathcal{M}$ the set of all bounded functions $u : X \times X \to R^1$. For each $w \in \mathcal{M}$ set

$$\|w\| = \sup\{|w(x, y)| : \ (x, y) \in X \times X\}.$$

Let Ω be a nonempty closed subset of $X \times X$.

A. J. Zaslavski, *Stability of the Turnpike Phenomenon in Discrete-Time Optimal Control Problems,* SpringerBriefs in Optimization,
DOI 10.1007/978-3-319-08034-5_2,

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called an (Ω)-program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \ge 0$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ where integers T_1, T_2 satisfy $0 \le T_1 < T_2$ is called an (Ω)-program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \in [T_1, T_2 - 1]$.

Let $v \in \mathcal{M}$ be an upper semicontinuous function.

We suppose that there exist $\bar{x} \in X$ and a constant $\bar{c} > 0$ such that the following assumptions hold.

(A1) $(\bar{x}, \bar{x})$ is an interior point of Ω (there is $\epsilon > 0$ such that $\{(x, y) \in X \times X : \rho(x, \bar{x}),\ \rho(y, \bar{x}) \le \epsilon\} \subset \Omega$) and v is continuous at $(\bar{x}, \bar{x})$.

(A2) For any integer $T \ge 1$ and any (Ω)-program $\{x_t\}_{t=0}^{T}$,

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \le T v(\bar{x}, \bar{x}) + \bar{c}.$$

Assumption (A2) implies the following result.

Proposition 2.1 *For each (Ω)-program $\{x_t\}_{t=0}^{\infty}$ either the sequence*

$$\left\{ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) - T v(\bar{x}, \bar{x}) \right\}_{T=1}^{\infty}$$

is bounded or $\lim_{T \to \infty} \left[\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - T v(\bar{x}, \bar{x}) \right] = -\infty.$

An (Ω)-program $\{x_t\}_{t=0}^{\infty}$ is called (v, Ω)-good if the sequence

$$\left\{ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) - T v(\bar{x}, \bar{x}) \right\}_{T=1}^{\infty}$$

is bounded [13, 17, 45, 55, 56].

In this chapter we suppose that the following assumption holds.

(A3) (the asymptotic turnpike property) For any (v, Ω)-good program $\{x_t\}_{t=0}^{\infty}$, $\lim_{t \to \infty} \rho(x_t, \bar{x}) = 0$.

Note that (A3) holds for many important infinite horizon optimal control problems. In particular, (A3) holds for a general model of economic dynamics considered in Example 1.7.

For each $x, y \in X$, each integer $T \ge 1$ and each $w \in \mathcal{M}$ set

$$\sigma(w, T, x, y) = \sup \left\{ \sum_{i=0}^{T-1} w(x_i, x_{i+1}) : \{x_i\}_{i=0}^{T} \text{is an } (\Omega) - \text{program and } x_0 = x,\ x_T = y \right\}.$$

(Here we use the convention that the supremum of an empty set is $-\infty$).

In Chap. 1 we considered the turnpike properties of approximate solutions of the problems

$$\sum_{i=0}^{T-1} v(x_i, x_{i+1}) \to \max,\ \{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega,\ x_0 = y,$$

and

$$\sum_{i=0}^{T-1} v(x_i, x_{i+1}) \to \max,\ \{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega,\ x_0 = y,\ x_T = z,$$

where $T \geq 1$ is an integer and the points $y, z \in X$.

In this chapter we show that these turnpike properties are stable under small perturbations of the objective function v and the set Ω. In order to meet this goal we introduce the following definitions.

By assumption (A1) there exists $\bar{r} \in (0, 1)$ such that

$$B(\bar{x}, \bar{r}) \times B(\bar{x}, \bar{r}) \subset \Omega. \tag{2.1}$$

Fix

$$\bar{\lambda} \in (0, \bar{r}). \tag{2.2}$$

For each $\lambda > 0$ denote by $\mathcal{E}(\lambda)$ the collection of all nonempty sets $\Omega' \subset X \times X$ such that

$$\rho_1(z, \Omega) \leq \lambda \text{ for each } z \in \Omega', \tag{2.3}$$

$$B(\bar{x}, \bar{\lambda}) \times B(\bar{x}, \bar{\lambda}) \subset \Omega'. \tag{2.4}$$

Let integers T_1, T_2 satisfy $0 \leq T_1 < T_2$ and let Ω_t, $t = T_1, \dots, T_2 - 1$ be nonempty subsets of $X \times X$.

