

# Preface

This monograph is a collection of articles contributed by friends, colleagues, and students of Paul Butzer, covering different topics to which Professor Butzer has made many contributions over the years. The idea of the monograph was conceived during the Summer School on “New Trends and Directions in Harmonic Analysis, Fractional Operator Theory, and Image Analysis” that was organized by B. Forster and P. Massopust and took place on September 17–21, 2012, Inzell, Germany, where both editors of this volume, Gerhard Schmeisser and Ahmed Zayed, as well as Paul Butzer, were invited speakers. The editors realized that Butzer’s 85th birthday was few months away. Thinking about a token of appreciation for Paul’s long-standing contribution to the field and support for his colleagues and students, it was soon concluded that a Festschrift in his honor would be an invaluable gift that would last for years to come.

The topics covered are sampling theory, compressed sampling and sensing, approximation theory, and various topics in harmonic analysis, to all of which Butzer and his school in Rheinisch-Westfälische Technische Hochschule (RWTH), Aachen, Germany, have contributed significantly. The chapters, which have been carefully refereed, are grouped together by themes. The first theme is *sampling theory, compressed sensing, and their applications in image processing*. This comprises the first ten chapters. The second theme is *approximation theory* and it consists of three chapters, Chaps. 11–13. The last theme is *harmonic analysis* and it consists of six chapters, Chaps. 14–19.

In what follows, we will give an overview of the content of the monograph so that the reader can focus on what is interesting for him/her.

In Chap. 1, M. M. Dodson presents an approximate sampling theorem with a sampling series inspired by that of Kluvánek. The underlying space of admissible functions is the Fourier algebra  $A(G)$  of a locally compact Abelian group  $G$ . In addition, a summability condition is needed for the sampling series to exist. Since  $A(G)$  is a Banach space whose elements need not be square integrable, the author’s results generalize various other abstractions of Kluvánek’s theorem and include an exact sampling theorem as a special case.

Closely related to the subject of Chap. 1 is sampling in Hilbert spaces. An account of sampling in the setting of reproducing kernel Hilbert (RKH) spaces is given by R. Higgins in Chap. 2. One of the main points of the chapter is to show that the sampling theory of Kluvánek, even though it is very general in some respects, is nevertheless a special case of the reproducing kernel theory. A Dictionary is provided as a handy summary of the essential steps. Another point the author emphasizes is that the RKH space theory does not always generate a sampling theorem as found in the Dictionary.

He shows that there are two different types of RKH spaces, one type has no associated sequence  $s_n$  such that the set,  $\{\phi_n(t) = k(t, s_n)\}$ , is complete, let alone being a basis or a frame, where  $k(t, x)$  is the reproducing kernel. On the other hand, the other type does have such a sequence, such as the Paley–Wiener space  $(PW)$ , for which  $s_n = n$ , and the set  $\{\sin \pi(t - n)/\pi(t - n)\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $PW$ .

Going from sampling in Hilbert spaces to Banach spaces seems to be a natural step. In Chap. 3, I. Z. Pesenson discusses sampling in Banach spaces. First, let us recall that Boas-type formulae represent  $f^{(n)}(t)$  for a bandlimited function  $f$  in terms of samples taken at uniformly spaced points located relative to  $t$ . These formulae are more powerful than the differentiated classical sampling series. They give easy access to Bernstein inequalities and are of interest in numerical differentiation. I. Z. Pesenson generalizes Boas-type formulae within a more abstract setting. Given an operator  $D$  (instead of differentiation) that generates a strongly continuous group of isometries  $e^{tD}$  in a Banach space  $E$ , he establishes analogous formulae for  $D^n f$  when  $f$  belongs to a subspace of  $E$  that corresponds to a Bernstein space. A sampling formula for  $e^{tD} D^n f$  is also derived. Finally, extensions of the results to compact homogeneous manifolds are given.

Sampling series, where the classical sinc function is replaced by a general kernel function, have been studied by P. Butzer and his school at RWTH Aachen since 1977. To study the  $L^p$ -convergence of these series, they introduced a class of functions  $\Lambda^p \subset L^p(\mathbb{R})$  and estimated the rate of approximation of these series in terms of  $\tau$ -modulus. In Chap. 4, A. Kivinukk and G. Tamberg study series with bandlimited kernel functions and show that in this case, the rate of approximation can be estimated via ordinary modulus of smoothness. Basics in the proof are Jackson- and Bernstein-type inequalities, as well as Zygmund-type sampling series. The main aim of this chapter is to give a short overview of results obtained previously by the authors and to extend the theory of P. Butzer and his school to generalized sampling operators where the kernel function is defined via the Fourier transform of a certain even window function.

In Chap. 5, C. Bardaro, I. Mantellini, R. Stens, J. Vautz, and G. Vinti establish a multivariate generalized sampling series and apply it to the approximation of not necessarily continuous functions. Error estimates are given in terms of an averaged modulus of smoothness whose properties are explored in a preliminary section. A multivariate sampling series built with the help of  $B$ -splines is studied

as an important special case. The power of the method is illustrated by concrete applications to image processing, including biomedical images as they appear in computer tomography.

