

Chapter 2

Sampling in Reproducing Kernel Hilbert Space

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Abstract An account of sampling in the setting of reproducing kernel spaces is given, the main point of which is to show that the sampling theory of Kluvánek, even though it is very general in some respects, is nevertheless a special case of the reproducing kernel theory. A Dictionary is provided as a handy summary of the essential steps.

Starting with the classical formulation, the notion of band-limitation is a key feature in these settings.

The present chapter is, by and large, self-contained and a specialist knowledge of reproducing kernel theory is not required.

Here is one of Ramanujan's beautiful Fourier integrals. Let Γ denote Euler's Gamma function as usual, and let $\alpha + \beta > 1$. Then

$$\int_{-\infty}^{\infty} \frac{e^{-ixt}}{\Gamma(\alpha+t)\Gamma(\beta-t)} dt = \begin{cases} \frac{\{2 \cos(x/2)\}^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{ix(\alpha-\beta)/2}, & |x| \leq \pi; \\ 0, & |x| \geq \pi. \end{cases} \quad (2.1)$$

One recognises qualities of simplicity and integrity in the nature of this remarkable formula. Simplicity appears first, in a left-hand side which involves only elementary functions and Gamma, the most basic transcendental function. By integrity I mean that the right-hand side stays within this regime. One might not have anticipated this!

As well as giving rise to several interesting special cases [22, p. 187], the formula provides an example of a function whose Fourier transform has support on a compact set. Such functions are usually called *band-limited*. Here the compact set, or *frequency band*, or *set of spectral support*, is $[-\pi, \pi]$.

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The band-limited nature of the transform (2.1) was noted by Jerri [10] in the context of sampling.

2.1 Beginnings

There are many ways of generalising the notion of band-limitation (e.g., [6, 11]), and when this has been done the possibilities of sampling and reconstruction are often just round the corner. In the present chapter we favour a reproducing kernel setting drawn from the works of Saitoh (e.g., [17–19] and many other places) and outlined in Sect. 2.2.

Definition 1. A separable Hilbert space $\mathbf{H}$ of complex-valued functions defined on a domain E and with inner product $\langle \cdot, \cdot \rangle$ is said to have a *reproducing kernel* if there exists a function $k : E \times E \rightarrow \mathbb{C}$, such that $k(\cdot, t) \in \mathbf{H}$ for every $t \in E$, and the *reproducing equation*

$$f(t) = \langle f, k(\cdot, t) \rangle$$

holds for every $f \in \mathbf{H}$. Such a Hilbert space is called a *reproducing kernel Hilbert space* and k is its *reproducing kernel*.

A function f is band-limited in a generalised sense if it belongs to the RKH space (reproducing kernel Hilbert space) denoted by R_k in Sect. 2.2 (below). This theory will be referred to throughout as ‘the RKH space theory’, and RK will be short for reproducing kernel. Insofar as it leads to a sampling theorem, an outline of this theory is in Sect. 2.2 and a synopsis is found in the left-hand column of the Dictionary in Sect. 2.3.

The purpose of this chapter is twofold. First, the RKH space theory is known to be an eminently natural setting for sampling; for example, it can contain a discrete reproducing formula (Dictionary, item 9) as well as a concrete reproducing formula (Dictionary, item 7). As a setting for sampling theorems it is also very general in that it can embrace a whole theory, for example, Kramer’s sampling theory (see, e.g., [8]). Our first purpose, then, is to emphasise this generality by showing that the sampling theory of Kluvánek in harmonic analysis can also be subsumed by the RKH space theory; this is explained in Sect. 2.3 and is the main purpose of the Dictionary found there.

Our second purpose is to point out that, while being very general in some directions, the RKH space theory does not always generate a sampling theorem as found in the Dictionary, item 9. Indeed, we give an example in Sect. 2.4 whose kernel does not admit a sequence $\{s_n\}$ satisfying a criterion of the kind in item 8.

Finally, in Sect. 2.5 these considerations lead to a question about whether it might be possible to classify RKH spaces into isomorphism types using properties of the kernel, and in particular whether sampling has a role to play here. The case for the importance of isomorphic classifications in mathematics is well made in [15].

Before taking up these questions in earnest it will be interesting to remark on the generality and wide applicability of RK theory. For example, Saitoh remarks “... we would like to show that the theory of reproducing kernels is fundamental, is beautiful and is applicable widely in mathematics.” See his Abstract to [19] where these admirable goals are fulfilled; see also [18, p. 23]. This is good news because it means that the foundations of the subject are not in ‘danger of becoming purely academic’.¹

A few references, selected from the large number available, and some areas of application of RKs are now mentioned; by no means all of these refer to sampling. See [17, p. 89] for a list of areas of application; see also [19, p. 138–140] where 14 areas of application are mentioned. Some of these areas are integral transforms, Green’s functions, ordinary and partial differential equations, linear transforms, norm inequalities, non-linear transforms, differential equations with variable coefficients, representation of inverse functions, special operators on Hilbert space, sampling theorems, Pick–Nevanlinna interpolation, analytic extension formulas and inversion of linear operators.

