

Chapter 2

Mathematical Foundations

Abstract This chapter presents basic mathematical concepts and tools for the development of finite element method. The concept of functional, associated with the variational formulation, and the function, associated with the boundary value problems, is discussed; the method of calculus of variation which transfers the variational formulation into the boundary value problem is also presented. A brief discussion of the numerical solution methods to handle the variational formulation and the boundary value problem is presented. Some numerical examples are given to show the convergence efficiency of the numerical methods.

2.1 Introduction

The fundamental problems of mechanics are governed by differential or partial differential equations which state the equilibrium and continuity conditions of the system. In particular cases, where the geometry, loading, and boundary conditions are simple, these governing differential equations are solved and the solutions are presented in the form of mathematical functions.

The problems of mechanics, in general, should satisfy the condition of the extremum of a functional at equilibrium condition. That is, a problem formulation is described by its proper differential equations of equilibrium as well as its associated energies at the extremal condition. While the formulation of problems based on the forced summation method and the variational method are entirely different, they are both related and yield identical results. That is, the equilibrium equations describing the system at equilibrium are associated with the minimum total work of the system.

2.2 Statement of Extremum Principle

Engineering problems are usually formulated in terms of a system of equilibrium equations which may be in the form of algebraic, ordinary, or partial differential equations. For many systems, the equilibrium equations are equivalent to a known extremum problem. The reality for physical problems relies on the basic laws of nature wherein the state of equilibrium is associated with a specific physical law in nature.

The mathematical statement of the extremum principle is as follows; *a certain class of allowable functions $\psi(t, x_i)$ $i = 1, 2, 3$ is fixed between time intervals $t_1 \leq t \leq t_2$ and in space domain $D(x_1, x_2, x_3)$, and a means for associating a value $\Phi(\psi)$ with each function ψ is defined. For a particular function ψ , there exists a single function for Φ such that as ψ ranges through the class of allowable functions, the corresponding functional value of Φ varies. The class of functions ψ are called **function** and their corresponding Φ values are called **functional**. The extremum problem is to locate the function ψ in which its functional Φ remains stationary.*

In the statement of the extremum problems and the relationship between the functional and function, the following two questions arise:

1. Given a function $\psi(t, x_i)$, $i = 1, 2, 3$, associated with a boundary value problem, does an equivalent extremum problem exist? and if so, what is the class of allowable functions and what is the functional Φ ?
2. Given an extremum problem, what is the equivalent function satisfying the boundary value problem?

We expect to answer the first question for the physical and engineering problems by the known extremum principle based on the laws of nature and physics. For mechanical problems, the examples are the law of entropy for thermodynamic problems, Hamilton's principle for the dynamic problems, and the law of minimum potential energy for the static problems. A more general treatment may be based on the principle of virtual work. The extremum values of entropy or work done under certain circumstances are always associated with the general equilibrium of the mechanical systems which are obtained by means of balance of forces, moments, and energies. There are no general mathematical treatments for obtaining the functional directly from its associated boundary value problem obtained through the balance of forces, moments, and energies.

The answer to the second question, on the other hand, is always positive and is based on the algorithm of *calculus of variations*. That is, it is always possible to find the associated boundary value problem of a given functional, using the method of calculus of variation. Applying this method, results in the boundary value problem expressing the state of equilibrium and natural boundary conditions, which are essentially obtained during the weak formulations of integral equations.

2.3 Method of Calculus of Variation

The general statement of the calculus of variation and the relationship between a functional and the function associated with its extremum is discussed in this section. In terms of physical problems, we try to obtain the equilibrium equation of a boundary value problem which governs the function $\psi(t, x_i)$, $i = 1, 2, 3$, through the minimization of its associated functional $\Phi(\psi)$.

Assume that the function $\psi(x_i)$, $i = 1, 2, 3$, is defined and is a function of the space variables x_1 , x_2 , and x_3 satisfy the equilibrium equation

$$L[\psi(x_1, x_2, x_3)] = 0. \quad (2.3.1)$$

The essential boundary condition which ψ has to satisfy is

$$B_i(\psi) = g_i \quad i = 1, 2, \dots \quad (2.3.2)$$

where L is a mathematical operation, B_i is a linear operation on ψ , and g_i is the non-homogeneous boundary condition applied on ψ . The associated functional of Eq. (2.3.1) is

$$\Phi = \Phi(\psi). \quad (2.3.3)$$

Now, consider a variational function u which meets the continuity conditions and satisfies the homogeneous boundary conditions

$$B_i(u) = 0 \quad i = 1, 2, \dots \quad (2.3.4)$$

Now, if ψ is the true solution of Eq. (2.3.1), $(\psi + \epsilon u)$ may be made to represent an arbitrary admissible function which satisfies the real non-homogeneous boundary conditions. For fixed u , the variational parameter ϵ may be changed to make a one-parameter family of admissible functions. Since u satisfies the homogeneous boundary conditions, it follows that

$$B_i[\psi + \epsilon u] = g_i \quad i = 1, 2, \dots \quad (2.3.5)$$

Substituting the family of admissible functions $\psi + \epsilon u$ in Eq. (2.3.3), the functional Φ may be changed by varying the variational parameter ϵ . The extremum of the functional Φ is obtained from the following rule:

$$\left(\frac{\partial \Phi[\psi + \epsilon u]}{\partial \epsilon} \right) \Big|_{\epsilon=0} = 0. \quad (2.3.6)$$

This rule holds for every possible family of $(\psi + \epsilon u)$ for the arbitrary variational function u . The rule expressed in Eq. (2.3.6) is the basic and formal procedure of the method of calculus of variation. The basic approach in the treatment of Eq. (2.3.6) is

integration by parts such that the arbitrary function u is factored out in the resulting equations. Since u is an arbitrary function, the remaining part is set to zero. This provides the boundary value problem and the associated natural boundary conditions.

To describe the method, a few general types of functional are considered, and, using the method of calculus of variation, their associated boundary value problems are obtained.

2.4 Function of One Variable, Euler Equation

Let us consider a functional Φ being a function of $y(x)$ and its first derivative $y'(x)$ as given below,

$$\Phi[y(x)] = \int_{x_1}^{x_2} F(x, y, y') dx. \quad (2.4.1)$$

The boundary conditions on $y(x)$ are assumed to be

$$y(x_1) = y_1 \quad y(x_2) = y_2. \quad (2.4.2)$$

It is further assumed that the function F is continuous in the interval x_1 and x_2 , and its derivative up to the first order exists and is continuous.

Among all functions $y(x)$ which satisfy the continuity conditions and the given boundary conditions, which we call the *class of admissible functions*, there is only one special function $y(x)$ which minimizes the functional Φ .

