

Chapter 2

Projective and Injective Representations

Projective representations and injective representations are key concepts in representation theory. A representation P is called *projective* if the functor $\text{Hom}(P, -)$ maps surjective morphisms to surjective morphisms. Dually a representation I is called *injective* if the functor $\text{Hom}(-, I)$ maps injective morphisms to injective morphisms. The terminology comes from the property that for any representation M there is a projective representation P_0 such that there exists a surjective morphism (a “projection”):

$$p_0 : P_0 \twoheadrightarrow M.$$

Dually, for any representation M , there is an injective representation I_0 such that there exists an injective morphism:

$$i_0 : M \hookrightarrow I_0.$$

If M is not projective itself, then the morphism p_0 above will have a kernel, and we can find another projective P_1 such that there exists a surjective morphism p_1 from P_1 to the kernel of p_0 . Iterating this procedure yields an exact sequence

$$\cdots \longrightarrow P_3 \xrightarrow{p_3} P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

where each P_i is a projective representation. Such a sequence is called a *projective resolution*. We think of projective resolutions as a way to approximate the representation M by projective representations. Often it is possible to deduce properties of M from a projective resolution of M .

Dually, we have *injective resolutions*, thus exact sequences of the form

$$0 \longrightarrow M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} I_2 \xrightarrow{i_3} I_3 \longrightarrow \dots$$

where each I_i is an injective representation.

For representations of quivers without oriented cycles the situation is very simple. We will see that every quiver representation has a projective resolution of the form

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 ,$$

and an injective resolution of the form

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0 .$$

We will use this result to show that every subrepresentation of a projective representation is projective. Categories with this property are called *hereditary*.

Moreover, it is very easy to write down all indecomposable projective representations of a quiver Q without oriented cycles. There is exactly one indecomposable projective representation $P(i)$ for each vertex $i \in Q_0$, and this representation $P(i)$ is given by the paths in Q starting at the vertex i . Dually, there is exactly one indecomposable injective representation $I(i)$ for each vertex $i \in Q_0$, and $I(i)$ is given by the paths ending at the vertex i .

We will see later in Chap. 4 that each quiver defines an algebra A , the *path algebra of the quiver*, whose basis consist of the set of all paths in Q . We will also see that we can consider the algebra A as a representation of Q and that this representation is isomorphic to the direct sum of the indecomposable projective representations, thus $A \cong \bigoplus_{i \in Q_0} P(i)$ as representations of Q .

In the current chapter, we also introduce the Auslander–Reiten translation τ , which is crucial to Auslander–Reiten theory and Auslander–Reiten quivers. It is defined in a rather curious way by taking the beginning of a projective resolution of M

$$P_1 \xrightarrow{p_1} P_0 \longrightarrow M \longrightarrow 0$$

and then setting $\tau M = \ker v p_1$, where v is the so-called Nakayama functor. This functor maps projective representations to injective representations, and therefore we obtain the beginning of an injective resolution:

$$0 \longrightarrow \tau M \longrightarrow v P_1 \xrightarrow{v p_1} v P_0 .$$

Throughout this chapter, and the rest of the book, the very natural notion of paths in a quiver will be essential. Here is a formal definition.

Definition 2.1. Let $Q = (Q_0, Q_1, s, t)$ be a quiver, $i, j \in Q_0$. A **path** c from i to j of length ℓ in Q is a sequence

$$c = (i | \alpha_1, \alpha_2, \dots, \alpha_\ell | j)$$

with $\alpha_h \in Q_1$ such that

$$\begin{aligned} s(\alpha_1) &= i, \\ s(\alpha_h) &= t(\alpha_{h-1}), \text{ for } h = 2, 3, \dots, \ell, \\ t(\alpha_\ell) &= j. \end{aligned}$$

Thus a path from i to j is a way to go from vertex i to vertex j in the quiver Q , where we are only allowed to walk along an arrow in the direction to which it is pointing.

Example 2.1. In the quiver

$$\alpha \curvearrowright 1 \xrightarrow{\beta} 2 \xleftarrow{\gamma} 3,$$

we have that $(1 | \alpha | 1)$, $(1 | \alpha, \beta | 2)$, $(1 | \alpha, \alpha, \beta | 2)$ are paths, but $(1 | \alpha, \beta, \gamma | 2)$ is not.

Example 2.2.

1. The **constant path** (or lazy path) $(i | i)$ at vertex i is the path of length $\ell = 0$ which never leaves the vertex i . We denote this path e_i .
2. An arrow $i \xrightarrow{\alpha} j$ is a path $(i | \alpha | j)$ of length one. If $i = j$ then

$$i \curvearrowright \alpha$$

is called a **loop**.

3. A path of the form

$$i \xleftarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{\ell-1}} \bullet \xleftarrow{\alpha_\ell} i$$

given by $(i | \alpha_1, \alpha_2, \dots, \alpha_\ell | i)$ is called an **oriented cycle**. Thus a loop is an oriented cycle of length one.

2.1 Simple, Projective, and Injective Representations

Let Q be a quiver without oriented cycles. For every vertex i in Q , we will now define three representations: the simple, the projective, and the injective representation at i .

We will show that in the category $\text{rep } Q$, these representations are respectively simple, projective, or injective objects in the categorical sense.

Definition 2.2. Let i be a vertex of Q . Define representations $S(i)$, $P(i)$, and $I(i)$ as follows:

(a) $S(i)$ is of dimension one at vertex i , and zero at every other vertex; thus

$$S(i) = (S(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}, \quad \text{where}$$

$$\begin{aligned} S(i)_j &= \begin{cases} k & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \\ \varphi_\alpha &= 0 \quad \text{for all arrows } \alpha. \end{aligned}$$

$S(i)$ is called the **simple representation** at vertex i .

(b)

$$P(i) = (P(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$$

where $P(i)_j$ is the k -vector space with basis the set of all paths from i to j in Q ; so the elements of $P(i)_j$ are of the form $\sum_c \lambda_c c$, where c runs over all paths from i to j , and $\lambda_c \in k$;

and if $j \xrightarrow{\alpha} \ell$ is an arrow in Q , then $\varphi_\alpha : P(i)_j \rightarrow P(i)_\ell$ is the linear map defined on the basis by composing the paths from i to j with the arrow $j \xrightarrow{\alpha} \ell$.

More precisely, the arrow α induces an injective map between the bases

$$\begin{aligned} \text{basis of } P(i)_j &\longrightarrow \text{basis of } P(i)_\ell \\ c = (i|\beta_1, \beta_2, \dots, \beta_s|j) &\longmapsto c\alpha = (i|\beta_1, \beta_2, \dots, \beta_s, \alpha|\ell) \end{aligned}$$

and φ_α is defined by

$$\varphi_\alpha \left(\sum_c \lambda_c c \right) = \sum_c \lambda_c c\alpha.$$

$P(i)$ is called the **projective representation** at vertex i .

(c)

$$I(i) = (I(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$$

where $I(i)_j$ is the k -vector space with basis the set of all paths from j to i in Q ; so the elements of $I(i)_j$ are of the form $\sum_c \lambda_c c$, where c runs over all paths from j to i , and $\lambda_c \in k$;

and if $j \xrightarrow{\alpha} \ell$ is an arrow in Q , then $\varphi_\alpha : I(i)_j \rightarrow I(i)_\ell$ is the linear map defined on the basis by deleting the arrow $j \xrightarrow{\alpha} \ell$ from those paths from j to i which start with α and sending to zero the paths that do not start with α .

More precisely, the arrow α induces a surjective map f between the bases

$$\begin{array}{ccc} \text{basis of } I(i)_j & \xrightarrow{f} & \text{basis of } I(i)_\ell \\ c = (j|\beta_1, \beta_2, \dots, \beta_s|i) & \mapsto & \begin{cases} (\ell|\beta_2, \dots, \beta_s|i) & \text{if } \beta_1 = \alpha, \\ 0 & \text{otherwise;} \end{cases} \end{array}$$

and φ_α is defined by

$$\varphi_\alpha \left(\sum_c \lambda_c c \right) = \sum_c \lambda_c f(c).$$

$I(i)$ is called the **injective representation** at vertex i .

Note that we need the hypothesis that Q has no oriented cycles, because, otherwise, there would be a vertex i such that $P(i)$ is infinite-dimensional, and hence not a representation in $\text{rep } Q$. For example, if Q is the quiver $1 \rightleftarrows 2$ then $P(1)$ and $P(2)$ would be infinite-dimensional.

The following remark will be very useful later on.

Remark 2.1. Let $P(i) = (P(i)_j, \varphi_\alpha)$ be the projective representation at vertex i and let c be a path starting at i , say

$$c = (i|\beta_1, \beta_2, \dots, \beta_\ell|j).$$

Then we can define the map

$$\varphi_c : P(i)_i \longrightarrow P(i)_j \quad \varphi_c = \varphi_{\beta_\ell} \cdots \varphi_{\beta_2} \varphi_{\beta_1}$$

as the composition of the maps in the representation $P(i)$ along the path c . Then, if e_i denotes the constant path at vertex i , it follows from the definition of $P(i)$ that

$$\varphi_c(e_i) = c. \quad (2.1)$$

Remark 2.2. Simple projectives and simple injectives:

- (1) The projective representation at vertex i is the simple representation at vertex i if and only if there is no arrow α in Q such that $s(\alpha) = i$. Such vertices are called *sinks* of the quiver Q . Thus

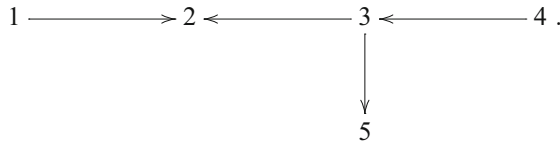
$$S(i) = P(i) \iff i \text{ is a sink in } Q.$$

- (2) The injective representation at vertex i is the simple representation at vertex i if and only if there is no arrow α in Q such that $t(\alpha) = i$. Such vertices are called *sources* of the quiver Q . Thus

$$S(i) = I(i) \iff i \text{ is a source in } Q.$$

In the following two examples, we use matrix notation to describe projective and injective representations. We use the isomorphism and not the equality symbol since there are many other possible descriptions for these representations; see Exercise 2.1.