Two quite different applications in approximation theory are found in [18, Chap. 4] and [3].

In [4] a natural place for RKH spaces is demonstrated, and how RKs link strong convergence and uniform convergence. Connections with Green’s functions are made and an interesting illustrative example given. See also [17, p. 81] and the several references found there. For another point of view, see, e.g., Nashed [12] and [13] with its very useful bibliography.

For use of RKs in probability theory, for example, in time series, detection, filtering and prediction, see [2]. See also [17, p. 4] reporting that RKs appear in work of Kolmogorov, Parzen and others.

Throughout, summations are indexed by $\mathbb{N}$ or $\mathbb{Z}$ when convergence is understood to be in the sense of symmetric partial sums.

2.1.1 Band-Limited Functions, the Classical Case

In this section the discussion is purely formal. Let a function f be band-limited to $[-\pi, \pi]$; this entails a *spectral function* φ for which

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \varphi(\omega) e^{i\omega t} d\omega. \quad (2.2)$$

¹In a penetrating and critical essay Taniyama [21] says: “In order to be meaningful it [mathematics] must be able to abstract, integrate and reconstruct classical results into broad perspectives.” It is certainly to be hoped that the RKH space theory does just that. Taniyama goes on: “... a subject that limits itself to a single abstract foundation is exposed to the danger of becoming purely academic.”

Let us formally invert this to obtain

$$\varphi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\omega t} dt. \quad (2.3)$$

It is clear from (2.2) that the Fourier coefficients for φ are values, or *samples*, of f taken at integer time points. On substituting this Fourier series into (2.2) one obtains

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin \pi(t - n)}{\pi(t - n)}. \quad (2.4)$$

This is the *classical sampling series*. Many scientists have taken part in its introduction and use; indeed it has a long and venerable history and has been widely generalised in both theory and practice.

The *reproducing formula* or *reproducing equation* (r.e.) for band-limited functions is

$$f(t) = \int_{\mathbb{R}} f(s) \frac{\sin \pi(t - s)}{\pi(t - s)} ds \quad (2.5)$$

and can be derived using the ‘tautology trick’ of representing f as the inverse Fourier transform of its Fourier transform (see, e.g., [7, p. 56]).

The Hilbert space of functions f for which (2.2) and (2.3) hold with some $\varphi \in L^2(-\pi, \pi)$ is denoted by PW (for Paley and Wiener). Clearly PW is an RKH space with kernel as in (2.5). These considerations will be generalised in Sect. 2.2.

2.1.2 A Modicum of History

The reader wishing to be informed about the early history of RKs can do no better than consult the Historical Introduction of the paper [1] by Aronszajn, which gives an excellent account of the historical origins of RKs; see also [17, p. 2 et seq]. In this paper Aronszajn goes on to establish reproducing kernel theory as a mathematical discipline in its own right. He recognises two ‘trends’ [1, p. 338] in early accounts of relevant types of kernel. In one of these trends a kernel K is considered to be ‘given’ and one studies its properties for members of an associated class F of functions introduced a posteriori and, perhaps, goes on to apply it in contexts such as integral equations, group theory, metric geometry, etc. In the present context a ‘reproducing property’ is particularly relevant.

In the second trend, it is the function class F that is assumed to be given and one attempts to associate with it a kernel K . A basic problem, then and now, is to construct or realise an appropriate kernel.

The first of these trends can be traced back to Hilbert's theory of integral equations in which context Mercer introduced the notion of positive definite kernel.² Later Moore called such kernels positive matrices:

Definition 2. A complex-valued function k defined on $E \times E$ is called a *positive matrix* in the sense of E.H. Moore (see, e.g., [18, p. 35]) if, for every sequence (t_i) , $i = 1, \dots, n$, of points of E and every sequence (c_i) , $i = 1, \dots, n$ of complex numbers:

$$\sum_i \sum_j \bar{c}_i c_j k(t_i, t_j) \geq 0. \quad (2.6)$$

The second trend commences with Zaremba in 1907 [23], often quoted as the starting point for reproducing kernel theory. Zaremba was working in the area of boundary value problems and was the first to introduce a kernel associated with a class of functions and obtain its reproducing property. However his work faded into obscurity until the matter was taken up again in three Berlin doctoral dissertations, those of Szegő (1921), Bergman (1922) and Bochner (1922).

