

Chapter 1

Introduction

Non-relativistic quantum mechanics was formulated in the first half of the twentieth century by various people, most prominently by Heisenberg [38] and Schrödinger [66] who independently followed, in the mid 1920s, on the preliminary work of Bohr [17]. The latter attempted to modify non-relativistic classical mechanics, as introduced by Newton in the seventeenth century and developed in the following two centuries by Euler, Lagrange, Hamilton, Jacobi and Poisson, among others,¹ in order to be compatible with the energy and momentum quantization postulates presented first by Planck and then by Einstein and de Broglie in the 1900s.

While the pioneering work of Heisenberg with the help of Born and Jordan introduced what at first became known as “matrix mechanics” [19, 20], the work of Schrödinger produced a partial differential equation for the “wave function”. Although Schrödinger was soon able to link his approach to Heisenberg’s, the two approaches were fully brought together a little later by von Neumann [76, 78]. His final formalism for non-relativistic quantum mechanics is based on the concepts of vectors and operators on a complex Hilbert space. In many cases these are infinite dimensional spaces and, in fact, the complete formalism had to expand on these concepts in order to accommodate distributions and functions which are not square integrable, thus leading to Gelfand’s rigged Hilbert spaces [31], quite later.

From the start, Bohr emphasized the importance of relating the measurable quantities in quantum mechanics to the measurable quantities in classical mechanics. However, his so-called “correspondence principle” was not so easily implemented at the level of relating the two mathematical formalisms in a coherent way. At first, the basic mathematical concepts for describing classical conservative dynamics, functions on a phase space of positions and momenta (a symplectic affine space), were brought in a very contrived way to the quantum formalism through a series of cooking recipes called “quantization”. Conversely, the “classical limit” of a quantum

¹Poincaré, for instance, came a little later and his name is perhaps better associated to two other contemporary revolutions in dynamics: chaos and relativity.

dynamical system, where classical dynamics should prevail, is often a singular limit and in the initial formulations of quantum mechanics it is not the phase space, but rather either the space of positions or the space of momenta, that is present in a more explicit way.

Some of these problems were addressed by Dirac [25, 26] already in his PhD thesis, but a clearer approach for relating the classical and the quantum mathematical formalisms was first introduced by Weyl [85] via a so-called symbol map, or symbol correspondence from bounded operators on Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}^n)$ to functions on phase space $\mathbb{R}^{2n}$, these latter functions depending on Planck's constant $\hbar$. Soon after, Wigner [90] expanded on Weyl's idea to produce an $\hbar$ -dependent function $\mathcal{W}_{\psi} \in L^1_{\mathbb{R}}(\mathbb{R}^{2n})$ as the phase-space representation of a wave function $\psi \in L^2_{\mathbb{C}}(\mathbb{R}^n)$. While real and integrating to 1 over phase space, $\mathcal{W}_{\psi}$ is only a pseudo probability distribution on $\mathbb{R}^{2n}$ because it can take on negative values, as opposed to $|\psi|^2$ which is a true probability distribution on $\mathbb{R}^n$.

Despite this shortcoming, Wigner's function inspired Moyal [50] to develop an alternative formulation of quantum mechanics as a "phase-space statistical theory", following on Weyl's correspondence, where the Poisson bracket of functions on $\mathbb{R}^{2n}$ was replaced by an $\hbar$ -dependent bracket of $\hbar$ -dependent functions on $\mathbb{R}^{2n}$, whose "classical limit" is the Poisson bracket. Moyal's skew-symmetric bracket still satisfies Jacobi's identity and, in fact, can be seen as the commutator of an $\hbar$ -dependent associative product on the space of $\hbar$ -dependent Weyl symbols. Although Moyal's product is written in terms of bi-differential operators, an integral formulation of this product had been previously developed, first by von Neumann [77] soon after Weyl's work, then re-discovered by Groenewold [35]. This Weyl-Moyal approach was further developed by Hörmander [27, 39, 40], among others, into the calculus of pseudo-differential and Fourier-integral operators. It also inspired the deformation quantization approach started by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer for general Poisson manifolds [9].