A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called an $(\{\Omega_t\}_{t=T_1}^{T_2-1})$-program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers $t \in [T_1, T_2 - 1]$.

For each $x, y \in X$ and each finite sequence $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}$ set

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, \{\Omega_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x) = \sup \Bigg\{ \sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is an } (\{\Omega_t\}_{t=T_1}^{T_2-1}) - \text{program and } x_{T_1} = x \Bigg\}, \tag{2.5}$$

$$\sigma\Big(\{u_t\}_{t=T_1}^{T_2-1}, \{\Omega_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y\Big) = \sup \Bigg\{ \sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is an } (\{\Omega_t\}_{t=T_1}^{T_2-1}) - \text{program, } x_{T_1} = x \text{ and } x_{T_2} = y \Bigg\}, \tag{2.6}$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, \{\Omega_t\}_{t=T_1}^{T_2-1}, T_1, T_2) = \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) :\right.$$

$$\left. \{x_t\}_{t=T_1}^{T_2} \text{is an} (\{\Omega_t\}_{t=T_1}^{T_2-1}) - \text{program}\right\}. \tag{2.7}$$

(Here we use the convention that the supremum of an empty set is $-\infty$).

Denote by $Y(\{\Omega_t\}_{t=T_1}^{T_2-1}, T_1, T_2)$ the set of all $x \in X$ for which there exists an $(\{\Omega_t\}_{t=T_1}^{T_2-1})$-program $\{x_t\}_{t=T_1}^{T_2}$ such that $x_{T_1} = \bar{x}$ and $x_{T_2} = x$.

Denote by $\bar{Y}(\{\Omega_t\}_{t=T_1}^{T_2-1}, T_1, T_2)$ the set of all $x \in X$ for which there exists an $(\{\Omega_t\}_{t=T_1}^{T_2-1})$-program $\{x_t\}_{t=T_1}^{T_2}$ such that $x_{T_1} = x$ and $x_{T_2} = \bar{x}$.

For sufficiently small positive numbers δ, we study the structure of approximate solutions of the problems

$$\sum_{i=0}^{T-1} u_i(x_i, x_{i+1}) \to \max,$$

$$\{x_i\}_{i=0}^{T} \text{is an } (\{\Omega_t\}_{t=0}^{T-1}) - \text{program and } x_0 = y,$$

and

$$\sum_{i=0}^{T-1} u_i(x_i, x_{i+1}) \to \max,$$

$$\{x_i\}_{i=0}^{T} \text{ is an } (\{\Omega_t\}_{t=0}^{T-1}) - \text{program and } x_0 = y,\ x_T = z,$$

where $T \geq 1$ is an integer, $y, z \in X$ and for all $t = 0, \dots, T-1$, we have

$$\Omega_t \in \mathcal{E}(\delta),\ u_t \in \mathcal{M} \text{ and } \|u_t - v\| \leq \delta.$$

In this chapter we prove the following four stability results.

Theorem 2.2 *Let ϵ be a positive number and let l_1, l_2 be natural numbers. Then there exist $\delta > 0$ and a natural number $L > l_1 + l_2$ such that for each integer $T > 2L$, each*

$$\Omega_t \in \mathcal{E}(\delta),\ t = 0, \dots, T-1,$$

each $u_t \in \mathcal{M},\ t = 0, \dots, T-1$ satisfying

$$\|u_t - v\| \leq \delta,\ t = 0, \dots, T-1$$

and each $(\{\Omega_t\}_{t=0}^{T-1})$-program $\{x_t\}_{t=0}^{T}$ which satisfies

$$x_0 \in \bar{Y}(\{\Omega_t\}_{t=0}^{l_1-1}, 0, l_1),\ x_T \in Y(\{\Omega_t\}_{t=T-l_2}^{T-1}, T-l_2, T),$$

$$\sigma(\{u_t\}_{t=0}^{T-1}, \{\Omega_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) \leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) + \delta$$

there exist integers $\tau_1 \in [0, L],\ \tau_2 \in [T - L, T]$ *such that*

$$\rho(x_t, \bar{x}) \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}) \le \delta$, *then* $\tau_1 = 0$ *and if* $\rho(x_T, \bar{x}) \le \delta$, *then* $\tau_2 = T$.