The next two theorems relate reproducing kernels to positive matrices; they enjoy a *rapprochement* with the two trends described above.

Theorem 1. Let $k : E \times E \rightarrow \mathbb{C}$ be the reproducing kernel for some Hilbert space of functions defined on E ; then it is a positive matrix.

Theorem 2 (Moore–Aronszajn). Suppose that k is a positive matrix on $E \times E$. Then there exists one and only one RKH space $\mathbf{H}$ whose reproducing kernel is k .

Classical reproducing kernels were treated by Szegő, Bergman and Bochner in the study of harmonic and analytic functions. In particular one may mention the work of Bergman and Schiffer from the period 1947–1953, in the context of conformal and pseudo-conformal mapping, and of solving boundary problems of partial differential equations, among many other topics.

It is worth mentioning that an early contribution of Bateman (1907) seems to have been missed by historians of the subject.³

For historical developments after Aronszajn see [20].

²In passing we may note that a function f is called *positive definite* if $f(s - t)$ is a positive matrix in the sense of Definition 2.

³In his paper: On an integral equation, Proc. London Math. Soc., series 2, vol. 7, pp. 445–472, (1909), Hardy reported that Bateman, in the paper: On the inversion of a definite integral, Proc. London Math. Soc., series 2, vol. 4, pp. 483 *et seq.*, (1907), introduced the classical r.e. (Sect. 2.5) and its kernel, and the band-limited property (2.2) (not with this terminology of course).

A slightly more general form of the topic was taken up by Hardy and others under the name ‘m-functions’.

2.2 Saitoh's Theory

This theory, as much as is needed, is outlined below and summarised in the left-hand column of the Dictionary.

Saitoh's theory (see [17, 18, Chap. 2]) starts with an abstract set E . We observe that three ingredients are basic for the present method and provide a convenient starting point for the development of sampling theory. They are E , $\mathfrak{H}$ and κ_t , where

- (i) E is an abstract set;
- (ii) $\mathfrak{H}$ is a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$;
- (iii) κ_t , ($t \in E$), denotes a mapping of E into $\mathfrak{H}$. These are the first three items in the left-hand column of the Dictionary.

We refer to $\{E, \mathfrak{H}, \kappa_t\}$ as a *Basic Triple*. Once a Basic Triple has been set in place we can move on to

Definition 3. The kernel k defined on $E \times E$ and given by

$$k(s, t) := \langle \kappa_t, \kappa_s \rangle_{\mathfrak{H}}$$

is called the kernel function of the map κ_t . It is obviously Hermitian symmetric.

The kernel k is a positive matrix in the sense of E.H. Moore (see Definition 2). It follows from Theorem 2 that there exists one and only one Hilbert space R_k having k as its reproducing kernel.

As an aid to the realisation of R_k , it can be shown that R_k is the image of $\mathfrak{H}$ under the transformation $\mathcal{L} : \mathfrak{H} \rightarrow \mathfrak{C}(E)$ given by

$$(\mathcal{L}\varphi)(s) := \langle \varphi, \kappa_s \rangle_{\mathfrak{H}}, \quad (\varphi \in \mathfrak{H}), \quad (2.7)$$

where $\mathfrak{C}(E)$ denotes the linear space of complex-valued functions on E . (see, e.g., [18, Chap. 2]).

For all $f \in R_k$ and $t \in E$, $k(\cdot, t) \in R_k$ and the reproducing equation

$$f(t) = \langle f, k(\cdot, t) \rangle_{R_k}$$

holds. (see, e.g., [16], [18, p. 21], [9]). Indeed, if $f = \mathcal{L}\varphi$,

$$f(t) = \langle \varphi, \kappa_t \rangle_{\mathfrak{H}} = \langle \mathcal{L}\varphi, \mathcal{L}\kappa_t \rangle_{R_k} = \langle f, k(\cdot, t) \rangle_{R_k}.$$

The following two theorems provide some more essential background.

Theorem 3. *With the notations established above, R_k is a Hilbert space which has the reproducing kernel k , where*

$$k(s, t) = \langle \kappa_t, \kappa_s \rangle_{\mathfrak{H}} = (\mathcal{L}\kappa_t)(s), \quad (2.8)$$

and it is uniquely determined by this kernel. When $f \in R_k$ there is a $\varphi \in \mathfrak{H}$ such that

$$f(s) := \langle \varphi, \kappa_s \rangle_{\mathfrak{H}}$$

and

$$\|f\|_{R_k} = \|\mathcal{L}\varphi\|_{R_k} \leq \|\varphi\|_{\mathfrak{H}}. \quad (2.9)$$

