Chapter 2
Simplicial (Co)homology

Simplicial homology was invented by Poincaré in 1899 [162] and its mod 2 version, presented in this chapter, was introduced in 1908 by Tietze [196]. It is the simplest homology theory to understand and, for finite complexes, it may be computed algorithmically. The mod 2 version permits rapid computations on easy but non-trivial examples, like spheres and surfaces (see Sect. 2.4).

Simplicial (co)homology is defined for a simplicial complex, but is an invariant of the homotopy type of its geometric realization (this result will be obtained in different ways using singular homology: see Sect. 3.6). The first section of this chapter introduces classical techniques of (abstract) simplicial complexes. Since simplicial homology was the only existing (co)homology theory until the 1930s, simplicial complexes played a predominant role in algebraic topology during the first third of the 12th century (see the Introduction of Sect. 5.1). Later developments of (co)homology theories, defined directly for topological spaces, made this combinatorial approach less crucial. However, simplicial complexes remain an efficient way to construct topological spaces, also largely used in computer science.

2.1 Simplicial Complexes

In this section we fix notations and recall some classical facts about (abstract) simplicial complexes. For more details, see [179, Chap. 3].

A *simplicial complex* K consists of

- a set $V(K)$, the set of *vertices* of K.
- a set $\mathcal{S}(K)$ of finite non-empty subsets of $V(K)$ which is closed under inclusion: if $\sigma \in \mathcal{S}(K)$ and $\tau \subset \sigma$, then $\tau \in \mathcal{S}(K)$. We require that $\{v\} \in \mathcal{S}(K)$ for all $v \in V(K)$.

An element σ of $\mathcal{S}(K)$ is called a *simplex* of K ("simplexes" and "simplices" are admitted as plural of "simplex"; we shall use "simplexes", in analogy with "complexes"). If $\sharp(\sigma) = m + 1$, we say that σ is of *dimension* m or that σ is an *m-simplex*. The set of m-simplexes of K is denoted by $\mathcal{S}_m(K)$. The set $\mathcal{S}_0(K)$ of 0-simplexes is

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in bijection with $V(K)$, and we usually identify $v \in V(K)$ with $\{v\} \in \mathcal{S}_0(K)$. We say that K is of *dimension* $\leq n$ if $\mathcal{S}_m(K) = \emptyset$ for $m > n$, and that K is of *dimension n* (or *n-dimensional*) if it is of dimension $\leq n$ but not of dimension $\leq n-1$. A simplicial complex of dimension ≤ 1 is called a *simplicial graph*. A simplicial complex K is called *finite* if $V(K)$ is a finite set.

If $\sigma \in \mathcal{S}(K)$ and $\tau \subset \sigma$, we say that τ is a *face* of σ. As $\mathcal{S}(K)$ is closed under inclusion, it is determined by it subset $\mathcal{S}_{\max}(K)$ of *maximal* simplexes (if K is finite dimensional). A *subcomplex* L of K is a simplicial complex such that $V(L) \subset V(K)$ and $\mathcal{S}(L) \subset \mathcal{S}(K)$. If $S \subset \mathcal{S}(K)$ we denote by $\bar{S}$ the subcomplex generated by S, i.e. the smallest subcomplex of K such that $S \subset \mathcal{S}(\bar{S})$. The *$m$-skeleton* K^m of K is the subcomplex of K generated by the union of $\mathcal{S}_k(K)$ for $k \leq m$.

Let $\sigma \in \mathcal{S}(K)$. We denote by $\bar{\sigma}$ the subcomplex of K formed by σ and all its faces ($\overline{\{\sigma\}}$ in the above notation). The subcomplex $\dot{\sigma}$ of $\bar{\sigma}$ generated by the proper faces of σ is called the *boundary of* σ.

2.1.1 *Geometric realization.* The *geometric realization* $|K|$ of a simplicial complex K is, as a set, defined by

$$|K| := \{\mu : V(K) \to [0,1] \,\big|\, \textstyle\sum_{v \in V(K)} \mu(v) = 1 \text{ and } \mu^{-1}((0,1]) \in \mathcal{S}(K)\} \,.$$

We can thus see $|K|$ as the set of probability measures on $V(K)$ which are supported by the simplexes (this language is just used for comments and only in this section). There is a distance on $|K|$ defined by

$$d(\mu, \nu) = \sqrt{\sum_{v \in V(K)} [\mu(v) - \nu(v)]^2}$$

which defines the *metric topology* on $|K|$. The set $|K|$ with the metric topology is denoted by $|K|_d$. For instance, if $\sigma \in \mathcal{S}_m(K)$, then $|\bar{\sigma}|_d$ is isometric to the standard Euclidean simplex $\Delta^m = \{(x_0, \dots, x_m) \in \mathbb{R}^{m+1} \mid x_i \geq 0 \text{ and } \sum x_i = 1\}$.

However, a more used topology for $|K|$ is the *weak topology*, for which $A \subset |K|$ is closed if and only if $A \cap |\bar{\sigma}|_d$ is closed in $|\bar{\sigma}|_d$ for all $\sigma \in \mathcal{S}(K)$. The notation $|K|$ stands for the set $|K|$ endowed with the weak topology. A map f from $|K|$ to a topological space X is then continuous if and only if its restriction to $|\bar{\sigma}|_d$ is continuous for each $\sigma \in \mathcal{S}(K)$. In particular, the identity $|K| \to |K|_d$ is continuous, which implies that $|K|$ is Hausdorff. The weak and the metric topology coincide if and only if K is locally finite, that is each vertex is contained in a finite number of simplexes. When K is not locally finite, $|K|$ is not metrizable (see e.g. [179, Theorem 3.2.8]).

When a simplicial complex K is locally finite, has countably many vertices and is finite dimensional, it admits a *Euclidean realization*, i.e. an embedding of $|K|$ into some Euclidean space $\mathbb{R}^N$ which is piecewise affine. A map $f : |K| \to \mathbb{R}^N$ is *piecewise affine* if, for each $\sigma \in \mathcal{S}(K)$, the restriction of f to $|\bar{\sigma}|$ is an affine map. Thus, for each simplex σ, the image of $|\bar{\sigma}|$ is an affine simplex of $\mathbb{R}^N$. If $\dim K \leq n$, such a realization exists in $\mathbb{R}^{2n+1}$ (see e.g. [179, Theorem 3.3.9]).

If $\sigma \in \mathcal{S}(K)$ then $|\bar{\sigma}| \subset |K|$. We call $|\bar{\sigma}|$ the *geometric simplex* associated to σ. Its *boundary* is $|\dot{\sigma}|$. The space $|\sigma| - |\dot{\sigma}|$ is the *geometric open simplex* associated to σ. Observe that $|K|$ is the disjoint union of its geometric open simplexes.

There is a natural injection $i : V(K) \hookrightarrow |K|$ sending v to the Dirac measure with value 1 on v. We usually identify v with $i(v)$, seeing a simplex v as a point of $|K|$ (a geometric vertex). In this way, a point $\mu \in |K|$ may be expressed as a convex combination of (geometric) vertices:

$$\mu = \sum_{v \in V(K)} \mu(v) v \,. \tag{2.1.1}$$

2.1.2 Let K and L be simplicial complexes. Their *join* is the simplicial complex $K * L$ defined by

(1) $V(K * L) = V(K) \dot{\cup} V(L)$.
(2) $\mathcal{S}(K * L) = \mathcal{S}(K) \cup \mathcal{S}(L) \cup \{\sigma \cup \tau \mid \sigma \in \mathcal{S}(K) \text{ and } \tau \in \mathcal{S}(L)\}$.

Observe that, if $\sigma \in \mathcal{S}_r(K)$ and $\tau \in \mathcal{S}_s(L)$, then $\sigma \cup \tau \in \mathcal{S}_{r+s+1}(K * L)$. Also, $\overline{\sigma \cup \tau} = \bar{\sigma} * \bar{\tau}$ and $|K * L|$ the topological join of $|K|$ and $|L|$ (see p. 171).

2.1.3 *Stars, links, etc.* Let K be a simplicial complex and $\sigma \in \mathcal{S}(K)$. The *star* $\mathrm{St}(\sigma)$ *of* σ is the subcomplex of K generated by all the simplexes containing σ. The *link* $\mathrm{Lk}(\sigma)$ *of* σ is the subcomplex of K formed by the simplexes $\tau \in \mathcal{S}(K)$ such that $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma \in \mathcal{S}(K)$. Thus, $\mathrm{Lk}(\sigma)$ is a subcomplex of $\mathrm{St}(\sigma)$ and

$$\mathrm{St}(\sigma) = \bar{\sigma} * \mathrm{Lk}(\sigma) \,.$$

More generally, if L is a subcomplex of K, the *star* $\mathrm{St}(L)$ *of* L is the subcomplex of K generated by all the simplexes containing a simplex of L. The *link* $\mathrm{Lk}(L)$ *of* L is the subcomplex of K formed by the simplexes $\tau \in \mathcal{S}(\mathrm{St}(L)) - \mathcal{S}(L)$. One has $\mathrm{St}(L) = L * \mathrm{Lk}(L)$. The *open star* $\mathrm{Ost}(L)$ of L is the open neighbourhood of $|L|$ in $|K|$ defined by

$$\mathrm{Ost}(L) = \{\mu \in |K| \mid \mu(v) > 0 \text{ if } v \in V(L)\} \,.$$

This is the interior of $|\mathrm{St}(L)|$ in $|K|$.

2.1.4 *Simplicial maps.* Let K and L be two simplicial complexes. A *simplicial map* $f\colon K \to L$ is a map $f\colon V(K) \to V(L)$ such that $f(\sigma) \in \mathcal{S}(L)$ if $\sigma \in \mathcal{S}(K)$, i.e. the image of a simplex of K is a simplex of L. Simplicial complexes and simplicial maps form a category, the *simplicial category*, denoted by **Simp**.

A simplicial map $f\colon K \to L$ induces a continuous map $|f|\colon |K| \to |L|$ defined, for $w \in V(L)$, by

$$|f|(\mu)(w) = \sum_{v \in f^{-1}(w)} \mu(v) \,.$$

In other words, $|f|(\mu)$ is the pushforward of the probability measure μ on $|L|$. The geometric realization is thus a covariant functor from the simplicial category **Simp** to the topological category **Top** of topological spaces and continuous maps.

2.1.5 *Components.* Let K be a simplicial complex. We define an equivalence relation on $V(K)$ by saying that $v \sim v'$ if there exists $x_0, \dots, x_m \in V(K)$ with $x_0 = v$, $x_m = v'$ and $\{x_i, x_{i+1}\} \in \mathcal{S}(K)$. A maximal subcomplex L of K such that $V(L)$ is an equivalence class is called a *component* of K. The set of components of K is denoted by $\pi_0(K)$. As the vertices of a simplex are all equivalent, K is the disjoint union of its components and $\pi_0(K)$ is in bijection with $V(K)/\sim$. The relationship with $\pi_0(|K|)$, the set of (path)-components of the topological space $|K|$, is the following.

Lemma 2.1.6 *The natural injection $j\colon V(K) \to |K|$ descends to a bijection $\bar{j}$: $\pi_0(K) \xrightarrow{\approx} \pi_0(|K|)$.*

Proof The definition of the relation $\sim$ makes clear that j descends to a map $\bar{j}$: $\pi_0(K) \to \pi_0(|K|)|$. Any point of $|K|$ is joinable by a continuous path to some vertex $j(v)$. Hence, $\bar{j}$ is surjective. To check the injectivity of $\bar{j}$, let $v, v' \in V(K)$ with $\bar{j}(v) = \bar{j}(v')$. There exists then a continuous path $c\colon [0, 1] \to |K|$ with $c(0) = j(v)$ and $c(1) = j(v')$. Consider the open cover $\{\mathrm{Ost}(w) \mid w \in V(K)\}$ of $|K|$. By compactness of $[0, 1]$, there exists $n \in \mathbb{N}$ and vertices $v_0, \dots, v_{n-1} \in V(K)$ such that $c([k/n, (k+1)/n]) \subset \mathrm{Ost}(v_k)$ for all $k = 0, \dots, n-1$. As $c(0) = j(v)$ and $c(1) = j(v')$, one deduces that $v_0 = v$ and $v_{n-1} = v'$. For $0 < k \le n-1$, one has $c(k/n) \in \mathrm{Ost}(v_{k-1}) \cap \mathrm{Ost}(v_k)$. This implies that $\{v_{k-1}, v_k\} \in \mathcal{S}(K)$ for all $k = 1, \dots, n-1$, proving that $v \sim v'$. □

A simplicial complex is called *connected* if it is either empty or has one component. Note that $|K|$ is locally path-connected for any simplicial complex K. Indeed, any point has a neighborhood of the form $|\mathrm{St}(v)|$ for some vertex v, and $|\mathrm{St}(v)|$ path-connected. Therefore, $|K|$ is path-connected if and only if $|K|$ is connected. Using Lemma 2.1.6, this proves the following lemma.

Lemma 2.1.7 *Let K be a simplicial complex. Then K is connected if and only if $|K|$ is a connected space.*

Finally, we note the functoriality of π_0. Let $f\colon K \to L$ be a simplicial map. If $v \sim v'$ for $v, v' \in V(K)$, then $f(v) \sim f(v')$, so f descends to a map $\pi_0 f$: $\pi_0(K) \to \pi_0(L)$. If $f\colon K \to L$ and $g\colon L \to M$ are two simplicial maps, then $\pi_0(g \circ f) = \pi_0 g \circ \pi_0 f$. Also, $\pi_0 \mathrm{id}_K = \mathrm{id}_{\pi_0(K)}$. Thus, π_0 is a covariant functor from the simplicial category **Simp** to the category **Set** of sets and maps.

2.1.8 *Simplicial order.* A *simplicial order* on a simplicial complex L is a partial order $\le$ on $V(L)$ such that each simplex is totally ordered. For example, a total order on $V(L)$, as in examples where vertices are labeled by integers, is a simplicial order. A simplicial order always exists, as a consequence of the well-ordering theorem.

2.1.9 *Triangulations.* A *triangulation* of a topological space X is a homeomorphism $h\colon |K| \to X$, where K is a simplicial complex. A topological space is *triangulable* if it admits a triangulation. It will be useful to have a good process to triangulate some subspaces of $\mathbb{R}^n$. A compact subspace A of $\mathbb{R}^n$ is a *convex cell* if it is the set of solutions of families of affine equations and inequalities

$$f_i(x) = 0,\ i = 1, \dots, r \quad \text{and} \quad g_j(x) \geq 0,\ j = 1, \dots, s\,.$$

A *face* B of A is a convex cell obtained by replacing some of the inequalities $g_j \geq 0$ by the equations $g_j = 0$. The *dimension* of B is the dimension of the smallest affine subspace of $\mathbb{R}^n$ containing B. A *vertex* of A is a cell of dimension 0. By induction on the dimension, one proves that a convex cell is the convex hull of its vertices (see e.g. [138, Theorem 5.2.2]).

A *convex-cell complex* P is a finite union of convex cells in $\mathbb{R}^n$ such that:

(i) if A is a cell of P, so are the faces of A;
(ii) the intersection of two cells of P is a common face of each of them.

The *dimension* of P is the maximal dimension of a cell of P. The *r-skeleton* P^r is the subcomplex formed by the cells of dimension $\leq r$. The 0-skeleton coincides with the set $V(P)$ of *vertices* of P.

A partial order $\leq$ on $V(P)$ is an *affine order for P* if any subset $R \in V(P)$ formed by affinely independent points is totally ordered. For instance, a total order on $V(P)$ is an affine order. The following lemma is a variant of [104, Lemma 1.4].

Lemma 2.1.10 *Let P be a convex-cell complex. An affine order $\leq$ for P determines a triangulation $h_\leq\colon |L_\leq| \xrightarrow{\approx} P$, where $L_\leq$ is a simplicial complex with $V(L_\leq) = V(P)$. The homeomorphism $h_\leq$ is piecewise affine and $\leq$ is a simplicial order on $L_\leq$.*

Proof The order $\leq$ being chosen, we drop it from the notations. For each subcomplex Q of P, we shall construct a simplicial complex $L(Q)$ and a piecewise affine homeomorphism $h_Q\colon |L(Q)| \to Q$ such that,

(i) $V(L(Q)) = V(Q)$;
(ii) if $Q' \subset Q$, then $L(Q') \subset L(Q)$ and $h_{Q'}$ is the restriction of h_Q to $|L(Q')|$.

The case $Q = P$ will prove the lemma. The construction is by induction on the dimension of Q, setting $L(Q) = Q$ and $h_Q = \mathrm{id}$ if $\dim Q = 0$.

Suppose that $L(Q)$ and h_Q have been constructed, satisfying (i) and (ii) above, for each subcomplex Q of P of dimension $\leq k-1$. Let A be a k-cell of K with minimal vertex a. Then A is the topological cone, with cone-vertex a, of the union B of faces of A not containing a. The triangulation $h_B\colon |L(B)| \to |B|$ being constructed by induction hypothesis, define $L(A)$ to be the join $L(B) * \{a\}$ and h_A to be the unique piecewise affine extension of h_B. Observe that, if C is a face of A, then h_C is the restriction to $L(C)$ of h_A. Therefore, this process may be used for each k-cell of P to construct $h_Q\colon |L(Q)| \to Q$ for each subcomplex Q of P with $\dim Q \leq k$. □

2.1.11 *Subdivisions.* Let Z be a set and $\mathcal{A}$ be a family of subsets of Z. A simplicial complex L such that

(a) $V(L) \subset Z$;
(b) for each $\sigma \in \mathcal{S}(L)$ there exists $A \in \mathcal{A}$ such that $\sigma \subset A$;

is called a $(Z, \mathcal{A})$-*simplicial complex*, or a Z-simplicial complex *supported by* $\mathcal{A}$.

Let K be a simplicial complex. Let N be a $(|K|, \mathcal{GS}(K))$-simplicial complex, where

$$\mathcal{GS}(K) = \{|\sigma| \mid \sigma \in \mathcal{S}(K)\}$$

is the family of geometric simplexes of K. A continuous map $j\colon |N| \to |K|$ is associated to N, defined by

$$j(\mu) = \sum_{w \in V(N)} \mu(w)w\,.$$

In other word, j is the piecewise affine map sending each vertex of N to to the corresponding point of $|K|$. A *subdivision* of a simplicial complex K is a $(|K|, \mathcal{GS}(K))$-simplicial complex N for which the associated map $j\colon |N| \to |K|$ is a homeomorphism (in other words, j is a triangulation of $|K|$).