Example 2.3. Let Q be the quiver

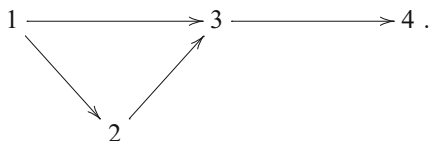


Then

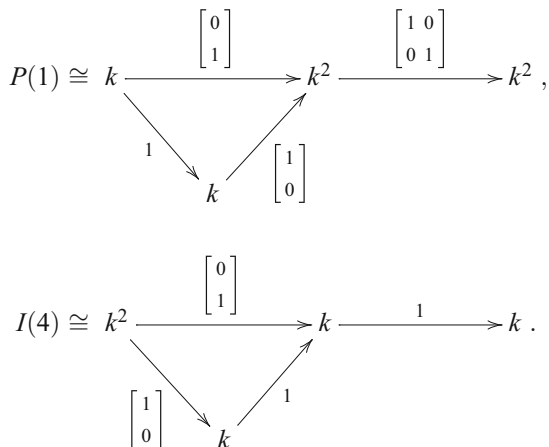
$$\begin{array}{l} S(3) \cong \begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longleftarrow & k & \longleftarrow & 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array} \\ P(3) \cong \begin{array}{ccccccc} 0 & \longrightarrow & k & \xleftarrow{1} & k & \longleftarrow & 0 \\ & & & & \downarrow 1 & & \\ & & & & k & & \end{array} \\ I(3) \cong \begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longleftarrow & k & \xleftarrow{1} & k \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array} \end{array}$$

The quiver in the next example contains parallel paths. As a result the indecomposable projective modules can be of dimension greater than 1 at a single vertex.

Example 2.4. Let Q be the quiver



Then



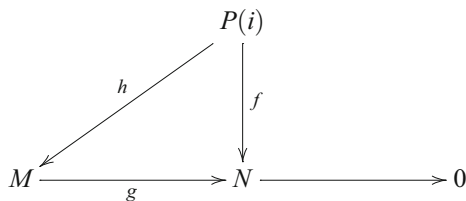
In category theory, a **projective** object is an object P such that the Hom functor $\text{Hom}(P, -)$ maps surjective morphisms to surjective morphisms. The following proposition shows that the representations $P(i)$ satisfy this condition, which is the reason why we call them projective.

Proposition 2.3. *Let $g: M \rightarrow N$ be a surjective morphism between representations of Q , and let $P(i)$ be the projective representation at vertex i . Then the map*

$$g_*: \text{Hom}(P(i), M) \longrightarrow \text{Hom}(P(i), N)$$

is surjective.

In other words, if $f: P(i) \rightarrow N$ is any morphism, then there exists a morphism $h: P(i) \rightarrow M$ such that the diagram



commutes, that is, $f = g \circ h = g_(h)$.*

Proof. Exercise 2.4. □

Corollary 2.4. *If P is projective, then any exact sequence of the form*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{g} P \longrightarrow 0$$

splits.

Proof. Use Proposition 2.3 with the identity morphism $f = 1_P$ to get the commutative diagram:

$$\begin{array}{ccccc} & & P & & \\ & \nearrow h & \parallel & & \\ M & \xrightarrow{g} & P & \longrightarrow & 0 \end{array}$$

Therefore $1 = g \circ h$, and g is a retraction. □

A dual statement holds for the injective representations. In category theory, an **injective** object is an object I such that the Hom functor $\text{Hom}(-, I)$ maps injective morphisms to surjective morphisms. The next proposition shows that the representations $I(i)$ satisfy this condition, which is the reason why we call them injective.

Proposition 2.5. *Let $g: L \rightarrow M$ be an injective morphism between representations of Q , and let $I(i)$ be the injective representation at vertex i . Then the map*

$$g^*: \text{Hom}(M, I(i)) \longrightarrow \text{Hom}(L, I(i))$$

is surjective.

In other words, if $f: L \rightarrow I(i)$ is any morphism, then there exists a morphism $h: M \rightarrow I(i)$ such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{g} & M \\ & & \downarrow f & \nearrow h & \\ & & I(i) & & \end{array}$$

commutes, that is, $f = h \circ g = g^(h)$.*

Proof. Exercise 2.5. □

Corollary 2.6. *If I is injective then any exact sequence of the form*

$$0 \longrightarrow I \xrightarrow{g} M \longrightarrow N \longrightarrow 0$$

splits.

Proof. Use Proposition 2.5 with the identity morphism $f = 1_I$ to get a commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{g} & M \\ & & \parallel & \searrow h & \\ & & I & & \end{array}$$

Thus $1_I = h \circ g$, and g is a section. □

Finally, a **simple** object in a category is a nonzero object S that has no proper subobjects. The representations $S(i)$ have this property, hence their name.

The next proposition states that sums of projective objects are projective and that summands of projective objects are projective. We state the result for the category $\text{rep } Q$, but the proof holds in any additive category.

Proposition 2.7. (1) *Let P and P' be representations of Q . Then*

$$P \oplus P' \text{ is projective} \iff P \text{ and } P' \text{ are projective.}$$

(2) *Let I and I' be representations of Q . Then*

$$I \oplus I' \text{ is injective} \iff I \text{ and } I' \text{ are injective.}$$

Proof. We only show (1) since the proof of (2) is similar.

(\Rightarrow) Let $g : M \rightarrow N$ be surjective in $\text{rep } Q$ and let $f : P \rightarrow N$ be any morphism in $\text{rep } Q$. Consider the following diagram:

$$\begin{array}{ccccc} & & P \oplus P' & & \\ & & \downarrow \text{pr}_1 & \nearrow i_1 & \\ & & P & & \\ & \nearrow \exists h & \downarrow f & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

where pr_1 denotes the projection on the first summand and i_1 is the canonical injection. Clearly, $pr_1 \circ i_1 = 1_P$. Since $P \oplus P'$ is projective, there exists a map $h : P \oplus P' \rightarrow M$ such that $g h = f pr_1$. Therefore

$$g h i_1 = f pr_1 i_1 = f 1_P = f.$$

Now we can define $h' : P \rightarrow M$ as $h' = h i_1$, and we have $g h' = f$. This shows that P is projective. One can show in a similar way that P' is projective.

(\Leftarrow) Let $g : M \rightarrow N$ be a surjective morphism in $\text{rep } Q$ and let $f : P \oplus P' \rightarrow N$ be any morphism in $\text{rep } Q$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & P & & \\
 & & \downarrow i_1 & & \\
 & \nearrow \exists h_1 & P \oplus P' & \searrow f & \\
 & & \downarrow & & \\
 M & \xrightarrow{g} & N & \xrightarrow{\quad} & 0
 \end{array}$$

where i_1 denotes the canonical injection and, since P is projective, there exists a morphism h_1 such that $g h_1 = f i_1$. By symmetry, there also exists a morphism $h_2 : P' \rightarrow M$ such that $g h_2 = f i_2$. Define $h = (h_1, h_2) : P \oplus P' \rightarrow M$ by $h(p + p') = h_1(p) + h_2(p')$. Then

$$g h(p + p') = g h_1(p) + g h_2(p') = f i_1(p) + f i_2(p') = f(p + p'),$$

which shows that $P \oplus P'$ is projective. \square

Proposition 2.7 implies that if we know the indecomposable projective, respectively injective, representations, then we know all projective, respectively injective, representations. The next step is to show that the representations $P(i)$ and $I(i)$ are in fact indecomposable. We will see later in Corollary 2.21 that there are no other indecomposable projective or injective representations.

Proposition 2.8. *The representations $S(i)$, $P(i)$, and $I(i)$ are indecomposable.*

Proof. For $S(i)$ this follows directly from the fact that $S(i)$ is simple. Let us prove the result for the projective representation $P(i) = (P(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$. Since Q has no oriented cycles, we have $P(i)_i = k$. Suppose that $P(i) = M \oplus N$ for some $M, N \in \text{rep } Q$. Then we may suppose without loss of generality that $P(i)_i = M_i$ and $N_i = 0$. Let ℓ be a vertex of Q such that $N_\ell \neq 0$. Now, $P(i)_\ell$ has a basis consisting of the paths from i to ℓ in Q . Let $c = (i|\beta_1, \dots, \beta_s|\ell)$ be such a path.

Let $\varphi_c = \varphi_{\beta_s} \cdots \varphi_{\beta_1}$ denote the composition of the linear maps of the representation $P(i)$ along the path c . Then, since $P(i)$ is the direct sum of M and N , the map

$$\varphi_c : M_i \oplus 0 \longrightarrow M_\ell \oplus N_\ell$$

sends the unique basis element e_i of M_i to an element $\varphi_c(e_i)$ of M_ℓ . But from Remark 2.1 we know that $\varphi_c(e_i) = c$; thus every basis element c of $P(i)_\ell$ lies in M_ℓ , a contradiction.

The proof for $I(i)$ is similar. \square

The following proposition shows that the simple representations $S(i)$ form a complete set of simple representations in $\text{rep } Q$, up to isomorphism.