Furthermore, there exists a unique member, φ_0 say, of the class of all φ 's satisfying (2.9) for which it will be true that

$$f(s) = \langle \varphi_0, \kappa_s \rangle_{\mathfrak{H}}, \quad (s \in E)$$

and

$$\|f\|_{R_k} = \|\varphi_0\|_{\mathfrak{H}}.$$

The reproducing equation for R_k is

$$f(t) = \langle f, k(\cdot, t) \rangle_{R_k}. \quad (2.10)$$

It will be supposed throughout that $\{\kappa_t\}$, ($t \in E$) is complete in $\mathfrak{H}$. As far as sampling theory goes this is no real restriction since, if the assumption in item 8 of the Dictionary is made, $\{\kappa_t\}$, ($t \in E$) is automatically complete. This means that the only possible φ in (2.9) is φ_0 , because from (2.7) the null space of $\mathcal{L}$ is $\{\theta\}$. Hence $\mathcal{L}$ is one-to-one. Then $\mathcal{L} : \mathfrak{H} \rightarrow R_k$ is an isometric isomorphism, since clearly $\mathcal{L}$ is linear and onto; it is isometric (as in Theorem 3 above) and hence bounded.

Theorem 4 (Sampling Theorem). *Let $\{s_n\} \subset E$ be such that $\{\varphi_n\} := \{\mu_n \kappa_{s_n}\}$ is an ON basis for $\mathfrak{H}$, where $\{\mu_n\}$ are normalizing factors. Then for every $f \in R_k$ and $t \in E$,*

$$f(t) = \sum_{n \in \mathbb{N}} f(t_n) \mu_n^2 k(t, s_n), \quad (2.11)$$

converging in the norm of R_k and pointwise over E .

Proof. The proof is a special case of the proof of Theorem 2 in [8], but is included here for reference.

Since $\mathcal{L}$ is an isometric isomorphism, $\{(\mathcal{L}\varphi_n)(t)\}$ is an ON basis of R_k . The n th coefficient of f in this basis is $\langle f(\cdot), (\mathcal{L}\varphi_n)(\cdot) \rangle_{R_k}$. But for every $t \in E$,

$$(\mathcal{L}\varphi_n)(t) = \langle \varphi_n, \kappa_t \rangle_{\mathfrak{H}} = \mu_n \langle \kappa_{s_n}, \kappa_t \rangle_{\mathfrak{H}} = \mu_n k(t, s_n).$$

Hence the coefficient is

$$\langle f(\cdot), (\mathcal{L}\varphi_n)(\cdot) \rangle_{R_k} = \langle f(\cdot), \mu_n k(\cdot, s_n) \rangle_{R_k} = \mu_n f(s_n).$$

So the expansion in the basis $\{(\mathcal{L}\varphi_n)(t)\}$ is

$$f(t) = \sum_{n \in \mathbb{N}} \mu_n f(s_n) (\mathcal{L}\varphi_n)(t) = \sum_{n \in \mathbb{N}} f(s_n) \mu_n^2 k(t, s_n).$$

Convergence is in the norm of R_k and pointwise by the general principle [18, p. 36]. $\square$

With only a few more technicalities this theorem can be extended to bases and frames (see, e.g., [8]).

Definition 4. Let k be a reproducing kernel and let $\{s_n\}$, $(n \in \mathbb{N})$ be a subset of E . We will say that k has the completeness property with respect to $\{s_n\}$ if $\{k(t, s_n)\}$ is complete in R_k . With trivial changes of wording, ‘basis’ or ‘frame’ can be substituted for ‘completeness’.

It is now apparent that Theorem 4 will hold as a consequence of a Basic Triple and the fact that k has the basis or frame property of Definition 4.

2.3 The Harmonic Analysis Case

The form that the classical sampling theorem takes in the context of locally compact abelian groups is due to Kluvánek [11], and there are several further generalisations (e.g., [6]). The purpose of this section is to show that the Kluvánek theorem is a special case of the RKH space theory. There follows some briefly sketched background in the theory of locally compact abelian groups suitable for the present context. Full details and references can be found in [5], including a theorem somewhat more general than of Kluvánek. In the present section we place the Kluvánek theory in the RKH setting, thereby emphasising how general this setting really is.

The details of this placement are summarised in the Dictionary to follow, after which there are some comments and explanations.

Throughout this section G will denote a locally compact abelian group endowed with a Hausdorff topology, and Γ will denote the dual group. Equality between groups is taken to mean that they are isomorphic.

Let Haar measure on G (unique up to constant multiples) be normalised so that the Weil coset decomposition formula holds ([14, Theorem 3.4.6], [11, p. 44] or [5, p. 238]) and be denoted by m_G .