Let N be a $(|K|, \mathcal{GS}(K))$-simplicial complex for a simplicial complex K. If L is a subcomplex of K, then

$$N_L = \{\sigma \in \mathcal{S}(N) | \sigma \subset |L|\}$$

is a $(|L|, \mathcal{GS}(L))$-simplicial complex. Its associated map $j_L\colon |N_L| \to |L|$ is the restriction of j to $|L|$. The following Lemma is useful to recognize a subdivision (compare [179, Chap. 3, Sect. 3, Theorem 4]).

Lemma 2.1.12 *Let N be a $(|K|, \mathcal{GS}(K))$-simplicial complex. Then N is a subdivision of K if and only if, for each $\tau \in \mathcal{S}(K)$, the simplicial complex $N_{\bar{\tau}}$ is finite and $j_{\bar{\tau}}\colon |N_{\bar{\tau}}| \to |\bar{\tau}|$ is bijective.*

Proof If N is a subdivision of K, then $j_{\bar{\tau}}$ is bijective since j is a homeomorphism. Also, $|N_{\bar{\tau}}| = j^{-1}(|\bar{\tau}|)$ is compact, so $N_{\bar{\tau}}$ is finite.

Conversely, The fact that $j_{\bar{\tau}}$ is bijective for each $\tau \in \mathcal{S}(K)$ implies that the continuous map j is bijective. If $N_{\bar{\tau}}$ is finite, then $j_{\bar{\tau}}$ is a continuous bijection between compact spaces, hence a homeomorphism. This implies that the map j^{-1}, restricted to each geometric simplex, is continuous. Therefore, j^{-1} is continuous since K is endowed with the weak topology. □

Seeing $V(K)$ as a subset of $|K|$, we get the following corollary.

Corollary 2.1.13 *Let N be a subdivision of K. Then $V(K) \subset V(N)$.*

A useful systematic subdivision process is the barycentric subdivision. Let $\sigma \in \mathcal{S}_m(K)$ be an m-simplex of a simplicial complex K. The *barycenter* $\hat{\sigma} \in |K|$ of σ is defined by

$$\hat{\sigma} = \frac{1}{m+1} \sum_{v \in \sigma} v \,.$$

The *barycentric subdivision* K' of K is the $(|K|, \mathcal{GS}(K))$-simplicial complex where

- $V(K') = \{\hat{\sigma} \in |K| \mid \sigma \in \mathcal{S}(K)\}$;
- $\{\hat{\sigma}_0, \dots, \hat{\sigma}_m\} \in \mathcal{S}_m(K')$ whenever $\sigma_0 \subset \cdots \subset \sigma_m$ ($\sigma_i \neq \sigma_j$ if $i \neq j$).

Using Lemma 2.1.12, the reader can check that K' is a subdivision of K. Observe that the partial order "$\leq$" defined by

$$\hat{\sigma} \leq \hat{\tau} \iff \sigma \subset \tau \tag{2.1.2}$$

is a simplicial order on K'.

2.2 Definitions of Simplicial (Co)homology

Let K be a simplicial complex. In this section, we give the definitions of the homology $H_*(K)$ and cohomology $H^*(K)$ of K under the various and peculiar forms available when the coefficients are in the field $\mathbb{Z}_2 = \{0, 1\}$.

Definition 2.2.1 (*subset definitions*)

(a) An *m-cochain* is a subset of $\mathcal{S}_m(K)$.
(b) An *m-chain* is a finite subset of $\mathcal{S}_m(K)$.

The set of m-cochains of K is denoted by $C^m(K)$ and that of m-chains by $C_m(K)$. By identifying $\sigma \in \mathcal{S}_m(K)$ with the singleton $\{\sigma\}$, we see $\mathcal{S}_m(K)$ as a subset of both $C_m(K)$ and $C^m(K)$. Each subset A of $\mathcal{S}_m(K)$ is determined by its characteristic function $\chi_A \colon \mathcal{S}_m(K) \to \mathbb{Z}_2$, defined by

$$\chi_A(\sigma) = \begin{cases} 1 & \text{if } \sigma \in A \\ 0 & \text{otherwise.} \end{cases}$$

This gives a bijection between subsets of $\mathcal{S}_m(K)$ and functions from $\mathcal{S}_m(K)$ to $\mathbb{Z}_2$. We see such a function as a colouring (0 = white and 1 = black). The following "colouring definition" is equivalent to the subset definition:

Definition 2.2.2 (*colouring definitions*)

(a) An *m-cochain* is a function $a \colon \mathcal{S}_m(K) \to \mathbb{Z}_2$.
(b) An *m-chain* is a function $\alpha \colon \mathcal{S}_m(K) \to \mathbb{Z}_2$ with finite support.

The colouring definition is used in low-dimensional graphical examples to draw (co)chains in black (bold lines for 1-(co)chains).

Definition 2.2.2 endow $C^m(K)$ and $C_m(K)$ with a structure of a $\mathbb{Z}_2$-vector space. The singletons provide a basis of $C_m(K)$, in bijection with $\mathcal{S}_m(K)$. Thus, Definition 2.2.2b is equivalent to

Definition 2.2.3 $C_m(K)$ is the $\mathbb{Z}_2$-vector space with basis $\mathcal{S}_m(K)$:

$$C_m(K) = \bigoplus_{\sigma \in \mathcal{S}_m(K)} \mathbb{Z}_2\, \sigma\,.$$

We shall pass from one of Definitions 2.2.1, 2.2.2 or 2.2.3 to another without notice; the context usually prevents ambiguity. We consider $C_*(K) = \oplus_{m\in\mathbb{N}} C_m(K)$ and $C^*(K) = \oplus_{m\in\mathbb{N}} C^m(K)$ as graded $\mathbb{Z}_2$-vector spaces. The convention $C_{-1}(K) = C^{-1}(K) = 0$ is useful.

We now define the *Kronecker pairing* on (co)chains

$$C^m(K) \times C_m(K) \xrightarrow{\langle\,,\,\rangle} \mathbb{Z}_2$$

by the equivalent formulae

$$\begin{aligned} \langle a, \alpha\rangle &= \sharp(a \cap \alpha) \pmod 2 && \text{using Definition 2.2.1a and b} \\ &= \textstyle\sum_{\sigma\in\alpha} a(\sigma) && \text{using Definitions 2.2.1a and 2.2.2b} \\ &= \textstyle\sum_{\sigma\in\mathcal{S}_m(K)} a(\sigma)\alpha(\sigma) && \text{using Definitions 2.2.2a and b.} \end{aligned} \tag{2.2.1}$$

Lemma 2.2.4 *The Kronecker pairing is bilinear and the map $a \mapsto \langle a, \;\rangle$ is an isomorphism between $C^m(K)$ and $C_m(K)^\sharp = hom(C_m(K), \mathbb{Z}_2)$.*

Proof The bilinearity is obvious from the third line of Eq. (2.2.1). Let $0 \neq a \in C^m(K)$. This means that, as a subset of $\mathcal{S}_m(K)$, a is not empty. If $\sigma \in a$, then $\langle a, \sigma\rangle \neq 0$, which proves the injectivity of $a \mapsto \langle a, \;\rangle$. As for its surjectivity, let $h \in \hom(C_m(K), \mathbb{Z}_2)$. Using the inclusion $\mathcal{S}_m(K) \hookrightarrow C_m(K)$ given by $\tau \mapsto \{\tau\}$, define

$$a = \{\tau \in \mathcal{S}_m(K) \mid h(\tau) = 1\}\,.$$

For each $\sigma \in \mathcal{S}_m(K)$ the equation $h(\sigma) = \langle a, \sigma\rangle$ holds true. As $\mathcal{S}_m(K)$ is a basis of $C_m(K)$, this implies that $h = \langle a, \;\rangle$. □

We now define the boundary and coboundary operators. The *boundary operator* ∂: $C_m(K) \to C_{m-1}(K)$ is the $\mathbb{Z}_2$-linear map defined by

$$\partial(\sigma) = \{(m-1)\text{-faces of } \sigma\} = \mathcal{S}_{m-1}(\bar{\sigma}), \quad \sigma \in \mathcal{S}_m(K). \tag{2.2.2}$$

Formula (2.2.2) is written in the language of Definition 2.2.1b. Using Definition 2.2.3, we get

$$\partial(\sigma) = \sum_{\tau \in \mathcal{S}_{m-1}(\bar{\sigma})} \tau\,. \tag{2.2.3}$$

The *coboundary operator* $\delta : C^m(K) \to C^{m+1}(K)$ is defined by the equation

$$\langle \delta a, \alpha \rangle = \langle a, \partial \alpha \rangle\,. \tag{2.2.4}$$

The last equation indeed defines δ by Lemma 2.2.4 and δ may be seen as the Kronecker adjoint of ∂. In particular, if $\sigma \in \mathcal{S}_m(K)$ and $\tau \in \mathcal{S}_{m-1}(K)$ then

$$\tau \in \partial(\sigma) \;\Leftrightarrow\; \tau \subset \sigma \;\Leftrightarrow\; \sigma \in \delta(\tau)\,. \tag{2.2.5}$$

The first equivalence determines the operator ∂ since $\mathcal{S}_m(K)$ is a basis for $C_m(K)$. The second equivalence determines δ if $\mathcal{S}_{m-1}(K)$ is finite. Note that the definition of δ may also be given as follows: if $a \in C^m(K)$, then

$$\delta(a) = \{\tau \in \mathcal{S}_{m+1}(K) \mid \sharp\,(a \cap \partial(\tau)) \text{ is odd}\}\,.$$

Let $\sigma \in \mathcal{S}_m(K)$. Each $\tau \in \mathcal{S}_{m-2}(K)$ with $\tau \subset \sigma$ belongs to the boundary of exactly two $(m-1)$-simplexes of σ. Using Eq. (2.2.3), this implies that $\partial\circ\partial = 0$. By Eq. (2.2.4) and Lemma 2.2.4, we get $\delta\circ\delta = 0$. We define the $\mathbb{Z}_2$-vector spaces

- $Z_m(K) = \ker(\partial : C_m(K) \to C_{m-1}(K))$, the *m-cycles* of K.
- $B_m(K) = \text{image}\,(\partial : C_{m+1}(K) \to C_m(K))$, the *m-boundaries* of K.
- $Z^m(K) = \ker(\delta : C^m(K) \to C^{m+1}(K))$, the *m-cocycles* of K.
- $B^m(K) = \text{image}\,(\delta : C^{m-1}(K) \to C^m(K))$, the *m-coboundaries* of K.

For example, Fig. 2.1 shows a triangulation K of the plane, with $V(K) = \mathbb{Z}\times\mathbb{Z}$. The bold line is a cocycle a which is a coboundary: $a = \delta B$, with $B = \{\{(m,n)\} \mid (m,n) \in V(K) \text{ and } m \le 0\}$, drawn in bold dots.

Since $\partial\circ\partial = 0$ and $\delta\circ\delta = 0$, one has $B_m(K) \subset Z_m(K)$ and $B^m(K) \subset Z^m(K)$. We form the quotient vector spaces

- $H_m(K) = Z_m(K)/B_m(K)$, the mth *-homology vector space of* K.
- $H^m(K) = Z^m(K)/B^m(K)$, the mth *-cohomology vector space of* K.

As for the (co)chains, the notations $H_*(K) = \oplus_{m\in\mathbb{N}} H_m(K)$ and $H^*(K) = \oplus_{m\in\mathbb{N}} H^m(K)$ stand for the (co)homology seen as graded $\mathbb{Z}_2$-vector spaces. By

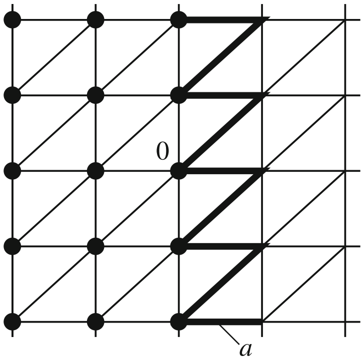


Fig. 2.1 A triangulation K of the plan, with $V(K) = \mathbb{Z} \times \mathbb{Z}$

convention, $H_{-1}(K) = H^{-1}(K) = 0$. Also, the homology and the cohomology are in duality via the Kronecker pairing:

Proposition 2.2.5 (Kronecker duality) *The Kronecker pairing on (co)chains induces a bilinear map*

$$H^m(K) \times H_m(K) \xrightarrow{\langle\,,\,\rangle} \mathbb{Z}_2 .$$

Moreover, the correspondence $a \mapsto \langle a, \ \rangle$ is an isomorphism

$$H^m(K) \xrightarrow[\approx]{\mathbf{k}} hom(H_m(K), \mathbb{Z}_2) .$$

Proof Instead of giving a direct proof, which the reader may do as an exercise, we will take advantage of the more general setting of *Kronecker pairs*, developed in the next section. In this way, Proposition 2.2.5 follows from Proposition 2.3.5. □

2.3 Kronecker Pairs

All the vector spaces in this section are over an arbitrary fixed field $\mathbb{F}$. The dual of a vector space V is denoted by $V^\sharp$.

A *chain complex* is a pair (C_*, ∂), where

- C_* is a graded vector space $C_* = \bigoplus_{m\in\mathbb{N}} C_m$. We add the convention that $C_{-1} = 0$.
- $\partial : C_* \to C_*$ is a linear map of degree -1, i.e. $\partial(C_m) \subset C_{m-1}$, satisfying $\partial\circ\partial = 0$. The operator ∂ is called the *boundary* of the chain complex.

A *cochain complex* is a pair (C^*, δ), where

- C^* is a graded vector space $C^* = \bigoplus_{m\in\mathbb{N}} C^m$. We add the convention that $C^{-1} = 0$.
- $\delta : C^* \to C^*$ is a linear map of degree $+1$, i.e. $\partial(C^m) \subset C^{m+1}$, satisfying $\delta\circ\delta = 0$. The operator δ is called the *coboundary* of the cochain complex.

A *Kronecker pair* consists of three items:

(a) a chain complex (C_*, ∂).
(b) a cochain complex (C^*, δ).
(c) a bilinear map

$$C^m \times C_m \xrightarrow{\langle\,,\,\rangle} \mathbb{F}$$

satisfying the equation

$$\langle \delta a, \alpha \rangle = \langle a, \partial \alpha \rangle \,. \tag{2.3.1}$$

for all $a \in C^m$ and $\alpha \in C_{m+1}$ and all $m \in \mathbb{N}$. Moreover, we require that the map $\mathbf{k}\colon C^m \to C_m^\sharp$, given by $\mathbf{k}(a) = \langle a, \ \rangle$, is an isomorphism.

Example 2.3.1 Let K be a simplicial complex. Its simplicial (co)chain complexes $(C^*(K), \delta)$, $(C_*(K), \partial)$, together with the pairing $\langle\,,\,\rangle$ of Sect. 2.2 is a Kronecker pair, with $\mathbb{F} = \mathbb{Z}_2$, as seen in Lemma 2.2.4 and Eq. (2.2.4).

Example 2.3.2 Let (C_*, ∂) be a chain complex. One can define a cochain complex (C^*, δ) by $C^m = C_m^\sharp$ and $\delta = \partial^\sharp$ and then get a bilinear map (pairing) $\langle,\rangle$ by the evaluation: $\langle a, \alpha \rangle = a(\alpha)$. These constitute a Kronecker pair. Actually, via the map $\mathbf{k}$, any Kronecker pair is isomorphic to this one. The reader may use this fact to produce alternative proofs of the results of this section.

We first observe that, as the Kronecker pairing is non-degenerate, chains and cochains mutually determine each other:

Lemma 2.3.3 *Let $\big((C^*, \delta), (C_*, \partial), \langle\,,\,\rangle\big)$ be a Kronecker pair.*

(a) *Let $a, a' \in C^m$. Suppose that $\langle a, \alpha \rangle = \langle a', \alpha \rangle$ for all $\alpha \in C_m$. Then $a = a'$.*
(b) *Let $\alpha, \alpha' \in C_m$. Suppose that $\langle a, \alpha \rangle = \langle a, \alpha' \rangle$ for all $a \in C^m$. Then $\alpha = \alpha'$.*
(c) *Let $\mathcal{S}_m$ be a basis for C_m and let $f\colon \mathcal{S}_m \to \mathbb{F}$ be a map. Then, there is a unique $a \in C^m$ such that $\langle a, \sigma \rangle = f(\sigma)$ for all $\sigma \in \mathcal{S}_m$.*

Proof In Point (a), the hypotheses imply that $\mathbf{k}(a) = \mathbf{k}(a')$. As $\mathbf{k}$ is injective, this shows that $a = a'$.

In Point (b), suppose that $\alpha \neq \alpha'$. Let $A \in (C_m)^\sharp$ such that $A(\alpha - \alpha') \neq 0$. Then, $\langle a, \alpha \rangle \neq \langle a, \alpha' \rangle$ for $a = \mathbf{k}^{-1}(A) \in C^m$.

Finally, the condition $\tilde{a}(\sigma) = f(\sigma)$ for all $\sigma \in \mathcal{S}_m$ defines a unique $\tilde{a} \in C_m^\sharp$ and $a = \mathbf{k}^{-1}(\tilde{a})$. □

As is Sect. 2.2, we consider the $\mathbb{Z}_2$-vector spaces

- $Z_m = \ker(\partial : C_m \to C_{m-1})$, the *m-cycles* (of C_*).
- $B_m = \text{image}\,(\partial : C_{m+1} \to C_m)$, the *m-boundaries*.
- $Z^m = \ker(\delta : C^m \to C^{m+1})$, the *m-cocycles*.
- $B^m = \text{image}\,(\delta : C^{m-1} \to C^m)$, the *m-coboundaries*.

Since $\partial \circ \partial = 0$ and $\delta \circ \delta = 0$, one has $B_m \subset Z_m$ and $B^m \subset Z^m$. We form the quotient vector spaces

- $H_m = Z_m / B_m$, the mth-*homology group* (*or vector space*).
- $H^m = Z^m / B^m$, the mth-*cohomology group* (*or vector space*).

We consider the (co)homology as graded vector spaces: $H_* = \oplus_{m \in \mathbb{N}} H_m$ and $H^* = \oplus_{m \in \mathbb{N}} H^m$.

The cocycles and coboundaries may be detected by the pairing:

Lemma 2.3.4 *Let $a \in C^m$. Then*

(i) $a \in Z^m$ *if and only if* $\langle a, B_m \rangle = 0$.
(ii) $a \in B^m$ *if and only if* $\langle a, Z_m \rangle = 0$.