Theorem 2.3 *Let* ϵ *be a positive number and let* l_1 *be a natural number. Then there exist* $\delta > 0$ *and a natural number* $L > l_1$ *such that for each integer* $T > 2L$, *each*

$$\Omega_t \in \mathcal{E}(\delta),\ t = 0, \dots, T-1,$$

each $u_t \in \mathcal{M},\ t = 0, \dots, T-1$ *satisfying*

$$\|u_t - v\| \le \delta,\ t = 0, \dots, T-1$$

and each $(\{\Omega_t\}_{t=0}^{T-1})$*-program* $\{x_t\}_{t=0}^{T}$ *which satisfies*

$$x_0 \in \bar{Y}(\{\Omega_t\}_{t=0}^{l_1-1}, 0, l_1),$$

$$\sigma(\{u_t\}_{t=0}^{T-1}, \{\Omega_t\}_{t=0}^{T-1}, 0, T, x_0) \le \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) + \delta$$

there exist integers $\tau_1 \in [0, L],\ \tau_2 \in [T - L, T]$ *such that*

$$\rho(x_t, \bar{x}) \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}) \le \delta$, *then* $\tau_1 = 0$ *and if* $\rho(x_T, \bar{x}) \le \delta$, *then* $\tau_2 = T$.

Denote by $\mathrm{Card}(B)$ the cardinality of a set B.

Theorem 2.4 *Let* ϵ, M *be positive numbers and let* l_1, l_2 *be natural numbers. Then there exist* $\delta > 0$ *and a natural number* $L > l_1 + l_2$ *such that for each integer* $T > L$, *each*

$$\Omega_t \in \mathcal{E}(\delta),\ t = 0, \dots, T-1,$$

each $u_t \in \mathcal{M},\ t = 0, \dots, T-1$ *satisfying*

$$\|u_t - v\| \le \delta,\ t = 0, \dots, T-1$$

and each $(\{\Omega_t\}_{t=0}^{T-1})$*-program* $\{x_t\}_{t=0}^{T}$ *which satisfies*

$$x_0 \in \bar{Y}(\{\Omega_t\}_{t=0}^{l_1-1}, 0, l_1),\ x_T \in Y(\{\Omega_t\}_{t=T-l_2}^{T-1}, T - l_2, T),$$

$$\sigma(\{u_t\}_{t=0}^{T-1}, \{\Omega_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) \le \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) + M$$

the inequality

$$\mathit{Card}(\{t \in \{0, \dots, T\} :\ \rho(x_t, \bar{x}) > \epsilon\}) \le L$$

holds.

Theorem 2.5 *Let ϵ, M be positive numbers and let l_1 be a natural number. Then there exist $\delta > 0$ and a natural number $L > l_1$ such that for each integer $T > L$, each*

$$\Omega_t \in \mathcal{E}(\delta),\ t = 0, \dots, T-1,$$

each $u_t \in \mathcal{M},\ t = 0, \dots, T-1$ satisfying

$$\|u_t - v\| \le \delta,\ t = 0, \dots, T-1$$

and each $(\{\Omega_t\}_{t=0}^{T-1})$-program $\{x_t\}_{t=0}^{T}$ which satisfies

$$x_0 \in \bar{Y}(\{\Omega_t\}_{t=0}^{l_1-1}, 0, l_1),$$

$$\sigma(\{u_t\}_{t=0}^{T-1}, \{\Omega_t\}_{t=0}^{T-1}, 0, T, x_0) \le \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) + M$$

the inequality

$$Card(\{t \in \{0, \dots, T\} :\ \rho(x_t, \bar{x}) > \epsilon\}) \le L$$

holds.