Proposition 2.9. *A representation of Q is simple if and only if it is isomorphic to $S(i)$, for some $i \in Q_0$.*

Proof. It is clear that the $S(i)$ are simple representations. Conversely, let $M = (M_i, \varphi_\alpha)$ be any representation of Q . We want to show that there is a vertex i such that $S(i)$ is a subrepresentation of M , and we have to choose this vertex i carefully. We do not want to have a nonzero map in the representation M that starts at the vertex i . For example, if i is a sink in the quiver, we have what we want. But on the other hand, we also need the representation M to be nonzero at the vertex i . This leads us to pick i as follows.

Let $i \in Q_0$ such that $M_i \neq 0$ and $M_j = 0$, whenever there is an arrow $i \xrightarrow{\alpha} j$ in Q . Note that such a vertex exists since Q has no oriented cycles. Choose any injective linear map $f_i : S(i)_i \cong k \rightarrow M_i$, and extend it trivially to a morphism $f : S(i) \rightarrow M$ by letting $f_j = 0$ if $i \neq j$. Note that f actually is a morphism since the diagram

$$\begin{array}{ccccc} 0 & \xrightarrow{\quad} & S(i)_i & \xrightarrow{\quad} & 0 \\ \downarrow & & \downarrow f_i & & \downarrow \\ M_\ell & \xrightarrow{\varphi_\alpha} & M_i & \xrightarrow{\varphi_\beta} & 0 \end{array}$$

commutes, for all arrows $\ell \xrightarrow{\alpha} i$ and $i \xrightarrow{\beta} j$ in Q . Since f is injective this shows that $S(i)$ is a subrepresentation of M , and therefore, either $M \cong S(i)$ or M is not simple. \square

Remark 2.10. Proposition 2.9 does not hold if the quiver has oriented cycles. For example, if Q is the quiver $\alpha \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} 1$ then for each $\lambda \in k$, there is a simple representation $f_\lambda \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} k$, where f_λ is given by multiplication by λ .

The vector space at vertex i of any representation can be described as a space of morphisms using the projective representation $P(i)$ as follows:

Theorem 2.11. *Let $M = (M_i, \psi_\alpha)$ be a representation of \mathcal{Q} . Then, for any vertex i in \mathcal{Q} , there is an isomorphism of vector spaces:*

$$\text{Hom}(P(i), M) \cong M_i.$$

Proof. Let $e_i = (i|i)$ be the constant path at i . Then $\{e_i\}$ is a basis of the vector space $P(i)_i$. Define a map

$$\begin{aligned} \phi : \text{Hom}(P(i), M) &\longrightarrow M_i \\ f = (f_j)_{j \in \mathcal{Q}_0} &\longmapsto f_i(e_i). \end{aligned}$$

If f is a morphism from $P(i)$ to M , then its component f_i is a linear map from $P(i)_i$ to M_i , which shows that the map ϕ is well defined, since $e_i \in P(i)_i$.

We will show that ϕ is an isomorphism of vector spaces. Let us use the notation $P(i) = (P(i)_j, \varphi_\alpha)$. First, we show that ϕ is linear. If $f, g \in \text{Hom}(P(i), M)$ are two morphisms, then $\phi(f + g) = (f + g)_i(e_i) = f_i(e_i) + g_i(e_i) = \phi(f) + \phi(g)$, and if $\lambda \in k$ then $\phi(\lambda f) = (\lambda f)_i(e_i) = \lambda f_i(e_i) = \lambda \phi(f)$.

Next, we show that ϕ is injective. If $0 = \phi(f) = f_i(e_i)$, then the linear map f_i sends the basis $\{e_i\}$ to zero, and thus f_i is the zero map. We will now show that $f_j : P(i)_j \rightarrow M_j$ is the zero map, for any vertex j , and this will show that ϕ is injective. By definition of $P(i)$, the vector space $P(i)_j$ has a basis consisting of all paths from i to j . Let $c = (i|\alpha_1, \dots, \alpha_l|j)$ be such a basis element, and consider the maps $\varphi_c = \varphi_{\alpha_l} \circ \dots \circ \varphi_{\alpha_1}$ and $\varphi'_c = \varphi'_{\alpha_l} \circ \dots \circ \varphi'_{\alpha_1}$ defined as the composition of the maps along the path c of the representation $P(i)$ and M , respectively. It follows from the definition of $P(i)$ that $\varphi_c(e_i) = c$. Since f is a morphism of representations, we have $f_j \varphi_c = \varphi'_c f_i$, and, since $f_i(e_i) = 0$, this implies that f_j maps c to zero. As c is an arbitrary basis element of $P(i)_j$, it follows that $f_j = 0$.

It remains to show that ϕ is surjective. Let $m_i \in M_i$. We want to construct a morphism $f : P(i) \rightarrow M$ such that $f_i(e_i) = m_i$. Let us start by fixing its component $f_i : P(i)_i \rightarrow M_i$ by requiring the condition we need, that is, $f_i(e_i) = m_i$. Since $\{e_i\}$ is a basis of $P(i)_i$, this condition defines the linear map f_i in a unique way. We can extend the map f_i to a morphism $f = (f_j)_{j \in \mathcal{Q}_0}$ by *following the paths* in \mathcal{Q} . More precisely, for any path c from i to a vertex j in \mathcal{Q} , put $f_j(c) = \varphi'_c(m_i)$. This defines each map f_j on a basis of $P(i)_j$, and we extend this map linearly to the whole vector space $P(i)_j$. It follows from our construction that f is a morphism of representations, thus $f \in \text{Hom}(P(i), M)$ and $\phi(f) = m_i$, so ϕ is surjective. \square

Note that in the proof of Theorem 2.11, the particular structure of the representation $P(i)$ is the essential ingredient to show the injectivity and surjectivity of the map ϕ .

As an immediate consequence of Theorem 2.11, we can describe the morphisms between projective representations as follows:

Corollary 2.12. *Let i and j be vertices in Q .*

- (1) *The vector space $\text{Hom}(P(i), P(j))$ has a basis consisting of all paths from j to i in Q . In particular,*

$$\text{End}(P(i)) = \text{Hom}(P(i), P(i)) \cong k.$$

- (2) *If $A = \bigoplus_{i \in Q_0} P(i)$, then the vector space $\text{End}(A) = \text{Hom}(A, A)$ has a basis consisting of all paths in Q .*

Remark 2.13. In Chap. 4, we will see that the so-called path algebra of the quiver Q is isomorphic to $\text{End}(A)$ as a vector space and as an algebra.

Proof. Theorem 2.11 implies that $\text{Hom}(P(i), P(j))$ is isomorphic to $P(j)_i$, and this vector space has a basis consisting of all paths from j to i in Q . The fact that Q has no oriented cycles implies that $\text{End}(P(i))$ is of dimension one; hence $\text{End}(P(i)) \cong k$. This proves (1), and (2) is a direct consequence. \square

Corollary 2.14. *The representation $P(j)$ is a simple representation if and only if $\text{Hom}(P(i), P(j)) = 0$ for all $i \neq j$.*

Proof. The representation $P(j)$ is simple if and only if j is a sink, which means that there are no paths from j to any other vertex i . The statement now follows from Corollary 2.12. \square

2.2 Projective Resolutions and Radicals of Projectives

We have seen above that the projective representations can be used to describe the vector spaces of an arbitrary representation using the Hom functor. Now we will introduce another way of describing arbitrary representations by means of projective representations: the projective resolutions. As usual, there is a dual notion, the injective resolutions.

Definition 2.3. Let M be a representation of Q .

- (1) A **projective resolution** of M is an exact sequence

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is a projective representation.

- (2) An **injective resolution** of M is an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \cdots,$$

where each I_i is an injective representation.

Theorem 2.15. *Let M be a representation of Q .*

(1) *There exists a projective resolution of M of the form*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

(2) *There exists an injective resolution of M of the form*

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0.$$

Proof. We will show only (1), and to achieve this, we will construct the so-called standard projective resolution of M . Let $M = (M_i, \varphi_i)$, and denote by d_i the dimension of M_i . Define

$$P_1 = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)) \quad P_0 = \bigoplus_{i \in Q_0} d_i P(i),$$

where $d_i P(i)$ stands for the direct sum of d_i copies of $P(i)$.

Before defining the morphisms of the projective resolution, let us examine the representations P_0 and P_1 . For every vector space M_i , we have $d_i = \dim M_i$ copies of $P(i)$ in P_0 . The natural map g from P_0 to M will send the d_i copies of the constant path e_i in P_0 to a basis of M_i . Now in each copy of $P(i)$, the kernel of the map g contains a copy of $P(t(\alpha))$ for every α that starts at i . So, for every arrow α with $s(\alpha) = i$, we have $d_{s(\alpha)}$ copies of $P(t(\alpha))$ in the kernel of g , which justifies the definition of P_1 .

To define the morphisms of the projective resolution, we introduce specific bases for each of the representations P_1, P_0 and M as follows: For each $i \in Q_0$, let $\{m_{i1}, \dots, m_{id_i}\}$ be a basis for M_i , and thus

$$B'' = \{m_{ij} \mid i \in Q_0, j = 1, 2, \dots, d_i\}$$

is a basis for M . Taking the standard bases for the projective representations, the set

$$B = \{c_{ij} \mid i \in Q_0, c_i \text{ a path with } s(c_i) = i, \text{ and } j = 1, \dots, d_i\}$$

is a basis for P_0 ; and the set

$$B' = \{b_{\alpha j} \mid \alpha \in Q_1, b_{\alpha} \text{ a path with } s(b_{\alpha}) = t(\alpha), \text{ and } j = 1, \dots, d_{s(\alpha)}\}$$

is a basis for P_1 . Define a map g on the basis B by

$$g(c_{ij}) = \varphi_{c_i}(m_{ij}) \in M_{t(c_i)}$$

and extend g linearly to P_1 . Define the map f on the basis B' by

$$f(b_{\alpha j}) = (\alpha b_{\alpha})_j - b_{\alpha}^M,$$

where αb_α is the path from $s(\alpha)$ to $t(b_\alpha)$ given by the composition of α and b_α , and $b_\alpha^M = \sum_{\ell=1}^{d_{t(\alpha)}} \theta_\ell b_{\alpha\ell}$, where the θ_ℓ are the scalars that occur when writing $\varphi_\alpha(m_{s(\alpha)j})$ in the basis $\{m_{t(\alpha)\ell} \mid \ell = 1, \dots, d_{t(\alpha)}\}$ of $M_{t(\alpha)}$; thus

$$\varphi_\alpha(m_{s(\alpha)j}) = \sum_{\ell=1}^{d_{t(\alpha)}} \theta_\ell m_{t(\alpha)\ell}. \quad (2.2)$$

We will now prove that the sequence

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)) \xrightarrow{f} \bigoplus_{i \in Q_0} d_i P(i) \xrightarrow{g} M \longrightarrow 0 \quad (2.3)$$

is exact.

g is surjective, because for any basis vector m_{ij} of M , we have $m_{ij} = g(e_{ij})$, where e_i is the constant path at vertex i .