Haar measures for other groups must be adopted as well, but their normalisations for sampling theory are not quite standard. See [5, p. 251] for a fully detailed account.

A character γ of G is defined to be a homomorphism $\gamma : G \rightarrow \mathbb{T}$, the multiplicative circle group, usually written $x \mapsto (x, \gamma)$. The continuous characters on G form a locally compact abelian group.

Let H denote a discrete subgroup of G with discrete annihilator $\Lambda = \{\gamma : (h, \gamma) = 1, (h \in H)\}$.

For $\gamma \in \Gamma$ let $[\gamma]$ be the coset of Λ which contains γ . That is, $[\gamma] = \gamma + \Lambda$. When $h \in H$, $(h, [\gamma])$ denotes the constant value of (h, γ) on the coset $[\gamma]$ ([11, p. 43]).

Let Ω be a complete set of coset representatives, or *transversal*, of Γ/Λ (assumed compact from now on), that is, Ω consists of exactly one point from each coset $[\gamma] = \Lambda + \gamma$, i.e., $\Omega \cap (\Lambda + \gamma)$ consists of a single point in Ω . Thus translates of Ω by non-zero elements in Λ are disjoint. We will always assume that Ω is measurable with $m_\Gamma(\Omega) < \infty$.

Every character of Γ/Λ may be written $(h, [\gamma])$ for some $h \in H$. The set of all characters is an orthonormal basis of $L^2(\Gamma/\Lambda)$. It follows that the characters (h, γ) , $(h \in H)$, where γ is restricted to Ω , form an ON basis of $L^2(\Omega)$ ([11, p. 45]).

The Fourier transform, denoted by $\mathcal{F}f$, is defined formally by

$$(\mathcal{F}f)(\gamma) := \int_G f(x)(x, -\gamma) dm_G(x). \quad (2.12)$$

See, e.g., [5, p. 242] for the Fourier transform of $f \in L^2(G)$ and its inverse.

2.3.1 Assumptions

It has been mentioned that the placement of sampling in the context of locally compact abelian groups is due to Kluvánek [11]. A more general version is in [5], but we shall only treat the Kluvánek theorem here. In order to show that the Kluvánek sampling theorem is a special case of Theorem 4 we need to build a Basic Triple for the harmonic analysis case. In fact the first three items in the right-hand column of the Dictionary are proposed as an acceptable Basic Triple. But it should be borne in mind that the first three items in the left-hand column, while sufficient to determine an RKH space, may not be sufficient to generate a sampling theorem (see Sect. 2.4).

Assumptions I and II (below) can be thought of as natural assumptions for a sampling theorem. Inspection of [5, 11] shows that three hypotheses are sufficient in order to achieve such a sampling theorem. They are that:

- (i) there exists a discrete subgroup H of G ;
- (ii) $\Lambda = H^\perp$ (the annihilator of H) be discrete;
- (iii) Γ/Λ be compact.

It is immediate that the three hypotheses (i), (ii) and (iii) hold under Either **Assumption I** ([11, p. 43]):

1. H is a discrete subgroup of G ;
2. $\Lambda := H^\perp$ is discrete.

These imply [5, p. 241] that Γ/Λ is compact, and mean that the set of sample points H is the starting point.

Or **Assumption II** ([5, p. 254]):

- 1'. Λ is a discrete subgroup of Γ ;
- 2'. Γ/Λ is compact.

These imply that $H = H^{\perp\perp} = \Lambda^\perp$ is compact, and mean that Λ is now the starting point; effectively, this means that the nature of the spectral support, Ω , is the starting point.

Indeed, under Assumption I, $H^\wedge = \Gamma/\Lambda$ is compact since H is discrete. Under Assumption II H is discrete since Γ/Λ is compact.

Consequences of the three hypotheses (i), (ii) and (iii) are summarised in the Dictionary,⁴ right-hand column.