2.2 Extensions

We use the notation, definitions, and assumptions introduced in Sect. 2.1. In this section we state the extensions of the turnpike results of the previous section. In these extensions we describe the structure of programs defined on an interval $[0, T]$ with sufficiently large T which are approximate solutions of the corresponding optimal problems on subintervals of the length L, where L is a constant which does not depend on T.

Theorem 2.6 *Let $\epsilon \in (0, \bar{\lambda})$ and M be a positive number. Then there exist $\gamma \in (0, \epsilon)$ and a natural number L_0 such that for each integer $L_1 \ge L_0$ there exists a positive number $\delta < \gamma$ such that the following assertion holds.*

Assume that an integer $T > 3L_1$,

$$\Omega_t \in \mathcal{E}(\delta),\ t = 0, \dots, T-1,$$

$u_t \in \mathcal{M},\ t = 0, \dots, T-1$ satisfy

$$\|u_t - v\| \le \delta,\ t = 0, \dots, T-1$$

and that an $(\{\Omega_t\}_{t=0}^{T-1})$-program $\{x_t\}_{t=0}^{T}$ and a finite sequence of integers $\{S_i\}_{i=0}^{q}$ satisfy

$$S_0 = 0,\ S_{i+1} - S_i \in [L_0, L_1],\ i = 0, \dots, q-1,\ S_q > T - L_1,$$

$$\sum_{t=S_i}^{S_{i+1}-1} u_t(x_t, x_{t+1}) \geq \sum_{t=S_i}^{S_{i+1}-1} u_t(\bar{x}, \bar{x}) - M$$

for each integer $i \in [0, q-1]$,

$$\sum_{t=S_i}^{S_{i+2}-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=S_i}^{S_{i+2}-1}, \{\Omega_t\}_{t=S_i}^{S_{i+2}-1}, S_i, S_{i+2}, x_{S_i}, x_{S_{i+2}}) - \gamma$$

for each integer $i \in [0, q-2]$ *and*

$$\sum_{t=S_{q-2}}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=S_{q-2}}^{T-1}, \{\Omega_t\}_{t=S_{q-2}}^{T-1}, S_{q-2}, T, x_{S_{q-2}}, x_T) - \gamma.$$

Then there exist integers $\tau_1 \in [0, L_1]$, $\tau_2 \in [T - 2L_1, T]$ *such that*

$$\rho(x_t, \bar{x}) \leq \epsilon \text{ for all } t = \tau_1, \ldots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}) \leq \gamma$, *then* $\tau_1 = 0$ *and if* $\rho(x_T, \bar{x}) \leq \gamma$, *then* $\tau_2 = T$.

Theorem 2.7 *Let* ϵ, M *be positive numbers and let* l_1, l_2 *be natural numbers. Then there exist* $\delta > 0$ *and a natural number* $L > l_1 + l_2$ *such that for each integer* $T > 2\,L$, *each*

$$\Omega_t \in \mathcal{E}(\delta),\ t = 0, \ldots, T-1,$$

each $u_t \in \mathcal{M},\ t = 0, \ldots, T-1$ *satisfying*

$$\|u_t - v\| \leq \delta,\ t = 0, \ldots, T-1$$

and each $(\{\Omega_t\}_{t=0}^{T-1})$*-program* $\{x_t\}_{t=0}^{T}$ *which satisfies*

$$x_0 \in \bar{Y}(\{\Omega_t\}_{t=0}^{l_1-1}, 0, l_1),\ x_T \in Y(\{\Omega_t\}_{t=T-l_2}^{T-1}, T - l_2, T),$$

$$\sigma(\{u_t\}_{t=0}^{T-1}, \{\Omega_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) \leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) + M$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \{\Omega_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau + L, x_\tau, x_{\tau+L}) - \delta$$

for each integer $\tau \in [0, T - L]$, *there exist integers* $\tau_1 \in [0, L]$, $\tau_2 \in [T - L, T]$ *such that*

$$\rho(x_t, \bar{x}) \leq \epsilon \text{ for all } t = \tau_1, \ldots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}) \leq \delta$, *then* $\tau_1 = 0$ *and if* $\rho(x_T, \bar{x}) \leq \delta$, *then* $\tau_2 = T$.