$\ker g \supset \text{im } f$: It suffices to show that $g \circ f(b_{\alpha j}) = 0$ for any $b_{\alpha j}$ in the basis B' . We compute

$$\begin{aligned} g(f(b_{\alpha j})) &= g((\alpha b_\alpha)_j - b_\alpha^M) \\ &= \varphi_{\alpha b_\alpha}(m_{s(\alpha)j}) - \varphi_{b_\alpha}(\sum_{\ell} \theta_\ell m_{t(\alpha)\ell}) \\ &= \varphi_{b_\alpha}(\varphi_\alpha(m_{s(\alpha)j}) - \sum_{\ell} \theta_\ell m_{t(\alpha)\ell}) \\ &= \varphi_{b_\alpha}(0) \\ &= 0, \end{aligned}$$

where the next to last equation follows from (2.2).

$\ker g \subset \text{im } f$: First note that any $x \in \bigoplus_{i \in Q_0} d_i P(i)$ can be written as a linear combination of the basis B ; thus

$$x = \sum_{c_{ij} \in B} \lambda_{c_{ij}} c_{ij} = x_0 + \sum_{c_{ij} \in B \setminus B_0} \lambda_{c_{ij}} c_{ij},$$

where B_0 is the subset of B consisting of constant paths (together with a choice of j),

$$B_0 = \{e_{ij} \mid i \in Q_0, j = 1, \dots, d_i\},$$

and $x_0 = \sum_{e_{ij} \in B_0} \lambda_{e_{ij}} e_{ij}$. Any nonconstant path is the product of an arrow and another path; thus

$$x = x_0 + \sum_{c_{ij}: c_i = \alpha b_\alpha} \lambda_{c_{ij}} (\alpha b_\alpha)_j,$$

and using the definition of f , we get

$$x = x_0 + \sum_{c_{ij}: c_i = \alpha b_\alpha} \lambda_{c_{ij}} f(b_{\alpha j}) + \lambda_{c_{ij}} b_\alpha^M. \quad (2.4)$$

Let $x_1 = x_0 + \sum_{c_i = \alpha b_\alpha} \lambda_{c_{ij}} b_\alpha^M$. Note that $x - x_1 \in \text{im } f$.

Define the degree of a linear combination of paths to be the length of the longest path that appears in it with nonzero coefficient. Note that $\deg x_1 < \deg x$ and $\deg x_0 = 0$.

Now let us suppose that $x \in \ker g$. We want to show that $x \in \text{im } f$. Using (2.4) and the fact that $g \circ f = 0$, we get

$$0 = g(x) = g(x_1).$$

Summarizing, we have $x_1 \in \ker g$, $\deg x_1 < \deg x$ and $x - x_1 \in \text{im } f$.

Now we repeat the argument with x_1 instead of x . We get $x_2 \in \ker g$, $\deg x_2 < \deg x_1 < \deg x$, and $x - x_2 \in \text{im } f$. Continuing like this, we will eventually get $x_h \in \ker g$, $x - x_h \in \text{im } f$, and $\deg x_h = 0$; thus x_h is a linear combination of constant paths, say $x_h = \sum_{i,j} \mu_{ij} e_{ij}$, for some $\mu_{ij} \in k$. By definition of g , we have

$$0 = g(x_h) = \sum_{i,j} \mu_{ij} m_{ij},$$

and since the m_{ij} form a basis of M , this implies that all μ_{ij} are zero. Hence $x_h = 0$ and thus $x \in \text{im } f$.

f is injective. Suppose that

$$0 = f\left(\sum \lambda_{b_{\alpha h}} b_{\alpha h}\right) = \sum \lambda_{b_{\alpha h}} ((\alpha b_\alpha)_h - b_\alpha^M).$$

Then

$$\sum \lambda_{b_{\alpha h}} (\alpha b_\alpha)_h = \sum \lambda_{b_{\alpha h}} b_\alpha^M = \sum \lambda_{b_{\alpha h}} \sum \theta_\ell b_{\alpha \ell}.$$

Let i_0 be a source in M , that is, i_0 is such that there is no arrow $j \rightarrow i_0$ with $d_j \neq 0$. Note that such an i_0 exists since Q has no oriented cycles. Then, since each of the paths b_α starts at the endpoint of the arrow α , none of the paths b_α can go through i_0 , and this shows that $\lambda_{b_{\alpha h}}$ must be zero, for all arrows α with $s(\alpha) = i_0$. Now let i_1 be a source in $M \setminus i_0$, that is, i_1 is such that there is no arrow $j \rightarrow i_1$ with $j \neq i_0$ and $d_j \neq 0$. Since $\lambda_{b_{\alpha h}} = 0$ for all arrows α with $s(\alpha) = i_0$, there is no path $b_{\alpha h}$ with $\lambda_{b_{\alpha h}} \neq 0$ that goes through i_1 . Continuing in this way, we show that every $\lambda_{b_{\alpha h}} = 0$, since Q has only finitely many arrows. Thus f is injective. This concludes the proof of the exactness of the standard resolution (2.3). \square

Remark 2.16. There are other projective resolutions than the standard resolution.

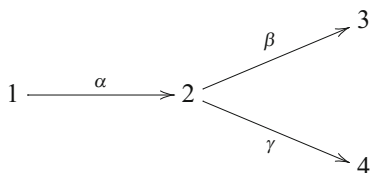
Example 2.5. Let Q be the quiver $1 \longrightarrow 2 \longleftarrow 3$ and consider the representations $M = S(3) = 3$ and $M' = {}_1 \frac{2}{3}$. Then we have the standard projective resolutions:

$$\begin{aligned} 0 &\longrightarrow 2 \longrightarrow \frac{3}{2} \longrightarrow 3 \longrightarrow 0 \\ 0 &\longrightarrow 2 \oplus 2 \longrightarrow \frac{1}{2} \oplus \frac{3}{2} \oplus 2 \longrightarrow \frac{1}{2} \frac{3}{2} \longrightarrow 0. \end{aligned}$$

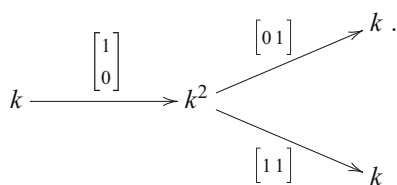
The second resolution is not minimal in the sense that one can eliminate a direct summand $S(2) = 2$ in each of the projective modules and still have a projective resolution:

$$0 \longrightarrow 2 \longrightarrow \frac{1}{2} \oplus \frac{3}{2} \longrightarrow \frac{1}{2} \frac{3}{2} \longrightarrow 0.$$

Example 2.6. Let Q be the quiver



and consider the representation $M = \begin{smallmatrix} 1 \\ 2 & 2 \\ 3 & 4 \end{smallmatrix}$ given by



Then we have the standard projective resolution:

$$0 \rightarrow \begin{smallmatrix} 2 \\ 3 & 4 \end{smallmatrix} \oplus (3 \oplus 3) \oplus (4 \oplus 4) \rightarrow \begin{smallmatrix} 1 \\ 2 & 4 \\ 3 & 4 \end{smallmatrix} \oplus \left(\begin{smallmatrix} 2 \\ 3 & 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 & 4 \end{smallmatrix} \right) \oplus 3 \oplus 4 \rightarrow \begin{smallmatrix} 1 \\ 2 & 2 \\ 3 & 4 \end{smallmatrix} \rightarrow 0.$$

Again, this resolution is not minimal in the sense that one can eliminate three direct summands in each of the projective modules and still have a projective resolution:

$$0 \longrightarrow 3 \oplus 4 \longrightarrow \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \longrightarrow 0.$$

To make this notion of minimality more precise, we need the definitions of projective covers and injective envelopes.

Definition 2.4. Let $M \in \text{rep } Q$. A **projective cover** of M is a projective representation P together with a surjective morphism $g: P \rightarrow M$ with the property that, whenever $g': P' \rightarrow M$ is a surjective morphism with P' projective, then there exists a *surjective* morphism $h: P' \twoheadrightarrow P$ such that the diagram

$$\begin{array}{ccccc} & & P' & & \\ & \swarrow h & \downarrow g' & & \\ P & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

commutes, that is, $gh = g'$.

An **injective envelope** of M is an injective representation I together with an injective morphism $f: M \rightarrow I$ with the property that, whenever $f': M \rightarrow I'$ is an injective morphism into an injective representation I' , then there exists an *injective* morphism $h: I \hookrightarrow I'$ such that the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{f} & I \\ & & \downarrow f' & \swarrow h & \\ & & I' & & \end{array}$$

commutes, that is, $hf = f'$.