2.3.2 The Dictionary

	Reproducing kernel theory	Harmonic analysis
1.	E , an abstract set	G , a locally compact abelian group with dual Γ
2.	$\mathfrak{H}$, a separable Hilbert space	$L^2(\Omega)$, Ω a transversal of Γ/Λ , where H is a discrete subgroup of G Λ is a discrete subgroup of Γ
3.	$\kappa_t(\cdot) \in \mathfrak{H}$, ($t \in E$)	$(-t, \gamma)$, γ restricted to Ω
4.	$k(s, t) := \langle \kappa_t, \kappa_s \rangle_{\mathfrak{H}}$	$\int_{\Omega} (-t, \gamma)(-s, \gamma) dm_{\Gamma}(\gamma)$ $= \int_{\Omega} (-t + s, \gamma) dm_{\Gamma}(\gamma) = (\mathcal{F}^{-1} \chi_{\Omega})(s - t)$
5.	$f(t) = (\mathcal{L}g)(t) := \langle g, \kappa_t \rangle_{\mathfrak{H}}$	$f(t) = \int_{\Omega} g(\gamma)(-t, \gamma) dm_{\Gamma}(\gamma) = (\mathcal{F}^{-1}g)(t)$
6.	R_k realised as $\{f = \mathcal{L}g, g \in \mathfrak{H}\}$ $\ni \ f\ _{R_k} = \ g\ _{\mathfrak{H}}$	$PW_{\Omega}(G)$ $= \{f : f = \mathcal{F}^{-1}g, g \in L^2(\Gamma), g \text{ null outside } \Omega\}$
7.	For $f \in R_k$ the r.e. is $f(t) = \langle f, k(\cdot, t) \rangle_{R_k}$	For $f \in PW_{\Omega}$ the r.e. is $f(t) = \int_G f(\tau)(\mathcal{F}^{-1} \chi_{\Omega})(t - \tau) dm_G(\tau)$
8.	There must exist $\{s_n\}_{n \in \mathbb{N}} \subset E \ni$ $\{\kappa_{s_n}\}$ is a basis or a frame of $\mathfrak{H}$	Characters $\{(h, \gamma)\}, (h \in H), \gamma \in \Omega$ form an ON basis of $L^2(\Omega)$
9.	$f(t) = \sum_{n \in \mathbb{N}} f(s_n)k(t, s_n)$	$f(t) = \sum_{h \in H} f(h)(\mathcal{F}^{-1} \chi_{\Omega})(t - h)$
	$\vdots$	$\vdots$

The left-hand column ‘Reproducing kernel theory’ is a *précis* of Saitoh’s approach to the theory (Sect. 2.2), leading up to the sampling theorem, Theorem 4. In its generality, based on rather few assumptions, it is a foundational approach to sampling theory.

⁴ The word ‘Dictionary’ is also used in other ways in mathematics; see, e.g., K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, 2001, p. 24.

The Basic Triple for the locally compact abelian group case, items 1, 2 and 3, is now settled, since item 1 establishes a choice for E , and in item 2 we choose Ω to be a measurable transversal of Γ/Λ . The remaining items confirm that the entry in the right-hand column, item 3, is the ‘right’ choice for κ_t for a sampling theory.

Note that in the left-hand column, item 3 and subsequently, ‘ $\cdot$ ’ is a free variable. It becomes a bound variable, γ , in the right-hand column, item 3, since there it has a definite range.

The remaining items can now be interpreted and we can finally read off the Kluvánek theorem in item 9. The following notes are intended to help these interpretations:

Items 4 and 5 are self-evident.

Item 6. The Paley–Wiener space PW_Ω appears in the right-hand column. It is the counterpart in harmonic analysis of PW in the classical case (Sect. 2.1.1). Its reproducing kernel is found in item 4.

Item 7. The reproducing equation is needed in the proof of Theorem 4.

Item 8. The basis property of these characters is a well-known result [11, p. 45] and is a consequence of the orthogonality of the characters and the Stone–Weierstrass theorem.

Item 9. In the left-hand column we find the sampling series of Theorem 4. With the appropriate identifications we find in the right-hand column the Kluvánek sampling series and its mode of convergence for functions f such that $f \in L^2(G)$ and $\mathcal{F}f = 0$ for almost all $\gamma \notin \Omega$.

A form of Parseval’s theorem (see, e.g., [11, p. 45, (7)]) could also have been incorporated here but is not strictly relevant.

2.3.3 Remarks

The comparison embodied in this Dictionary might have taken shape earlier than it did. On page 44 of his famous paper Kluvánek shows that the function φ of his equation (4), or rather the shifted form of it (his equation (6), found in the Dictionary, right-hand column, items (4) and (9)) is positive definite. But, continuing on to page 45 he says “The positive definiteness will not be used in the following.”

Kluvánek did not observe that the Moore–Aronszajn theorem applies to $\varphi(x - y)$. Had he done so, a door to RKH space theory might have opened, leading at once, e.g., to the fact that the Paley–Wiener space PW_Ω is a reproducing kernel Hilbert space.

While a wide background in RKH spaces was already in place at that time, it must be appreciated that Kluvánek’s paper appeared well before sampling in the context of reproducing kernel theory was in existence. Nevertheless, that open door might have led to more general consequences.

2.4 An RKH Space for Which the Completeness Property Fails

In the paper [6] a treatment can be found of the ‘discretisation of kernel’ problem at a high level of generality. At a lower level, a related problem is that of asking whether there exists a set $\{s_n\} \subset E$ that makes $\{k(t, s_n)\}$ a basis of R_k .