In order to determine $y(x)$, an arbitrary function $u(x)$ is selected such that it is continuous in the interval x_1 and x_2 along with its first derivative and satisfies the homogeneous boundary conditions

$$u(x_1) = u(x_2) = 0. \quad (2.4.3)$$

Now, the function \bar{y} is constructed as

$$\bar{y}(x) = y(x) + \epsilon u(x) \quad (2.4.4)$$

where $\bar{y}(x)$ satisfies all the continuity conditions and the given boundary conditions. The parameter ϵ is a positive arbitrary variational parameter and is selected as sufficiently small so that the function $\bar{y}(x)$ is as close as possible to the function $y(x)$. Therefore, since $y(x)$ makes the functional at a relative minimum, for $\epsilon \neq 0$ the following inequality exists:

$$\Phi(y + \epsilon u) \geq \Phi(y). \quad (2.4.5)$$

The functional Φ is a function of ϵ and is at a relative minimum for $\epsilon = 0$. Calling $f(\epsilon) = \Phi(y + \epsilon u)$, $f(\epsilon)$ is a function of ϵ , and according to Eq. (2.4.5),

$$f(\epsilon) \geq f(0). \quad (2.4.6)$$

This suggests that $f(\epsilon)$ is at a relative minimum when $\epsilon = 0$, and since $f(\epsilon) = \Phi(y + \epsilon u)$ is differentiable, the necessary condition for $f(\epsilon)$ to be at a relative minimum is therefore

$$\frac{\partial f(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} = 0. \quad (2.4.7)$$

Introducing Eq. (2.4.4) into Eq. (2.4.1) yields

$$\Phi(y + \epsilon u) = \int_{x_1}^{x_2} F[x, (y + \epsilon u), (y' + \epsilon u')] dx. \quad (2.4.8)$$

Differentiating with respect to ϵ gives

$$\begin{aligned} \frac{\partial \Phi(y + \epsilon u)}{\partial \epsilon} &= \int_{x_1}^{x_2} \left[\frac{\partial}{\partial \bar{y}} F(x, y + \epsilon u, y' + \epsilon u') u + \frac{\partial}{\partial \bar{y}'} F(x, y + \epsilon u, y' + \epsilon u') \right. \\ &\quad \left. + \epsilon u' u' \right] dx. \end{aligned} \quad (2.4.9)$$

Integrating the last integral by parts gives

$$\begin{aligned} \frac{\partial \Phi(y + \epsilon u)}{\partial \epsilon} &= \int_{x_1}^{x_2} \left[\frac{\partial F(x, y + \epsilon u, y' + \epsilon u')}{\partial \bar{y}} \right. \\ &\quad \left. - \frac{d}{dx} \frac{\partial F(x, y + \epsilon u, y' + \epsilon u')}{\partial \bar{y}'} \right] u dx + \frac{\partial F}{\partial \bar{y}'}(x, y + \epsilon u, y' + \epsilon u') u(x) \Big|_{x_1}^{x_2}. \end{aligned} \quad (2.4.10)$$

Setting $\epsilon = 0$ yields

$$\begin{aligned} \frac{\partial \Phi(y + \epsilon u)}{\partial \epsilon} \Big|_{\epsilon=0} &= \int_{x_1}^{x_2} \left[\frac{\partial F(x, y, y')}{\partial y} - \frac{d}{dx} \frac{\partial F(x, y, y')}{\partial y'} \right] u(x) dx \\ &\quad + \frac{\partial F}{\partial y'} u \Big|_{x_1}^{x_2} = 0. \end{aligned} \quad (2.4.11)$$

The function $u(x)$ is an arbitrary function satisfying the homogeneous boundary conditions. The expression in Eq. (2.4.11) should be zero for all values of $u(x)$. This leads to the Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \quad (2.4.12)$$

subjected to the *natural boundary condition*

$$\begin{aligned}\frac{\partial F}{\partial y'} &= 0 \quad \text{at } x = x_1 \\ \frac{\partial F}{\partial y'} &= 0 \quad \text{at } x = x_2.\end{aligned}\tag{2.4.13}$$

Equation (2.4.12) is equivalent to the boundary value problem. Expanding the differential term yields

$$\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - \frac{dy}{dx} \frac{\partial^2 F}{\partial y \partial y'} - \frac{d^2 y}{dx^2} \frac{\partial^2 F}{\partial y'^2} = 0.\tag{2.4.14}$$

Equation (2.4.14) is the necessary condition for a function $y(x)$ to minimize the functional $\Phi(y)$ given by Eq. (2.4.1).

2.5 Higher Order Derivatives

Consider a functional as a function of the variable x , function $y(x)$, and higher order derivatives of function $y^n(x)$, defined in the interval $[x_1, x_2]$ as

$$\Phi[y(x)] = \int_{x_1}^{x_2} F(x, y, y', \dots, y^n) dx.\tag{2.5.1}$$

The function $y(x)$ is (n) times differentiable with respect to x . The boundary conditions are given for the function and its derivatives up to the order $n - 1$, as

$$\begin{aligned}y(x_1) &= y_1 \quad \dots \quad y^k(x_1) = y_1^k \\ y(x_2) &= y_2 \quad \dots \quad y^k(x_2) = y_2^k \quad k = 1, 2, \dots, (n - 1)\end{aligned}\tag{2.5.2}$$

where y_1, y_2, \dots, y_1^k , and y_2^k are known functions on the boundary. That is, the boundary conditions are specified for the function itself and its derivatives up to order $(n - 1)$. Consider a variational function $u(x)$, and construct the function $\bar{y}(x)$ as

$$\bar{y}(x) = y(x) + \epsilon u(x).\tag{2.5.3}$$

The variational function $u(x)$ is an arbitrary function with the following properties:

1. The function $u(x)$ and its derivatives up to order n are continuous in the interval (x_1, x_2) .
2. The function $u(x)$ and all of its derivatives up to order $(n - 1)$ satisfy the homogeneous boundary conditions.

$$\begin{aligned} u(x_1) &= 0 \cdots u^k(x_1) = 0 \\ u(x_2) &= 0 \cdots u^k(x_2) = 0 \quad k = 1, 2, \dots, (n-1). \end{aligned}$$

Substitution of Eq. (2.5.3) in Eq. (2.5.1) yields

$$\Phi(y + \epsilon u) = \int_{x_1}^{x_2} F(x, y + \epsilon u, y' + \epsilon u', \dots, y^n + \epsilon u^n) dx. \quad (2.5.4)$$

The derivative of Φ with respect to ϵ gives

$$\frac{\partial \Phi}{\partial \epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \epsilon} + \frac{\partial F}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \epsilon} + \dots + \frac{\partial F}{\partial \bar{y}^n} \frac{\partial \bar{y}^n}{\partial \epsilon} \right) dx. \quad (2.5.5)$$

Let $\epsilon = 0$, yields

$$\frac{\partial \Phi}{\partial \epsilon} \Big|_{\epsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} u + \frac{\partial F}{\partial y'} u' + \frac{\partial F}{\partial y''} u'' + \dots + \frac{\partial F}{\partial y^n} u^n \right) dx. \quad (2.5.6)$$

Using integration by parts, gives

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} u' dx = \frac{\partial F}{\partial y'} u(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) u dx \quad (2.5.7)$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} u'' dx = \frac{\partial F}{\partial y''} u'(x) \Big|_{x_1}^{x_2} - \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) u(x) \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) u dx. \quad (2.5.8)$$