Definition 2.5. A projective resolution

$$\cdots \longrightarrow P_3 \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

is called **minimal** if $f_0: P_0 \rightarrow M$ is a projective cover and $f_i: P_i \rightarrow \ker f_{i-1}$ is a projective cover, for every $i > 0$.

An injective resolution

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \xrightarrow{f_3} I_3 \longrightarrow \cdots$$

is called **minimal** if $f_0: M \rightarrow I_0$ is an injective envelope and, for every $i > 0$, $f_i: \operatorname{coker} f_{i-1} \rightarrow I_i$ is an injective envelope.

The next two propositions show that projective covers are unique up to isomorphism.

Proposition 2.17. *Let $g: P \rightarrow M$ be a projective cover of M and let $g': P' \rightarrow M$ be a surjective morphism with P' projective. Then P is isomorphic to a direct summand of P' .*

Proof. From the definition of projective covers, we see that there exists a surjective morphism $h: P' \rightarrow P$. This morphism gives rise to an exact sequence:

$$0 \longrightarrow \ker h \longrightarrow P' \xrightarrow{h} P \longrightarrow 0.$$

Since P is projective, Corollary 2.4 implies that this sequence splits, and then the result follows from Proposition 1.8. \square

Proposition 2.18. *Let $g: P \rightarrow M$ and $g': P' \rightarrow M$ be projective covers of M . Then P is isomorphic to P' .*

Proof. From Proposition 2.17, we conclude that P is isomorphic to a direct summand of P' , and P' is isomorphic to a direct summand of P . Thus $P \cong P'$. \square

Remark 2.19. The dual statements to the propositions 2.17 and 2.18 about injective envelopes hold too. We leave the statements and their proofs as an exercise.

We introduce now the concept of a free representation. The prototype of a free representation is the direct sum of the indecomposable projective representations, and in general, free representations are direct sums of this prototype.

Definition 2.6. Let $A = \bigoplus_{i \in Q_0} P(i)$. A representation $F \in \operatorname{rep} Q$ is called **free** if $F \cong A \oplus \cdots \oplus A$.

Proposition 2.20. *A representation $M \in \operatorname{rep} Q$ is projective if and only if there exists a free representation $F \in \operatorname{rep} Q$ such that M is isomorphic to a direct summand of F .*

Proof.

- (\Leftarrow) By the Krull–Schmidt Theorem 1.2 and Proposition 2.8, every direct summand of F is a direct sum of $P(i)$ ’s, hence projective, by Proposition 2.7.
- (\Rightarrow) Suppose that M is projective and $\underline{\dim} M = (d_i)_{i \in Q_0}$. The standard projective resolution of M gives a surjective morphism $g: \bigoplus d_i P(i) \rightarrow M$. Thus there is a short exact sequence:

$$0 \longrightarrow \ker g \longrightarrow \bigoplus d_i P(i) \xrightarrow{g} M \longrightarrow 0$$

Since M is projective, this sequence splits, and therefore M is isomorphic to a direct summand $\bigoplus d_i P(i)$. \square

Corollary 2.21. *Any projective representation $P \in \text{rep } Q$ is a direct sum of $P(i)$ ’s, that is,*

$$P \cong P(i_1) \oplus \cdots \oplus P(i_t),$$

with i_1, \dots, i_t not necessarily distinct.

Proof. This follows directly from Proposition 2.20 \square

Our next goal is to show that, in $\text{rep } Q$, subrepresentations of projective representations are projective. We start by introducing a particular subrepresentation of $P(i)$.

Definition 2.7. Let $P(i) = (P(i)_j, \varphi_\alpha)$ be the projective representation at vertex i . The **radical** of $P(i)$ is the representation $\text{rad } P(i) = (R_j, \varphi'_\alpha)$ defined by

$$R_i = 0, \quad R_j = P(i)_j \text{ if } i \neq j, \quad \text{and} \quad \varphi'_\alpha = \begin{cases} 0 & \text{if } s(\alpha) = i \\ \varphi_\alpha & \text{otherwise.} \end{cases}$$

The next lemma shows that the radical of $P(i)$ is the maximal proper subrepresentation of $P(i)$.

Lemma 2.22. *Any proper subrepresentation of $P(i)$ is contained in $\text{rad } P(i)$.*

Proof. Suppose $f: M \hookrightarrow P(i)$ is an injective morphism of representations. Let $M = (M_i, \psi_\alpha)$ and $P(i) = (P(i)_j, \varphi_\alpha)$. It is clear that if $M_i = 0$, we have that $f(M_i) \subset \text{rad } P(i)$, so let us suppose that $M_i \neq 0$. We will show that this implies that the morphism f is an isomorphism. Since $P(i)_i \cong k$, it follows that $M_i \cong k$, and there is an element $m_i \in M_i$ such that $f_i(m_i) = e_i$. Now let j be any vertex, and let c be a path from i to j . Then

$$c = \varphi_c(e_i) = \varphi_c(f_i(m_i)) = f_j(\psi_c(m_i)) \in \text{im } f_j,$$

where the first identity is shown in Remark 2.1, and the third identity holds because f is a morphism of representations. Thus we see that the arbitrary element c of the basis of $P(i)_j$ lies in the image of f_j , which implies that f is surjective, hence an isomorphism, and so M is not a proper subrepresentation of $P(i)$. \square

Lemma 2.23. *If $P(i)$ is simple, then $\text{rad } P(i) = 0$. If $P(i)$ is not simple, then the radical of $P(i)$ is projective.*

Proof. We will show that $\text{rad } P(i)$ is isomorphic to the projective representation $P = \bigoplus_{\alpha: s(\alpha)=i} P(t(\alpha))$. If $i \neq j$, then $(\text{rad } P(i))_j = P(i)_j$ has as a basis the set of paths from i to j . Define a morphism $f = (f_j)_{j \in Q_0} : \text{rad } P(i) \rightarrow P$ on this basis by

$$f_j(i|\alpha, \beta_1, \dots, \beta_s|j) = (t(\alpha)|\beta_1, \dots, \beta_s|j).$$

Then f_j sends the basis of $(\text{rad } P(i))_j$ to a basis of P_j , for each $j \in Q_0$, and thus f is an isomorphism. \square

Theorem 2.24. *Subrepresentations of projective representations in $\text{rep } Q$ are projective.*

Remark 2.25. The subrepresentation inherits the projectivity. Categories with this property are called **hereditary**.

Proof. Suppose that P is a projective representation with dimension vector $(d_i)_{i \in Q_0}$. We will prove the theorem by induction on $d = \sum_{i \in Q_0} d_i$, the dimension of P .

If $d = 1$, then P is simple and there is nothing to prove. So suppose that $d > 1$. Let M be a subrepresentation of P and let $u : M \rightarrow P$ be the inclusion morphism. By Corollary 2.21, we have $P \cong P(i_1) \oplus \dots \oplus P(i_t)$ for some vertices i_1, \dots, i_t , and thus the inclusion u is of the form

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}$$

with $\text{im } u_j \subset P(i_j)$. It follows that $M \cong \text{im } u_1 \oplus \dots \oplus \text{im } u_t$, and, by Proposition 2.7, it suffices to show that $\text{im } u_j$ is projective for each j . This is obvious in the case where $\text{im } u_j = P(i_j)$, so let us suppose that $\text{im } u_j$ is a proper subrepresentation of $P(i_j)$. Then $\text{im } u_j$ is a subrepresentation of $\text{rad } P(i_j)$, and $\text{rad } P(i_j)$ is projective, by Lemma 2.23. Moreover, the dimension of $\text{rad } P(i_j)$ is strictly smaller than d , and, by induction, we conclude that $\text{im } u_j$ is projective, which completes the proof. \square

As a consequence of Theorem 2.24, we obtain the following result on morphisms into projective modules:

Corollary 2.26. *Let $f: M \rightarrow P$ be a nonzero morphism from an indecomposable representation M to a projective representation P . Then M is projective, and f is injective.*

Proof. Since the image of f is a subrepresentation of P , it is projective, by Theorem 2.24. Therefore, the short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \operatorname{im} f \longrightarrow 0$$

splits, and then Proposition 1.8 implies that $\operatorname{im} f$ is isomorphic to a direct summand of M . But M is indecomposable, so $M \cong \operatorname{im} f$ is projective and $\ker f = 0$. \square

Corollary 2.26 shows that when we construct the Auslander–Reiten quiver of \mathcal{Q} , we must start with the projective representations and that the projective representations are partially ordered by inclusion.

2.3 Auslander–Reiten Translation

In this section, we will define the Auslander–Reiten translation, which is fundamental for the Auslander–Reiten theory and Auslander–Reiten quivers. We consider at this point only the Auslander–Reiten translation in the category of quiver representations. Later, we will also consider the more general situation of bound quiver representations and modules.

We start with another notion from category theory.

Categories 5 *If \mathcal{C}, \mathcal{D} are two categories. We say that two functors $F_1, F_2: \mathcal{C} \rightarrow \mathcal{D}$ are functorially isomorphic, and we write $F_1 \cong F_2$, if for every object $M \in \mathcal{C}$, there exists an isomorphism $\eta_M: F_1(M) \rightarrow F_2(M) \in \mathcal{D}$ such that, for every morphism $f: M \rightarrow N$ in \mathcal{C} , the following diagram commutes:*

$$\begin{array}{ccc} F_1(M) & \xrightarrow{F_1(f)} & F_1(N) \\ \eta_M \downarrow & & \downarrow \eta_N \\ F_2(M) & \xrightarrow{F_2(f)} & F_2(N) \end{array}$$

A covariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called an **equivalence of categories** if there exists a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$. The functor G is called a quasi-inverse functor for F .

A contravariant functor F that has a (contravariant) quasi-inverse is called a **duality**.