Proof Point (i) directly follows from Eq. (2.3.1) and the fact that $\mathbf{k}$ is injective. Also, if $a \in B^m$, Eq. (2.3.1) implies that $\langle a, Z_m \rangle = 0$. It remains to prove the converse (this is the only place in this lemma where we need vector spaces over a field instead just module over a ring). We consider the exact sequence

$$0 \to Z_m \to C_m \to B_{m-1} \stackrel{\partial}{\to} 0 \,. \tag{2.3.2}$$

Let $a \in C^m$ such that $\langle a, Z_m \rangle = 0$. By (2.3.2), there exists $a_1 \in B^\sharp_{m-1}$ such that $\langle a, \ \rangle = a_1 \circ \partial$. As we are dealing with vector spaces, B_{m-1} is a direct summand of C_{m-1}. We can thus extend a_1 to $a_2 \in C^\sharp_{m-1}$. As $\mathbf{k}$ is surjective, there exists $a_3 \in C^{m-1}$ such that $\langle a_3, \ \rangle = a_2$. For all $\alpha \in C_m$, one then has

$$\langle \delta a_3, \alpha \rangle = \langle a_3, \partial \alpha \rangle = a_2(\partial \alpha) = a_1(\partial \alpha) = \langle a, \alpha \rangle \,.$$

As $\mathbf{k}$ is injective this implies that $a = \delta a_3 \in B^m$. □

Let us restrict the pairing $\langle \, , \rangle$ to $Z^m \times Z_m$. Formula (2.3.1) implies that

$$\langle Z^m, B_m \rangle = \langle B^m, Z_m \rangle = 0 \,.$$

Hence, the pairing descends to a bilinear map $H^m \times H_m \xrightarrow{\langle \, , \rangle} \mathbb{F}$, giving rise to a linear map $\mathbf{k}\colon H^m \to H^\sharp_m$, called the *Kronecker pairing on* (*co*)*homology*. We see H_* and H^* as (co)chain complexes by setting $\partial = 0$ and $\delta = 0$.

Proposition 2.3.5 $(H_*, H^*, \langle \, , \rangle)$ *is a Kronecker pair.*

Proof Equation (2.3.1) holds trivially since ∂ and δ both vanish. It remains to show that $\mathbf{k}\colon H^m \to H^\sharp_m$ is bijective.

Let $a_0 \in H^\sharp_m$. Pre-composing a_0 with the projection $Z_m \twoheadrightarrow H_m$ produces $a_1 \in Z^\sharp_m$. As Z_m is a direct summand in C_m, one can extend a_1 to $a_2 \in C^\sharp_m$. Since $(C_*, C^*, \langle \, , \rangle)$ is a Kronecker pair, there exists $a \in C^m$ such that $\langle a, \ \rangle = a_2$. The cochain a satisfies $\langle a, B_m\rangle = a_2(B_m) = 0$ which, by Lemma 2.3.4, implies that $a \in Z^m$. The cohomology class $[a] \in H^m$ of a then satisfies $\langle [a], \rangle = a_0$. Thus, $\mathbf{k}$ is surjective.

For the injectivity of $\mathbf{k}$, let $b \in H^m$ with $\langle b, H_m\rangle = 0$. Represent b by $\tilde{b} \in Z^m$, which then satisfies $\langle \tilde{b}, Z_m\rangle = 0$. By Lemma 2.3.4, $\tilde{b} \in B^m$ and thus $b = 0$. □

Let (C_*, ∂) and $(\bar{C}_*, \bar{\partial})$ be two chain complexes. A map $\varphi\colon C_* \to \bar{C}_*$ is a *morphism of chain complexes* or a *chain map* if it is linear map of degree 0 (i.e. $\varphi(C_m) \subset \bar{C}_m$) such that $\varphi\circ\partial = \bar{\partial}\circ\varphi$. This implies that $\varphi(Z_m) \subset \bar{Z}_m$ and $\varphi(B_m) \subset \bar{B}_m$. Hence, φ induces a linear map $H_*\varphi : H_m \to \bar{H}_m$ for all m.

In the same way, let (C^*, δ) and $(\bar{C}^*, \bar{\delta})$ be two cochain complexes. A linear map $\phi\colon \bar{C}^* \to C^*$ of degree 0 is a *morphism of cochain complexes* or a *cochain map* if $\phi\circ\bar{\delta} = \delta\circ\phi$. Hence, ϕ induces a linear map $H^*\phi : \bar{H}^m \to H^m$ for all m.

Let $\mathcal{P} = (C_*, \partial, C^*, \delta, \langle\,,\rangle)$ and $\bar{\mathcal{P}} = (\bar{C}_*, \bar{\partial}, \bar{C}^*, \bar{\delta}, \langle\,,\rangle^-)$ be two Kronecker pairs. A *morphism of Kronecker pairs*, from $\mathcal{P}$ to $\bar{\mathcal{P}}$, consists of a pair (φ, ϕ) where $\varphi\colon C_* \to \bar{C}_*$ is a morphism of chain complexes and $\phi\colon \bar{C}^* \to C^*$ is a morphism of cochain complexes such that

$$\langle a, \varphi(\alpha)\rangle^- = \langle \phi(a), \alpha\rangle\,. \tag{2.3.3}$$

Using the isomorphisms $\mathbf{k}$ and $\bar{\mathbf{k}}$, Eq. (2.3.3) is equivalent to the commutativity of the diagram

$$\begin{tikzcd} \bar{C}^* \arrow[r, "\phi"] \arrow[d, "\bar{\mathbf{k}}", "\approx"'] & C^* \arrow[d, "\mathbf{k}", "\approx"'] \\ \bar{C}_*^\sharp \arrow[r, "\varphi^\sharp"] & C_*^\sharp \end{tikzcd}\,. \tag{2.3.4}$$

Lemma 2.3.6 *Let $\mathcal{P}$ and $\bar{\mathcal{P}}$ be Kronecker pairs as above. Let $\varphi\colon C_* \to \bar{C}_*$ be a morphism of chain complex. Define $\phi\colon \bar{C}^* \to C^*$ by Eq.* (2.3.3) (*or Diagram* (2.3.4)). *Then the pair (φ, ϕ) is a morphism of Kronecker pairs.*

Proof Obviously, ϕ is a linear map of degree 0 and Eq. (2.3.3) is satisfied. It remains to show that ϕ is a morphism of cochain-complexes. But, if $b \in C_m(\bar{K})$ and $\alpha \in C_{m+1}(K)$, one has

$$\begin{aligned}\langle \delta\phi(b), \alpha\rangle &= \langle \phi(b), \partial\alpha\rangle = \langle b, \varphi(\partial\alpha)\rangle^- = \langle b, \bar{\partial}\varphi(\alpha)\rangle^- \\ &= \langle \bar{\delta}b, \varphi(\alpha)\rangle^- = \langle \phi(\bar{\delta}b), \alpha\rangle\,,\end{aligned}$$

which proves that $\delta\phi(b) = \phi(\bar{\delta}b)$. □

A morphism (φ, ϕ) of Kronecker pairs determines a morphism of Kronecker pairs $(H_*\varphi, H^*\phi)$ from $(H_*, H^*, \langle\,,\rangle)$ to $(\bar{H}_*, \bar{H}^*, \langle\,,\rangle^-)$. This process is functorial:

Lemma 2.3.7 *Let (φ_1, ϕ_1) be a morphism of Kronecker pairs from $\mathcal{P}$ to $\bar{\mathcal{P}}$ and let (φ_2, ϕ_2) be a morphism of Kronecker pairs from $\bar{\mathcal{P}}$ to $\dot{\mathcal{P}}$. Then*

$$(H_*\varphi_2 \circ H_*\varphi_1, H^*\phi_1 \circ H^*\phi_2) = (H_*(\varphi_2 \circ \varphi_1), H^*(\phi_2 \circ \phi_1))$$

Proof That $H_*\varphi_2 \circ H_*\varphi_1 = H_*(\varphi_2 \circ \varphi_1)$ is a tautology. For the cohomology equality, we use that

$$\begin{aligned}\langle H^*\phi_1 \circ H^*\phi_2(a), \alpha\rangle &= \langle H^*\phi_2(a), H_*\varphi_1(\alpha)\rangle = \langle a, H_*\varphi_2 \circ H_*\varphi_1(\alpha)\rangle \\ &= \langle a, H_*(\varphi_2 \circ \varphi_1)(\alpha)\rangle = \langle H^*(\phi_2 \circ \phi_1))(a), \alpha\rangle\end{aligned}$$

holds for all $a \in \bar{H}^*$ and all $\alpha \in H_*$. □

We finish this section with some technical results which will be used later.

Lemma 2.3.8 *Let $f\colon U \to V$ and $g\colon V \to W$ be two linear maps between vector spaces. Then, the sequence*

$$U \xrightarrow{f} V \xrightarrow{g} W \tag{2.3.5}$$

is exact at V if and only if the sequence

$$U^\sharp \xleftarrow{f^\sharp} V^\sharp \xleftarrow{g^\sharp} W^\sharp \tag{2.3.6}$$

is exact at $V^\sharp$.

Proof As $f^\sharp \circ g^\sharp = (g \circ f)^\sharp$, then $f^\sharp \circ g^\sharp = 0$ if and only if $g \circ f = 0$.

On the other hand, suppose that $\ker g \subset \operatorname{image} f$. We shall prove that $\ker f^\sharp \subset \operatorname{image} g^\sharp$. Indeed, let $a \in \ker f^\sharp$. Then, $a(\operatorname{image} f) = 0$ and, using the inclusion $\ker g \subset \operatorname{image} f$, we deduce that $a(\ker g) = 0$. Therefore, a descends to a linear map $\bar{a}\colon V/\ker g \to \mathbb{F}$. The quotient space $V/\ker g$ injects into W, so there exists $b \in W^\sharp$ such that $a = b \circ g = g^\sharp(b)$, proving that $a \in \operatorname{image} g^\sharp$.

Finally, suppose that $\ker g \not\subset \operatorname{image} f$. Then there exists $a \in V^\sharp$ such that $a(\operatorname{image} f) = 0$, i.e., $a \in \ker f^\sharp$, and $a(\ker g) \neq 0$, i.e. $a \notin \operatorname{image} g^\sharp$. This proves that $\ker f^\sharp \not\subset \operatorname{image} g^\sharp$. □

Lemma 2.3.9 *Let (φ, ϕ) be a morphism of Kronecker pairs from $\mathcal{P} = (C_*, \partial, C^*, \delta, \langle\,,\rangle)$ to $\bar{\mathcal{P}} = (\bar{C}_*, \bar{\partial}, \bar{C}^*, \bar{\delta}, \langle\,,\rangle^-)$. Then the pairings $\langle\,,\rangle$ and $\langle\,,\rangle^-$ induce bilinear maps*

$$\operatorname{coker}\phi \times \ker\varphi \xrightarrow{\langle,\rangle} \mathbb{F} \quad \textit{and} \quad \ker\phi \times \operatorname{coker}\varphi \xrightarrow{\langle,\rangle^-} \mathbb{F}$$

such that the induced linear maps

$$\operatorname{coker}\phi \xrightarrow{\mathbf{k}} (\ker\varphi)^\sharp \quad \textit{and} \quad \ker\phi \xrightarrow{\bar{\mathbf{k}}} (\operatorname{coker}\varphi)^\sharp$$

are isomorphisms.

Proof Equation (2.3.3) implies that $\langle\phi(C^*), \ker\varphi\rangle = 0$ and $\langle\ker\phi, \varphi(C_*)\rangle^- = 0$, whence the induced pairings. Consider the exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow C_* \xrightarrow{\varphi} \bar{C}_* \longrightarrow \operatorname{coker}\varphi \longrightarrow 0\,.$$

By Lemma 2.3.8, passing to the dual preserves exactness. Using Diagram (2.3.4), one gets a commutative diagram

$$\begin{tikzcd}
0 & (\ker\varphi)^\sharp \arrow[l] & C_*^\sharp \arrow[l] & \bar{C}_*^\sharp \arrow[l, "\varphi^\sharp"'] & (\operatorname{coker}\varphi)^\sharp \arrow[l] & 0 \arrow[l] \\
0 & \operatorname{coker}\phi \arrow[l] \arrow[u, "\mathbf{k}"'] & H^* \arrow[l] \arrow[u, "\mathbf{k}"', "\approx"] & \bar{C}^* \arrow[l, "\phi"'] \arrow[u, "\bar{\mathbf{k}}"', "\approx"] & \ker\phi \arrow[l] \arrow[u, "\bar{\mathbf{k}}"'] & 0 \arrow[l]
\end{tikzcd} \quad . \tag{2.3.7}$$

By diagram-chasing, the two extreme up-arrows are bijective (one can also invoke the famous *five-lemma*: see e.g. [179, Chap. 4, Sect. 5, Lemma 11]). □

Corollary 2.3.10 *Let* (φ, ϕ) *be a morphism of Kronecker pairs from* $(C_*, \partial, C^*, \delta, \langle\,,\rangle)$ *to* $(\bar{C}_*, \bar{\partial}, \bar{C}^*, \bar{\delta}, \langle\,,\rangle^-)$. *Then the pairings* $\langle\,,\rangle$ *and* $\langle\,,\rangle^-$ *on (co)homology induce bilinear maps*

$$\operatorname{coker} H^*\phi \times \ker H_*\varphi \xrightarrow{\langle,\rangle} \mathbb{F} \quad \textit{and} \quad \ker H^*\phi \times \operatorname{coker} H_*\varphi \xrightarrow{\langle,\rangle^-} \mathbb{F}$$

such that the induced linear maps

$$\operatorname{coker} H^*\phi \xrightarrow{\mathbf{k}} (\ker H_*\varphi)^\sharp \quad \textit{and} \quad \ker H^*\phi \xrightarrow{\bar{\mathbf{k}}} (\operatorname{coker} H_*\varphi)^\sharp$$

are isomorphisms.

Proof The morphism (ϕ, φ) induces a morphism of Kronecker pairs $(H^*\phi, H_*\varphi)$ from $(H^*, H_*, \langle\,,\rangle)$ to $(\bar{H}^*, \bar{H}_*, \langle\,,\rangle^-)$. Corollary 2.3.10 follows then from Lemma 2.3.9 applied to $(H^*\phi, H_*\varphi)$. □

Corollary 2.3.10 implies the following

Corollary 2.3.11 *Let* (φ, ϕ) *be a morphism of Kronecker pairs from* $(C_*, \partial, C^*, \delta, \langle\,,\rangle)$ *to* $(\bar{C}_*, \bar{\partial}, \bar{C}^*, \bar{\delta}, \langle\,,\rangle^-)$. *Then*

(a) *$H^*\phi$ is surjective if and only if $H_*\varphi$ is injective.*
(b) *$H^*\phi$ is injective if and only if $H_*\varphi$ is surjective.*
(c) *$H^*\phi$ is bijective if and only if $H_*\varphi$ is bijective.*

2.4 First Computations

2.4.1 Reduction to Components

Let K be a simplicial complex. We have seen in 2.1.5 that K is the disjoint union of its components, whose set is denoted by $\pi_0(K)$. Therefore, $\mathcal{S}_m(K) = \coprod_{L\in\pi_0(K)} \mathcal{S}_m(L)$ which, by Definition 2.2.3, gives a canonical isomorphism

$$\bigoplus_{L\in\pi_0(K)} C_m(L) \xrightarrow{\approx} C_m(K)\,.$$

This direct sum decomposition commutes with the boundary operators, giving a canonical isomorphism

$$\bigoplus_{L\in\pi_0(K)} H_*(L) \xrightarrow{\approx} H_*(K)\,. \tag{2.4.1}$$

As for the cohomology, seeing an m-cochain as a map $\alpha\colon \mathcal{S}_m(K) \to \mathbb{Z}_2$ (Definition 2.2.2) the restrictions of α to $\mathcal{S}_m(L)$ for all $L \in \pi_0(K)$ gives an isomorphism

$$C^m(K) \xrightarrow{\approx} \prod_{L\in\pi_0(K)} C^m(L)$$

commuting with the coboundary operators. This gives an isomorphism

$$H^*(K) \xrightarrow{\approx} \prod_{L\in\pi_0(K)} H^*(L)\,. \tag{2.4.2}$$

The isomorphisms of (2.4.1) and (2.4.2) permit us to reduce (co)homology computations to connected simplicial complexes. They are of course compatible with the Kronecker duality (Proposition 2.2.5). A formulation of these isomorphisms using simplicial maps is given in Proposition 2.5.3.

2.4.2 0-Dimensional (Co)homology

Let K be a simplicial complex. The *unit cochain* $\mathbf{1} \in C^0(K)$ is defined by $\mathbf{1} = \mathcal{S}_0(K)$, using the subset definition. In the language of colouring, one has $\mathbf{1}(v) = 1$ for all

$v \in V(K) = \mathcal{S}_0(K)$, that is all vertices are black. If $\beta = \{v, w\} \in \mathcal{S}_1(K)$, then

$$\langle \delta \mathbf{1}, \beta \rangle = \langle \mathbf{1}, \partial \beta \rangle = \mathbf{1}(v) + \mathbf{1}(w) = 0\,,$$

which proves that $\delta(\mathbf{1}) = 0$ by Lemma 2.2.4. Hence, $\mathbf{1}$ is a cocycle, whose cohomology class is again denoted by $\mathbf{1} \in H^0(K)$.

Proposition 2.4.1 *Let K be a non-empty connected simplicial complex. Then,*

(i) *$H^0(K) = \mathbb{Z}_2$, generated by $\mathbf{1}$ which is the only non-vanishing 0-cocycle.*
(ii) *$H_0(K) = \mathbb{Z}_2$. Any 0-chain α is a cycle, which represents the non-zero element of $H_0(K)$ if and only if $\sharp\alpha$ is odd.*

Proof If K is non-empty the unit cochain does not vanish. As $C^{-1}(K) = 0$, this implies that $\mathbf{1} \neq 0$ in $H^0(K)$.

Let $a \in C^0(K)$ with $a \neq 0, \mathbf{1}$. Then there exists $v, v' \in V(K)$ with $a(v) \neq a(v')$. Since K is connected, there exists $x_0, \dots, x_m \in V(K)$ with $x_0 = v$, $x_m = v'$ and $\{x_i, x_{i+1}\} \in \mathcal{S}(K)$. Therefore, there exists $0 \leq k < m$ with $a(x_k) \neq a(x_{k+1})$. This implies that $\{x_k, x_{k+1}\} \in \delta a$, proving that $\delta a \neq 0$. We have thus proved (i).