Higher order derivatives are similarly reduced to factors of $u(x)$ and a series of terms which should be evaluated at the boundaries $x = x_1$ and $x = x_2$. Since the arbitrary variational function $u(x)$ and all its derivatives up to (n) are continuous in the interval (x_1, x_2) and vanish at x_1 and x_2 , Eq. (2.5.6) therefore reduces to

$$\begin{aligned} \frac{\partial \Phi(y + \epsilon u)}{\partial \epsilon} \Big|_{\epsilon=0} &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y''} \right. \\ &+ \dots (-1)^n \frac{d^n}{dx^n} \frac{\partial F}{\partial y^n} \Big] u(x) dx + \frac{\partial F}{\partial y'} u \Big|_{x_1}^{x_2} + \frac{\partial F}{\partial y''} u' \Big|_{x_1}^{x_2} - \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) u \Big|_{x_1}^{x_2} \\ &+ \dots \end{aligned} \quad (2.5.9)$$

This equation is valid for all possible arbitrary functions $u(x)$ with the given properties. Therefore, if the integral equation should be zero for all possible functions $u(x)$, the following expressions must be identically equal to zero :

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y''} + \dots (-1)^n \frac{d^n}{dx^n} \frac{\partial F}{\partial y^n} = 0. \quad (2.5.10)$$

The natural boundary conditions are

$$\begin{aligned}
 & \frac{\partial F}{\partial y'} - \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y'''} \right) - \cdots + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} \frac{\partial F}{\partial y^n} = 0 \\
 & \frac{\partial F}{\partial y''} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'''} \right) + \cdots + (-1)^{n-2} \frac{d^{n-2}}{dx^{n-2}} \frac{\partial F}{\partial y^n} = 0 \\
 & \dots \\
 & \dots \\
 & \dots \\
 & \frac{\partial F}{\partial y^{n-1}} = 0 \quad \text{at } x = x_1, \text{ and } x = x_2.
 \end{aligned} \tag{2.5.11}$$

Since the function F is known, Eq. (2.5.10) results in the boundary value problem governing the function $y(x)$. This function minimizes the functional Φ given by Eq. (2.5.1). Equation (2.5.11) are the natural boundary conditions derived through the integrations by parts of the functional.

2.6 Minimization of Functions of Several Variables

Consider a function $u(x, y)$ defined in the domain D enclosed by the boundary curve C . The functional Φ is assumed to be proportional to the function u and its first partial derivatives with respect to the variables x and y as defined:

$$\Phi[u(x, y)] = \int_D F(x, y, u, u_x, u_y) dx dy \tag{2.6.1}$$

where u_x and u_y are the partial derivatives of the function u with respect to x and y . We assume the functional F to be at least differentiable up to the second order and the extremizing function $u(x, y)$ differentiable up to the first order. We further assume that the class of admissible functions $u(x, y)$ has the following properties;

a- $u(x, y)$ is prescribed on boundary curve C .

b- $u(x, y)$ and its partial derivatives with respect to x and y up to the order one, are continuous in the domain D .

We assume that the function $u(x, y)$ is the only function among the class of admissible functions which minimizes the functional Φ . Now, the variational function $g(x, y)$ with the following properties is considered

a- $g(x, y) = 0$ for all the boundary points on C .

b- $g(x, y)$ is continuous and differentiable in D .

We construct the variational function as

$$\tilde{u}(x, y) = u(x, y) + \epsilon g(x, y). \tag{2.6.2}$$

Substituting in Eq. (2.6.1) and carrying out the partial derivatives gives

$$\frac{\partial \Phi}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \int_D F[x, y, (u + \epsilon g), (u + \epsilon g)_x, (u + \epsilon g)_y] dx dy. \quad (2.6.3)$$

Carrying out the integration and setting $\epsilon = 0$ yields

$$\frac{\partial \Phi}{\partial \epsilon} |_{\epsilon=0} = \int_D \left(g \frac{\partial F}{\partial u} + g_x \frac{\partial F}{\partial u_x} + g_y \frac{\partial F}{\partial u_y} \right) dx dy. \quad (2.6.4)$$

The subscripts indicate differentiating with respect to x or y . To evaluate the integral, consider the expressions

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} g \right) &= \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) g + \frac{\partial F}{\partial u_x} g_x \\ \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} g \right) &= \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) g + \frac{\partial F}{\partial u_y} g_y. \end{aligned} \quad (2.6.5)$$

Substituting Eq. (2.6.5) in Eq. (2.6.4), the last two terms become

$$\begin{aligned} \int_D \left(\frac{\partial F}{\partial u_x} g_x + \frac{\partial F}{\partial u_y} g_y \right) dx dy &= \int_D \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} g \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} g \right) \right] dx dy \\ &\quad - \int_D \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \right] g dx dy. \end{aligned} \quad (2.6.6)$$

Here, $\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right)$ is called the total partial derivative with respect to x , and in performing the partial derivatives with respect to x , the variable y remains constant, that is

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) = \frac{\partial^2 F}{\partial x \partial u_x} + \frac{\partial^2 F}{\partial u_x \partial u} u_x + \frac{\partial^2 F}{\partial u_x^2} \frac{\partial u_x}{\partial x} + \frac{\partial^2 F}{\partial u_x \partial u_y} \frac{\partial u_y}{\partial x} \quad (2.6.7)$$

and

$$\frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) = \frac{\partial^2 F}{\partial y \partial u_y} + \frac{\partial^2 F}{\partial u_y \partial u} u_y + \frac{\partial^2 F}{\partial u_y \partial u_x} \frac{\partial u_x}{\partial y} + \frac{\partial^2 F}{\partial u_y^2} \frac{\partial u_y}{\partial y}. \quad (2.6.8)$$

Now, using Green's integral theorem, the area integral is transferred into the line integral as

$$\int_D \left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} \right) dx dy = \int_C (N dy - M dx). \quad (2.6.9)$$

Using this rule, we obtain

$$\int_D \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} g \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} g \right) \right] dx dy = \int_C \left(\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx \right) g. \quad (2.6.10)$$

The right-hand side of this equation is integrated over the boundary curve C . From Eqs. (2.6.10) and (2.6.6), we get

$$\begin{aligned} \int_D \left(\frac{\partial F}{\partial u_x} g_x + \frac{\partial F}{\partial u_y} g_y \right) dx dy &= - \int_D \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \right] g dx dy \\ &+ \int_C \left(\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx \right) g. \end{aligned} \quad (2.6.11)$$

Substituting Eq. (2.6.11) into Eq. (2.6.4), and letting the expression Eq. (2.6.4) be zero, we obtain

$$\begin{aligned} \int_D \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \right] g dx dy \\ + \int_C \left(\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx \right) g = 0. \end{aligned} \quad (2.6.12)$$

Since the function $g(x, y)$ is arbitrary,

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial u_y} = 0 \quad \text{in } D. \quad (2.6.13)$$

This equation is called the Euler equation, which is the boundary value problem associated with the functional Eq. (2.6.1). The natural boundary condition is

$$\int_C \left(\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx \right) = 0 \quad \text{on } C \quad (2.6.14)$$

2.7 Cantilever Beam

Consider a cantilever beam of arbitrary cross-sectional area and length L and bending stiffness EI subjected to a bending force F , as shown in Fig. 2.1.

