

Chapter 2

The Jordan Basis and Special Subspaces

In this chapter Jordan basis and subspaces related to it will be discussed as well as various Jordan cells and basis vectors which play a different role at accidental numeration of separated basis vectors. The theory of subspace related with linear operators has been disposed.

2.1 The Special Numeration of the Jordan Basis

One calls the *Jordan cell* (of dimension q with eigenvalue λ) a matrix of the form

$$J = \begin{pmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}. \quad (2.1.1)$$

Such a matrix may be represented in the form $J = \lambda I + N$ where I is the identity matrix, and matrix

$$N = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

is nilpotent: $N^q = 0$. Under the action of the matrix N we have $Ne_1 = 0$ and $Ne_k = e_{k-1}$ for $k > 1$. This means that the vector e_1 is an eigenvector with eigenvalue 0, and all other vectors e_k are adjoint. Note that in the representation of a linear operator in the form of the Jordan cell, numeration of the vectors from the basis is essential. The Jordan cell may be of dimension 1, with no adjoint vectors.

One also considers lower-triangular Jordan cells having the form

$$J = \begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ 1 & \lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & \dots & 1 & \lambda \end{pmatrix}.$$

That matrix is similar to (2.1.1) and their difference is only in the method of numeration of the elements of the basis.

As we know, from the standard courses in linear algebra, if X is finite-dimensional complex vector space, then for any linear operator $A : X \rightarrow X$ there exists the *Jordan form*, i.e. there exists the basis (*Jordan basis*), with respect to which the matrix of the operator has block-diagonal form, with the Jordan cells on the diagonal. The Jordan form is uniquely determined up to the order of the Jordan cells, and these cells are usually numbered randomly. But, in a number of problems, including those discussed below, different Jordan cells and different basis vectors have different roles. In the case of random numeration, the extraction of basis vectors, needed for some concrete situation, is carried out usually with cumbersome verbal descriptions. The corresponding objects may be given by formulas, if one employs the following specific numeration of the vectors of such a basis, using four indices. This numeration of the vectors of the Jordan basis was introduced in [1].

The given a linear operator $A : X \rightarrow X$, where X is a m -dimensional complex vector space, denote by q the number of different moduli of eigenvalues of this operator, and numerate these moduli in the increasing order

$$0 \leq r_1 < r_2 < \dots < r_q.$$

Let $q(k)$ be the number of eigenvalues with modulus r_k . Let us numerate all different eigenvalues with the given modulus r_k , using the second index j . We obtain a numeration of the set of different eigenvalues using the two indices k and j .

$$\{\lambda_{kj} : 1 \leq k \leq q, 1 \leq j \leq q(k)\}.$$

Here, for definiteness, we may assume that for any k , numbers λ_{kj} are numerated according to its location on the circle $|\lambda| = r_k$, i.e. in the increasing order with respect to the argument of the complex number. Otherwise speaking, the eigenvalues are presented in the form $\lambda_{kj} = r_k e^{ih_{kj}}$ where $0 \leq h_{kj} < 2\pi$, and we assume that the numeration is chosen in such a way that $h_{k,j} < h_{k,j+1}$. Note here that the method of numeration is not essential.

Let $q(k, j)$ be the number of Jordan cells corresponding to the eigenvalue λ_{kj} ; for any admissible couple (k, j) numerate those cells by the third index i , and denote these cells by $J(k, j, i)$. We obtain a numeration of all Jordan cells by three indices:

$$\{J(k, j, i) : 1 \leq k \leq q; 1 \leq j \leq q(k), 1 \leq i \leq q(k, j)\}.$$

We also assume that for given k and j , the Jordan cells are numerated in the nondecreasing order with respect to their dimensions. The cells of equal dimensions with equal eigenvalues are of equal importance and they are numerated arbitrarily.