Now, $H_0(K) = \mathbb{Z}_2$ since $H^0(K) \approx H_0(K)^\sharp$. Any $\alpha \in C_0(K)$ is a cycle since $C_{-1}(K) = 0$. It represents the non-zero homology class if and only if $\langle \mathbf{1}, \alpha \rangle = 1$, that is if and only if $\sharp\alpha$ is odd. □

Corollary 2.4.2 *Let K be a simplicial complex. Then $H^0(K) \approx \mathbb{Z}_2^{\pi_0(K)}$.*

Here, $\mathbb{Z}_2^{\pi_0(K)}$ denotes the set of maps from $\pi_0(K)$ to $\mathbb{Z}_2$. The isomorphism of Corollary 2.4.2 is natural for simplicial maps (see Corollary 2.5.6).

Proof By Proposition 2.4.1 and its proof, $H^0(K) = Z^0(K)$ is the set of maps from $V(K)$ to $\mathbb{Z}_2$ which are constant on each component. Such a map is determined by a map from $\pi_0(K)$ to $\mathbb{Z}_2$ and conversely. □

2.4.3 Pseudomanifolds

An *n-dimensional pseudomanifold* is a simplicial complex M such that

(a) every simplex of M is contained in an n-simplex of M.
(b) every $(n-1)$-simplex of M is a face of exactly two n-simplexes of M.
(c) for any $\sigma, \sigma' \in \mathcal{S}_n(M)$, there exists a sequence $\sigma = \sigma_0, \dots, \sigma_m = \sigma'$ of n-simplexes such that σ_i and σ_{i+1} have an $(n-1)$-face in common for $i \leq 1 < m$.

Example 2.4.3 (1) Let m be an integer with $m \geq 3$. The *polygon* $\mathcal{P}_m$ is the 1-dimensional pseudomanifold for which $V(\mathcal{P}_m) = \{0, 1, \dots, m-1\} = \mathbb{Z}/m\mathbb{Z}$ and $\mathcal{S}_1(\mathcal{P}_m) = \{\{k, k+1\} \mid k \in V(\mathcal{P}_m)\}$. It can be visualized in the complex plane as the equilateral m-gon whose vertices are the mth roots of the unity.

(2) Consider the triangulation of S^2 given by an icosahedron. Choose one pair of antipodal vertices and identify them in a single point. This gives a quotient simplicial complex K which is a 2-dimensional pseudomanifold. Observe that $|K|$ is not a topological manifold.

Pseudomanifolds have been introduced in 1911 by Brouwer [22, p. 477], for his work on the degree and on the invariance of the dimension. They are also called *n-circuits* in the literature. Proposition 2.4.4 below and its proof, together with Proposition 2.4.1, shows that n-dimensional pseudomanifolds satisfy Poincaré duality in dimensions 0 and n.

Let M be a finite n-dimensional pseudomanifold. The n-chain $[M] = \mathcal{S}_n(M) \in C_n(M)$ is called the *fundamental cycle* of M (it is a cycle by Point (b) of the above definition). Its homology class, also denoted by $[M] \in H_n(M)$ is called the *fundamental class* of M.

Proposition 2.4.4 *Let M be a finite non-empty n-dimensional pseudomanifold. Then,*

(i) *$H_n(M) = \mathbb{Z}_2$, generated by $[M]$ which is the only non-vanishing n-cycle.*
(ii) *$H^n(M) = \mathbb{Z}_2$. Any n-cochain a is a cocycle, and $[a] \neq 0$ in $H^n(M)$ if and only if $\sharp a$ is odd.*

Proof We define a simplicial graph L with $V(L) = \mathcal{S}_n(M)$ by setting $\{\sigma, \sigma'\} \in \mathcal{S}_1(L)$ if and only if σ and σ' have an $(n-1)$-face in common. The identification $\mathcal{S}_n(M) = V(L)$ produces isomorphisms

$$\tilde{F}_n : C_n(M) \xrightarrow{\approx} C^0(L) \text{ and } \tilde{F}^n : C^n(M) \xrightarrow{\approx} C_0(L)\,. \tag{2.4.3}$$

(As M is finite, so is L and $C_*(L)$ is equal to $C^*(L)$, using Definition 2.2.2) On the other hand, by Point (b) of the definition of a pseudomanifold, one gets a bijection $\tilde{F}\colon \mathcal{S}_{n-1}(M) \xrightarrow{\approx} \mathcal{S}_1(L)$. It gives rise to isomorphisms

$$\tilde{F}_{n-1} : C_{n-1}(M) \xrightarrow{\approx} C^1(L) \text{ and } \tilde{F}^{n-1} : C^{n-1}(M) \xrightarrow{\approx} C_1(L)\,. \tag{2.4.4}$$

The isomorphisms of (2.4.3) and (2.4.4) satisfy

$$\tilde{F}_{n-1} \circ \partial = \delta \circ \tilde{F}_n \text{ and } \partial \circ \tilde{F}^{n-1} = F^n \circ \delta\,.$$

Since $C_{n+1}(M) = 0$ by Point (a) of the definition of a pseudomanifold, the above isomorphisms give rise to isomorphisms

$$F_* : H_n(M) \xrightarrow{\approx} H^0(L) \text{ and } F^* : H^n(M) \xrightarrow{\approx} H_0(L)$$

with $F_*([M]) = \mathbf{1}$. By Point (c) of the definition of a pseudomanifold, the graph L is connected. Therefore, Proposition 2.4.4 follows from Proposition 2.4.1. □

The proof of Proposition 2.4.4 actually gives the following result.

Proposition 2.4.5 *Let M be a finite non-empty simplicial complex satisfying Conditions (a) and (b) of the definition of an n-dimensional pseudomanifold. Then, M is a pseudomanifold if and only if $H_n(M) = \mathbb{Z}_2$.*

2.4.4 Poincaré Series and Polynomials

A graded $\mathbb{Z}_2$-vector space $A_* = \bigoplus_{i\in\mathbb{N}} A_i$ is of *finite type* if A_i is finite dimensional for all $i \in \mathbb{N}$. In this case, the *Poincaré series* of A_* is the formal power series defined by

$$\mathfrak{P}_t(A_*) = \sum_{i\in\mathbb{N}} \dim A_i t^i \in \mathbb{N}[[t]].$$

When $\dim A_* < \infty$, the series $\mathfrak{P}_t(A_*)$ is a polynomial, also called the *Poincaré polynomial* of A_*.

A simplicial complex K is of *finite (co)homology type* if $H_*(K)$ (or, equivalently, $H^*(K)$) is of finite type. In this case, the *Poincaré series* of K is that of $H_*(K)$. The (co)homology of a simplicial complex of finite (co)homology type is, up to isomorphism, determined by its Poincaré series, which is often the shortest way to describe it. The number $\dim H_m(K)$ is called the m-th *Betti number of K*. The vector space $C_*(K)$ is endowed with the basis $\mathcal{S}(K)$ for which the matrix of the boundary operator is given explicitly. Thus, the Betti numbers may be effectively computed by standard algorithms of linear algebra.

2.4.5 (Co)homology of a Cone

The simplest non-empty simplicial complex is a point whose (co)homology is obviously

$$H^m(pt) \approx H_m(pt) \approx \begin{cases} 0 & \text{if } m > 0 \\ \mathbb{Z}_2 & \text{if } m = 0\,. \end{cases} \tag{2.4.5}$$

In terms of Poincaré polynomial: $\mathfrak{P}_t(pt) = 1$. Let L be a simplicial complex. The *cone on L* is the simplicial complex CL defined by $V(CL) = V(L) \cup \{\infty\}$ and

$$\mathcal{S}_m(CL) = \mathcal{S}_m(L) \cup \{\sigma \cup \{\infty\} \mid \sigma \in \mathcal{S}_{m-1}(L)\}\,.$$

Note that CL is the join $CL \approx L * \{\infty\}$.

Proposition 2.4.6 *The cone CL on a simplicial complex L has its (co)homology isomorphic to that of a point. In other words, $\mathfrak{P}_t(CL) = 1$.*

Proof By Kronecker duality, it is enough to prove the result on homology. The cone CL is obviously connected and non-empty (it contains ∞), so $H_0(CL) = \mathbb{Z}_2$.

Define a linear map $D\colon C_m(CL) \to C_{m+1}(CL)$ by setting, for $\sigma \in \mathcal{S}_m(CL)$:

$$D(\sigma) = \begin{cases} \sigma \cup \{\infty\} & \text{if } \infty \notin \sigma \\ 0 & \text{if } \infty \in \sigma\,. \end{cases}$$

Hence, $D \circ D = 0$. If $\infty \notin \sigma$, the formula

$$\partial D(\sigma) = D(\partial\sigma) + \sigma \tag{2.4.6}$$

holds true in $C_m(CL)$ (and has a clear geometrical interpretation). Suppose that $\infty \in \sigma$ and $\dim \sigma \geq 1$. Then $\sigma = D(\tau)$ with $\tau = \sigma - \{\infty\}$. Using Formula (2.4.6) and that $D \circ D = 0$, one has

$$D(\partial\sigma) + \sigma = D(\partial D(\tau)) + \sigma = D(D(\partial\tau) + \tau) + D(\tau) = 0\,.$$

Therefore, Formula (2.4.6) holds also true if $\infty \in \sigma$, provided $\dim \sigma \geq 1$. This proves that

$$\partial D(\alpha) = D(\partial\alpha) + \alpha \quad \textit{for all } \alpha \in C_m(CL) \textit{ with } m \geq 1\,. \tag{2.4.7}$$

Now, if $\alpha \in C_m(CL)$ satisfies $\partial\alpha = 0$, Formula (2.4.7) implies that $\alpha = \partial D(\alpha)$, which proves that $H_m(CL) = 0$ if $m \geq 1$. □

As an application of Proposition 2.4.6, let A be a set. The *full complex* $\mathcal{F}A$ on A is the simplicial complex for which $V(\mathcal{F}A) = A$ and $\mathcal{S}(\mathcal{F}A)$ is the family of *all* finite non-empty subsets of A. If A is finite and non-empty, then $\mathcal{F}A$ is isomorphic to a simplex of dimension $\sharp A - 1$. Denote by $\dot{\mathcal{F}}A$ the subcomplex of $\mathcal{F}A$ generated by the proper (i.e. $\neq A$) subsets of A. For instance, $\dot{\mathcal{F}}A = \mathcal{F}A$ if A is infinite.

Corollary 2.4.7 *Let A be a non-empty set. Then*

(i) *$\mathcal{F}A$ has its (co)homology isomorphic to that of a point, i.e. $\mathfrak{P}_t(\mathcal{F}A) = 1$.*
(ii) *If $3 \leq \sharp A \leq \infty$, then $\mathfrak{P}_t(\dot{\mathcal{F}}A) = 1 + t^{\sharp A - 1}$.*
(iii) *If $\sharp A = 2$, then $\mathfrak{P}_t(\dot{\mathcal{F}}A) = 2$.*

Proof As A is not empty, $\mathcal{F}A$ is isomorphic to the cone over $\mathcal{F}A$ deprived of one of its elements. Point (i) then follows from Proposition 2.4.6. Let $n = \sharp A - 1$. The chain complex of $\mathcal{F}A$ looks like a sequence

$$0 \to C_n(\mathcal{F}A) \xrightarrow{\partial_n} C_{n-1}(\mathcal{F}A) \xrightarrow{\partial_{n-1}} \cdots \to C_0(\mathcal{F}A) \to 0\,,$$

which, by (i), is exact except at $C_0(\mathcal{F}A)$. One has $C_n(\mathcal{F}A) = \mathbb{Z}_2$, generated by the $A \in \mathcal{S}_n(\mathcal{F}A)$. Hence, $\ker \partial_{n-1} \approx \mathbb{Z}_2$. As the chain complex $C_*(\dot{\mathcal{F}}A)$ is the same as that of $\mathcal{F}A$ with C_n replaced by 0, this proves (ii). If $\sharp A = 2$, then $\dot{\mathcal{F}}A$ consists of two 0-simplexes and Point (iii) follows from (2.4.5) to (2.4.1). □

2.4.6 The Euler Characteristic

Let K be a finite simplicial complex. Its *Euler characteristic* $\chi(K)$ is defined as

$$\chi(K) = \sum_{m\in\mathbb{N}} (-1)^m \, \sharp\, \mathcal{S}_m(K) \in \mathbb{Z}\,.$$

Proposition 2.4.8 *Let K be a finite simplicial complex. Then*

$$\chi(K) = \sum_{m\in\mathbb{N}} (-1)^m \dim H_m(K) = \sum_{m} (-1)^m \dim H^m(K)\,.$$

As in the definition of the Poincaré polynomial, the number $\dim H_m(K)$ is the dimension of $H_m(K)$ as a $\mathbb{Z}_2$-vector space. In other words, $\dim H_m(K)$ is the m-th Betti number of K. Proposition 2.4.8 holds true for the (co)homology with coefficients in any field $\mathbb{F}$, though the Betti numbers depend individually on $\mathbb{F}$.

Proof By Kronecker duality, only the first equality requires a proof. Let c_m, z_m, b_m and h_m be the dimensions of $C_m(K)$, $Z_m(K)$, $B_m(K)$ and $H_m(K)$. Elementary linear algebra gives the equalities

$$\begin{cases} c_m = z_m + b_{m-1} \\ z_m = b_m + h_m\,. \end{cases}$$

We deduce that

$$\chi(K) = \sum_{m\in\mathbb{N}} (-1)^m c_m = \sum_{m\in\mathbb{N}} (-1)^m h_m + \sum_{m\in\mathbb{N}} (-1)^m b_m + \sum_{m\in\mathbb{N}} (-1)^m b_{m-1}\,.$$

As $b_{-1} = 0$, the last two sums cancels each other, proving Proposition 2.4.8. □

Corollary 2.4.9 *Let K be a finite simplicial complex. Then*

$$\chi(K) = \mathfrak{P}_t(K)_{t=-1}.$$

The following additive formula for the Euler characteristic is useful.

Lemma 2.4.10 *Let K be a simplicial complex. Let K_1 and K_2 be two subcomplexes of K such that $K = K_1 \cup K_2$. Then,*

$$\chi(K) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2)\,.$$

Proof The formula follows directly from the equations $\mathcal{S}_m(K) = \mathcal{S}_m(K_1) \cup \mathcal{S}_m(K_2)$ and $\mathcal{S}_m(K_1 \cap K_2) = \mathcal{S}_m(K_1) \cap \mathcal{S}_m(K_2)$. □

2.4.7 Surfaces

A *surface* is a manifold of dimension 2. In this section, we give examples of triangulations of surfaces and compute their (co)homology. Strictly speaking, the results would hold only for the given triangulations, but we allow us to formulate them in more general terms. For this, we somehow admit that

- a connected surface is a pseudomanifold of dimension 2. This will be established rigorously in Corollary 5.2.7 but the reader may find a proof as an exercise and this is easy to check for the particular triangulations given below.
- up to isomorphism, the (co)homology of a simplicial complex K depends only of the homotopy type of $|K|$. This will be proved in Sect. 3.6. In particular, the Euler characteristic of two triangulations of a surface coincide.

The 2-Sphere

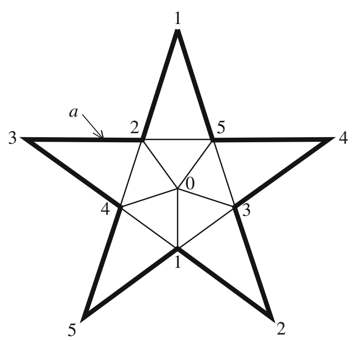
The 2-sphere S^2 being homeomorphic to the boundary of a 3-simplex, it follows from Corollary 2.4.7 that:

$$\mathfrak{P}_t(S^2) = 1 + t^2 .$$

The Projective Plane

The *projective plane* $\mathbb{R}P^2$ is the quotient of S^2 by the antipodal map. The triangulation of S^2 as a regular icosahedron being invariant under the antipodal map, it gives a triangulation of $\mathbb{R}P^2$ given in Fig. 2.2. Note that the border edges appear twice,

Fig. 2.2 A triangulation of $\mathbb{R}P^2$



showing as expected that $\mathbb{R}P^2$ is the quotient of a 2-disk modulo the antipodal involution on its boundary.

Being a quotient of an icosahedron, the triangulation of Fig. 2.2 has 6 vertices, 15 edges and 10 facets, thus $\chi(\mathbb{R}P^2) = 1$. Using that $\mathbb{R}P^2$ is a connected 2-dimensional pseudomanifold, we deduce that

$$\mathfrak{P}_t(\mathbb{R}P^2) = 1 + t + t^2 . \tag{2.4.8}$$

To identify the generators of $H^1(\mathbb{R}P^2) \approx \mathbb{Z}_2$ and $H_1(\mathbb{R}P^2)$, we define

$$a = \alpha = \big\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\big\} \subset \mathcal{S}_1(\mathbb{R}P^2) . \tag{2.4.9}$$

We see $a \in C^1(\mathbb{R}P^2)$ and $\alpha \in C_1(\mathbb{R}P^2)$. The cochain a is drawn in bold on Fig. 2.2, where it looks as the set of border edges, since each of its edges appears twice on the figure. It is easy to check that $\delta(a) = 0$ and $\partial(\alpha) = 0$. As $\sharp\alpha = 5$ is odd, one has $\langle a, \alpha\rangle = 1$, showing that a is the generator of $H^1(\mathbb{R}P^2) = \mathbb{Z}_2$ and α is the generator of $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$.