In this section we take a very down-to-earth approach and show that there exist RKH spaces for which no uniformly discrete set $\{s_n\} \subset E$ makes $\{k(t, s_n)\}$ complete in R_k . The uniformly discrete condition is common throughout sampling theory; without it a pathological case would arise which we will not discuss here. The proof consists of the following counter-example.

The following RKH space, denoted by M , can be found in the books [17, p. 89–90] and [18, p. 55–57, 242, 253] by Saitoh, which contain many further interesting examples of RKH spaces. Here we derive the example from a Basic Triple. The example is as follows:

Let

$$M := \{f : f(0) = 0; f \in AC(0, \infty); f' \in L^2(\mathbb{R}_0^+)\}.$$

Then with the inner product

$$\langle f, g \rangle_M := \int_0^\infty f'(x) \overline{g'(x)} dx,$$

M is an RKH space with reproducing kernel $\min(s, t)$.

There are two approaches to this RKH space; they depend on what is considered to be given. Prompted by [18, p. 55–57] we assume that we are given the following Basic Triple:

$$\begin{cases} E & \text{taken to be } \mathbb{R}_0^+; \\ \mathfrak{H} & \text{taken to be } L^2(\mathbb{R}_0^+, \frac{dx}{x^2}); \\ \kappa_t & \text{taken to be } \sqrt{\frac{2}{\pi}} \sin xt, \quad x, t \in \mathbb{R}_0^+. \end{cases}$$

Then as in Definition 3 we form the kernel function

$$k(s, t) = \langle \kappa_t, \kappa_s \rangle_{\mathfrak{H}} := \frac{2}{\pi} \int_0^\infty \frac{\sin xs}{x} \frac{\sin xt}{x} dx = \min(t, s),$$

using a special integral.

It is a routine matter to check that $\min(t, s)$ is a positive matrix; hence by the Moore–Aronszajn theorem there exists one and only one Hilbert space, which we have already called M , having $\min(t, s)$ as its reproducing kernel.

In fact, $t \mapsto \min(t, s)$ belongs to $\mathbf{M}$ for each fixed value of s , and a reproducing equation for $\mathbf{M}$ is

$$\begin{aligned} \langle f, \min(t, \cdot) \rangle_{\mathbf{M}} &= \int_0^\infty f'(x) \left\{ \overline{\frac{d}{dx} \min(t, x)} \right\} dx \\ &= \int_0^\infty f'(x) \chi_{[0, t)}(x) dx = \int_0^t f'(x) dx = f(t), \end{aligned}$$

so that $\min(t, s)$ is *the* reproducing kernel for $\mathbf{M}$.

It will be shown that there exists a $T \in \mathbf{M}$ for which $T \perp \min(t, s_n)$ no matter what choice for $\{s_n\}$ is made, i.e., $\{\min(t, s_n)\}$ is incomplete in $\mathbf{M}$ and thus cannot be a basis or a frame. Therefore $\mathbf{M}$ has no standard sampling theorem.

2.4.1 Failure of the Completeness Property

Without loss of generality let $s_0 = 0$ and let $\{s_n\} \subset \mathbb{R}_0^+$, ($n \in \mathbb{N}_0$), satisfy the usual ‘uniformly discrete’ spacing condition $s_n - s_{n-1} \geq \sigma$ for some $\sigma > 0$, for every $n \in \mathbb{N}$.

For interest, we show first that there do not exist points $\{s_n\} \subset \mathbb{R}$ such that $\{\min(t, s_n)\}$ is orthogonal in $\mathbf{M}$. This is because, if $m, n \neq 0$, $m \neq n$, orthogonality fails:

$$\begin{aligned} \langle \min(\cdot, s_m), \min(\cdot, s_n) \rangle_{\mathbf{M}} &= \int_0^\infty \chi_{[0, s_m]}(t) \chi_{[0, s_n]}(t) dt \\ &= \int_0^{\min(s_m, s_n)} dt = \min(s_m, s_n) \neq 0. \end{aligned}$$

Next we show that no set $\{s_n\}$ makes $\{\min(t, s_n)\}$ complete in $\mathbf{M}$. Let us define T as follows. First, for $n = 1, 2, \dots$ let

$$T_n(t) := \begin{cases} \frac{2a_n}{s_n - s_{n-1}}(t - s_{n-1}), & s_{n-1} \leq t \leq \frac{s_{n-1} + s_n}{2}; \\ -\frac{2a_n}{s_n - s_{n-1}}(t - s_n), & \frac{s_{n-1} + s_n}{2} \leq t < s_n; \\ 0 & \text{otherwise,} \end{cases} \quad (2.13)$$

where $\{a_n\}$ is to be chosen.