Chapter 6 by H. Boche and U. J. Mönich is motivated by digital signal processing. The authors consider a linear, bounded, time-invariant operator  $T$  that maps the Paley–Wiener space  $PW$  into itself. Their aim is to study the approximation of  $(Tf)(t)$  by a symmetrically truncated sampling series that involves a sequence of measurements of  $f$ . They show that for point evaluations of  $f$ , convergence of the approximation process cannot be guaranteed even if oversampling is used and not necessarily equidistant sequences of sampling points are admitted. However, for certain classes of measurement functionals, a stable approximation is possible in case of oversampling, which turns out to be a necessary condition. Without oversampling, there may exist convergent subsequence of the approximation process.

The next two chapters deal with sparse sampling and compression. In Chap. 7, J. Benedetto and Nava-Tudela discuss sparse solutions of underdetermined systems of linear equations and show how they can be used to describe images in a compact form. They develop this approach in the context of sampling theory and for problems in image compression. The idea briefly goes as follows: Suppose that we have a full-rank matrix  $A \in R^{m \times n}$ , where  $n < m$ , and we want to find solutions to the equation,

$$Ax = b, \tag{1}$$

where  $b$  is a given “signal.” Since the matrix  $A$  is full rank and there are more unknowns than equations, there are infinitely many solutions to Eq. (1). The focus is on finding the “sparsest” solution  $x_0$  that is the one having the least number of nonzero entries. If the number of nonzero entries in  $x_0$  happens to be less than the number of nonzero entries in  $b$ , we could store  $x_0$  instead of  $b$ , achieving a representation  $x_0$  of the original signal  $b$  in a compressed form. If there is a unique “sparsest” solution to Eq. (1), how does one find such a solution? What are the practical implications of this approach to image compression? This notion of compression and its applications in image processing is the authors’ focus.

Conventional sampling techniques are based on the Shannon–Nyquist theorem which states that the required sampling rate for perfect recovery of a bandlimited signal is at least twice its bandwidth. Although the class of bandlimited signals is a very important one, many signals in practice are not necessarily bandlimited. A low-pass filter is applied to the signal prior to its sampling for the purpose of anti-aliasing. Many of these signals are sparse, which means that they have a small number of nonzero coefficients in some domain such as time, discrete cosine transform, discrete wavelet transform, or discrete Fourier transform. This property of such a signal is the foundation of a new sampling theory called compressed sampling (CS). In Chap. 8, M. Azghani and F. Marvasti give an overview of compressed sensing and sampling, as well as popular recovery techniques. The CS recovery techniques and some notions of random sampling are investigated. Furthermore, the block sparse recovery problem is discussed and illustrated by an example.

In Chap. 9, M. Pawlak investigates the joint nonparametric signal sampling and detection problem when noisy samples of a signal are observed. Two distinct detection schemes are examined. In the first scheme, which deals with what is called the off-line testing problem, the complete data set is given in advance and one is interested in testing the null hypothesis that a signal  $f$  takes a certain parametric form. That is, given a class of signals  $\mathfrak{F}$  and a data set, we wish to test the null hypothesis  $H_0 : f \in \mathfrak{F}$  against an arbitrary alternative  $H_a : f \notin \mathfrak{F}$ .

In the second scheme, which deals with what is called the on line detection, the data are collected in a sequential fashion and one would like to detect a possible departure from a reference signal as quickly as possible. In such a scheme, one makes a decision at every new observed data point and stops the procedure when a detector finds that the null hypothesis is false. In practice, only discrete and often noisy samples are available, which makes it difficult to verify whether a transmitted signal is bandlimited, time limited, and parametric or belongs to some general nonparametric function space. The author suggests a joint nonparametric detection/reconstruction scheme to verify the type of the signal and simultaneously recover it.

P.J.S.G. Ferreira in Chap. 10 gives a brief account of a phenomenon called “superoscillation.” Briefly, a superoscillating function is a bandlimited function that oscillates faster than its maximal Fourier component, and it may do so over arbitrarily long intervals. Superoscillating functions provide direct refutation of the statement that a bandlimited function contains no frequencies above a limit, say  $\mu/2$ , and so it cannot change to substantially new values in a time less than one-half cycle of its highest frequency, that is,  $1/\mu$ . Although the theoretical interest in superoscillating functions is relatively recent, a number of applications are already known in quantum physics, super-resolution, sub-wavelength imaging, and antenna theory.