The 2-Torus

The 2-torus $T^2 = S^1 \times S^1$ is the quotient of a square whose opposite sides are identified. A triangulation of T^2 is described (in two copies) in Fig. 2.3. This triangulation has 9 vertices, 27 edges and 18 facets, which implies that $\chi(T^2) = 0$. Since T^2 is a connected 2-dimensional pseudomanifold, we deduce that

$$\mathfrak{P}_t(T^2) = (1 + t)^2 .$$

In Fig. 2.3 are drawn two chains $\alpha, \beta \in C_1(T^2)$ given by

$$\alpha = \big\{\{3, 8\}, \{8, 9\}, \{9, 3\}\big\} \ \text{ and } \ \beta = \big\{\{5, 7\}, \{7, 9\}, \{9, 5\}\big\} .$$

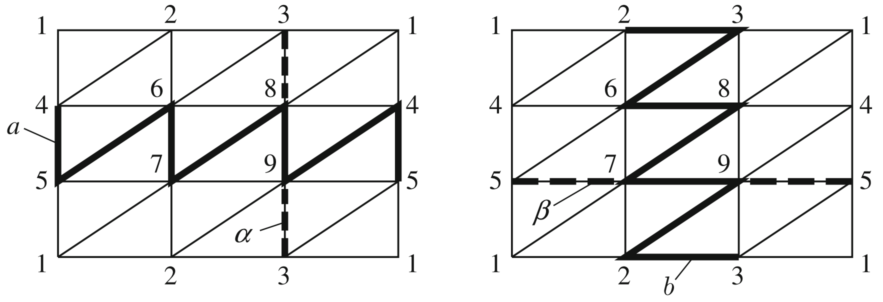


Fig. 2.3 Two copies of a triangulation of the 2-torus T^2, showing generators of $H^1(T^2)$ and $H_1(T^2)$

We also drew two cochains $a, b \in C^1(T^2)$ defined as

$$a = \big\{\{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 9\}, \{9, 4\}\big\}$$

and

$$b = \big\{\{2, 3\}, \{3, 6\}, \{6, 8\}, \{8, 7\}, \{7, 9\}, \{9, 2\}\big\} .$$

One checks that $\partial\alpha = \partial\beta = 0$ and that $\delta a = \delta b = 0$. Therefore, they represent classes $a, b \in H^1(T^2)$ and $\alpha, \beta \in H_1(T^2)$. The equalities

$$\langle a, \alpha\rangle = 1 \ , \ \langle a, \beta\rangle = 0 \ , \ \langle b, \alpha\rangle = 0 \ , \ \langle b, \beta\rangle = 1$$

imply that a, b is a basis of $H^1(T^2)$ and α, β is a basis of $H_1(T^2)$.

If we consider a and b as 1-chains (call them $\tilde{a}$ and $\tilde{b}$), we also have $\partial\tilde{a} = \partial\tilde{b} = 0$. Note that

$$\langle a, \tilde{b}\rangle = 1 \ , \ \langle a, \tilde{a}\rangle = 0 \ , \ \langle b, \tilde{b}\rangle = 0 \ , \ \langle b, \tilde{a}\rangle = 1$$

This proves that $\tilde{a} = \beta$ and $\tilde{b} = \alpha$ in $H_1(T^2)$.

The Klein Bottle

A triangulation of the Klein bottle K is pictured in Fig. 2.4. As the 2-torus, the Klein bottle is the quotient of a square with opposite side identified, one of these identifications "reversing the orientation". One checks that $\chi(K) = 0$. Since K is a connected 2-dimensional pseudomanifold, the (co)homology of K is abstractly isomorphic to that of T^2:

$$\mathfrak{P}_t(K) = (1 + t)^2$$

(In Chap. 3, $H^*(T^2)$ and $H^*(K)$ will be distinguished by their cup product: see p. 138). In Fig. 2.4 the dotted lines show two 1-chains $\alpha, \beta \in C_1(K)$ given by

$$\alpha = \big\{\{3, 8\}, \{8, 9\}, \{9, 3\}\big\} \ \text{ and } \ \beta = \big\{\{5, 7\}, \{7, 9\}, \{9, 5\}\big\} . \tag{2.4.10}$$

The bold lines describe two 1-cochains $a, b \in C^1(K)$ defined as

$$a = \big\{\{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 9\}, \{9, 5\}\big\} \tag{2.4.11}$$

and

$$b = \big\{\{2, 3\}, \{3, 6\}, \{6, 8\}, \{8, 7\}, \{7, 9\}, \{9, 2\}\big\} . \tag{2.4.12}$$

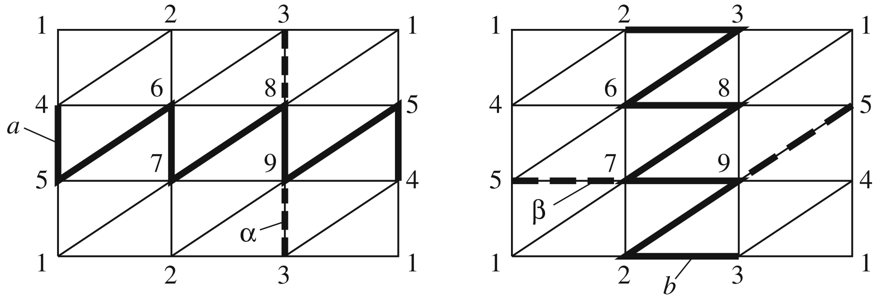


Fig. 2.4 Two copies of a triangulation of the Klein bottle K, showing generators of $H^1(K)$ and $H_1(K)$

One checks that $\partial\alpha = \partial\beta = 0$ and that $\delta a = \delta b = 0$. Therefore, they represent classes $a, b \in H^1(K)$ and $\alpha, \beta \in H_1(K)$. The equalities

$$\langle a, \alpha\rangle = 1\ ,\ \langle a, \beta\rangle = 1\ ,\ \langle b, \alpha\rangle = 0\ ,\ \langle b, \beta\rangle = 1$$

imply that a, b is a basis of $H^1(K)$ and α, β is a basis of $H_1(K)$.

As in the case of T^2, we may regard a and b as 1-chains (call them $\tilde{a}$ and $\tilde{b}$). Here $\partial\tilde{b} = 0$ but $\partial\tilde{a} = \{4\} + \{5\} \neq 0$.

Other Surfaces

Let K_1 and K_2 be two simplicial complexes such that $|K_1|$ and $|K_2|$ are surfaces. A simplicial complex L with $|L|$ homeomorphic to the connected sum $|K_1|\sharp|K_2|$ may be obtained in the following way: choose 2-simplexes $\sigma_1 \in K_1$ and $\sigma_2 \in K_2$. Let $L_i = K_i - \sigma_i$ and let L be obtained by taking the disjoint union of L_1 and L_2 and identifying $\dot{\sigma}_1$ with $\dot{\sigma}_2$. Thus, $L = L_1 \cup L_2$ and $L_0 = L_1 \cap L_2$ is isomorphic to the boundary of a 2-simplex.

By Lemma 2.4.10, one has

$$\begin{aligned}\chi(L) &= \chi(L_1) + \chi(L_2) - \chi(L_0)\\ &= \chi(K_1) - 1 + \chi(K_2) - 1 - 0\\ &= \chi(K_1) + \chi(K_2) - 2\,. \end{aligned} \tag{2.4.13}$$

The *orientable surface* Σ_g *of genus* g is defined as the connected sum of g copies of the torus T^2. By Formula (2.4.13), one has

$$\chi(\Sigma_g) = 2 - 2g\,. \tag{2.4.14}$$

As Σ_g is a 2-dimensional connected pseudomanifold, one has

$$\mathfrak{P}_t(\Sigma_g) = 1 + 2gt + t^2 .$$

The *nonorientable surface* $\bar{\Sigma}_g$ *of genus* g is defined as the connected sum of g copies of $\mathbb{R}P^2$. For instance, $\bar{\Sigma}_1 = \mathbb{R}P^2$ and $\bar{\Sigma}_2$ is the Klein bottle. Formula (2.4.13) implies

$$\chi(\bar{\Sigma}_g) = 2 - g . \tag{2.4.15}$$

As $\bar{\Sigma}_g$ is a 2-dimensional connected pseudomanifold, one has

$$\mathfrak{P}_t(\bar{\Sigma}_g) = 1 + gt + t^2 .$$

2.5 The Homomorphism Induced by a Simplicial Map

Let $f : K \to L$ be a simplicial map between the simplicial complexes K and L. Recall that f is given by a map $f : V(K) \to V(L)$ such that $f(\sigma) \in \mathcal{S}(L)$ if $\sigma \in \mathcal{S}(K)$, i.e. the image of an m-simplex of K is an n-simplex of L with $n \leq m$. We define $C_* f : C_*(K) \to C_*(L)$ as the degree 0 linear map such that, for all $\sigma \in \mathcal{S}_m(K)$, one has

$$C_* f(\sigma) = \begin{cases} f(\sigma) & \text{if } f(\sigma) \in \mathcal{S}_m(L) \quad \text{(i.e. if } f_{|\sigma} \text{ is injective)} \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.1}$$

We also define $C^* f : C^*(L) \to C^*(K)$ by setting, for $a \in C^m(L)$,

$$C^* f(a) = \big\{ \sigma \in \mathcal{S}_m(K) \mid f(\sigma) \in a \big\} . \tag{2.5.2}$$

In the following lemma, we use the same notation for the (co)boundary operators ∂ and δ and the Kronecker product $\langle\ ,\ \rangle$, both for K of for L.

Lemma 2.5.1 *Let $f : K \to L$ be a simplicial map. Then*

(a) $C_* f \circ \partial = \partial \circ C_* f$.
(b) $\delta \circ C^* f = C^* f \circ \delta$.
(c) $\langle C^* f(b), \alpha \rangle = \langle b, C_* f(\alpha) \rangle$ *for all* $b \in C^*(L)$ *and all* $\alpha \in C_*(K)$.

In other words, the couple $(C_* f, C^* f)$ is a morphism of Kronecker pairs.

Proof To prove (a), let $\sigma \in \mathcal{S}_m(K)$. If f restricted to σ is injective, it is straightforward that $C_* f \circ \partial(\sigma) = \partial \circ C_* f(\sigma)$. Otherwise, we have to show that $C_* f \circ \partial(\sigma) = 0$. Let us label the vertices $v_0, v_1, \ldots, v_m$ of σ in such a way that $f(v_0) = f(v_1)$. Then, $C_* f \circ \partial(\sigma)$ is a sum of two terms: $C_* f \circ \partial(\sigma) = C_* f(\tau_0) + C_* f(\tau_1)$, where $\tau_0 = \{v_1, v_2, \ldots, v_m\}$ and $\tau_1 = \{v_0, v_2, \ldots, v_m\}$. As $C_* f(\tau_0) = C_* f(\tau_1)$, one has

$C_* f \circ \partial(\sigma) = 0$. Thus, Point (a) is established. Point (c) can be easily deduced from Definitions (2.5.1) and (2.5.2), taking for α a simplex of K. Point (b) then follows from Points (a) and (c), using Lemma 2.3.6 and its proof. □

By Lemma 2.5.1 and Proposition 2.3.5, the couple $(C_* f, C^* f)$ determines linear maps of degree zero

$$H_* f : H_*(K) \to H_*(L) \text{ and } H^* f : H^*(L) \to H^*(K)$$

such that

$$\langle H^* f(a), \alpha\rangle = \langle a, H_* f(\alpha)\rangle \text{ for all } a \in H^*(L) \text{ and } \alpha \in H_*(K)\,. \tag{2.5.3}$$

Lemma 2.5.2 (Functoriality) *Let $f\colon ZK \to L$ and $g\colon L \to M$ be simplicial maps. Then $H_*(g \circ f) = H_* g \circ H_* f$ and $H^*(g \circ f) = H^* f \circ H^* g$. Also $H_*\mathrm{id}_K = \mathrm{id}_{H_*(K)}$ and $H^*\mathrm{id}_K = \mathrm{id}_{H^*(K)}$*

In other words, H^* and H_* are functors from the simplicial category **Simp** to the category **GrV** of graded vector spaces and degree 0 linear maps. The cohomology is contravariant and the homology is covariant.

Proof For $\sigma \in \mathcal{S}(K)$, the formula $C_*(g \circ f)(\sigma) = C_* g \circ C_* f(\sigma)$ follows directly from Definition (2.5.1). Therefore $C_*(g \circ f) = C_* g \circ C_* f$ and then $H_*(g \circ f) = H_* g \circ H_* f$. The corresponding formulae for cochains and cohomology follow from Point (c) of Lemma 2.5.1. The formulae for id_K is obvious. □

Simplicial maps and components. Let K be a simplicial complex. For each component $L \in \pi_0(K)$ of K, the inclusion $i_L\colon L \to K$ is a simplicial map. The results of Sect. 2.4.1 may be strengthened as follows.

Proposition 2.5.3 *Let K be a simplicial complex. The family of simplicial maps i_L: $L \to K$ for $L \in \pi_0(K)$ gives rise to isomorphisms*

$$H^*(K) \xrightarrow[\approx]{(H^* i_L)} \prod_{L \in \pi_0(K)} H^*(L)$$

and

$$\bigoplus_{L \in \pi_0(K)} H_*(L) \xrightarrow[\approx]{\sum H_* i_L} H_*(K)\,.$$

The homomorphisms $H^0 f$ and $H_0 f$. We use the same notation $\mathbf{1} \in H^0(K)$ and $\mathbf{1} \in H^0(L)$ for the classes given by the unit cochains.

Lemma 2.5.4 *Let $f\colon K \to L$ be a simplicial map. Then $H^0 f(\mathbf{1}) = \mathbf{1}$.*

Proof The formula $C^0 f(\mathbf{1}) = \mathbf{1}$ in $C^0(K)$ follows directly from Definition (2.5.2). □

Corollary 2.5.5 *Let $f\colon K \to L$ be a simplicial map with K and L connected. Then*

$$H^0 f : \mathbb{Z}_2 = H^0(L) \to H^0(K) = \mathbb{Z}_2$$

and

$$H_0 f : \mathbb{Z}_2 = H_0(K) \to H_0(L) = \mathbb{Z}_2$$

are the identity isomorphism.

Proof By Proposition 2.4.1, the generator of $H^0(L)$ (or $H^0(K)$) is the unit cocycle **1**. By Lemma 2.5.4, this proves the cohomology statement. The homology statement also follows from Proposition 2.4.1, since $H_0(K)$ and $H_0(L)$ are generated by a cycle consisting of a single vertex. □

More generally, one has $H^0(L) \approx \mathbb{Z}_2^{\pi_0(L)}$ and $H^0(K) \approx \mathbb{Z}_2^{\pi_0(K)}$ by Corollary 2.4.2. Using this and Lemma 2.5.4, one gets the following corollary.

Corollary 2.5.6 *Let $f\colon K \to L$ be a simplicial map. Then $H^0 f\colon \mathbb{Z}_2^{\pi_0(L)} \to \mathbb{Z}_2^{\pi_0(K)}$ is given by $H^0 f(\lambda) = \lambda \circ \pi_0 f$.*

The degree of a map. Let $f\colon K \to L$ be a simplicial map between two finite connected n-dimensional pseudomanifolds. Define the *degree* $\deg(f) \in \mathbb{Z}_2$ by

$$\deg(f) = \begin{cases} 0 & \text{if } H^n f = 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2.5.4}$$

By Proposition 2.4.4, $H^n(K) \approx H^n(L) \approx \mathbb{Z}_2$. Thus, $\deg(f) = 1$ if and only if $H^n f$ is the (only possible) isomorphism between $H^n(K)$ and $H^n(L)$. By Kronecker duality, the homomorphism $H_n f$ may be used instead of $H^n f$ in the definition of $\deg(f)$. Our degree is sometimes called the mod 2 *degree*, since, for oriented pseudomanifolds, it is the mod 2 reduction of a degree defined in $\mathbb{Z}$ (see, e.g. [179, Exercises of Chap. 4]).

Let $f\colon K \to L$ be a simplicial map between two finite n-dimensional pseudomanifolds. For $\sigma \in \mathcal{S}_n(L)$, define

$$d(f, \sigma) = \sharp\{\tau \in \mathcal{S}_n(K) \mid f(\tau) = \sigma\} \in \mathbb{N}. \tag{2.5.5}$$

As an example, let $K = L = \mathcal{P}_4$, the polygon of Example 2.4.3 with 4 edges. Let f be defined by $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, $f(3) = 1$. Then, $d(f, \{0, 1\}) = d(f, \{1, 2\}) = 2$, $d(f, \{2, 3\}) = d(f, \{3, 0\}) = 0$ and $\deg(f) = 0$. This example illustrates the following proposition.

Proposition 2.5.7 *Let $f\colon K \to L$ be a simplicial map between two finite n-dimensional pseudomanifolds which are connected. For any $\sigma \in \mathcal{S}_n(L)$, one has*

$$\deg(f) = d(f, \sigma) \bmod 2\,.$$

Proof By Proposition 2.4.4, $H^n(L) = \mathbb{Z}_2$ is generated by the cocycle formed by the singleton σ and $C^n f(\sigma)$ represents the non-zero element of $H^n(K)$ if and only if $\sharp C^n f(\sigma) = d(f, \sigma)$ is odd. □

The interest of Proposition 2.5.7 is 2-fold: first, it tells us that $\deg(f)$ may be computed using any $\sigma \in \mathcal{S}_m(L)$ and, second, it asserts that $d(f, \sigma)$ is independent of σ. Proposition 2.5.7 is the mod 2 context of the identity between the degree introduced by Brouwer in 1910, [22, p.419], and its homological interpretation due to Hopf in 1930, [98, Sect. 2]. For a history of the notion of the degree of a map, see [40, pp.169–175].

Example 2.5.8 Let $f: T^2 \to K$ be the two-fold cover of the Klein bottle K by the 2-torus T^2, given in Fig. 2.5. In formulae: $f(i) = i = f(\bar{i})$ for $i = 1, \dots, 9$.

The 1-dimensional (co)homology vector spaces of T^2 and K admit the bases:

(i) $\tilde{\mathcal{V}} = \{[\tilde{a}], [\tilde{b}]\} \subset H^1(T^2)$, where $\tilde{a}$ is drawn in Fig. 2.5 and

$$\tilde{b} = \big\{\{2,3\}, \{3,6\}, \{6,8\}, \{8,7\}, \{7,9\}, \{9,2\}\big\}.$$

(ii) $\tilde{\mathcal{W}} = \{[\tilde{\alpha}], [\tilde{\beta}]\} \subset H_1(T^2)$, where $\tilde{\alpha}$ is drawn in Fig. 2.5 and

$$\tilde{\beta} = \big\{\{5,7\}, \{7,9\}, \{9,\bar{4}\}, \{\bar{4},\bar{7}\}, \{\bar{7},\bar{9}\}, \{\bar{9},5\}\big\}.$$

(iii) $\mathcal{V} = \{[a], [b]\} \subset H^1(K)$, where a and b are defined in Eqs. (2.4.11) and (2.4.12) (a drawn in Fig. 2.5).
(iv) $\mathcal{W} = \{[\alpha], [\beta]\} \subset H_1(K)$, where α and β are defined in Eq. (2.4.10) (α drawn in Fig. 2.5).