Let $q(k, j, i)$ be the dimension of the cell $J(k, j, i)$. We numerate all basis vectors corresponding to this cell, using the additional fourth index l , where here the numeration is uniquely determined by the structure of the Jordan cell, according to the rule formulated earlier.

In this way, vectors from the basis in which the matrix of the operator has the Jordan form, are naturally numerated using four indices:

$$e(k, j, i, l) : 1 \leq k \leq q; 1 \leq j \leq q(k), 1 \leq i \leq q(k, j), 1 \leq l \leq q(k, j, i).$$

This method of numeration is convenient since different indices represent different characteristics of the corresponding basis vector, and this numeration simplifies recording of constructions discussed below.

The coordinates of a vector x in the constructed bases will be denoted by $x(k, j, i, l)$.

We will write below the corresponding indices of basis vectors (and also of coordinates, operators, eigenvalues etc.) in parenthesis and not in the form of lower or upper indices. This notation is more convenient in the case where one uses indices of vectors given by complicated expressions in the considered formulas.

Proposition 2.1.1 *Under the chosen numeration of elements of the Jordan bases, the action of the operator A on the basis vectors is given by formulas:*

$$Ae(k, j, i, 1) = \lambda(k, j, i)e(k, j, i, 1);$$

$$Ae(k, j, i, l) = \lambda(k, j, i)e(k, j, i, l) + e(k, j, i, l-1) \text{ if } l > 1.$$

In particular, in this notation, the vector $e(k, j, i, 1)$ is an eigenvector with the eigenvalue $\lambda(k, j)$, vectors $e(k, j, i, l)$ for $l > 1$ are adjoint vectors of order $l - 1$.

Without loss of generality, following this, it is assumed that there is fixed a scalar product on the given vector space such that the chosen basis vectors are orthonormal.

Let us note that for a given operator there are many bases for which the matrix of that operator has the Jordan form, and the numeration discussed above is related to the specific basis. Therefore, some objects (subspaces, projectors etc.) appearing in the following constructions, depend on the choice of the basis and some of them are canonical—they are uniquely determined by the operator itself and they do not depend on the choice of a basis. Naturally, the final results should not depend on the choice of a basis.

Let us give another remark on the notion of a matrix. Ordinarily, in the courses on linear algebra, by a matrix of dimension $m \times n$ one calls a collection of numbers $\{a_{ij}\}$, indexed by two indices $1 \leq i \leq m, 1 \leq j \leq n$. In other words, a matrix is defined as a function on the product $M \times N$ of two sets $M = \{1, 2, \dots, m\}, N = \{1, 2, \dots, n\}$. A numeration of the elements of the sets M and N by natural numbers is convenient

for the presentation of the matrix in the form of a table, on a piece of paper. But, in most cases, the actual numeration of elements of the sets M and N is of no importance and Bourbaki [2] defines a matrix as a function on the product $M \times N$ of two arbitrary finite sets. In particular, in the previous construction the set M , containing m elements, is realized as the set consisting of 4-tuples of integers:

$$M = \{(k, j, i, l) : 1 \leq k \leq q; 1 \leq j \leq q(k), 1 \leq i \leq q(k, j), 1 \leq l \leq q(k, j, i)\}.$$

2.2 Subspaces Related to a Linear Operator

In the following, we need a number of subspaces generated by specific collections of basis vectors from the Jordan basis and projectors onto these spaces.

Let us denote by $P(k, j, i, l)$ the operator, acting by the formula

$$P(k, j, i, l)x = x(k, j, i, l)e(k, j, i, l).$$

This operator is the orthogonal projector onto the one-dimensional subspace $L(k, j, i, l)$, generated by the basis vector $e(k, j, i, l)$.