Definition 5.

$$T(t) := \sum_{n \in \mathbb{N}}^{\infty} T_n(t). \quad (2.14)$$

It follows that $T'(t) = \sum_{n \in \mathbb{N}}^{\infty} T'_n(t)$, where

$$T'_n(t) = \begin{cases} \frac{2a_n}{s_n - s_{n-1}}, & s_{n-1} \leq t < \frac{s_{n-1} + s_n}{2}; \\ -\frac{2a_n}{s_n - s_{n-1}}, & \frac{s_{n-1} + s_n}{2} \leq t < s_n; \\ 0 & \text{otherwise.} \end{cases} \quad (2.15)$$

First we need to show that $T \in \mathbf{M}$. Now $T_1(0) = 0$, and T is the indefinite integral of its derivative and is therefore absolutely continuous.

To show that $T' \in L^2(\mathbb{R}_0^+)$,

$$\begin{aligned} \int_0^{\infty} |T'(t)|^2 dt &= \int_0^{\infty} \left| \sum_{n \in \mathbb{N}} T'_n(t) \right|^2 dt = \int_0^{\infty} \sum_{n \in \mathbb{N}} T'_n(t)^2 dt \quad (\text{by Pythagoras}) \\ &= \sum_{n \in \mathbb{N}} \int_0^{\infty} T'_n(t)^2 dt = \sum_{n \in \mathbb{N}} \left(\frac{2a_n}{s_n - s_{n-1}} \right)^2 (s_n - s_{n-1}) \\ &= 4 \sum_{n \in \mathbb{N}} \frac{a_n^2}{s_n - s_{n-1}} \leq \frac{4}{\sigma} \sum_{n \in \mathbb{N}} a_n^2, \end{aligned}$$

by hypothesis. The interchange is clearly valid by the standard criterion, after choosing $\{a_n\}$ to make the last summation convergent.

Now we want to prove that $T(t) \perp \min(t, s_n)$, that is,

$$(T, \min(\cdot, s_n))_{\mathbf{M}} = 0, \quad (n = 1, 2, \dots).$$

We have

$$\int_0^{\infty} T'(t) \chi_{[0, s_n)}(t) dt = \int_0^{s_n} T'(t) dt = \int_0^{s_n} \sum_{i=1}^n T'_i(t) dt = \sum_{i=1}^n \int_{s_{i-1}}^{s_i} T'_i(t) dt.$$

But (2.15) shows clearly that each summand here is zero. The required result follows.

2.5 Conclusions

We look towards a taxonomy, or system of classification, of Hilbert spaces in terms of reproducing kernels. It can be seen first that there is a crude classification of Hilbert spaces into two types, one type consists of those spaces that have, and the other those that do not have an RK. An L^2 space, for example, does not have an RK since its members are not functions with a well-defined value at each point.

It is the second type, Hilbert spaces that do have an RK, that now claims our attention.

We have already seen that the RKH spaces M of Sect. 2.4 and PW of Sect. 2.1 exhibit an obvious difference. Indeed, we now have two different types of RKH space, one type has no associated sequence (s_n) such that the kernels, e.g., $\{\min(t, s_n)\}$ in the case of M , have even a completeness property (see Definition 4), let alone a basis or frame property. On the other hand, another type, such as PW , does have such an associated (s_n) , namely $\mathbb{Z}$, for which the kernels $[\sin \pi(t - n)]/[\pi(t - n)]$, $(n \in \mathbb{Z})$ do form an orthonormal basis of PW .

More generally, it appears from the literature that mathematics can exhibit many problems of isomorphic classification, (see [15] where an excellent introduction to this interesting topic is to be found). Let $\mathcal{A}$ denote a class of mathematical objects. One tries to find an explicit listing of all isomorphism types of members of $\mathcal{A}$, any two members of a type being isomorphic under some suitably chosen notion of isomorphism. Undoubtedly the most famous example is when $\mathcal{A}$ is taken to be the class of finite simple groups.

There are other ways of achieving a classification (see, e.g., [15, p. 1251]), but the present one may be said to be typical.

At present our crude classification of $\mathcal{A}$ when it is taken to be the class of all separable Hilbert spaces is, first of all, into two types, one being all RKH spaces and the other its complement in $\mathcal{A}$. This leads on to a consideration of RKH spaces distinguished by properties of their RK.

We can now ask the following:

Question. Can these ideas be refined into a *bona fide* isomorphic classification of RKH spaces based on the presence, or not, of a sequence $(s_n) \subset E$ such that $\{k(s_n, t)\}$ has some completeness or basis property (Definition 4) and how would the presence, or not, of a sampling theorem of the type Theorem 4 be relevant?

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