However, while the Hilbert space used to describe the dynamics of particles in 3-dimensional configuration-space of positions is generally infinite dimensional, if one restricts attention to rotations around a point, only, the corresponding Hilbert space is finite dimensional. Historically, the necessity to introduce an independent, or intrinsic finite dimensional Hilbert space for studying the dynamics of atomic and subatomic particles stemmed from the understanding that a particle such as an electron has an intrinsic "spinning" which is independent of its dynamics in 3-space. An extra degree of freedom, therefore.

After the hypothesis of an extra degree of freedom was first posed in 1924 by Pauli [52] and 1 year later identified by Goudsmit and Uhlenbeck [71, 72] as an intrinsic "electron spin", the spin theory was developed around 1930, mostly by Wigner [88, 89, 91], but also by Weyl [86], through a careful study of the group of rotations and its simply-connected double cover $SU(2)$, and their representations.

Because this intrinsic quantum dynamics in a finite-dimensional Hilbert space had no obvious classical counterpart, at that time, the necessity to relate it with a corresponding classical dynamics was not present from the start. In fact, to the extra degree of freedom of spin, there should correspond a 2-dimensional phase space

and, by $SU(2)$ invariance, this phase space must be the homogeneous 2-sphere S^2 . Thus, the fact that its classical configuration-space cannot be products of euclidean 3-space or the real 2-sphere was at first interpreted as an indication that a physical correspondence principle for spin systems was impossible.

This may explain why the mathematical formulation of a Weyl's style correspondence for spin systems was developed much later. Thus, while Weyl's to Moyal's works are dated from the mid 1920s to the late 1940s, an incomplete version of a symbol correspondence for spin systems was first set forth in mid 1950s by Stratonovich [67] and a first more complete version was presented in mid 1970s by Berezin [10–12]. Then, in the late 1980s, the work of Varilly and Gracia-Bondia [73] finally completed and expanded the draft of Stratonovich. Its contemporary work of Wildberger [92] was followed by the relevant works of Madore [47], in the 1990s, and of Freidel and Krasnov [30], in early 2000s. All these works produced a very good, but in our view, still incomplete understanding of symbol correspondences and symbol products, for spin systems.

From the mathematical point of view, such a late historical development is surprising because quantum mechanics in finite-dimensional Hilbert space, on top of being far closer to Heisenberg's original matrix mechanics, is far simpler than quantum mechanics in infinite-dimensional Hilbert space. On the other hand, the corresponding phase space for the classical dynamics of spin systems, the 2-sphere S^2 , has nontrivial topology, as opposed to $\mathbb{R}^{2n}$. Nonetheless, one should expect that a mathematical formulation of Bohr's correspondence principle "à la Weyl" would have been developed for spin systems in a more systematic and complete way than it had been achieved for mechanics of particles in k -dimensional Euclidean configuration space. However, despite the many developments in the last 50 years, such a systematic mathematical formulation of correspondences in spin systems was still lacking. In particular, we wanted a solid mathematical presentation unifying the main scattered papers on the subject (some of them based also on heuristic arguments) and completing the various gaps, so as to clarify the whole landscape and open new paths to still unexplored territories.

Here we attempt to fulfill this goal, at least to some extent. This monograph, written to be as self-contained as possible, is organized as follows.

In Chap. 2 we review generalities on Lie groups and their representations, with special attention to the groups $SU(2)$ and $SO(3)$.

In Chap. 3 we present the elements for quantum dynamics of spin systems, that is, $SU(2)$ -symmetric quantum mechanical systems. First, we carefully review its representation theory which defines spin- j systems and their standard basis of Hilbert space. Then, after reviewing the tensor product and presenting the space of operators, with its irreducible summands and standard coupled basis, we recall how the operator product is related to quantum dynamics via Heisenberg's equation involving the commutator and then we present in a very detailed way the $SO(3)$ -invariant decomposition of the operator product, its multiplication rule in the coupled basis of operators, with its parity property.