Let $L(k, j, i)$ be the subspace generated by vectors $e(k, j, i, l)$, with fixed k, j, i and arbitrary l . This subspace corresponds to the Jordan cell $J(k, j, i)$. By $P(k, j, i)$, we denote the orthogonal projector onto the subspace $L(k, j, i)$. We shall denote by $x(k, j, i)$ the projection of the vector x onto the subspace $L(k, j, i)$:

$$x(k, j, i) = P(k, j, i)x = \sum_l x(k, j, i, l)e(k, j, i, l).$$

Note that the subspaces $L(k, j, i)$ and projectors $P(k, j, i)$ we have just introduced are not canonical, since they depend on the choice of a basis, and in the general case, they are not uniquely determined for the given operator.

By $L(k, j)$, we denote the subspace generated by all the vectors $e(k, j, i, l)$, for fixed k and j . This subspace is independent of a choice of a basis and it is canonical—this is the so-called *root subspace* corresponding to the eigenvalue $\lambda(k, j)$ and it can be given directly using the operator A by:

$$L(k, j) = \ker(A - \lambda(k, j)I)^{p(k, j)},$$

where $p(k, j) = \max_i q(k, j, i)$.

Let us denote the projection of the vector x onto the subspace $L(k, j)$ by $x(k, j)$, and corresponding projector by $P(k, j)$:

$$x(k, j) = P(k, j)x = \sum_i \sum_l x(k, j, i, l)e(k, j, i, l).$$

Let

$$L(k) = \bigoplus_j L(k, j),$$

i.e. this is the subspace generated by all basis vectors $e(k, j, i, l)$, for a given k . These subspaces are also canonical—they do not depend on choice of a basis.

The operator

$$P(k) = \sum_j P(k, j)$$

is the orthogonal projector onto the subspace $L(k)$. The notion of the orthogonal projector is related to the given scalar product. Nevertheless, the operator $P(k)$ may be introduced as a projector onto the subspace $L(k)$, whose kernel contains all subspaces $L(j)$ for $j \neq k$, therefore the operators $P(k)$ are also canonical.

We denote by $x(k)$ the projection of a vector x onto the subspace $L(k)$:

$$x(k) = P(k) x = \sum_j \sum_i \sum_l x(k, j, i, l) e(k, j, i, l).$$

It is obvious that all subspaces $L(k)$, $L(k, j)$ and $L(k, j, i)$ are invariant with respect to the action of A , and there is a decomposition

$$X = \bigoplus_{k=1}^q L(k) = \bigoplus_{k=1}^q \bigoplus_{j=1}^{q(k)} L(k, j) = \bigoplus_{k=1}^q \bigoplus_{j=1}^{q(k)} \bigoplus_{i=1}^{q(k, j)} L(k, j, i). \quad (2.2.2)$$

If by $A(k)$, $A(k, j)$, $A(k, j, i)$, one denotes the restriction of the operator A on the corresponding subspace, then

$$A = \bigoplus_{k=1}^q A(k) = \bigoplus_{k=1}^q \bigoplus_{j=1}^{q(k)} A(k, j) = \bigoplus_{k=1}^q \bigoplus_{j=1}^{q(k)} \bigoplus_{i=1}^{q(k, j)} A(k, j, i).$$

Let us also introduce a collection of subspaces

$$M(k) = \bigoplus_{s=1}^k L(s). \quad (2.2.3)$$

These subspaces are also invariant with respect to the action of the operator A . In addition to it, it is obvious that

$$M(1) \subset M(2) \subset \cdots \subset M(q),$$

i.e. the subspaces $M(k)$ form a pyramid.

Note that the sequence $M = (0 = M_0, M_1, M_2, \dots, M_p)$ of vector subspaces of the space X is called a *pyramid*, if $M_j \subset M_{j+1}$ and $M_p = X$ (one also uses the term *flag*) [3]. The subspaces M_j are called *elements of the pyramid*.

The set-theoretical difference $S_j = M_j \setminus M_{j-1}$ is called a *level* of the pyramid M , and number $\dim M_j - \dim M_{j-1}$ is called the *weight* of the level. The subspace $M_0 = \{0\}$ is called the null level with weight 0. The set of levels is ordered by the order relation, generated by the index of the level: a level with index j follows a level with index k if $j > k$.