2.3.1 Duality

Let Q be a quiver without oriented cycles, and let Q^{op} be the quiver obtained from Q by reversing each arrow. Thus $Q_0^{\text{op}} = Q_0$ and $Q_1^{\text{op}} = \{\alpha^{\text{op}} \mid \alpha \in Q_1\}$ with $s(\alpha^{\text{op}}) = t(\alpha)$ and $t(\alpha^{\text{op}}) = s(\alpha)$.

In this section, we need to work with projective and injective representations of both Q and Q^{op} . To distinguish between these, we will often use the notation $P_Q(i)$, $I_Q(i)$ for the representations of Q and $P_{Q^{\text{op}}}(i)$, $I_{Q^{\text{op}}}(i)$ for the representations of Q^{op} .

The **duality**

$$D = \text{Hom}_k(-, k) : \text{rep } Q \longrightarrow \text{rep } Q^{\text{op}}$$

is the contravariant functor defined as follows:

- On objects $M = (M_i, \varphi_\alpha)$, we have

$$DM = (DM_i, D\varphi_{\alpha^{\text{op}}})_{i \in Q_0, \alpha \in Q_1},$$

where DM_i is the dual vector space of the vector space M_i , and thus $DM_i = \text{Hom}_k(M_i, k)$ is the space of linear maps $M_i \rightarrow k$; and if α is an arrow in Q then $D\varphi_{\alpha^{\text{op}}}$ is the pullback of φ_α , and thus

$$\begin{aligned} D\varphi_{\alpha^{\text{op}}} : DM_{t(\alpha)} &\longrightarrow DM_{s(\alpha)} \\ u &\longmapsto u \circ \varphi_\alpha. \end{aligned}$$

- On morphisms $f : M \rightarrow N$ in $\text{rep } Q$, we have $Df : DN \rightarrow DM$ in $\text{rep } Q^{\text{op}}$ defined by $Df(u) = u \circ f$:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow u \circ f & \downarrow u \\ & & k \end{array}.$$

If we compose the duality of Q with the duality of Q^{op} , we get the identity functor $1_{\text{rep } Q}$; thus the quasi-inverse of D_Q is $D_{Q^{\text{op}}}$.

Let $\text{proj } Q$ be the category of projective representations of Q and let $\text{inj } Q$ be category of injective representations of Q . Thus the objects in $\text{proj } Q$ are the projective representations of Q , and the morphisms are the morphisms between projective representations.

Proposition 2.27. *We have $D(P_Q(i)) = I_{Q^{\text{op}}}(i)$, for all vertices $i \in Q_0$, in particular, the duality restricts to a duality $\text{proj } Q \rightarrow \text{inj } Q^{\text{op}}$.*

Proof. Exercise 2.10. □

Example 2.7. Let Q be the quiver $1 \longrightarrow 2 \longleftarrow 3$ as in Example 1.14. The indecomposable representations of the subcategory $\text{proj } Q$ are the three projective representations:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & & 2 \end{array}.$$

The quiver Q^{op} is $1 \longleftarrow 2 \longrightarrow 3$, and the indecomposable representations of the subcategory $\text{inj } Q^{\text{op}}$ are

$$\begin{array}{ccc} 2 & 2 & 2 \\ 1 & & 3 \end{array}.$$

2.3.2 Nakayama Functor

Let A be the free representation given as the direct sum of the indecomposable projective representations of Q , that is, $A = \bigoplus_{j \in Q_0} P(j)$.

Consider the contravariant functor $\text{Hom}(-, A)$. We know already from Sect. 1.4 that the Hom functors map representations of Q to vector spaces, and thus $\text{Hom}(X, Y)$ is a vector space for every pair of representations X, Y . But now, instead of the arbitrary representation Y , we use the special free representation A , and, in this case, we can give $\text{Hom}(X, A)$ the structure of a representation $(M_i, \varphi_{\alpha^{\text{op}}})$ of the opposite quiver Q^{op} as follows: Define the vector space at vertex i as $M_i = \text{Hom}(X, P(i))$ for every $i \in Q_0$, and for an arrow α from i to j in Q , define a linear map $\varphi_{\alpha^{\text{op}}} : \text{Hom}(X, P(j)) \rightarrow \text{Hom}(X, P(i))$ as $\varphi_{\alpha^{\text{op}}}(f) = \alpha \circ f$; thus we have the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & P(j) \\ & \searrow \varphi_{\alpha^{\text{op}}}(f) & \downarrow \alpha \\ & & P(i) \end{array}$$

where we use the fact that α , being a path from i to j , gives a morphism from $P(j)$ to $P(i)$, by Corollary 2.12. Thus $\text{Hom}(X, A)$ is a representation of Q^{op} .

To show that $\text{Hom}(-, A)$ is a functor from $\text{rep } Q$ to $\text{rep } Q^{\text{op}}$, we must check that the image under $\text{Hom}(-, A)$ of any morphism $g : X \rightarrow X'$ of representations of Q is a morphism of representations of Q^{op} . This means we must check for every arrow $i \xrightarrow{\alpha} j$ in Q that the following diagram commutes:

$$\begin{array}{ccc}
 \text{Hom}(X', P(j)) & \xrightarrow{\varphi'_{\alpha^{\text{op}}}} & \text{Hom}(X', P(i)) \\
 \downarrow g^* = \text{Hom}(g, P(j)) & & \downarrow g^* = \text{Hom}(g, P(i)) \\
 \text{Hom}(X, P(j)) & \xrightarrow{\varphi_{\alpha^{\text{op}}}} & \text{Hom}(X, P(i)).
 \end{array}$$

To do so, let $f \in \text{Hom}(X', P(j))$, then $g^* \varphi'_{\alpha^{\text{op}}}(f) = g^*(\alpha \circ f) = (\alpha \circ f) \circ g$, whereas $\varphi_{\alpha^{\text{op}}} \circ g^*(f) = \varphi_{\alpha^{\text{op}}}(f \circ g) = \alpha \circ (f \circ g) = (\alpha \circ f) \circ g$, so the diagram commutes. We have shown the following:

Proposition 2.28. *$\text{Hom}(-, A)$ is a functor from $\text{rep } Q$ to $\text{rep } Q^{\text{op}}$.*

Composing the two contravariant functors D and $\text{Hom}(-, A)$, we get the following important covariant functor:

Definition 2.8. The functor $\nu = D\text{Hom}(-, A) : \text{rep } Q \rightarrow \text{rep } Q$ is called the **Nakayama functor**:

$$\begin{array}{ccccc}
 \text{rep } Q & \xrightarrow{\text{Hom}(-, A)} & \text{rep } Q^{\text{op}} & \xrightarrow{D} & \text{rep } Q \\
 & \searrow & & \nearrow & \\
 & & \nu & &
 \end{array}$$

Corollary 2.26 implies that the functor $\text{Hom}(-, A)$ is zero on all representations which have no projective direct summands. Therefore we must study the behavior of the Nakayama functor when applied to an indecomposable projective representation, say, $P_Q(i)$. Let us first consider the functor $\text{Hom}(-, A)$ only. Let $M = \text{Hom}(P_Q(i), A)$, and use the habitual notation $M = (M_j, \varphi_{\alpha^{\text{op}}})_{j \in Q_0, \alpha \in Q_1}$ to denote the representation. Then the vector space M_j is $\text{Hom}(P_Q(i), P_Q(j))$; thus, according to Corollary 2.14, the space M_j has a basis consisting of all paths from j to i in Q . In terms of Q^{op} this can be rephrased as M_j has a basis consisting of all paths from i to j in Q^{op} .

Moreover, for any arrow $h \xrightarrow{\alpha} j$ in Q , the linear map $\varphi_{\alpha^{\text{op}}} : M_j \rightarrow M_h$ maps the basis element c , which is a path from i to j in Q^{op} , to the basis element $c\alpha^{\text{op}}$, which is a path from i to h in Q^{op} . This shows that $M = \text{Hom}(P_Q(i), A)$ is the indecomposable projective Q^{op} -representation $P_{Q^{\text{op}}}(i)$ at vertex i .

Thus the restriction of $\text{Hom}(-, A)$ to the subcategory $\text{proj } Q$ gives a duality of categories $\text{proj } Q \rightarrow \text{proj } Q^{\text{op}}$, whose quasi-inverse is given by $\text{Hom}_{Q^{\text{op}}}(-, A^{\text{op}})$, where A^{op} is the sum of all indecomposable projective Q^{op} -representations.

To obtain the Nakayama functor ν , we must now form the composition with the duality D . Note that $DA^{\text{op}} = \bigoplus_{i \in Q_0} I_Q(i)$.

Proposition 2.29. *The restriction of ν to $\text{proj } Q$ is an equivalence of categories $\text{proj } Q \rightarrow \text{inj } Q$ whose quasi-inverse is given by*

$$\nu^{-1} = \text{Hom}(DA^{\text{op}}, -): \text{inj } Q \longrightarrow \text{proj } Q.$$

Moreover, for any vertex i ,

$$\nu P(i) = I(i),$$

and if c is a path from i to j , and $f_c \in \text{Hom}(P(j), P(i))$ is the corresponding morphism, then

$$\nu f_c : I(j) \rightarrow I(i)$$

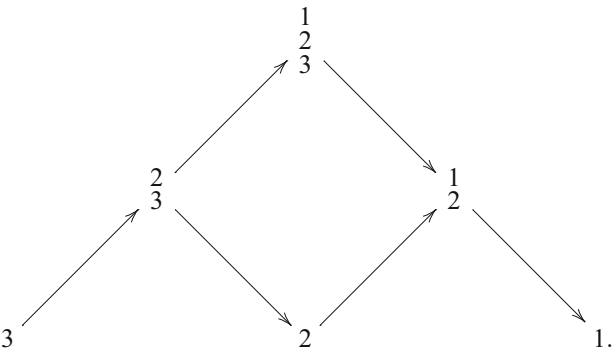
is the morphism given by the cancellation of the path c .