We will now write the potential energy function of the beam under the applied force F , and by minimization of this function, using the method of calculus of variation, we obtain the Euler equation for the equilibrium of the beam.

The potential energy of the beam is the sum of two parts, *internal strain energy*, and the *strain energy of the external forces*, as

$$V = U + \Omega \quad (2.7.1)$$

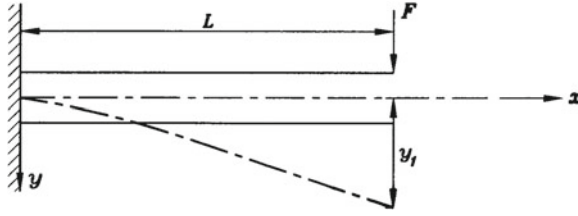


Fig. 2.1 A cantilever beam under transverse force F

where V is the total potential energy of the beam, U is the internal strain energy and Ω is the strain energy of the external forces.

From the strength of the material, it is recalled that the internal strain energy of a beam subjected to a bending force is obtained from the following relation

$$U = \int_0^L \frac{M^2 dx}{2EI} \quad (2.7.2)$$

where M is the bending moment distribution along the beam and dx is an element of the length of the beam the associated strain energy of which is dU . From the elementary beam theory, the bending moment M and the curvature $1/R$ are related by

$$M = \frac{EI}{R}. \quad (2.7.3)$$

Substituting M from Eq. (2.7.3) into Eq. (2.7.2) yields

$$U = \int_0^L \frac{EI}{2R^2} dx. \quad (2.7.4)$$

The radius of curvature R and the beam's elastic deflection equation are related as

$$R = \frac{(1 + y'^2)^{3/2}}{|y''|}. \quad (2.7.5)$$

Since for the small deformation, assumption $y' \ll 1$, y' is neglected compared to 1 in Eq. (2.7.5). Substituting the reduced form of Eq. (2.7.5) in Eq. (2.7.4) gives

$$U = \frac{1}{2} \int_0^L EI (y'')^2 dx. \quad (2.7.6)$$

This equation is valid for an arbitrarily small deflection, provided that plastic deformation does not occur.

The potential energy of external force F is

$$\Omega = -F y_1 \quad (2.7.7)$$

where y_1 is the deflection under the force F in y -direction and the negative sign indicates that work is done on the system. The total potential energy from Eq. (2.7.1), after substituting from Eqs. (2.7.7) and (2.7.6), becomes

$$V = -F y_1 + \frac{1}{2} \int_0^L EI (y'')^2 dx. \quad (2.7.8)$$

In Eq. (2.7.8) y , the deflection function is a function of the variable x . As y takes on different values, the functional V varies. The equilibrium state of the beam occurs when the functional V has its minimum value, and at this condition, $y(x)$ represents the deflection equation of the beam at equilibrium. The function $y(x)$ has to satisfy certain conditions. The function $y(x)$ and its first derivative $y'(x)$ have to be continuous over the interval $(0, L)$. Furthermore, the second derivative y'' must exist, and must be an integrable function. The function $y(x)$ must also satisfy the boundary conditions at the beam's boundary.

In this case, the boundary conditions at the clamped edge for a cantilever beam are $y(0) = y'(0) = 0$. These conditions are called the *essential boundary conditions* since they are physical constraints of the problem.

Now, we may apply the method of calculus of variation to obtain the minimum of the potential energy function and, thus, the equilibrium equation for the deflection of the beam. Let us define the variational function $\eta(x)$ and the variational parameter ϵ . The variational function $\eta(x)$ is arbitrary, and both itself and its derivative with respect to x are continuous functions between the interval $(0, L)$. The potential energy of the beam, corresponding to the deflection $\bar{y} = y + \epsilon\eta$, is

$$V(\bar{y}) = -F(y_1 + \epsilon\eta_1) + \frac{1}{2} EI \int_0^L (y'' + \epsilon\eta'')^2 dx \quad (2.7.9)$$

or

$$\begin{aligned} V(\bar{y}) = & -F(y_1 + \epsilon\eta_1) + \frac{1}{2} EI \left[\int_0^L (y'')^2 dx + \int_0^L (\epsilon^2 (\eta'')^2 dx \right. \\ & \left. + 2 \int_0^L \epsilon y'' (\eta'')^2 dx \right] \end{aligned}$$

differentiating with respect to ϵ gives

$$\frac{\partial V}{\partial \epsilon} = -F\eta_1 + \frac{1}{2} EI \left[2\epsilon \int_0^L (\eta'')^2 dx + 2 \int_0^L y'' \eta'' dx \right]$$

Setting $\epsilon = 0$ yields

$$\frac{\partial V}{\partial \epsilon}|_{\epsilon=0} = -F\eta_1 + EI \int_0^L y'' \eta'' dx = 0$$

where in the above equation ϵ is set equal to zero, and according to the rule of calculus of variation, the remaining expression is equal to zero. Two times integrations by parts give

$$-F\eta(L) + EI y'' \eta'|_0^L - EI y''' \eta|_0^L + EI \int_0^L y^{IV} \eta dx = 0.$$

This may be written as

$$-F\eta(L) + EI[y''(L)\eta'(L) - y'''(L)\eta(L) + \int_0^L y^{IV} \eta dx] = 0. \quad (2.7.10)$$

In order that Eq. (2.7.10) vanishes for all the admissible functions $\eta(x)$, the following conditions must hold

$$y^{IV} = 0 \quad (2.7.11)$$

and

$$\begin{aligned} F + EI y'''(L) &= 0 & y'''(0) &= 0 \\ y''(L) &= 0 & EI y''(0) &= 0. \end{aligned} \quad (2.7.12)$$

The conditions Eq. (2.7.12) are known as the *natural boundary conditions*, since they are necessary to make the potential energy a minimum. Equation (2.7.11) is known as the Euler equation and is the equilibrium equation of the beam which minimizes the total potential energy equation under the given boundary conditions.

A general solution of Eq. (2.7.11) is

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 \quad (2.7.13)$$

where C_0 , C_1 , C_2 and C_3 are the constants of integration. For the given essential boundary conditions, we have

$$C_0 = C_1 = 0. \quad (2.7.14)$$

The other two force boundary conditions related to the moment and shear force on $x = L$ give

$$C_2 = \frac{FL}{2EI} \quad C_3 = \frac{-F}{6EI} \quad (2.7.15)$$

and therefore, the deflection equation of the beam becomes

$$y = \frac{FL}{2EI} x^2 - \frac{F}{6EI} x^3 \quad (2.7.16)$$

or

$$y = \frac{Fx^2}{2EI} \left(L - \frac{x}{3}\right) \quad (2.7.17)$$

2.8 Approximate Techniques

A system under equilibrium condition is considered, in which its equilibrium equation is described by the general form

$$L_{2m}[\psi] = f \quad (2.8.1)$$

where L_{2m} is a general type of mathematical operation of order $2m$ applied to the function ψ , and f is a known function of the given variables. The function ψ satisfies the general form of the boundary conditions given as

$$B_i[\psi] = g_i \quad i = 1, 2, \dots, 2m \quad (2.8.2)$$

where B_i is a linear mathematical operator describing the boundary conditions on ψ . Here, the known functions g_i are the given boundary conditions.