The matrices for C^*f and C_*f in these bases are

$$C^*f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } C_*f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that, under the isomorphism $\mathbf{k}: H^1(-) \xrightarrow{\approx} H_1(-)^\sharp$, the bases $\tilde{\mathcal{V}}$ and $\mathcal{V}$ are dual of $\tilde{\mathcal{W}}$ and $\mathcal{W}$; therefore, the matrix of C^*f is the transposed of that of C_*f.

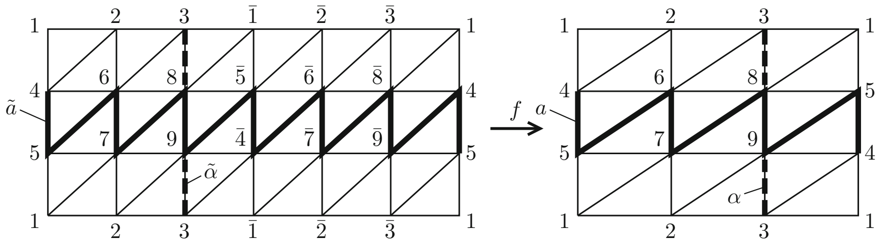


Fig. 2.5 Two-fold cover $f: T^2 \to K$ over the triangulation K of the Klein bottle given in Fig. 2.4

Now, T^2 and K are 2-dimensional pseudomanifolds and $d(f, \sigma) = 2$ for each $\sigma \in \mathcal{S}_2(K)$. By Proposition 2.5.7, $\deg(f) = 0$ and both $H^* f : H^2(K) \to H^2(T^2)$ and $H_* f : H_2(T^2) \to H_2(K)$ vanish.

Contiguous maps. Two simplicial maps $f, g \colon K \to L$ are called *contiguous* if $f(\sigma) \cup g(\sigma) \in \mathcal{S}(L)$ for all $\sigma \in \mathcal{S}(K)$. We denote by $\tau(\sigma)$ the subcomplex of L generated by the simplex $f(\sigma) \cup g(\sigma) \in \mathcal{S}(L)$. For example, the inclusion $K \hookrightarrow CK$ of a simplicial complex K into its cone and the constant map of K onto the cone vertex of CK are contiguous.

Proposition 2.5.9 *Let $f, g \colon K \to L$ be two simplicial maps which are contiguous. Then $H_* f = H_* g$ and $H^* f = H^* g$.*

Proof By Kronecker Duality, using Diagram (2.3.4), it is enough to prove that $H_* f = H_* g$. By induction on m, we shall prove the following property:

Property $\mathcal{H}(m)$: there exists a linear map $D \colon C_m(K) \to C_{m+1}(L)$ such that:

(i) $\partial D(\alpha) + D(\partial \alpha) = C_* f(\alpha) + C_* g(\alpha)$ for each $\alpha \in C_m(K)$.
(ii) for each $\sigma \in \mathcal{S}_m(K)$, $D(\sigma) \in C_{m+1}(\tau(\sigma)) \subset C_{m+1}(L)$.

We first prove that Property $\mathcal{H}(m)$ for all m implies that $H_* f = H_* g$. Indeed, we would then have a linear map $D \colon C_*(K) \to C_{*+1}(L)$ satisfying

$$C_* f + C_* g = \partial \circ D + D \circ \partial \,. \tag{2.5.6}$$

Such a map D is called a *chain homotopy* from $C_* f$ to $C_* g$. Let $\beta \in Z_*(K)$. By Eq. (2.5.6), one has $C_* f(\beta) + C_* g(\beta) = \partial D(\beta)$ which implies that $H_* f([\beta]) + H_* g([\beta])$ in $H_*(L)$.

We now prove that $\mathcal{H}(0)$ holds true. We define $D \colon C_0(K) \to C_1(L)$ as the unique linear map such that, for $v \in V(K)$:

$$D(\{v\}) = \begin{cases} \{f(v), g(v)\} = \tau(\{v\}) & \text{if } f(v) \neq g(v) \\ 0 & \text{otherwise.} \end{cases}$$

Formula (i) being true for any $\{v\} \in \mathcal{S}_0(K)$, it is true for any $\alpha \in C_0(K)$. Formula (ii) is obvious.

Suppose that $\mathcal{H}(m-1)$ holds true for $m \geq 1$. We want to prove that $\mathcal{H}(m)$ also holds true. Let $\sigma \in \mathcal{S}_m(K)$. Observe that $D(\partial \sigma)$ exists by $\mathcal{H}(m-1)$. Consider the chain $\zeta \in C_m(L)$ defined by

$$\zeta = C_* f(\sigma) + C_* g(\sigma) + D(\partial \sigma)$$

Using $\mathcal{H}(m-1)$, one has

$$\begin{aligned} \partial \zeta &= \partial C_* f(\sigma) + \partial C_* g(\sigma) + \partial D(\partial \sigma) \\ &= C_* f(\partial \sigma) + C_* g(\partial \sigma) + D(\partial \partial \sigma) + C_* f(\partial \sigma) + C_* g(\partial \sigma) \\ &= 0 \,. \end{aligned}$$

On the other hand, $\zeta \in C_m(\tau(\sigma))$. As $m \geq 1$, $H_m(\tau(\sigma)) = 0$ by Corollary 2.4.7. There exists then $\eta \in C_{m+1}(\tau(\sigma))$ such that $\zeta = \partial\eta$. Choose such an η and set $D(\sigma) = \eta$. This defines $D\colon C_m(K) \to C_{m+1}(L)$ which satisfies (i) and (ii), proving $\mathcal{H}(m)$. □

Remark 2.5.10 The chain homotopy D in the proof of Proposition 2.5.9 is not explicitly defined. This is because several of these exist and there is no canonical way to choose one (see [155, p. 68]). The proof of Proposition 2.5.9 is an example of the technique of acyclic carriers which will be developed in Sect. 2.9.

Remark 2.5.11 Let $f, g\colon K \to L$ be two simplicial maps which are contiguous. Then $|f|, |g|\colon |K| \to |L|$ are homotopic. Indeed, the formula $F(\mu, t) = (1-t)|f|(\mu) + t|g|(\mu)$ ($t \in [0, 1]$) makes sense and defines a homotopy from $|f|$ to $|g|$.

2.6 Exact Sequences

In this section, we develop techniques to obtain long (co)homology exact sequences from short exact sequences of (co)chain complexes. The results are used in several forthcoming sections. All vector spaces in this section are over a fixed arbitrary field $\mathbb{F}$.

Let (C_1^*, δ_1), (C_2^*, δ_2) and (C^*, δ) be cochain complexes of vector spaces, giving rise to cohomology graded vector spaces H_1^*, H_2^* and H^*. We consider morphisms of cochain complexes $J\colon C_1^* \to C^*$ and $I\colon C^* \to C_2^*$ so that

$$0 \to C_1^* \xrightarrow{J} C^* \xrightarrow{I} C_2^* \to 0 \tag{2.6.1}$$

is an exact sequence. We call (2.6.1) a *short exact sequence of cochain complexes*. Choose a **GrV**-morphism $S\colon C_2^* \hookrightarrow C^*$ which is a section of I. The section S cannot be assumed in general to be a morphism of cochain complexes. The linear map $\delta \circ S\colon C_2^m \to C^{m+1}$ satisfies

$$I \circ \delta \circ S(a) = \delta_2 \circ I \circ S(a) = \delta_2(a)\,,$$

thus $\delta \circ S(Z_2^m) \subset J(C_1^{m+1})$. We can then define a linear map $\tilde{\delta}^*\colon Z_2^m \to C_1^{m+1}$ by the equation

$$J \circ \tilde{\delta}^* = \delta \circ S\,. \tag{2.6.2}$$

If $a \in Z_2^m$, then $J \circ \delta_1(\tilde{\delta}^*(a)) = \delta \circ \delta(S(a)) = 0$. Therefore, $\tilde{\delta}^*(Z_2^m) \subset Z_1^{m+1}$. Moreover, if $b \in C_2^{m-1}$ and $a = \delta_2(b)$, then

$$I \circ \delta \circ S(b) = \delta_2 \circ I \circ S(b) = a\,,$$

whence $\delta \circ S(b) = S(a) + J(c)$ for some $c \in C_1^m$. Therefore $\tilde{\delta}^*(a) = \delta_1(c)$, which shows that $\tilde{\delta}^*(B_2^*) \subset B_1^*$. Hence, $\tilde{\delta}^*$ induces a linear map

$$\delta^* : H_2^* \to H_1^{*+1}$$

which is called the *cohomology connecting homomorphism* for the short exact sequence (2.6.1).

Lemma 2.6.1 *The connecting homomorphism $\delta^* : H_2^* \to H_1^{*+1}$ does not depend on the linear section S.*

Proof Let $S' \colon C_2^m \to C^m$ be another section of I, giving rise to $\tilde{\delta}'^* \colon Z_2^m \to Z_1^{m+1}$, via the equation $J \circ \tilde{\delta}'^* = \delta \circ S'$. Let $a \in Z_2^m$. Then

$$S'(a) = S(a) + J(u)$$

for some $u \in C_1^m$. Therefore, the equations

$$J \circ \tilde{\delta}'^*(a) = \delta(S(a)) + \delta(J(u)) = \delta(S(a)) + J(\delta_1(u))$$

hold in C^{m+1}. This implies that $\tilde{\delta}'^*(a) = \tilde{\delta}^*(a) + \delta_1(u)$ in Z_1^{m+1}, and then $\delta'^*(a) = \delta^*(a)$ in H_1^{m+1}. □

Proposition 2.6.2 *The long sequence*

$$\cdots \to H_1^m \xrightarrow{H^*J} H^m \xrightarrow{H^*I} H_2^m \xrightarrow{\delta^*} H_1^{m+1} \xrightarrow{H^*J} \cdots$$

is exact.

The exact sequence of Proposition 2.6.2 is called the *cohomology exact sequence*, associated to the short exact of cochain complexes (2.6.1).

Proof The proof involves 6 steps.

1. $H^*I \circ H^*J = 0$ As $H^*I \circ H^*J = H^*(I \circ J)$, this comes from that $I \circ J = 0$.
2. $\delta^* \circ H^*I = 0$ Let $b \in Z^m$. Then $I(b + S(I(b))) = 0$. Hence, $b + S(I(b)) = J(c)$ for some $c \in C_1^m$. Therefore,

   $$J \circ \tilde{\delta}^* \circ I(b) = \delta(S(I(b)) = \delta(b + J(c)) = \delta(b) + J \circ \delta_1(c) = J \circ \delta_1(c)\,,$$

   which proves that $\tilde{\delta}^* \circ I(b) = \delta_1(c)$, and then $\delta^* \circ H^*I = 0$ in H_1^*.
3. $H^*J \circ \delta^* = 0$ Let $a \in Z_2^m$. Then, $J \circ \tilde{\delta}^*(a) = \delta(S(a)) \subset B^{m+1}$, so $H^*J \circ \delta^*([a]) = 0$ in $H^{m+1}(K)$.
4. $\ker H^*J \subset \mathrm{Image}\, \delta^*$ Let $a \in Z_1^{m+1}$ representing $[a] \in \ker H^*J$. This means that $J(a) = \delta(b)$ for some $b \in C^m$. Then, $I(b) \in Z_2^m$ and $S(I(b)) = b + J(c)$

for some $c \in C_1^m$. Therefore,

$$\delta \circ S \circ I(b) = \delta(b) + \delta(J(c)) = J(a) + J(\delta_1(c)) .$$

As J is injective, this implies that $\tilde{\delta}^*(I(c)) = a + \delta_1(c)$, proving that $\delta^*([I(c)]) = [a]$.

5. $\ker H^*I \subset \operatorname{Image} H^*J$ Let $a \in Z^m$ representing $[a] \in \ker H^*I$. This means that $I(a) = \delta_2(b)$ for some $b \in C_2^{m-1}$. Let $c = \delta(S(b)) \in C^m$. One has $I(a+c) = 0$, so $a+c = J(e)$ for some $e \in C_1^m$. As $\delta(a+c) = 0$ and J is injective, the cochain e is in Z_1^m. As $c \in B^m$, $H^*J([e]) = [a]$ in H^m.
6. $\ker \delta^* \subset \operatorname{Image} H^*I$ Let $a \in Z_2^m$ representing $[a] \in \ker \delta^*$. This means that $\tilde{\delta}^*(a) = \delta_1(b)$ for some $b \in C_1^m$. In other words,

$$\delta(S(a)) = J(\delta_1(b)) = \delta(J(b)) .$$

Hence, $c = J(b) + S(a) \in Z^m$ and $H^*I([c]) = [a]$. □

We now prove the naturality of the connecting homomorphism in cohomology. We are helped by the following intuitive interpretation of δ^*: first, we consider C_1^* a cochain subcomplex of C^* via the injection J. Second, a cocycle $a \in Z_2^m$ may be represented by a cochain in $\tilde{a} \in C^m$ such that $\delta(\tilde{a}) \in C_1^*$. Then, $\delta^*([a]) = [\delta(\tilde{a})]$. More precisely:

Lemma 2.6.3 *Let*

$$0 \to C_1^* \xrightarrow{J} C^* \xrightarrow{I} C_2^* \to 0$$

be a short exact sequence of cochain complexes. Then

(a) $I^{-1}(Z_2^m) = \{b \in C^m \mid \delta(b) \in J(C_1^{m+1})\}$.
(b) *Let* $a \in Z_2^m$ *representing* $[a] \in H_2^m$. *Let* $b \in C^m$ *with* $I(b) = a$. *Then* $\delta^*([a]) = [J^{-1}(\delta(b))]$ *in* H_1^{m+1}.

Proof Point (a) follows from the fact that I is surjective and from the equality $\delta_2 \circ I = I \circ \delta$. For Point (b), choose a section $S\colon C_2^m \to C_m$ of I. By Lemma 2.6.1, $\delta^*([a]) = [J^{-1}(\delta(S(a))]$. The equality $I(b) = a$ implies that $b = S(a) + J(c)$ for some $c \in C_1^m$. Therefore,

$$[J^{-1}(\delta(b))] = [J^{-1}(\delta \circ S(a))] + [\delta_1(c)] = \delta^*([a]) . \qquad \square$$

Let us consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{C}_1^* & \xrightarrow{\bar{J}} & \bar{C}^* & \xrightarrow{\bar{I}} & \bar{C}_2^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F_2} & & \\ 0 & \longrightarrow & C_1^* & \xrightarrow{J} & C^* & \xrightarrow{I} & C_2^* & \longrightarrow & 0 \end{array} \tag{2.6.3}$$

of morphisms of cochain complexes, where the horizontal sequences are exact. This gives rise to two connecting homomorphisms $\bar\delta^*\colon \bar H_2^* \to \bar H_1^{*+1}$ and $\delta^*\colon H_2^* \to H_1^{*+1}$.

Lemma 2.6.4 (Naturality of the cohomology exact sequence) *The diagram*

$$
\begin{tikzcd}
\cdots \arrow[r] & \bar H_1^m \arrow[r, "H^*\bar J"] \arrow[d, "H^*F_1"] & \bar H^m \arrow[r, "H^*\bar I"] \arrow[d, "H^*F"] & \bar H_2^m \arrow[r, "\bar\delta^*"] \arrow[d, "H^*F_2"] & \bar H_1^{m+1} \arrow[r, "H^*\bar J"] \arrow[d, "H^*F_1"] & \cdots \\
\cdots \arrow[r] & H_1^m \arrow[r, "H^*J"] & H^m \arrow[r, "H^*I"] & H_2^m \arrow[r, "\delta^*"] & H_1^{m+1} \arrow[r, "H^*J"] & \cdots
\end{tikzcd}
$$

is commutative.

Proof The commutativity of two of the square diagrams follows from the functoriality of the cohomology: $H^*F \circ H^*\bar J = H^*J \circ H^*F_1$ since $F \circ \bar J = J \circ F_1$ and $H^*F_2 \circ H^*\bar I = H^*I \circ H^*F$ since $F_2 \circ \bar I = I \circ F$. It remains to prove that $H^*F_1 \circ \bar\delta^* = H^*\delta^* \circ F_2$.

Let $a \in \bar Z_2^m$ representing $[a] \in \bar H_2^m$. Let $b \in \bar C^m$ with $\bar I(b) = a$. Then, $I \circ F(b) = F_2(a)$. Using Lemma 2.6.3, one has

$$
\begin{aligned}
\delta^* \circ H^*F_2([a]) &= [J^{-1} \circ \delta \circ F(b)] \\
&= [J^{-1} \circ F \circ \bar\delta(b)] \\
&= [F_1 \circ \bar J^{-1} \circ \bar\delta(b)] \\
&= H^*F_1 \circ \bar\delta^*([a]) .
\end{aligned}
$$
□

We are now interested in the case where the cochain complexes (C_i^*, δ_i) and (C^*, δ) are parts of Kronecker pairs

$$
\mathcal{P}_1 = \big((C_1^*, \delta_1), (C_{*,1}, \partial_1), \langle\ ,\ \rangle_1\big) \quad , \quad \mathcal{P}_2 = \big((C_2^*, \delta_2), (C_{*,2}, \partial_2)\langle\ ,\ \rangle_2\big)
$$

and

$$
\mathcal{P} = \big((C^*, \delta), (C_*, \partial), \langle\ ,\ \rangle\big) .
$$

Let us consider two morphism of Kronecker pairs, (J, j) from $\mathcal{P}$ to $\mathcal{P}_1$ and (I, i) from $\mathcal{P}_2$ to $\mathcal{P}$. We suppose that the two sequences

$$
0 \to C_1^* \xrightarrow{J} C^* \xrightarrow{I} C_2^* \to 0 \tag{2.6.4}
$$

and

$$
0 \to C_{*,2} \xrightarrow{i} C_* \xrightarrow{j} C_{*,1} \to 0 \tag{2.6.5}
$$

are exact sequences of (co)chain complexes. Note that, by Lemma 2.3.8, (2.6.4) is exact if and only if (2.6.5) is exact. Exact sequence (2.6.4) gives rise to the cohomology connecting homomorphism $\delta^*\colon H_2^* \to H_1^{*+1}$. We construct a homology connecting homomorphism in the same way. Choose a linear section $s\colon C_{*,1} \to C_*$ of j, not required to be a morphism of chain complexes. As in the cohomology setting, one can defines $\tilde{\partial}_*\colon Z_{m+1,1} \to Z_{m,2}$ by the equation

$$i \circ \tilde{\partial}_* = \partial \circ s\,. \tag{2.6.6}$$

We check that $\tilde{\partial}_*(B_{m+1,1}) \subset B_{m,2}$. Hence $\tilde{\partial}_*$ induces a linear map

$$\partial_* : H_{*+1,1} \to H_{*,2}$$

called the *homology connecting homomorphism* for the short exact sequence (2.6.5).