Proof. The functor ν is an equivalence because it is the composition of the two dualities D and $\text{Hom}(-, A)$. Its quasi-inverse ν^{-1} is the composition of the quasi-inverses of D and $\text{Hom}(-, A)$, thus $\nu^{-1} = \text{Hom}_{Q^{\text{op}}}(-, A^{\text{op}}) \circ D$. Note that since $\text{Hom}_{Q^{\text{op}}}(DX, DY) \cong \text{Hom}_Q(Y, X)$ for all $X, Y \in \text{rep } Q$, we have in particular that $\text{Hom}_{Q^{\text{op}}}(DX, A^{\text{op}}) \cong \text{Hom}_Q(DA^{\text{op}}, X)$, whence $\nu^{-1} = \text{Hom}(DA, -)$. Finally,

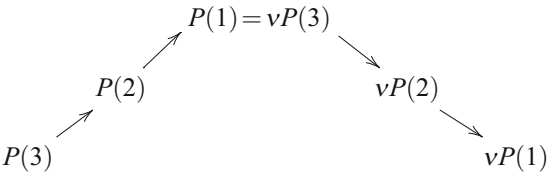
$$\nu P_Q(i) = D\text{Hom}(P_Q(i), A) = D(P_{Q^{\text{op}}}(i)) = I_Q(i).$$

To show the last statement, let c be a path from i to j , and let $f_c: P_Q(j) \rightarrow P_Q(i)$ be defined by $f_c(x) = cx$ as in the proposition. Let f_c^* be the image of f_c under the functor $\text{Hom}(-, A)$. Thus $f_c^*: \text{Hom}(P_Q(i), A) \rightarrow \text{Hom}(P_Q(j), A)$ maps a morphism g to the pullback $g \circ f_c^*$. Now using $\text{Hom}(P_Q(x), A) \cong P_{Q^{\text{op}}}(x)$, we see that $f_c^*: P_{Q^{\text{op}}}(i) \rightarrow P_{Q^{\text{op}}}(j)$ is given by $f_c^*(y) = c^{\text{op}}y$, where c^{op} denotes the opposite of the path c in Q^{op} . Finally, $\nu f_c = Df_c^*$ is the map sending $D(c^{\text{op}}y)$ to $D(y)$; thus the result follows from $D(c^{\text{op}}y) = D(y)c$. \square

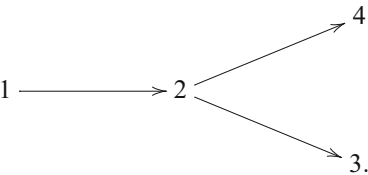
Example 2.8. Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. Its Auslander–Reiten quiver is



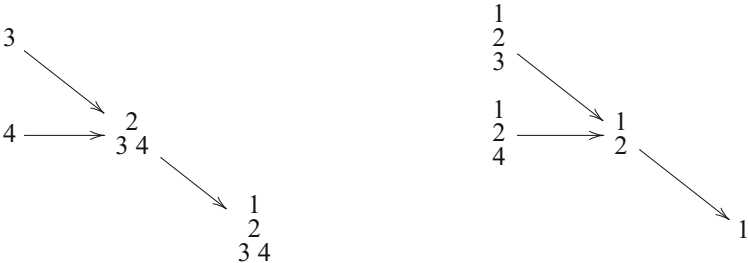
Note that the Nakayama functor ν sends $\text{proj } Q$ to $\text{inj } Q$ as follows:



Example 2.9. Let Q be the quiver



Then $\text{proj } Q$ and $\text{inj } Q$ are as follows:



Before we can state the next property of the Nakayama functor, we need to introduce the notion of exactness for functors.

Categories 6 *Let \mathcal{C} and \mathcal{D} be abelian categories. A (covariant or contravariant) functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called exact if it maps exact sequences in \mathcal{C} to exact sequences in \mathcal{D} . For example, every equivalence or duality of abelian categories is exact. Many nice functors, the Hom-functors for example, are not exact but have the weaker property of being left exact or right exact, which we define below. The definition of these notions is different for covariant and contravariant functors.*

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a covariant functor. F is called *left exact* if for any exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$$

the sequence

$$0 \longrightarrow F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N)$$

is exact.

F is called *right exact* if for any exact sequence

$$L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

the sequence

$$F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \longrightarrow 0$$

is exact.

Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a contravariant functor. G is called *left exact* if for any exact sequence

$$L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

the sequence

$$0 \longrightarrow G(N) \xrightarrow{G(g)} G(M) \xrightarrow{G(f)} G(L)$$

is exact.

G is called *right exact* if for any exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$$

the sequence

$$G(N) \xrightarrow{G(g)} G(M) \xrightarrow{G(f)} G(L) \longrightarrow 0$$

is exact.

We have shown in Sect. 1.4 that the Hom functors $\text{Hom}(X, -)$ and $\text{Hom}(-, X)$ are left exact. Since the Nakayama functor is the composition of the left exact functor $\text{Hom}(-, A)$ and the exact contravariant functor D , we get the following proposition:

Proposition 2.30. *The Nakayama functor ν is right exact.*

Example 2.10. In the setting of Example 2.8 there is a short exact sequence

$$0 \longrightarrow 3 \xrightarrow{f} \frac{1}{2} \xrightarrow{g} \frac{1}{2} \longrightarrow 0,$$

and applying ν yields the exact sequence

$$\frac{1}{2} \xrightarrow{\nu f} 1 \xrightarrow{\nu g} 0 \longrightarrow 0.$$

This confirms that ν is right exact. Since the morphism νf is clearly not injective, this also shows that ν is not exact.

2.3.3 The Auslander–Reiten Translations τ, τ^{-1}

Let Q be a quiver without oriented cycles, and let M be an indecomposable representation of Q .

Definition 2.9. Let

$$0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be a minimal projective resolution. Applying the Nakayama functor, we get an exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \longrightarrow 0,$$

where $\tau M = \ker \nu p_1$ is called the Auslander–Reiten translate of M and τ the **Auslander–Reiten translation**.

Let

$$0 \longrightarrow M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_i \longrightarrow 0$$

be a minimal injective resolution. Applying the inverse Nakayama functor, we get an exact sequence

$$0 \longrightarrow \nu^{-1}M \xrightarrow{\nu^{-1}i_0} \nu^{-1}I_0 \xrightarrow{\nu^{-1}i_1} \nu^{-1}I_i \longrightarrow \tau^{-1}M \longrightarrow 0,$$

where $\tau^{-1}M = \operatorname{coker} \nu^{-1}i_1$ is called the inverse Auslander–Reiten translate of M and τ^{-1} the inverse Auslander–Reiten translation.

Example 2.11. Continuing Example 2.10, we compute $\tau \frac{1}{2}$. We have already constructed the minimal projective resolution and applied the Nakayama functor to it. It only remains to compute the kernel of νf ; thus

$$\tau \frac{1}{2} = \ker \nu f = \frac{2}{3}.$$

Remark 2.31. The Auslander–Reiten translation has been introduced by Auslander and Reiten in [10].

2.4 Extensions and Ext

In this section, Q always denotes a quiver without oriented cycles. We give here a short account on the Ext^1 -groups; for further information we refer to [53, Sect. 7.2].

Let $M \in \operatorname{rep} Q$ and take a projective resolution

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

of M in $\operatorname{rep} Q$. Thus P_0 and P_1 are projective representations and the above sequence is exact. Let N be any representation in $\operatorname{rep} Q$. Then we can apply the functor $\operatorname{Hom}(-, N)$ to this projective resolution, and as a result we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}(M, N) \xrightarrow{g^*} \operatorname{Hom}(P_0, N) \xrightarrow{f^*} \operatorname{Hom}(P_1, N) \longrightarrow \operatorname{Ext}^1(M, N) \longrightarrow 0,$$

where $\operatorname{Ext}^1(M, N) = \operatorname{coker} f^*$ is called the **first group of extensions** of M and N .

Remark 2.32. In arbitrary categories, projective resolutions do not necessarily stop after two steps; in fact, they might not even stop at all. Thus a projective resolution in a general category is of the form

$$\cdots \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

and applying $\text{Hom}(-, N)$ yields a so-called *cochain complex*

$$0 \longrightarrow \text{Hom}(M, N) \xrightarrow{f_0^*} \text{Hom}(P_0, N) \xrightarrow{f_1^*} \cdots \xrightarrow{f_n^*} \text{Hom}(P_n, N) \longrightarrow \cdots,$$

which means that $f_i^* f_{i-1}^* = 0$, for all i . One then defines the i th extension group $\text{Ext}^i(M, N)$ for $i \geq 1$ to be the i th cohomology group of this complex, that is,

$$\text{Ext}^i(M, N) = \ker f_{i+1}^* / \text{im } f_i^*.$$

One can show that this definition does not depend on the choice of the projective resolution; see, for example, [53, Proposition 6.4].

In the category $\text{rep } Q$, all the Ext^i -groups, with $i \geq 2$, vanish, because the minimal projective resolutions are of the form

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0.$$

On the other hand, the Ext^1 -groups provide very interesting information.

Our next goal is to show that the vector space $\text{Ext}^1(M, N)$ is isomorphic to the vector space of extensions of M by N .