We further assume that the equilibrium equation Eq.(2.8.1) is associated with a variational problem such that the general expression for the functional Φ is

$$\Phi = \Phi(\psi). \quad (2.8.3)$$

An approximate solution of Eq.(2.8.1) may have the following linear form

$$\psi^* = \phi_0 + \sum_{j=1}^n C_j \phi_j \quad (2.8.4)$$

where the functions ϕ_j are linearly independent known functions of the variables in the solution domain D satisfying the homogeneous boundary conditions. Function ϕ_0 is a known function of the variables satisfying the nonhomogeneous boundary conditions, and the constants C_j are the undetermined parameters. With the above definitions, the functions ϕ_j in Eq.(2.8.4) satisfy the boundary conditions

$$\begin{aligned} B_i[\phi_0] &= g_i \quad i = 1, \dots, 2m \\ B_i[\phi_j] &= 0 \quad i = 1, \dots, 2m, \quad j = 1, \dots, n. \end{aligned} \quad (2.8.5)$$

Thus, the function ψ of Eq.(2.8.1) satisfies all the boundary conditions for arbitrary values of the constant coefficients C_j .

We find the undetermined parameters C_j so that they make the functional Φ , related to system (2.8.1), stationary. In this case, a set of n simultaneous equations for the constants C_j must be obtained. This method may therefore be considered as a means for reducing a continuous equilibrium problem to an approximately equivalent equilibrium problem with n degrees of freedom. There are, however, two different approaches for finding the undetermined coefficients C_j : the weighted residual methods and the variational method. The weighted residual methods are based on four different techniques. These approaches are discussed in the following section.

2.8.1 A: Weighted Residual Methods

When the trial solution Eq.(2.8.4), which satisfies Eq.(2.8.5), is inserted into Eq.(2.8.1), the *residual equation* R is

$$R = f - L_{2m}[\psi^*] = f - L_{2m}[\phi_0 + \sum_{j=1}^r C_j \phi_j]. \quad (2.8.6)$$

For the exact solution, the residual R has to be identically zero. For a proper approximate solution, it should be restricted within a small tolerance. The classical weighted residual methods are as follows:

2.8.1.1 Collocation

The solution domain D is considered and n arbitrary points are selected inside the domain, usually with a known geometric pattern. The residual R of equation Eq.(2.8.6) is set equal to zero at n points in the domain D . That is

$$R = f - L_{2m}[\phi_0 + \sum_{j=1}^n C_j \phi_j] = 0. \quad (2.8.7)$$

This provides n simultaneous algebraic equations for the constants C_j . The locations of the points are arbitrary but, as mentioned, are usually such that D is covered more or less uniformly by a simple pattern.

2.8.1.2 Subdomain

The solution domain D is subdivided into n subdomains D_i , $i = 1, 2, \dots, n$, usually according to a simple pattern. Then, the integral of the residual Eq.(2.8.6)

over each subdomain D_i is set equal to zero, as

$$\int_{D_i} R dD = 0 \quad i = 1, \dots, n. \quad (2.8.8)$$

This equation provides a system of n algebraic equations to be solved for n constant coefficients C_j .

2.8.1.3 Galerkin

The complete solution domain is considered, and the residue R is made orthogonal with respect to the approximating functions ϕ_j over the whole domain as;

$$\int_D \phi_k R dD = 0 \quad k = 1, \dots, n. \quad (2.8.9)$$

This equation provides a system of n algebraic equations for the n constant coefficients C_j .

The main difference between the collocation and subdomain methods and the Galerkin method is that, in the collocation and subdomain methods, the solution domain is divided into a number of elements and nodal points, while in the Galerkin method the solution domain is considered as a whole. This is called the *traditional Galerkin method*.

2.8.1.4 Least Square

Similar to the Galerkin method, the complete solution domain is Considered, and the integral of the square of the residue is minimized with respect to the constant coefficients C_j as

$$\frac{\partial}{\partial C_k} \int_D R^2 dD = 0 \quad k = 1, \dots, n. \quad (2.8.10)$$

This equation provides a system of n algebraic equations with n unknowns C_j , which may be solved for C_j . If L_{2m} is a linear mathematical operator, then Eq. (2.8.10) is simplified as

$$-2 \int_D R L_{2m}[\phi_k] dD = 0 \quad k = 1, \dots, n \quad (2.8.11)$$

2.8.2 B: Stationary Functional Method

Let Φ be a functional such that the extremum problem for Φ is equivalent to the equilibrium problem. The *Ritz method* consists of treating the extremum problem directly by inserting the trial family Eq. (2.8.6) into Φ and setting

$$\frac{\partial \Phi}{\partial C_j} = 0 \quad j = 1, \dots, n. \quad (2.8.12)$$

These n equations are solved for the constants C_j , and when multiplied by their corresponding functions ψ , represent an approximate solution to the extremum problem. It is an approximate solution, because it gives Φ a stationary value only for the class of functions ψ which are part of the trial family Eq. (2.8.5).

The most important step in the above discussion is the selection of the trial family Eq. (2.8.5). The purpose of the above criterion is merely to pick the *best* approximation from a given family.

2.9 Further Notes on the Ritz and Galerkin Methods

In discussion of the boundary-value and extremum problems, it was concluded that any boundary-value problem representing a mechanical system in equilibrium is associated with an equivalent extremum problem in which it's corresponding functional is in a relative minimum condition.

Now, let us assume that the potential energy of a mechanical system is represented by the following double integral

$$\Phi(\psi) = \iint_D F(x, y, \psi, \psi_x, \psi_y) dx dy \quad (2.9.1)$$

subjected to the condition

$$\psi = \phi(s) \quad \text{on} \quad \Gamma \quad (2.9.2)$$

where Γ is the contour bounding the region D and the subscript in Eq. (2.9.1) indicates the derivative with respect to x or y . Let ψ be the exact solution to this problem, and $\Phi(\psi) = m$ the value of the minimum. If we can find a function $\bar{\psi}(x, y)$ which satisfies the boundary condition Eq. (2.9.2) and for which the value of functional $\Phi(\bar{\psi})$ is very close to m , then $\bar{\psi}$ is a good approximation for the minimum of functional Eq. (2.9.1). On the other hand, if we can find a minimizing sequence $\bar{\psi}$, i.e., a sequence of functions satisfying the condition Eq. (2.9.2), and for which $\Phi(\bar{\psi}_n)$ approaches m , it would be expected that such a sequence would converge to the solution.