Theorem 2.8 *Let ϵ, M be positive numbers and let l_1 be a natural number. Then there exist $\delta > 0$ and a natural number $L > l_1$ such that for each integer $T > 2L$, each*

$$\Omega_t \in \mathcal{E}(\delta),\ t = 0, \dots, T-1,$$

each $u_t \in \mathcal{M},\ t = 0, \dots, T-1$ satisfying

$$\|u_t - v\| \le \delta,\ t = 0, \dots, T-1$$

and each $(\{\Omega_t\}_{t=0}^{T-1})$-program $\{x_t\}_{t=0}^{T}$ which satisfies

$$x_0 \in \bar{Y}(\{\Omega_t\}_{t=0}^{l_1-1}, 0, l_1),$$

$$\sigma(\{u_t\}_{t=0}^{T-1}, \{\Omega_t\}_{t=0}^{T-1}, 0, T, x_0) \le \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) + M$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \{\Omega_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau + L, x_\tau, x_{\tau+L}) - \delta$$

for each integer $\tau \in [0, T-L]$, there exist integers $\tau_1 \in [0, L],\ \tau_2 \in [T-L, T]$ such that

$$\rho(x_t, \bar{x}) \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}) \le \delta$, then $\tau_1 = 0$ and if $\rho(x_T, \bar{x}) \le \delta$, then $\tau_2 = T$.

2.3 Three Lemmata

In order to prove our stability results we need the following useful lemmas. Lemmas 2.9 and 2.10 were obtained in [46] while Lemma 2.11 was proved in [47].

Lemma 2.9 *Let $\epsilon > 0$ and $M_0 > 0$. Then there exists a natural number T such that for each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \ge T v(\bar{x}, \bar{x}) - M_0$$

the relation

$$\min\{\rho(x_i, \bar{x}) :\ i = 1, \dots, T\} \le \epsilon$$

holds.

Proof Let us assume the contrary. Then for each natural number k there exists an (Ω)-program $\{x_t^{(k)}\}_{t=0}^{k}$ which satisfies

$$\sum_{t=0}^{k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \ge k v(\bar{x}, \bar{x}) - M_0, \tag{2.8}$$

$$\rho(x_t^{(k)}, \bar{x}) > \epsilon \text{ for all integers } t = 1, \ldots, k. \tag{2.9}$$

Let $k \geq 1$ be an integer. By (2.8) and (A2) for each integer j satisfying $0 < j < k$

$$\begin{aligned}\sum_{t=0}^{j-1} v(x_t^{(k)}, x_{t+1}^{(k)}) &= \sum_{t=0}^{k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) - \sum_{t=j}^{k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \\ &\geq kv(\bar{x}, \bar{x}) - M_0 - \sum_{t=j}^{k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \\ &\geq kv(\bar{x}, \bar{x}) - M_0 - (k-j)v(\bar{x}, \bar{x}) - \bar{c}.\end{aligned}$$

Together with (2.8) this inequality implies that for each integer $k \geq 1$ and each $j \in \{1, \ldots, k\}$

$$\sum_{t=0}^{j-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \geq jv(\bar{x}, \bar{x}) - \bar{c} - M_0. \tag{2.10}$$

There exists a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that for each integer $t \geq 0$ there exists

$$x_t = \lim_{i \to \infty} x_t^{(k_i)}. \tag{2.11}$$

Clearly, $\{x_t\}_{t=0}^{\infty}$ is an (Ω)-program. In view of (2.11) and (2.9)

$$\rho(x_t, \bar{x}) \geq \epsilon \text{ for all integers } t \geq 1. \tag{2.12}$$

It follows from (2.11) and (2.10) that for each integer $T \geq 1$ we have

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq Tv(\bar{x}, \bar{x}) - M_0 - \bar{c}.$$

This implies that $\{x_t\}_{t=0}^{\infty}$ is a (v, Ω)-good program. By (A3) $\lim_{t\to\infty} \rho(x_t, \bar{x}) = 0$. This equality contradicts (2.12). The contradiction we have reached proves Lemma 2.9.