Lemma 2.6.5 *The connecting homomorphism $\partial_*\colon H_{*+1,1} \to H_{*,2}$ does not depend on the linear section s.*

Proof The proof is analogous to that of Lemma 2.6.1 and is left as an exercise to the reader. □

Lemma 2.6.6 *The connecting homomorphisms $\delta^*\colon H_2^m \to H_1^{m+1}$ and $\partial_*\colon H_{m+1,1} \to H_{m,1}$ satisfy the equation*

$$\langle \delta^*(a), \alpha\rangle_1 = \langle a, \partial_*(\alpha)\rangle_2$$

for all $a \in H_2^m$, $\alpha \in H_{m+1,1}$ and all $m \in \mathbb{N}$. In other words, (δ^*, ∂_*) is a morphism of Kronecker pairs from $(H_1^*, H_{*,1}, \langle\,,\rangle_1)$ to $(H_2^*, H_{*,2}, \langle\,,\rangle_2)$.

Proof Let $\tilde{a} \in Z_2^m$ represent a and $\tilde{\alpha} \in Z_{m+1,1}$ represent α. Choose linear sections S and s of I and j. Using Formulae (2.6.2) and (2.6.6), one has

$$\begin{aligned}
\langle \delta^*(a), \alpha\rangle_1 &= \langle \tilde{\delta}^*(\tilde{a}), \tilde{\alpha}\rangle_1 \\
&= \langle \tilde{\delta}^*(\tilde{a}), j \circ s(\tilde{\alpha})\rangle_1 \\
&= \langle J \circ \tilde{\delta}^*(\tilde{a}), s(\tilde{\alpha})\rangle \\
&= \langle S(\tilde{a}), \partial \circ s(\tilde{\alpha})\rangle \\
&= \langle S(\tilde{a}), i \circ \tilde{\partial}_*(\tilde{\alpha})\rangle \\
&= \langle I \circ S(\tilde{a}), \tilde{\partial}_*(\tilde{\alpha})\rangle_2 \\
&= \langle \tilde{a}, \tilde{\partial}_*(\tilde{\alpha})\rangle_2 = \langle a, \partial_*(\alpha)\rangle_2\,.
\end{aligned}$$

□

Proposition 2.6.7 *The long sequence*

$$\cdots \to H_{m,2} \xrightarrow{H_* i} H_m \xrightarrow{H_* j} H_{m,1} \xrightarrow{\partial_*} H_{m-1,2} \xrightarrow{H_* i} \cdots$$

is exact.

The exact sequence of Proposition 2.6.7 is called the *homology exact sequence* associated to the short exact of chain complexes (2.6.5). It can be established directly, in an analogous way to that of Proposition 2.6.2. To make a change, we shall deduce Proposition 2.6.7 from Proposition 2.6.2 by Kronecker duality.

Proof By our hypotheses couples (I, i) and (J, j) are morphisms of Kronecker pairs, and so is (δ^*, ∂_*) by Lemma 2.6.6. Using Diagram (2.3.4), we get a commutative diagram

$$\begin{tikzcd}
\cdots & (H_{m,2})^\sharp \arrow[l] & (H_m)^\sharp \arrow[l, "(H_*i)^\sharp"'] & (H_{m,1})^\sharp \arrow[l, "(H_*j)^\sharp"'] & H^\sharp_{m-1,2} \arrow[l, "\partial_*^\sharp"'] & \cdots \arrow[l] \\
\cdots & H_1^m \arrow[l] \arrow[u, "\approx"', "\mathbf{k}"] & H^m \arrow[l, "H^*I"'] \arrow[u, "\approx"', "\mathbf{k}"] & H_1^m \arrow[l, "H^*J"'] \arrow[u, "\approx"', "\mathbf{k}"] & H_2^{m-1} \arrow[l, "\delta^*"'] \arrow[u, "\approx"', "\mathbf{k}"] & \cdots \arrow[l]
\end{tikzcd}.$$

By Proposition 2.6.2, the bottom sequence of the above diagram is exact. Thus, the top sequence is exact. By Lemma 2.3.8, the sequence of Proposition 2.6.7 is exact. □

Let us consider commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \bar{C}_1^* \arrow[r, "\bar{J}"] & \bar{C}^* \arrow[r, "\bar{I}"] & \bar{C}_2^* \arrow[r] & 0 \\
0 \arrow[r] & C_1^* \arrow[r, "J"] \arrow[u, leftarrow, "F_1"] & C^* \arrow[r, "I"] \arrow[u, leftarrow, "F"] & C_2^* \arrow[r] \arrow[u, leftarrow, "F_2"] & 0
\end{tikzcd} \tag{2.6.7}$$

and

$$\begin{tikzcd}
0 & \bar{C}_{*,1} \arrow[l] & \bar{C}_* \arrow[l, "\bar{j}"'] & \bar{C}_{*,2} \arrow[l, "\bar{i}"'] & 0 \arrow[l] \\
0 & C_{*,1} \arrow[l] \arrow[u, "f_1"'] & C_* \arrow[l, "j"'] \arrow[u, "f"'] & C_{*,2} \arrow[l, "i"'] \arrow[u, "f_2"'] & 0 \arrow[l]
\end{tikzcd} \tag{2.6.8}$$

such that the horizontal sequences are exact, F_i and F are morphisms of cochain complexes and f_i and f are morphisms of cochain complexes.

Lemma 2.6.8 (Naturality of the homology exact sequence) *Suppose that* (F_i, f_i) *and* (F, f) *are morphisms of Kronecker pairs. Then, the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H_{m,2} \arrow[r, "H_*i"] \arrow[d, "H_*f_2"] & H_m \arrow[r, "H_*j"] \arrow[d, "H_*f"] & H_{m,1} \arrow[r, "\partial_*"] \arrow[d, "H_*f_1"] & H_{m-1,2} \arrow[r, "H_*i"] \arrow[d, "H_*f_2"] & \cdots \\
\cdots \arrow[r] & \bar{H}_{m,2} \arrow[r, "H_*\bar{i}"] & \bar{H}_m \arrow[r, "H_*\bar{j}"] & \bar{H}_{m,1} \arrow[r, "\bar{\partial}_*"] & \bar{H}_{m-1,2} \arrow[r, "H_*\bar{i}"] & \cdots
\end{tikzcd}$$

is commutative.

Proof By functoriality of the homology, the square diagrams not involving ∂_* commute. It remains to show that $\bar{\partial}_* \circ H_* f_1 = H_* f_2 \circ \partial_*$. As $H^* F_1 \circ \bar{\delta}^* = \delta^* \circ H^* F_2$ by Lemma 2.6.4, one has

$$\langle a, \bar{\partial}_* \circ H_* f_1(\alpha)\rangle_2 = \langle H^* F_1 \circ \bar{\delta}^*(a), \alpha\rangle_1 = \langle \delta^* \circ H^* F_2(a), \alpha\rangle_1 = \langle a, H_* f_1 \circ \partial_*(\alpha)\rangle_2$$

for all $a \in \bar{H}_2^{m-1}$ and $\alpha \in H_{m,1}$. By Lemma 2.3.3, this implies that $\bar{\partial}_* \circ H_* f_1 = H_* f_2 \circ \partial_*$. □

2.7 Relative (Co)homology

A *simplicial pair* is a couple (K, L) where K is a simplicial complex and L is a subcomplex of K. The inclusion $i \colon L \hookrightarrow K$ is a simplicial map. Let $a \in C^m(K)$. If, using Definition 2.2.1a of Sect. 2.2, we consider a as a subset of $\mathcal{S}_m(K)$, then $C^* i(a) = a \cap \mathcal{S}_m(L)$. If we see a as a map $a \colon \mathcal{S}_m(K) \to \mathbb{Z}_2$, then $C^* i(a)$ is the restriction of a to $\mathcal{S}_m(L)$. We see that $C^* i \colon C^*(K) \to C^*(L)$ is surjective. Define

$$C^m(K, L) = \ker\left(C^m(K) \xrightarrow{C^* i} C^m(L)\right)$$

and $C^*(K, L) = \oplus_{m \in \mathbb{N}} C^m(K, L)$. This definition implies that

- $C^m(K, L)$ is the set of subsets of $\mathcal{S}_m(K) - \mathcal{S}_m(L)$;
- if K is a finite simplicial complex, $C^m(K, L)$ is the vector space with basis $\mathcal{S}_m(K) - \mathcal{S}_m(L)$.

As $C^* i$ is a morphism of cochain complexes, the coboundary $\delta \colon C^*(K) \to C^*(K)$ preserves $C^*(K, L)$ and gives rise to a coboundary $\delta \colon C^*(K, L) \to C^*(K, L)$ so that $(C^*(K, L), \delta)$ is a cochain complex. The cocycles $Z^*(K, L)$ and the coboundaries $B^*(K, L)$ are defined as usual, giving rise to the definition

$$H^m(K, L) = Z^m(K, L)/B^m(K, L)\,.$$

The graded $\mathbb{Z}_2$-vector space $H^*(K, L) = \oplus_{m \in \mathbb{N}} H^m(K, L)$ is the *simplicial relative cohomology* of the simplicial pair (K, L).

When useful, the notations δ_K, δ_L and $\delta_{K,L}$ are used for the coboundaries of the cochain complexes $C^*(K)$, $C^*(L)$ and $C^*(K, L)$. We denote by j^* the inclusion $j^* \colon C^*(K, L) \hookrightarrow C^*(K)$, which is a morphism of cochain complexes, and use the same notation j^* for the induced linear map $j^* \colon H^*(K, L) \to H^*(K)$ on cohomology. We also use the notation i^* for both $C^* i$ and $H^* i$. We get thus a short exact sequence of cochain complexes

$$0 \to C^*(K, L) \xrightarrow{j^*} C^*(K) \xrightarrow{i^*} C^*(L) \to 0\,. \tag{2.7.1}$$

If $a \in C^m(L)$, any cochain $\bar{a} \in C^m(K)$ with $i^*(\bar{a}) = a$ is called a *extension of* a as a cochain in K. For instance, the 0-*extension* of a is defined by $\bar{a} = a \in \mathcal{S}_m(L) \subset \mathcal{S}_m(K)$. Using Sect. 2.6, Exact sequence (2.7.1) gives rise to a (*simplicial cohomology*) *connecting homomorphism*

$$\delta^* : H^*(L) \to H^{*+1}(K, L)\,.$$

It is induced by a linear map $\tilde{\delta}^*\colon Z^m(L) \to Z^{m+1}(K, L)$ characterized by the equation $j^* \circ \tilde{\delta}^* = \delta_K \circ S$ for some (or any) linear section $S\colon C^m(L) \to C^m(K)$ of i^*, not required to be a morphism of cochain complex. For instance, one can take $S(a)$ to be the 0-extension of a. Using that $C^*(K, L)$ is a chain subcomplex of $C^*(K)$, the following statement makes sense and constitutes a useful recipe for computing the connecting homomorphism δ^*.

Lemma 2.7.1 *Let $a \in Z^m(L)$ and let $\bar{a} \in C^m(K)$ be any extension of a as an m-cochain of K. Then, $\delta_K(\bar{a})$ is an $(m+1)$-cocycle of (K, L) representing $\delta^*(a)$.*

Proof Choose a linear section $S\colon C^m(L) \to C^m(K)$ such that $S(a) = \bar{a}$. The equation $j^* \circ \tilde{\delta}^* = \delta_K \circ S$ proves the lemma. □

We can now use Proposition 2.6.2 and get the following result.

Proposition 2.7.2 *The long sequence*

$$\cdots \to H^m(K, L) \xrightarrow{j^*} H^m(K) \xrightarrow{i^*} H^m(L) \xrightarrow{\delta^*} H^{m+1}(K, L) \xrightarrow{j^*} \cdots$$

is exact.

The exact sequence of Proposition 2.7.2 is called the *simplicial cohomology exact sequence*, or just the *simplicial cohomology sequence*, of the simplicial pair (K, L).

We now turn our interest to homology. The inclusion $L \hookrightarrow K$ induces an inclusion $i_*\colon C_*(L) \hookrightarrow C_*(K)$ of chain complexes. We define $C_m(K, L)$ as the quotient vector space

$$C_m(K, L) = \operatorname{coker}\big(i_* : C_m(L) \hookrightarrow C_m(K)\big)\,.$$

As i_* is a morphism of chain complexes, $C_*(K, L) = \oplus_{m\in\mathbb{N}} C_m(K, L)$ inherits a boundary operator $\partial = \partial_{K,L}\colon C_*(K, L) \to C_{*-1}(K, L)$. The projection $j_*\colon C_*(K) \to C_*(K, L)$ is a morphism of chain complexes and one gets a short exact sequence of chain complexes

$$0 \to C_*(L) \xrightarrow{i_*} C_*(K) \xrightarrow{j_*} C_*(K, L) \to 0\,. \tag{2.7.2}$$

The cycles and boundaries $Z_*(K, L)$ and $B_*(K, L)$ are defined as usual, giving rise to the definition

$$H_m(K, L) = Z_m(K, L)/B_m(K, L)\,.$$

The graded $\mathbb{Z}_2$-vector space $H_*(K, L) = \oplus_{m\in\mathbb{N}} H_m(K, L)$ is the *relative homology* of the simplicial pair (K, L). As before, the notations ∂_K and ∂_L may be used for the boundary operators in $C_*(K)$ and $C_*(L)$ and i_* and j_* are also used for the induced maps in homology.

Since the linear map $i_*\colon C_*(L) \hookrightarrow C_*(K)$ is induced by the inclusion of bases $\mathcal{S}(L) \hookrightarrow \mathcal{S}(K)$, the quotient vector space $C_*(K, L)$ may be considered as the vector space with basis $\mathcal{S}(K) - \mathcal{S}(L)$. This point of view provides a tautological linear map $s\colon C_*(K, L) \to C_*(K)$, which is a section of j_* but not a morphism of chain complexes.

The Kronecker pairings for K and L are denoted by $\langle\, ,\, \rangle_K$ and $\langle\, ,\, \rangle_L$, both at the levels of (co)chains and of (co)homology. As $\langle j^*(K, L), i_*(L)\rangle_K = 0$, we get a bilinear map

$$C^m(K, L) \times C_m(K, L) \xrightarrow{\langle,\rangle_{K,L}} \mathbb{Z}_2\,.$$

The formula

$$\langle a, \alpha\rangle_{K,L} = \langle j^*(a), s(\alpha)\rangle_K \tag{2.7.3}$$

holds for all $a \in C^m(K, L)$, $\alpha \in C_m(K, L)$ and all $m \in \mathbb{N}$. Observe also that the formula

$$\langle S(b), i_*(\beta)\rangle_K = \langle b, \beta\rangle_L \tag{2.7.4}$$

holds for all $b \in C^m(L)$, $\beta \in C_m(L)$ and all $m \in \mathbb{N}$.

Lemma 2.7.3 $\big(C^*(K, L), \delta_{K,L}, C_*(K, L), \partial_{K,L}, \langle\, ,\, \rangle_{K,L}\big)$ *is a Kronecker pair.*

Proof We first prove that $\langle \delta_{K,L}(a), \alpha\rangle_{K,L} = \langle a, \partial_{K,L}(\alpha)\rangle_{K,L}$ for all $a \in C^m(K, L)$ and all $\alpha \in C_{m+1}(K, L)$ and all $m \in \mathbb{N}$. Indeed, one has

$$\begin{aligned}\langle \delta_{K,L}(a), \alpha\rangle_{K,L} &= \langle j^*\circ\delta_{K,L}(a), s(\alpha)\rangle_K\\ &= \langle \delta_K\circ j^*(a), s(\alpha)\rangle_K\\ &= \langle j^*(a), \partial_K\circ s(\alpha)\rangle_K\end{aligned} \tag{2.7.5}$$

Observe that $j_*\circ\partial_K\circ s(\alpha) = \partial_{K,L}(\alpha)$ and therefore $\partial_K\circ s(\alpha) = s\circ\partial_{K,L}(\alpha) + i_*(c)$ for some $c \in C_m(L)$. Hence, the chain of equalities in (2.7.5) may be continued

$$\begin{aligned}\langle \delta_{K,L}(a), \alpha\rangle_{K,L} &= \langle j^*(a), \partial_K\circ s(\alpha)\rangle_K\\ &= \langle j^*(a), s\circ\partial_{K,L}(\alpha) + i_*(c)\rangle_K\\ &= \langle j^*(a), s\circ\partial_{K,L}(\alpha)\rangle_K + \underbrace{\langle j^*(a), i_*(c)\rangle_K}_{0}\\ &= \langle a, \partial_{K,L}(\alpha)\rangle_{K,L}\,.\end{aligned} \tag{2.7.6}$$

It remains to prove that the linear map $\mathbf{k}\colon C^*(K,L) \to C_*(K,L)^\sharp$ given by $\mathbf{k}(a) = \langle a, \ \rangle$ is an isomorphism. As the inclusion $i\colon L \hookrightarrow K$ is a simplicial map, the couple (C^*i, C_*i) is a morphism of Kronecker pairs by Lemma 2.5.1 and the result follows from Lemma 2.3.9. □

Passing to homology then produces three Kronecker pairs with vanishing (co)boundary operators:

$$\mathcal{P}_L = (H^*(L), H_*(L), \langle , \rangle_L), \quad \mathcal{P}_K = (H^*(K), H_*(K), \langle , \rangle_K)$$

and

$$\mathcal{P}_{K,L} = (H^*(K,L), H_*(K,L), \langle , \rangle_{K,L}) .$$

Using Sect. 2.6, short exact sequence (2.7.2) gives rise to the (*simplicial homology*) *connecting homomorphism*

$$\partial_* : H_*(K,L) \to H_{*-1}(L) .$$

It is induced by a linear map $\tilde{\partial}\colon Z_m(K,L) \to Z_{m-1}(L)$ characterized by the equation

$$j^* \circ \tilde{\partial}_* = \partial_K \circ s ,$$

using the section s of j_* defined above (or any other one).