Ritz proposed a classical method in which one can find $\bar{\psi}$, a function which minimizes the integral Eq. (2.9.1), systematically. To describe the Ritz method, let us

assume ψ to be a function of the variables x and y with n coefficients a_1, a_2, \dots, a_n .

$$\psi = \psi(x, y, a_1, a_2, \dots, a_n). \quad (2.9.3)$$

This function is chosen in such a way that, regardless of the values of a_n , ψ satisfies the boundary condition Eq. (2.9.2). The Ritz method is then based on calculating the coefficients a_1 through a_n for which ψ of equation Eq. (2.9.3) minimizes the integral Eq. (2.9.1). Upon substitution of equations Eq. (2.9.3) into Eq. (2.9.1), and performing the necessary differentiation and integration, we find that Φ is converted into a function of the coefficients a_1, a_2, \dots, a_n . That is, $\Phi = \Phi(a_1, a_2, \dots, a_n)$. To minimize this function, Ritz proved that the coefficients a_n must satisfy the following system of equations:

$$\frac{\partial \Phi}{\partial a_k} = 0 \quad k = 1, 2, \dots, n. \quad (2.9.4)$$

Let us assume that the solution to Eq. (2.9.4) for n coefficients a_n is $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$. Substituting this solution into Eq. (2.9.3) for ψ , we obtain

$$\bar{\psi}(x, y) = \bar{\psi}(x, y, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_n) \quad (2.9.5)$$

for which $\bar{\psi}$ is now the minimum of integral Eq. (2.9.1).

Now, let us apply the Ritz method to obtain a close approximation to the actual minimum using a family of functions

$$\psi(x, y) = \psi_n(x, y, a_1, a_2, \dots, a_n) \quad n = 1, 2, \dots \quad (2.9.6)$$

Let $\bar{\psi}_n$ be the n th. approximation giving the last value for integral Φ in comparison with all the functions up to the n th. family. Since each successive family contains all the functions of the preceding, i.e., for each successive problem the class of admissible functions is broader, it is clear that the successive minimums are non-increasing,

$$\Phi(\bar{\psi}_1) \geq \Phi(\bar{\psi}_2) \geq \Phi(\bar{\psi}_n). \quad (2.9.7)$$

The Galerkin method is an approximate numerical technique which directly solves the boundary-value problems. This method is particularly suitable for nonlinear problems due to its fast rate of convergence. In order to describe the method, we assume a boundary value problem represented by the following differential equation:

$$L(y) = f(x) \quad (2.9.8)$$

subjected to homogeneous boundary conditions

$$\begin{aligned} y(x_1) &= 0 \\ y(x_2) &= 0 \end{aligned} \quad (2.9.9)$$

where L is a mathematical operator and $f(x)$ is a known function. Note that nonhomogeneous boundary conditions of

$$\begin{aligned} y(x_1) &= y_1 \\ y(x_2) &= y_2 \end{aligned} \quad (2.9.10)$$

can be transformed to the homogeneous conditions Eq. (2.9.9) with a proper change of variables.

Let us choose a set of continuous linearly independent functions $w_i(x)$ in the interval $(x_1 - x_2)$ that satisfy the boundary conditions Eq. (2.9.9), that is,

$$w_i(x_1) = w_i(x_2) = 0 \quad i = 1, 2, \dots, n. \quad (2.9.11)$$

We seek the solution of the Eq. (2.9.8) in the form of

$$y_n = \sum_{i=1}^n a_i w_i(x) \quad (2.9.12)$$

where a_i 's are constant coefficients to be determined.

Galerkin suggested that, in order to find the coefficients a_i , the following orthogonality condition must be satisfied by the functions $w_i(x)$ in the interval (x_1, x_2)

$$\int_{x_1}^{x_2} [L(\sum_{i=1}^n a_i w_i(x)) - f(x)] w_i(x) dx = 0 \quad i = 1, 2, \dots, n. \quad (2.9.13)$$

When the number of functions $w_i(x)$ tends to infinity, $(n \rightarrow \infty)$, the solution tends to the exact solution. In order to solve for the coefficients a_i , the linear set of Eq. (2.9.13) has to be solved for the unknowns a_i (for discussion and solution of such a system of equations, one may refer to Kantovich and Krylov [1]). In practical cases, a finite number of series Eq. (2.9.12) are considered from which, upon substitution in Eq. (2.9.13), a finite set of linear equations are obtained to solve for a_i .

The functions $w_i(x)$ are usually selected in polynomial or trigonometric forms as

$$\begin{aligned} &(x - x_1)(x - x_2) && (x - x_1)^2(x - x_2) && (x - x_1)^n(x - x_2) \\ &\sin \frac{n \pi (x - x_1)}{x_2 - x_1} && n = 1, 2, \dots \end{aligned} \quad (2.9.14)$$

It is obvious that the origin of the coordinate system can be transformed to x_1 and, thus, in Eq. (2.9.14) $x_1 = 0$.

The Galerkin method is a powerful tool for obtaining an approximate solution for the ordinary differential equations of any order n , systems of differential or partial differential equations.

2.10 Application of the Ritz Method

We will now apply the Ritz method to the solution of an ordinary differential equation of the second order [1]

$$L(y) = \frac{d}{dx}(py') - qy - f = 0 \quad (2.10.1)$$

under the homogeneous boundary conditions

$$y(0) = 0, \quad y(L) = 0. \quad (2.10.2)$$

It may be verified that Eq. (2.10.1) is the minimum of the functional

$$\Phi(y) = \int_0^L [py'^2 + qy^2 + 2fy]dx \quad (2.10.3)$$

subjected to the boundary conditions Eq. (2.10.2). We furthermore assume that in the given interval the following inequalities are satisfied:

$$p(x) > 0 \quad q(x) \geq 0 \quad 0 \leq x \leq L. \quad (2.10.4)$$

Let us now take a series of linearly independent functions $\phi_k(x)$, $k = 1, 2, \dots, n$, continuous in the interval $[0, L]$ together with their first derivatives and satisfying the conditions Eq. (2.10.2). Such a series of functions may be taken as, for example,

$$\begin{aligned} \phi_k &= \sin \frac{k\pi x}{L} \\ \phi_k &= (L - x)x^k \quad k = 1, 2, \dots, n. \end{aligned} \quad (2.10.5)$$

We now apply the Ritz method to obtain the minimum of the functional Eq. (2.10.3) using the series of linear combinations of the functions ϕ_k . We seek a solution in the form of

$$y_n = \sum_{k=1}^n a_k \phi_k. \quad (2.10.6)$$

Substituting y_n in Eq. (2.10.3), gives

$$\begin{aligned} \Phi(y_n) &= \int_0^L [py_n'^2 + qy_n^2 + 2fy_n]dx \\ &= \int_0^L [p(\sum_{k=1}^n a_k \phi_k')^2 + q(\sum_{k=1}^n a_k \phi_k)^2 + 2f(\sum_{k=1}^n a_k \phi_k)]dx. \end{aligned} \quad (2.10.7)$$