Lemma 2.10 *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each integer $T \geq 1$ and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$\rho(x_0, \bar{x}),\ \rho(x_T, \bar{x}) \leq \delta, \tag{2.13}$$

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - \delta \tag{2.14}$$

the inequality $\rho(x_t, \bar{x}) \leq \epsilon$ holds for all $t = 0, \ldots, T$.

Proof Since v is continuous at $(\bar{x}, \bar{x})$ for each natural number k there exists

$$\delta_k \in (0, 2^{-k}\bar{r}) \tag{2.15}$$

such that

$$|v(x, y) - v(\bar{x}, \bar{x})| \leq 2^{-k} \tag{2.16}$$

for each $x, y \in X$ satisfying

$$\rho(x, \bar{x}),\ \rho(y, \bar{x}) \leq \delta_k. \tag{2.17}$$

Assume that the lemma is wrong. Then for each natural number k there exist an integer $T_k \geq 1$ and an (Ω)-program $\{x_t^{(k)}\}_{t=0}^{T_k}$ such that

$$\rho\left(x_0^{(k)}, \bar{x}\right),\ \rho\left(x_{T_k}^{(k)}, \bar{x}\right) \leq \delta_k, \tag{2.18}$$

$$\sum_{t=0}^{T_k-1} v\left(x_t^{(k)}, x_{t+1}^{(k)}\right) \geq \sigma\left(v, T_k, x_0^{(k)}, x_{T_k}^{(k)}\right) - \delta_k, \tag{2.19}$$

$$\max\{\rho\left(x_t^{(k)}, \bar{x}\right) :\ t = 0, \ldots, T_k\} > \epsilon. \tag{2.20}$$

Let $k \geq 1$ be an integer. Define a sequence $\{z_t\}_{t=0}^{T_k} \subset X$ as follows:

$$z_0 = x_0^{(k)},\ z_{T_k} = x_{T_k}^{(k)},\ z_t = \bar{x},\ t \in \{0, \ldots, T_k\} \setminus \{0, T_k\}. \tag{2.21}$$

By (2.21), (2.18), (2.15), and (2.1), $\{z_t\}_{t=0}^{T_k}$ is an (Ω)-program. It follows from (2.19) and (2.21) that

$$\sum_{t=0}^{T_k-1} v\left(x_t^{(k)}, x_{t+1}^{(k)}\right) \geq \sigma\left(v, T_k, x_0^{(k)}, x_{T_k}^{(k)}\right) - \delta_k \geq \sum_{t=0}^{T_k-1} v(z_t, z_{t+1}) - \delta_k. \tag{2.22}$$

In view of (2.18), (2.21), and the choice of δ_k (see (2.15)–(2.17))

$$\begin{aligned} &|v(z_0, z_1) - v(\bar{x}, \bar{x})| \leq 2^{-k}, |v(z_{T_{k-1}}, z_{T_k}) - v(\bar{x}, \bar{x})| \leq 2^{-k}, \\ &v(z_t, z_{t+1}) = v(\bar{x}, \bar{x}),\ t \in \{0, \ldots, T_{k-1}\} \setminus \{0, T_k - 1\}. \end{aligned} \tag{2.23}$$

Relations (2.23) and (2.22) imply that

$$\sum_{t=0}^{T_k-1} v\left(x_t^{(k)}, x_{t+1}^{(k)}\right) \geq T_k v(\bar{x}, \bar{x}) - 2 \cdot 2^{-k} - \delta_k. \tag{2.24}$$

Set

$$S_0 = 0,\ S_k = \sum_{i=1}^{k} (T_i + 1) - 1 \text{ for all integers } k \geq 1. \tag{2.25}$$

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Stability of the Turnpike Phenomenon in Discrete-Time
Optimal Control Problems
Zaslavski, A.J.
2014, X, 109 p., Softcover
ISBN: 978-3-319-08033-8