Definition 2.10. An **extension** ζ of M by N is a short exact sequence $0 \rightarrow N \xrightarrow{f} E \xrightarrow{g} M \rightarrow 0$. Two extensions ζ and ζ' are called *equivalent* if there is a commutative diagram:

$$\begin{array}{ccccccc} \zeta : 0 & \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M \longrightarrow 0 \\ & & \downarrow = & & \downarrow \cong & & \downarrow = \\ \zeta' : 0 & \longrightarrow & N & \xrightarrow{f'} & E' & \xrightarrow{g'} & M \longrightarrow 0 \end{array}$$

Example 2.12. Let Q be the quiver

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2,$$

let $N = S(2)$, $M = S(1)$ be the two simple modules and let

$$E = k \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{0} \end{array} k \quad \text{and} \quad E' = k \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} k.$$

Then the short exact sequences

$$\begin{aligned} \zeta : 0 &\longrightarrow S(2) \xrightarrow{f} E \xrightarrow{g} S(1) \longrightarrow 0 \\ \zeta' : 0 &\longrightarrow S(2) \xrightarrow{f'} E' \xrightarrow{g'} S(1) \longrightarrow 0 \end{aligned}$$

are not equivalent, because E and E' are not isomorphic.

An extension is *split* if the short exact sequence is split, that is, if the extension is equivalent to the short exact sequence:

$$0 \longrightarrow N \longrightarrow N \oplus M \longrightarrow M \longrightarrow 0.$$

Given two extensions ζ and ζ' of M by N , we define their sum $\zeta + \zeta'$ as follows: Let $E'' = \{(x, x') \in E \times E' \mid g(x) = g'(x')\}$ be the so-called *pull back* of g and g' , and define F to be the quotient of E'' by the subspace $\{(f(n), -f'(n)) \in E \oplus E' \mid n \in N\}$, compare with Exercises 1.8 and 1.9 in Chap. 1. Then $\zeta + \zeta'$ is

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0.$$

The set of equivalence classes $\mathcal{E}(M, N)$ of extensions of M by N together with the sum of extensions is an abelian group, and the class of the split extension is the zero element of that group.

There is an isomorphism of groups $\mathcal{E}(M, N) \rightarrow \text{Ext}^1(M, N)$ which is defined as follows. Let $\zeta : 0 \rightarrow N \xrightarrow{u'} E \xrightarrow{v'} M \rightarrow 0$ be a representative of a class in $\mathcal{E}(M, N)$, and let $0 \rightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \rightarrow 0$ be a projective resolution. Then since P_0 is projective, it follows that there exists a morphism $f' \in \text{Hom}(P_0, E)$ such that $g = v' f'$. Now since ζ is exact, the universal property of the kernel implies that there exists also a morphism $u \in \text{Hom}(P_1, N)$ such that the following diagram commutes:

$$\begin{array}{ccccccc} P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ \downarrow u & & \downarrow f' & & \downarrow = & & \\ \zeta : 0 & \longrightarrow & N & \xrightarrow{u'} & E & \xrightarrow{v'} & M \longrightarrow 0. \end{array}$$

Recall that $\text{Ext}^1(M, N)$ is the cokernel of f^* , which is the quotient $\text{Hom}(P_1, N)/\text{im } f^*$. The isomorphism $\mathcal{E}(M, N) \rightarrow \text{Ext}^1(M, N)$ is sending the class of ζ to the class of u .

Example 2.13. Let us compute the sum of the two short exact sequences ζ and ζ' in Example 2.12. Thus Q is the quiver

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2,$$

$N = S(2)$, $M = S(1)$ and

$$E = k \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{0} \end{array} k \quad \text{and} \quad E' = k \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} k.$$

Using our notation $E = (E_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $E' = (E'_i, \varphi'_\alpha)_{i \in Q_0, \alpha \in Q_1}$, we have

$$\begin{aligned} E_1 &\cong E_2 \cong k & \varphi_\alpha &= 1 & \varphi_\beta &= 0 \\ E'_1 &\cong E'_2 \cong k & \varphi'_\alpha &= 0 & \varphi'_\beta &= 1 \end{aligned}$$

Let us denote the elements of E as pairs $(e_1, e_2) \in E_1 \oplus E_2$ and those of E' as $(e'_1, e'_2) \in E'_1 \oplus E'_2$.

To compute the sum $\zeta + \zeta'$, we first need to compute the pull back E'' . By definition

$$E'' = \{((e_1, e_2), (e'_1, e'_2)) \in E \times E' \mid g(e_1, e_2) = g'(e'_1, e'_2)\}.$$

Since g and g' are both projections on the first component, we have

$$E'' = \{((e_1, e_2), (e_1, e'_2)) \in E \times E'\}.$$

We want to write E'' as a representation $E'' = (E''_i, \varphi''_\alpha)_{i \in Q_0, \alpha \in Q_1}$. Our computation above shows that $E''_1 \cong k$ and $E''_2 \cong k^2$. Let's compute φ''_α and φ''_β . We have

$$\begin{aligned} \varphi''_\alpha((e_1, e_2), (e_1, e'_2)) &= (\varphi_\alpha(e_1, e_2), \varphi'_\alpha(e_1, e'_2)) = ((0, e_1), (0, 0)) \\ \varphi''_\beta((e_1, e_2), (e_1, e'_2)) &= (\varphi_\beta(e_1, e_2), \varphi'_\beta(e_1, e'_2)) = ((0, 0), (0, e_1)) \end{aligned}$$

This shows that

$$E'' = k \begin{array}{c} \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \\ \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \end{array} k^2.$$

Now we compute F . By definition and using the fact that both f and f' are inclusions in the second component, we have

$$F = E'' / \{((0, n), (0, -n)) \mid n \in k\}.$$

Writing $F = (F_i, \psi_\alpha)$, we see that $F_1 \cong F_2 \cong k$. Moreover, if $((e_1, e_2), (e_1, e'_2)) \in E''$ and $\overline{((e_1, e_2), (e_1, e'_2))}$ denotes its class in F , then

$$\psi_\alpha(\overline{((e_1, e_2), (e_1, e'_2))}) = \overline{\varphi''_\alpha((e_1, e_2), (e_1, e'_2))} = \overline{((0, e_1), (0, 0))}$$

and

$$\psi_\beta(\overline{((e_1, e_2), (e_1, e'_2))}) = \overline{\varphi''_\beta((e_1, e_2), (e_1, e'_2))} = \overline{((0, 0), (0, e_1))}.$$

In particular, $\psi_\alpha(\overline{((e_1, e_2), (e_1, e'_2))}) = -\psi_\beta(\overline{((e_1, e_2), (e_1, e'_2))})$. This shows that

$$F \cong k \begin{smallmatrix} \xrightarrow{1} \\ \xrightarrow{-1} \end{smallmatrix} k.$$

Finally, we see that the sum $\zeta + \zeta'$ is the short exact sequence

$$0 \longrightarrow S(2) \xrightarrow{f''} F \xrightarrow{g''} S(1) \longrightarrow 0$$

with f'' the inclusion in the second component and g'' the projection on the first component.

Problems

Exercises for Chap. 2

2.1. Let Q be the quiver $1 \longrightarrow 3 \longrightarrow 4$. Prove that

$$\begin{array}{ccccc}
 1 & \longrightarrow & 3 & \longrightarrow & 4 \\
 & \searrow & \nearrow & & \\
 & 2 & & &
 \end{array}$$

$$\begin{array}{ccccc}
 P(1) \cong k & \xrightarrow{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} & k^2 & \xrightarrow{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} & k^2 \\
 & \searrow z & \nearrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} & & \\
 & k & & &
 \end{array}$$

if and only if

1. all four maps have maximal rank, that is, $ad - bc \neq 0$, $(x_1, x_2) \neq (0, 0)$, $(y_1, y_2) \neq (0, 0)$, $z \neq 0$ and
2. the vectors $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are linearly independent.

2.2. Compute the indecomposable projective representations $P(i)$ and the indecomposable injective representations $I(i)$ for the following quivers:

1. $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longleftarrow 5$.
 $6 \longrightarrow 3 \longrightarrow 7$
2. $1 \rightleftarrows 2 \rightleftarrows 3$.
3. $1 \xrightarrow{\quad} 2 \xrightarrow{\quad} 3$.
 $2 \downarrow 4$
 $3 \searrow 4$

- 2.3.** 1. Compute a projective resolution for each simple representation $S(i)$ for each of the quivers in (2.2).
 2. Compute the dimension vector of $\tau S(i)$ for each simple representation $S(i)$ of the quivers in (2.2).

2.4. Prove Proposition 2.3

2.5. Prove Proposition 2.5

2.6. Show that for each $i \in Q_0$, the sequence

$$0 \longrightarrow \text{rad} P(i) \longrightarrow P(i) \longrightarrow S(i) \longrightarrow 0$$

is a minimal projective resolution.

2.7. Let $M = (M_i, \psi_\alpha)$ be a representation of Q . Prove that for any vertex i in Q , there is an isomorphism of vector spaces:

$$\text{Hom}(M, I(i)) \cong M_i.$$

2.8. Let i and j be vertices in the quiver Q . Prove that the vector space $\text{Hom}(I(i), I(j))$ has a basis consisting of all paths from j to i in Q . In particular, $\text{End}(I(i)) \cong k$.

2.9. Prove that the representation $I(j)$ is a simple representation if and only if $\text{Hom}(I(j), I(i)) = 0$ for all $i \neq j$.

2.10. Prove Proposition 2.27.

2.11. Prove that P is projective if and only if $\text{Ext}^1(P, N) = 0$ for all representations N .

2.12. Let Q be the quiver

$$1 \begin{array}{c} \xleftarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2 ,$$

and let M_λ be the representation defined in Exercise 1.6 of Chap. 1.

1. Show that the short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & 1 & \longrightarrow & M_\lambda & \longrightarrow & 2 \longrightarrow 0, \\ 0 & \longrightarrow & 1 & \longrightarrow & M_\mu & \longrightarrow & 2 \longrightarrow 0, \end{array}$$

are not equivalent if $\lambda \neq \mu$.

2. Show that $\tau M_\lambda = M_\lambda$, for all λ .



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