But

$$\left(\sum_{k=1}^n a_k \phi'_k\right)^2 = \sum_{k=1}^n \sum_{s=1}^n a_k a_s \phi'_k \phi'_s. \quad (2.10.8)$$

Therefore

$$\Phi(y_n) = \int_0^L \left[p \sum_{k=1}^n \sum_{s=1}^n a_k a_s \phi'_k \phi'_s + q \sum_{k=1}^n \sum_{s=1}^n a_k a_s \phi_k \phi_s + 2f \sum_{k=1}^n a_k \phi_k \right] dx. \quad (2.10.9)$$

Calling

$$\begin{aligned} \alpha_{k,s} &= \alpha_{s,k} = \int_0^L (p \phi'_k \phi'_s + q \phi_k \phi_s) dx \\ \beta_k &= \int_0^L f \phi_k dx. \end{aligned} \quad (2.10.10)$$

We have

$$\Phi(y_n) = \sum_{k=1}^n \sum_{s=1}^n \alpha_{k,s} a_k a_s + 2 \sum_{k=1}^n \beta_k a_k. \quad (2.10.11)$$

Taking the derivative with respect to a_s

$$\frac{1}{2} \frac{d\Phi(y_n)}{da_s} = \sum_{k=1}^n \alpha_{k,s} a_k + \beta_s = 0 \quad (2.10.12)$$

or

$$\frac{1}{2} \frac{d\Phi(y_n)}{da_s} = \sum_{k=1}^n \int_0^L (p \phi'_k \phi'_s + q \phi_k \phi_s) a_k dx + \int_0^L f \phi_s dx = 0. \quad (2.10.13)$$

Multiplying a_k through the parentheses

$$\frac{1}{2} \frac{d\Phi(y_n)}{da_s} = \int_0^L \left(\sum_{k=1}^n a_k p \phi'_k \phi'_s + \sum_{k=1}^n q \phi_k \phi_s a_k + f \phi_s \right) dx = 0 \quad (2.10.14)$$

or, finally

$$\int_0^L (p y'_n \phi'_s + q y_n \phi_s + f \phi_s) dx = 0 \quad s = 1, 2, \dots, n. \quad (2.10.15)$$

Equation (2.10.15) represents a set of n integral equations to be solved for a_k . Using the rule of integration by parts gives

$$\int_0^L p y_n' \phi_s' dx = [p y_n' \phi_s]_0^L - \int_0^L \frac{d}{dx} (p y_n') \phi_s dx. \quad (2.10.16)$$

The first term in the right-hand side of the above equation vanishes, as ϕ_s vanishes at 0 and L , and thus

$$\int_0^L p y_n' \phi_s' dx = - \int_0^L \frac{d}{dx} (p y_n') \phi_s dx. \quad (2.10.17)$$

Substituting in Eq. (2.10.15), yields

$$\int_0^L \left[\frac{d}{dx} (p y_n') - q y_n - f \right] \phi_s dx = 0 \quad (2.10.18)$$

or, finally

$$\int_0^L L(y_n) \phi_s dx = 0. \quad (2.10.19)$$

Noticed that application of the Ritz method in this case reduced the problem to that of the Galerkin method.

2.10.1 Non-homogeneous Boundary Conditions

In the previous section, we discussed application of the Ritz method to problems with homogeneous boundary conditions. Now, let us consider a problem with a general non-homogeneous boundary condition as

$$\begin{aligned} y(x_1) &= y_1 \\ y(x_2) &= y_2. \end{aligned} \quad (2.10.20)$$

For this case, we will take the solution in the form

$$y_n = \sum_{k=1}^n a_k \phi_k + \phi_0(x) \quad (2.10.21)$$

where $\phi_0(x)$ satisfies the given nonhomogeneous boundary conditions. Since the known functions $\phi_j(x)$ satisfy the homogeneous boundary conditions

$$\phi_k(x_1) = \phi_k(x_2) = 0 \quad k = 1, 2, \dots, n \quad (2.10.22)$$

thus, the function ϕ_0 must satisfy the nonhomogeneous conditions as

$$\begin{aligned}\phi_0(x_1) &= y_1 \\ \phi_0(x_2) &= y_2.\end{aligned}\tag{2.10.23}$$

As an example, considering a linear approximation, function $\phi_0(x)$ has the following form:

$$\phi_0(x) = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) + y_1 \tag{2.10.24}$$

where y_1 and y_2 are given values.

Example 1 Consider an ordinary differential equation such as

$$y'' + y + x = 0 \tag{2.10.25}$$

subject to the boundary conditions

$$y(0) = y(1) = 0. \tag{2.10.26}$$

It is required to find an approximate solution of the equation using the Ritz method.

Solution: The exact solution of the above differential equation, using the classical method for the solution of a differential equation with constant coefficients, is

$$y = \frac{\sin x}{\sin 1} - x. \tag{2.10.27}$$

Now, the approximate solution of Eq.(2.10.25) is found and compared with Eq.(2.10.27).

The corresponding expression for the functional of Eq.(2.10.25) is

$$I = \int_0^1 (y'^2 + y^2 - 2xy)dx. \tag{2.10.28}$$

Comparing Eq.(2.10.28) with Eq.(2.10.3) reveals that $p = 1$, $q = -1$, and $f = -x$.

The solution is approximated with one term of the series Eq.(2.10.5) as

$$y_1 = a_1\phi_1 = a_1x(1 - x). \tag{2.10.29}$$

Substituting the approximate solution Eq.(2.10.29) in the expression for the functional, using Eq.(2.10.19), gives

$$\int_0^1 L(y_1)\phi_1 dx = \int_0^1 [-2a_1 + a_1x(1 - x) + x]x(1 - x)dx = 0.$$

Multiplying and integrating gives

$$\begin{aligned}
& \left[\frac{a_1}{5} x^5 - \frac{1+2a_1}{4} x^4 + \frac{1+3a_1}{3} x^3 - a_1 x^2 \right]_0^1 = 0 \\
& \frac{a_1}{5} - \frac{1+2a_1}{4} + \frac{1+3a_1}{3} - a_1 = 0 \\
& a_1 = \frac{5}{18}
\end{aligned}$$

and thus, the solution is

$$y_1 = \frac{5}{18} x(1-x) \quad (2.10.30)$$

Example 2 Consider again the same problem as in Example (1), but with an approximate solution with two terms of the series being considered as

$$\phi_1 = x(1-x) \quad , \quad \phi_2 = x^2(1-x)$$

and

$$y_2 = x(1-x)(a_1 + a_2 x).$$

Substituting in Eq. (2.10.19) gives

$$\int_0^1 L(y_2) \phi_1 dx = 0 \quad (2.10.31)$$

and

$$\int_0^1 L(y_2) \phi_2 dx = 0. \quad (2.10.32)$$

Substituting for y_2 , ϕ_1 , and ϕ_2 in Eqs. (2.10.31) and (2.10.32) yields

$$\begin{aligned}
& \int_0^1 [y_2'' + x(1-x)(a_1 + a_2 x) + x] x(1-x) dx = 0 \\
& \int_0^1 [y_2'' + x(1-x)(a_1 + a_2 x) + x] x^2(1-x) dx = 0
\end{aligned}$$

or

$$\begin{aligned}
& \int_0^1 [2(a_2 - a_1) - (6a_2 - 1 - a_1)x + (a_2 - a_1)x^2 - a_2 x^3](x - x^2) dx = 0 \\
& \int_0^1 [2(a_2 - a_1) + (a_1 + 1 - 6a_2)x + (a_2 - a_1)x^2 - a_2 x^3](x^2 - x^3) dx = 0.
\end{aligned}$$

