

Preface

The book deals with dynamical systems, generated by linear mappings of finite dimensional spaces. These systems have a relatively simple structure from the point of view of the modern dynamical systems theory. However, for dynamical systems of this kind, it is possible to obtain explicit answers to specific questions that are useful in applications.

The central place is occupied by the following problem. Let $A : \mathbb{C}^m \rightarrow \mathbb{C}^m$ be a linear nondegenerate mapping. The question is to investigate the trajectory of an arbitrary vector x under the action of the powers of linear mapping, that is, to give a description of the behavior of the sequence of vectors $A^n x$. In essence, this is a problem of investigation of the dynamical system given by linear mapping. These objects and questions arise in a series of applied and theoretical issues, first of all in iteration processes theory, and there are a significant number of publications on the theme.

At the same time linear mapping $A : \mathbb{C}^m \rightarrow \mathbb{C}^m$ generates a series of other dynamical systems. For example, under mapping A each d -dimensional subspace V is mapped to a d -dimensional subspace $A(V)$. The set of d -dimensional subspaces in $\mathbb{C}^m$ is the Grassmann manifold $G(m, d)$; thus the operator A generates the mapping φ_A of the Grassmann manifold $G(m, d)$ into itself, and generates a dynamical system with the phase space $G(m, d)$.

The monograph presents the results of investigation of dynamical systems of this kind and their applications.

The problems considered are natural and look rather simple (in essence their study is accessible to first-year university students), but in reality in the course of investigation, we are faced with a plenty of subtle questions and their detailed analysis needs a substantial effort.

The problems arising are related to linear algebra and dynamical systems theory, and therefore, the book can be considered as a natural amplification, refinement, and supplement to linear algebra and dynamical systems theory textbooks.

In Part I of the book the problem on the behavior of the sequence of vectors $A^n x$ is systematically considered for the first time. The main result is a complete

decomposition of the trajectory of a vector, making it possible to obtain exhaustive information on a concrete trajectory (existence of the limit for $n \rightarrow +\infty$ and for $n \rightarrow -\infty$, description of the set of limit points and so on), as well as on the behavior of the direction vectors $\frac{1}{\|A^n x\|} A^n x$, and on the behavior of one-dimensional spaces generated by the vectors $A^n x$.

These results form a base for further investigation.

A linear mapping $A : \mathbb{C}^m \rightarrow \mathbb{C}^m$ generates a number of dynamical systems whose dynamics has not been previously investigated at all. The most important among these associated dynamical systems is the system generated by the aforementioned mapping φ_A on the Grassmann manifold $G(m, d)$. The question on the behavior of the trajectory $A^n(V)$ of a subspace V under the action of a linear mapping is also important in a number of problems, but previously there has been no investigation in this direction carried out. The complexity of the problem on the behavior of the trajectory of a subspace V is caused, first of all, by the fact that construction of the limit $\tilde{V}$ of such trajectory cannot be reduced to evaluation of the limits of trajectories of vectors or one-dimensional subspaces from V . The point is that in general, limits of trajectories of one-dimensional subspaces generate only a part of the limit subspace $\tilde{V}$.

In Part II, two approaches to the solution of this problem are proposed. These approaches are arbitrarily called algebraic and geometric. They are based on the original work of the authors and their collaborators.

The algebraic approach exploits the embedding of the Grassmann manifold into the projective space generated by the exterior power $\bigwedge^d X$ of the initial vector space. Under this embedding, the action of mapping φ_A is given by operator $\bigwedge^d A$ —the exterior power of operator A . Thus the investigation of the trajectory of a d -dimensional subspace is reduced to investigation of the trajectory of a one-dimensional subspace under the action of a linear operator $\bigwedge^d A$. This allows us to apply the results of Part I.

The geometric approach rests on construction of linear combinations of vectors from subspace $A^n(V)$ that have nonzero limits. This enables one to construct the limit space. The mentioned linear combinations are built by means of the so-called renormalization on the base of a subtle analysis of the behavior of the trajectories of vectors.

The limit of the trajectory of a subspace is an invariant subspace. In the case of an operator with one point spectrum we have the most extensive family of invariant subspaces. Therefore, a description of the trajectory of a subspace under the action of an operator with one point spectrum, as well as evaluation of its limit, turned out to be the most difficult result.

Note that in the linear algebra and matrix theory there are plenty of subtle questions, and in spite of intensive investigation, a lot of answers to these questions are far from being answered, and the results obtained in this direction go far beyond the frames of the linear algebra courses and form the theme of special

individual monographs. For example, the results on invariant subspaces for linear mappings of finite dimensional spaces comprise a voluminous monograph Gohberg I., Lancaster P., Rodman L. *Invariant subspace of matrices with applications*. New York–Chichester–Toronto–Singapore. Wiley & Sons. 1986.

The problems considered in the book belong to the family of these subtle questions. The results obtained can be applied in different areas of Mathematics. In Part III, one of the possible applications is presented. Here, we apply the results obtained for investigation of the spectral properties of weighted shift operators, being operators in functional spaces of the form

$$Bu(x) = a(x)u(\alpha(x)),$$

where α is a certain mapping of the domain of the functional space under consideration into itself.

In Chaps. 13 and 14, we consider weighted shift operators generated by linear mappings in the spaces of scalar functions. In their investigation, one naturally comes across a new class of dynamical systems, associated with linear mappings. These are dynamical systems having the phase space being a certain compactification of $\mathbb{R}^n$, where the action is given by an extension of the linear mapping onto this compactification.

On the basis of the results on the behavior of the trajectories of vectors, we obtain an explicit description of the spectrum and find the properties of operators $B - \lambda I$ for spectral values λ . In particular, the necessary and sufficient conditions for one-sided invertibility of such operators have been obtained. These conditions depend essentially on the dynamics of the corresponding extension of the linear mapping. Weighted shift operators in spaces of vector functions are more complexly structured and are more difficult in investigation. In Chap. 15, it is shown how in the investigation of weighted shift operators in spaces of vector functions the problems on behavior of the trajectories of subspaces arise. By means of model examples, it is demonstrated how the results on behavior of the trajectories of subspaces obtained in Part II help to derive an explicit description of spectral properties of weighted shift operators in spaces of vector functions.

In the course of investigation, we also discuss a series of auxiliary problems and structures. In particular, we present an original presentation of principal facts on exterior powers of vector spaces and the Grassmann manifolds.

In addition, we clarify relations between the problems under consideration and some classical mathematical theories such that the Lyapunov exponents theory, nonarchimedean normalization theory, vector bundles theory (in particular, the known Hopf bundle), commutative C^* -algebras and the Gelfand transform, graded-linear mappings, trajectories behavior on the torus under the standard shift action.

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