In this terminology, each subspace M_j is the set-theoretical union of levels contained in it, the levels are disjoint and the dimension of the subspace M_j is the sum of weights of the levels contained in it.

For the given pyramid (2.2.3) and each vector x , let us introduce a number characterizing the location of the vector with respect to the pyramid: this number $k(x)$ is the index of the level $M(k) \setminus M(k-1)$, containing the given vector x . This number may also be described as the largest of the indices k of basis vectors $e(k, j, i, l)$, actually (with a non-zero coefficient) appearing in the decomposition of the vector x .

We also need some other subspaces.

For given k, j and l , let us introduce the subspace $L^l(k, j)$, generated by basis vectors $e(k, j, i, l)$, for given k, j, l and different i (if they exist). The projector onto the subspace $L^l(k, j)$ is denoted by $Q(k, j; l)$:

$$Q(k, j; l)x = \sum_i x(k, j, i, l)e(k, j, i, l). \quad (2.2.4)$$

The subspace $L^1(k, j)$ is generated by eigenvectors with eigenvalues $\lambda(k, j)$, the subspace $L^2(k, j)$ is generated by adjoint vectors of the first order, etc.

Dimensions of subspaces $L^l(k, j)$ do not increase with the increase of l :

$$\dim L^{l+1}(k, j) \leq \dim L^l(k, j),$$

and the number of such subspaces is the number $p(k, j) = \max_i \{q(k, j, i)\}$ —the largest dimension of the Jordan cells with eigenvalue $\lambda(k, j)$.

Let us also introduce the subspace

$$L^l(k) = \bigoplus_j L^l(k, j)$$

and corresponding projector $Q(k; l)$:

$$Q(k; l)x = \sum_j Q(k, j; l)x = \sum_j \sum_i x(k, j, i, l)e(k, j, i, l). \quad (2.2.5)$$

Obviously, one has decompositions

$$L(k, j) = \bigoplus_{l=1}^{p(k,j)} L^l(k, j),$$

$$L(k) = \bigoplus_{l=1}^{p(k)} L^l(k), \quad p(k) = \max_j p(k, j).$$

In the following, the fundamental role will be played the family of subspaces indexed by two indices (k, s) , where $s \leq p(k)$:

$$M^s(k) = \left[\bigoplus_{i=1}^{k-1} L(i) \right] \bigoplus \left[\bigoplus_{l=1}^s L^l(k) \right] = M(k-1) \bigoplus \left[\bigoplus_{l=1}^s L^l(k) \right]. \quad (2.2.6)$$

These subspaces are ordered by inclusion

$$M^s(k) \subset M^{s_1}(k_1) \iff \text{if } k < k_1 \text{ and if } k = k_1, s \leq s_1.$$

This ordering induces *lexicographic ordering* on the set of pairs (k, s) :

$$(k_1, s_1) \preccurlyeq (k_2, s_2), \text{ if } k_1 < k_2 \text{ and if } k_1 = k_2 \text{ and } s_1 \leq s_2. \quad (2.2.7)$$

In this way, the pyramid formed by these subspaces has the form

$$M^1(1) \subset M^2(1) \subset \dots \subset M^{p(1)}(1) \subset M^1(2) \subset M^2(2) \subset \dots \subset M^{p(q)}(q). \quad (2.2.8)$$

Note that $M^{p(k)} = M(k)$, i.e. the pyramid (2.2.3) is a part of the pyramid (2.2.8). The pyramid formed by subspaces $M^s(k)$ shall called the *thickened pyramid* generated by operator A .

The location of the vector x , with respect to the pyramid, will be characterized by the pair of numbers $(k(x), s(x))$ —the index of the level of the thickened pyramid (2.2.8), containing the vector x . As it will be shown below, this pair of numbers plays a fundamental role in the description of the behavior of the trajectory of the vector x .

References

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