Almost all choices, conventions and notations used in this chapter are standard ones used in the vast mathematics and physics literature on this subject. In particular, we introduce and manipulate Clebsch-Gordan coefficients and the various Wigner symbols in a traditional way, without resorting to the more modern language of “spin networks” which, in our understanding, would require the introduction of additional definitions, etc., which nonetheless would not contribute significantly to the main results to be used in the rest of the book.

In Chap. 4 we present the elements of the classical $SU(2)$ -symmetric mechanical system. After reviewing some basic facts about the 2-sphere as a symplectic manifold and presenting the key concepts of classical Hamiltonian dynamics and Poisson algebra of smooth functions on S^2 , defining the classical spin system, we present in some detail the space of polynomial functions on S^2 , the spherical harmonics, followed by a detailed presentation of the $SO(3)$ -invariant decomposition of the pointwise product and the Poisson bracket of functions on S^2 .

Chapter 5, Intermision, pauses the study of spin systems. Here we present a brief historical overview of symbol correspondences in affine mechanical systems, in preparation for the remaining and most novel chapters of the book.

Thus, in Chap. 6 we present the $SO(3)$ -equivariant symbol correspondences between operators on a finite-dimensional Hilbert space and (polynomial) functions on S^2 . After defining general symbol correspondences and determining their moduli space, we distinguish the isometric ones, the so-called Stratonovich-Weyl symbol correspondences. Then, we present explicit constructions of general symbol correspondences, particularly the ones via coupled-standard basis and via an operator kernel, introducing the key concept of characteristic numbers of a symbol correspondence, and the general covariant-contravariant duality for non-isometric ones. Besides correspondences of Stratonovich-Weyl type, special attention is devoted to the non-isometric correspondence defined by Berezin via Hermitian metric, which is then generalized to correspondences defined via coherent states.

In Chap. 7 we study in detail the products of symbols induced from the operator product via symbol correspondences, the so-called twisted products. After definition and basic properties, we produce explicit expressions for some twisted products of cartesian symbols, valid for all $n = 2j \in \mathbb{N}$. Then, we describe the formulae for general twisted products of spherical harmonics, Y_l^m , discussing some of their common properties. Next, we present a detailed study of integral trikernels, which define twisted products via integral equations. We state the general properties and produce various explicit formulae (including some integral ones) for these trikernels. In so doing, we arrive at formulae for various functional transforms that generalize the Berezin transform and we also see that it is not so easy to infer a simple closed formula for the Stratonovich trikernel in terms of midpoint triangles (in a form first inquired by Weinstein [83] based on the analogy with the Groenewold-vonNeumann trikernel) unless, perhaps, asymptotically.

Then, in Chap. 8 we start the study of the asymptotic $j \rightarrow \infty$ limit of symbol correspondence sequences and their sequences of twisted products. In this monograph, we focus on the high- j asymptotics for finite l , here called low- l high- j -asymptotics, leaving high- l -asymptotics to a later opportunity. We show that

Poisson (anti-Poisson) dynamics emerge in the asymptotic $j \rightarrow \infty$ limit of the standard (alternate) Stratonovich twisted product as well as the standard (alternate) Berezin twisted product, but this is not the generic case for sequences of twisted products, not even in the restricted subclass of twisted products induced from isometric correspondence sequences. Thus, we characterize some kinds of symbol correspondence sequences based on their asymptotic properties and also discuss some measurable consequences.

In Chap. 9 we present some concluding thoughts. For spin systems, adding to Rieffel's old theorem on $SO(3)$ -invariant strict deformation quantizations of the 2-sphere [56], one now also has to take into consideration the fact that generic symbol correspondence sequences do not yield Poisson dynamics in the asymptotic limit of high spin numbers. In light of these results, old and new, we reflect on the peculiar nature of the classical-quantum correspondence.

Finally, in the Appendix we gather proofs of some of the propositions and a theorem, which were stated in the main text.

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