In the second group of chapters, which is dedicated to topics in approximation theory, we have Chap. 11 by K. Runovski and H. J. Schmeisser in which they consider  $L^p$  spaces of  $2\pi$ -periodic functions. They introduce a general modulus of smoothness, called the  $\theta$ -modulus, which is generated by an arbitrary periodic multiplier  $\theta$ , and then explore its properties. They also establish a Jackson-type estimate and a Bernstein-type estimate for this modulus. The authors also introduce a constructive method of trigonometric approximation and describe its quality in terms of the  $\theta$ -modulus. Moduli related to Weyl derivatives, Riesz derivatives, and fractional Riesz derivatives are covered by this approach.

Chapter 12 by L. Angeloni and G. Vinti deals with approximation by a family of integral operators  $T_\omega$  of Mellin type in a multivariate setting on  $\mathbb{R}_+^N$ , where  $\omega \in \mathbb{R}_+$ . The authors introduce the notion of multidimensional variation  $V$  in the sense of Tonelli and study convergence and order of approximation in the variation for certain classes of functions  $f$ , that is, they investigate the convergence to zero of  $V[T_\omega f - f]$  as  $\omega \rightarrow \infty$ .

The chapter 13 by F. Stenger, Hany A.M. El-Sharkawy, and G. Baumann is a contribution to error estimates for Sinc approximation, that is, for approximation by a truncated classical sampling series also known as cardinal series. The authors

determine the exact asymptotic representation up to an  $O(n^{-2})$  term for the Lebesgue constant, where  $n$  is the number of consecutive terms in the truncated cardinal series. In numerical experiments, they compute the Lebesgue function and the Lebesgue constant when the cardinal series is transformed to the interval  $(-1, 1)$  by a conformal mapping and compare these quantities with the corresponding ones for three other approximation procedures that have been proposed in the literature.

The third group of chapters is a collection of articles on different topics in harmonic analysis. In Chap. 14, O. Christensen describes some open problems in frame theory and presents partial results. First, he introduces the necessary background for frame and operator theory, as well as wavelet and Gabor analysis. The problems under consideration deal with extension of dual Bessel sequences to dual frames, generalization of the duality principle in Gabor analysis, construction of wave packet frames, generation of Gabor frames by  $B$ -splines, and determination of good frame bounds. He also includes a conjecture by Heil–Ramanathan–Topiwala and a conjecture by Feichtinger. The latter has been answered affirmatively while the author was preparing his contribution.

Closely related to Chap. 14 is Chap. 15 by B. Forster in which she provides an overview of the importance of complex-valued transforms in image processing. Interpretations and applications of the complex-valued mathematical image analysis are given for Fourier, Gabor, Hilbert, and wavelet transforms. Relationships to physical imaging techniques such as holography are also described. The role of amplitude and phase of the Fourier coefficients in images is illustrated by several instructive examples.

Chapter 16 by D. Mugler and S. Clary introduces a new method to compute the frequencies in a digital signal that is based on the discrete Hermite transform that was developed by the authors in previous papers. Particularly for an input signal that is a linear combination of general sinusoids, this method provides more accurate estimations of both frequencies and amplitudes than the usual DFT. The method is based primarily on the property of the discrete Hermite functions being eigenvectors of the centered Fourier matrix, analogous to the classical result that the continuous Hermite functions are eigenfunctions of the Fourier transform.

Using this method for frequency determination, a new time-frequency representation based on the Hermite transform is developed and shown to provide clearer interpretations of frequency and amplitude content of a signal than the corresponding spectrograms or scalograms.

In Chap. 17, P. Massopust considers three quite distinct mathematical areas, namely, (1) fractional differential and integral operators, (2) Dirichlet averages, and (3) splines. He shows that if these areas are generalized to complex settings and infinite-dimensional spaces, then an interesting and unexpected relationship between them can be established.

In Chap. 18, H. G. Feichtinger and W. Hörmann propose a concept of generalized stochastic processes within a functional analytic setting. It is based on the unpublished thesis of the second-named author. Although the thesis was written several years ago, it has gained new interest because of recent developments. Considering

the Segal algebra  $S_0(G)$  of a locally compact Abelian group  $G$ , the authors define a generalized stochastic process  $\rho$  as a bounded linear mapping of  $S_0(G)$  into a Hilbert space and obtain various results that generalize those on classical stochastic processes. This concept has some advantages over other generalizations of stochastic processes.

Finally, in Chap. 19, E. E. Berdysheva and H. Berens study a multivariate generalization of an extremal problem for bandlimited functions due to Paul Turán, known as the Turán problem. A simple equivalent formulation of this problem is as follows: Let  $F$  be a nonnegative function on  $\mathbb{R}^d$  with  $\int_{\mathbb{R}^d} F(\mathbf{x}) d\mathbf{x} = 1$  such that its Fourier transform is  $\ell_1$ -radial and supported in the  $\ell_1$ -ball of radius  $\pi$ . Determine the supremum of  $F(0)$  over all such  $F$ . The authors consider a discrete multivariate Turán problem and state a conjecture for any positive integer dimension  $d$ . The proof of the conjecture for  $d = 2$  was given in one of their earlier papers. Here they settle the case for  $d = 3, 5, 7$ .

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