Multiplying and integrating, yields

Table 2.1 Comparison of the exact solution with one- and two-term approximate solutions

	y	y_1	y_2
$x=1/4$	0.044	0.052	0.044
$x=1/2$	0.070	0.069	0.069
$x=3/4$	0.060	0.052	0.060

$$18a_1 + 9a_2 - 5 = 0$$

$$\frac{3}{20}a_1 + \frac{13}{105}a_2 - \frac{1}{20} = 0.$$

Solving for a_1 and a_2 gives

$$a_1 = \frac{71}{369}, \quad a_2 = \frac{7}{41}$$

or

$$y_2 = x(1-x)\left(\frac{71}{369} + \frac{7}{41}x\right). \quad (2.10.33)$$

Now, the exact solution of the differential equation Eq.(2.10.25) is compared with the one-term and two-term approximate solutions. The exact solution from Eq.(2.10.27) is called y , the one-term approximate solution from Eq.(2.10.30) is called y_1 , and the two-term approximate solution from Eq.(2.10.33) is called y_2 . All three solutions satisfy the given boundary conditions at $x = 0$ and $x = 1$. To compare the three solutions, their values at $x = 1/4$, $x = 1/2$, and $x = 3/4$ are calculated and shown in Table 2.1.

Comparing the results, it is seen that the error of the first approximation is about %15 and that of the second approximation about %1.

Example 3 Consider the Bessel differential equation

$$x^2y'' + xy' + (x^2 - 1)y = 0 \quad (2.10.34)$$

defined in the interval $1 \leq x \leq 2$. The boundary conditions at $x = 1$ and $x = 2$ are assumed as

$$y(1) = 1 \quad y(2) = 2. \quad (2.10.35)$$

The exact solution of the assumed Bessel differential equation under the given boundary conditions is

$$y = 3.6072I_1(x) + 0.75195Y_1(x) \quad (2.10.36)$$

where I_1 and Y_1 are the modified Bessel functions of the first and second types and of order 1.

Now, the solution of Eq.(2.10.34) may be approximately obtained using the Galerkin method. Let us change the dependent function y to z by the transformation

Table 2.2 Comparison of the exact solution with a one-term approximate solution

x	y	y ₁
1.3	1.4706	1.4703
1.5	1.7026	1.7027
1.8	1.9294	1.9297

$y = z + x$. Then, Eq. (2.10.34) transforms into the following form:

$$xz'' + z' + \frac{x^2 - 1}{x} z + x^2 = 0. \quad (2.10.37)$$

The boundary conditions in terms of the function z become $z(1) = z(2) = 0$. We assume the solution by one-term approximation $z = a_1 \phi_1$, with $\phi_1 = (x - 1)(2 - x)$. Applying the Galerkin method to Eq. (2.10.37) gives

$$\int_1^2 [xz_1'' + z_1' + \frac{x^2 - 1}{x} z_1 + x^2] \phi_1 dx = 0. \quad (2.10.38)$$

Substituting for z_1 yields

$$\int_1^2 [-2a_1x + (3 - 2x)a_1 + \frac{x^2 - 1}{x}(x - 1)(2 - x)a_1 + x^2] (x - 1)(2 - x) dx = 0. \quad (2.10.39)$$

Solving for a_1 gives

$$a_1 = 0.8110 \quad (2.10.40)$$

and the approximate solution of Eq. (2.10.34) with one-term approximation for y_1 becomes

$$y_1 = 0.8110(x - 1)(2 - x) + x \quad (2.10.41)$$

The exact solution Eq. (2.10.36) is compared with the one-term approximate solution Eq. (2.10.41) in the following at three different locations (Table 2.2).

It should be noted that a very close agreement is reached with even the one-term approximation.

2.11 Problems

1. When the equilibrium problems Eq. (2.8.1) and Eq. (2.8.2) are linear, the weighted-residual methods to the trial family Eq. (2.8.4) all lead to equations for the C_j having the following form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_r \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}$$

Show that for collocation

$$a_{kj} = L_{2m}[\phi_j(P_k)] \quad b_k = f(P_k) - L_{2m}[\phi_0(P_k)]$$

where the P_k are the r locations arbitrarily selected. Show that for the subdomain method

$$a_{kj} = \int_{D_k} L_{2m}[\phi_j] dD \quad b_k = \int_{D_k} \{f - L_{2m}[\phi_0]\} dD$$

where the D_k are the r selected subdomains. Show that for the Galerkin method

$$a_{kj} = \int_D \phi_k L_{2m}[\phi_j] dD \quad b_k = \int_D \phi_k (f - L_{2m}[\phi_0]) dD$$

and for the least-square method

$$a_{kj} = \int_D L_{2m}[\phi_k] L_{2m}[\phi_j] dD \quad b_k = \int_D L_{2m}[\phi_k] (f - L_{2m}[\phi_0]) dD.$$

Note that in every case the matrix A has to do with the characteristics of the system and that the matrix B is related to the *loading* in the domain and acting on the boundary.

2. Show that the equation applying to the unknown value of an approximate solution to Poisson's equation at a nodal point is the same by either the finite element or finite difference method (solve this problem after the introduction to the finite element method).
3. Employing the Galerkin method, solve the following differential equation:

$$\begin{aligned} y'''' + (Ax + B)y &= C \\ y(x_1) &= 0 \\ y(x_2) &= 0 \end{aligned}$$

where A , B and C are constants. Solve this problem first by taking $n = 1$ in Eq. (2.9.12), that is, take only one term of the series. Then, solve the problem by taking $n = 2$ and compare the results. Any numerical values may be assumed for the constants A , B , and C .

4. Consider a functional given in the form

$$\Phi[u(x, y)] = \int_D F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) dx dy$$

where u_x and u_y are the first partial derivatives of the function u , and u_{xx} , u_{yy} , and u_{xy} are the second partial derivatives with respect to x and y . We assume the functional F to be at least differentiable up to the third order, and the extremizing function $u(x, y)$ differentiable up to the second order. We further assume that the class of admissible function $u(x, y)$ has the following properties;

a- $u(x, y)$ is prescribed on boundary curve C .

b- $u(x, y)$ and it's partial derivatives with respect to x and y up to the second order are continuous in the domain D .

Using the method of calculus of variations, obtain the associated Euler equation and the natural boundary conditions.

Further Readings

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