Lemma 2.7.4 *The following couples are morphisms of Kronecker pairs:*

(a) (i^*, i_*), *from* $\mathcal{P}_L$ *to* $\mathcal{P}_K$.
(b) (j^*, j_*), *from* $\mathcal{P}_K$ *to* $\mathcal{P}_{K,L}$.
(c) (δ^*, ∂_*), *from* $\mathcal{P}_{K,L}$ *to* $\mathcal{P}_L$.

Proof As the inclusion $L \hookrightarrow K$ is a simplicial map, Point (a) follows from Lemma 2.5.1. Point (c) is implied by Lemma 2.6.6. To prove Point (b), let $a \in C^m(K,L)$ and $\alpha \in C_m(K)$. Observe that $s(j_*(\alpha)) = \alpha + i_*(\beta)$ for some $\beta \in C_m(L)$ and that $\langle j^*(a), i_*(\beta)\rangle_K = 0$. Therefore:

$$\langle a, j_*(\alpha)\rangle_{K,L} = \langle j^*(a), s \circ j_*(\alpha)\rangle_K = \langle j^*(a), \alpha\rangle_K$$ □

Proposition 2.6.7 now gives the following result.

Proposition 2.7.5 *The long sequence*

$$\cdots \to H_m(L) \xrightarrow{i_*} H_m(K) \xrightarrow{j_*} H^m(K,L) \xrightarrow{\partial_*} H_{m-1}(L) \xrightarrow{i_*} \cdots$$

is exact.

The exact sequence of Proposition 2.7.5 is called the (*simplicial*) *homology exact sequence*, or just the (*simplicial*) *cohomology sequence*, of the simplicial pair (K, L).

We now study the naturality of the (co)homology sequences. Let (K, L) and (K', L') be simplicial pairs. A *simplicial map f of simplicial pairs* from (K, L) to (K', L') is a simplicial map $f_K \colon K \to K'$ such that the restriction of f to L is a simplicial map $f_L \colon L \to L'$. The morphism $C^* f_K \colon C^*(K') \to C^*(K)$ then restricts to a morphism of cochain complexes $C^* f \colon C^*(K', L') \to C^*(K, L)$ and the morphism $C_* f_K \colon C_*(K) \to C_*(K')$ descends to a morphism of chain complexes $C_* f \colon C_*(K, L) \to C_*(K', L')$. The couples $(C^* f_K, C_* f_K)$ and $(C^* f_L, C_* f_L)$ are morphisms of Kronecker pairs by Lemma 2.5.1. We claim that $(C_* f, C_* f)$ is a morphism of Kronecker pair from $(C^*(K, L), \dots)$ to $(C^*(K', L'), \dots)$. Indeed, let $a \in C^m(K', L')$ and $\alpha \in C_m(K, L)$. One has

$$\begin{aligned} \langle C^* f(a), \alpha\rangle_{K,L} &= \langle j^* \circ C^* f(a), s(\alpha)\rangle_K \\ &= \langle C^* f_K \circ j'^*(a), s(\alpha)\rangle_K \\ &= \langle j'^*(a), C_* f_K \circ s(\alpha)\rangle_{K'} \\ &= \langle j'^*(a), C_* f_K \circ s(\alpha)\rangle_{K'} \end{aligned} \tag{2.7.7}$$

and

$$\langle a, C_* f(\alpha)\rangle_{K',L'} = \langle j'^*(a), s' \circ C_* f(\alpha)\rangle_{K'} \tag{2.7.8}$$

The equation $j'_* \circ s' \circ C_* f(\alpha) = j'_* \circ C_* f_K \circ s(\alpha) = *f(\alpha)$ implies that $s' \circ C_* f(\alpha) = C_* f_K \circ s(\alpha) + i'_*(\beta)$ for some $\beta \in C_m(L')$. As $\langle j'^*(a), i'_*(\beta)\rangle_{K'} = 0$, Equations (2.7.7) and (2.7.8) imply that $\langle C^* f(a), \alpha\rangle_{K,L} = \langle a, C_* f(\alpha)\rangle_{K',L'}$.

Lemmas 2.6.4 and 2.6.8 then imply the following

Proposition 2.7.6 *The cohomology and homology sequences are natural with respect to simplicial maps of simplicial pairs. In other words, given a simplicial map of simplicial pairs $f \colon (K, L) \to (K', L')$, the following diagrams*

$$\begin{tikzcd} \cdots \arrow[r] & H^m(K', L') \arrow[r, "j'^*"] \arrow[d, "H^* f"] & H^m(K') \arrow[r, "i'^*"] \arrow[d, "H^* f_K"] & H^m(L') \arrow[r, "\delta'^*"] \arrow[d, "H^* f_L"] & H^{m+1}(K', L') \arrow[r, "j'^*"] \arrow[d, "H^* f"] & \cdots \\ \cdots \arrow[r] & H^m(K, L) \arrow[r, "j^*"] & H^m(K) \arrow[r, "i^*"] & H^m(L) \arrow[r, "\delta^*"] & H^{m+1}(K, L) \arrow[r, "j^*"] & \cdots \end{tikzcd}$$

and

$$\begin{tikzcd} \cdots \arrow[r] & H_m(L) \arrow[r, "i^*"] \arrow[d, "H_* f_L"] & H_m(K) \arrow[r, "j^*"] \arrow[d, "H_* f_K"] & H_m(K, L) \arrow[r, "\partial_*"] \arrow[d, "H_* f"] & H_{m-1}(L) \arrow[r, "i^*"] \arrow[d, "H_* f_L"] & \cdots \\ \cdots \arrow[r] & H_m(L') \arrow[r, "i'_*"] & H^m(K') \arrow[r, "j'_*"] & H_m(K', L') \arrow[r, "\partial'_*"] & H_{m-1}(L') \arrow[r, "i'_*"] & \cdots \end{tikzcd}$$

are commutative.

We finish this section by the exact sequences for a triple. A *simplicial triple* is a triplet (K, L, M) where K is a simplicial complex, L is a subcomplex of K and M is a subcomplex of L. A *simplicial map f of simplicial triples*, from (K, L, M) to (K', L', M') is a simplicial map $f_K \colon K \to K'$ such that the restrictions of f_K to L and M are simplicial maps $f_L \colon L \to L'$ and $f_M \colon M \to M'$.

A simplicial triple $T = (K, L, M)$ gives rise to pair inclusions

$$(L, M) \xrightarrow{i} (K, M) \xrightarrow{j} (K, L)$$

and to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C^*(K,L) \arrow[r, "j^*_{K,L}"] \arrow[d, "C^*j"] & C^*(K) \arrow[r, "i^*_{K,L}"] \arrow[d, "=", "\mathrm{id}"', leftrightarrow] & C^*(L) \arrow[r] \arrow[d, "i^*_{L,M}"] & 0 \\ 0 \arrow[r] & C^*(K,M) \arrow[r, "j^*_{K,M}"] & C^*(K) \arrow[r, "i^*_{K,M}"] & C^*(M) \arrow[r] & 0 \end{tikzcd} \tag{2.7.9}$$

where the horizontal lines are exact sequences of cochain complexes. A diagram-chase shows that the morphism $i^*_{K,L} \circ j^*_{K,M}$, which sends $C^*(K, M)$ to $C^*(L)$, has image $C^*(L, M)$ and kernel the image of C^*j. This morphism coincides with C^*i. We thus get a short exact sequence of cochain complexes

$$0 \to C^*(K, L) \xrightarrow{C^*j} C^*(K, M) \xrightarrow{C^*i} C^*(L, M) \to 0\,. \tag{2.7.10}$$

The same arguments with the chain complexes gives a short exact sequence

$$0 \to C_*(L, M) \xrightarrow{C_*i} C_*(K, M) \xrightarrow{C_*j} C_*(K, L) \to 0\,. \tag{2.7.11}$$

As above in this section, short exact sequences (2.7.10) and (2.7.11) produces connecting homomorphisms $\delta_T \colon H^*(L, M) \to H^{*+1}(K, L)$ and $\partial_T : H_*(K, L) \to C_{*-1}(L, M)$. They satisfy $\langle \delta_T(a), \alpha \rangle = \langle a, \partial_T(\alpha) \rangle$ as well as following proposition.

Proposition 2.7.7 ((Co)homology sequences of a simplicial triple) *Let $T = (K, L, M)$ be a simplicial triple. Then,*

(a) *the sequences*

$$\cdots \to H^m(K, L) \xrightarrow{H^*j} H^m(K, M) \xrightarrow{H^*i} H^m(L, M) \xrightarrow{\delta_T} H^{m+1}(K, L) \xrightarrow{H^*j} \cdots$$

and

$$\cdots \to H_m(L, M) \xrightarrow{H_*i} H_m(K, M) \xrightarrow{H_*j} H_m(K, L) \xrightarrow{\partial_T} H_{m-1}(L, M) \xrightarrow{H_*i} \cdots$$

are exact.

(b) *the exact sequences of Point* (*a*) *are natural for simplicial maps of simplicial triples.*

Remark 2.7.8 As $H^*(\emptyset) = 0$, we get a canonical **GrV**-isomorphisms $H^*(K, \emptyset) \xrightarrow{\approx} H^*(K)$, etc. Thus, the (co)homology sequences for the triple $(K, L, \emptyset)$ give back those of the pair (K, L)

$$\cdots \to H^m(K, L) \xrightarrow{H^*j} H^m(K) \xrightarrow{H^*i} H^m(L) \xrightarrow{\delta^*} H^{m+1}(K, L) \xrightarrow{H^*j} \cdots \tag{2.7.12}$$

and

$$\cdots \to H_m(L) \xrightarrow{H_*i} H_m(K) \xrightarrow{H_*j} H^m(K, L) \xrightarrow{\partial_*} H_{m-1}(L) \xrightarrow{H_*i} \cdots \tag{2.7.13}$$

where $i\colon L \to K$ and $j\colon (K, \emptyset) \to (K, L)$ denote the inclusions. This gives a more precise description of the morphisms j^* and j_* of Propositions 2.7.2 and 2.7.5.

2.7.9 *Historical note.* The relative homology was introduced by S. Lefschetz in 1927 in order to work out the Poincaré duality for manifolds with boundary (see, e.g. [40, p. 58], [51, p. 47]). The use of exact sequences occurred in several parts of algebraic topology after 1941 (see, e.g. [40, p. 86], [51, p. 47]). The (co)homology exact sequences play an essential role in the axiomatic approach of Eilenberg-Steenrod, [51].

2.8 Mayer-Vietoris Sequences

Let K be a simplicial complex with two subcomplexes K_1 and K_2. We suppose that $K = K_1 \cup K_2$ (i.e. $\mathcal{S}(K) = \mathcal{S}(K_1) \cup \mathcal{S}(K_2)$). We call (K, K_1, K_2) a *simplicial triad*. Then, $K_0 = K_1 \cap K_2$ is a subcomplex of K_1, K_2 and K, with $\mathcal{S}(K_0) = \mathcal{S}(K_1) \cap \mathcal{S}(K_2)$. The Mayer-Vietoris sequences relate the (co)homology of X to that of X_i, generalizing Lemma 2.4.10. The various inclusions are denoted as follows

$$\begin{tikzcd} K_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & K_1 \arrow[d, "j_1"] \\ K_2 \arrow[r, "j_2"] & K\,. \end{tikzcd} \tag{2.8.1}$$

The notations i_1^*, j_1^*, ..., stand for both C^*i_1, C^*j_1, etc, and H^*i_1, H^*j_1, etc. The same holds for chains and homology: i_{1*} for both C_*i_1 and H_*i_1, etc. Diagram (2.8.1) induces two diagrams

$$\begin{tikzcd} C^*(K) \arrow[r, two heads, "j_1^*"] \arrow[d, two heads, "j_2^*"'] & C^*(K_1) \arrow[d, two heads, "i_1^*"] \\ C^*(K_2) \arrow[r, two heads, "i_2^*"] & C^*(K_0) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} C_*(K_0) \arrow[r, tail, "i_{1*}"] \arrow[d, tail, "i_{2*}"'] & C_*(K_1) \arrow[d, tail, "j_{1*}"] \\ C_*(K_2) \arrow[r, tail, "j_{2*}"] & C_*(K)\,. \end{tikzcd}$$

The cohomology diagram is Cartesian (pullback) and the homology diagram is co-Cartesian (pushout). Therefore, the sequence

$$0 \to C^*(K) \xrightarrow{(j_1^*, j_2^*)} C^*(K_1) \oplus C^*(K_2) \xrightarrow{i_1^* + i_2^*} C^*(K_0) \to 0 \tag{2.8.2}$$

is an exact sequence of cochain complexes and the sequence

$$0 \to C_*(K_0) \xrightarrow{(i_{1*}, i_{2*})} C_*(K_1) \oplus C_*(K_2) \xrightarrow{j_{1*} + j_{2*}} C_*(K) \to 0 \tag{2.8.3}$$

is an exact sequence of chain complexes.

Consider the Kronecker pairs $(C^*(K_i), C_*(K_i), \langle\,,\rangle_i)$ for $i = 0, 1, 2$, and the Kronecker pair $(C^*(K), C_*(K), \langle\,,\rangle)$. A bilinear map

$$\langle\,,\rangle_\oplus : \big[C^*(K_1) \oplus C^*(K_2)\big] \times \big[C_*(K_1) \oplus C_*(K_2)\big] \to \mathbb{Z}_2$$

is defined by

$$\langle (a_1, a_2), (\alpha_1, \alpha_2)\rangle_\oplus = \langle a_1, \alpha_1\rangle_1 + \langle a_2, \alpha_2\rangle_2\,.$$

We check that $(C^*(K_1) \oplus C^*(K_2), C_*(K_1) \oplus C_*(K_2), \langle\,,\rangle_\oplus)$ is a Kronecker pair and that the couples $((j_1^*, j_2^*), j_1^* + j_2^*)$ and $(i_1^* + i_2^*, (i_1^*, i_2^*))$ are morphisms of Kronecker pairs. By Sect. 2.6, there exist linear maps $\delta_{MV}\colon H^*(K_0) \to H^{*+1}(K)$ and $\partial_{MV}\colon H_*(K) \to H_{*-1}(K_0)$ which, by Propositions 2.6.2 and 2.6.7, give the following proposition.

Proposition 2.8.1 (Mayer-Vietoris sequences) *The long sequences*

$$\cdots \to H^m(K) \xrightarrow{(j_1^*, j_2^*)} H^m(K_1) \oplus H^m(K_2) \xrightarrow{i_1^* + i_2^*} H^m(K_0) \xrightarrow{\delta_{MV}} H^{m+1}(K) \to \cdots$$

and

$$\cdots \to H_m(K_0) \xrightarrow{(i_{1*}, i_{2*})} H_m(K_1) \oplus H_m(K_2) \xrightarrow{j_{1*} + j_{2*}} H_m(K) \xrightarrow{\partial_{MV}} H_{m-1}(K_0) \to \cdots$$

are exact.

The homomorphisms δ_{MV} and ∂_{MV} are called the *Mayer-Vietoris connecting homomorphisms* in (co)homology. By Lemma 2.6.6, they satisfy $\langle \delta_{MV}(a), \alpha\rangle =$

$\langle a, \partial_{MV}(\alpha)\rangle_0$ for all $a \in H^m(K_0)$, all $\alpha \in H_{m+1}(k)$ and all $m \in \mathbb{N}$. To define the connecting homomorphisms, one must choose a linear section S of $i_1^* + i_2^*$ and s of $j_{1*} + j_{2*}$. One can choose $S(a) = (S_1(a), 0)$, where $S_1 \colon C^*(K) \to C^*(K_1)$ is the tautological section of i_1^* given by the inclusion $\mathcal{S}(K_0) \hookrightarrow \mathcal{S}(K_1)$ (see Sect. 2.7). A choice of s is given, for $\sigma \in \mathcal{S}(K)$, by

$$s(\sigma) = \begin{cases} (\sigma, 0) & \text{if } \sigma \in \mathcal{S}(K_1) \\ (0, 0) & \text{if } \sigma \notin \mathcal{S}(K_1)\,. \end{cases}$$

These choices produce linear maps $\tilde{\delta}_{MV} \colon Z^*(K_0) \to Z^{*+1}(K)$ and $\tilde{\partial}_{MV} \colon Z_*(K) \to Z_{*-1}(K_0)$, representing δ_{MV} and ∂_{MV} and defined by the equations

$$(j_1^*, j_2^*) \circ \tilde{\delta}_{MV} = (\delta_1, \delta_2) \circ S \quad \text{and} \quad (i_{1*}, i_{2*}) \circ \tilde{\partial}_{MV} = (\partial_1, \partial_2) \circ s\,.$$

(The apparent asymmetry of the choices has no effect by Lemma 2.6.1 and its homology counterpart: exchanging 1 and 2 produces other sections, giving rise to the same connecting homomorphisms.)

Finally, the Mayer-Vietoris sequences are natural for maps of simplicial triads. If $\mathcal{T} = (K, K_1, K_2)$ and $\mathcal{T}' = (K', K_1', K_2')$ are simplicial triads and if $f \colon K \to K'$ is a simplicial map such that $f(K_i) \subset K_i'$, then the Mayer Vietoris sequences of $\mathcal{T}$ and $\mathcal{T}'$ are related by commutative diagrams, as in Proposition 2.7.6. This is a direct consequence of Lemmas 2.6.4 and 2.6.8.

2.9 Appendix A: An Acyclic Carrier Result

The powerful technique of acyclic carriers was introduced by Eilenberg and MacLane in 1953 [50], after earlier work by Lefschetz. Proposition 2.9.1 below is a very particular example of this technique, adapted to our needs. For a full development of acyclic carriers, see, e.g., [155, Chap. 1,Sect. 13].

Let (C_*, ∂) and $(\bar{C}_*, \bar{\partial})$ be two chain complexes and let $\varphi \colon C_* \to \bar{C}_*$ be a morphism of chain complexes. We suppose that C_m is equipped with a basis $\mathcal{S}_m$ for each m and denote by $\mathcal{S}$ the union of all $\mathcal{S}_m$. An *acyclic carrier* A_* for φ with respect to the basis $\mathcal{S}$ is a correspondence which associates to each $s \in \mathcal{S}$ a subchain complex $A_*(s)$ of $\bar{C}_*$ such that

(a) $\varphi(s) \in A_*(s)$.
(b) $H_0(A_*(s)) = \mathbb{Z}_2$ and $H_m(A_*(s)) = 0$ for $m > 0$.
(c) let $s \in \mathcal{S}_m$ and $t \in \mathcal{S}_{m-1}$ such that t occurs in the expression of ∂s in the basis $\mathcal{S}_{m-1}$. Then $A_*(t)$ is a subchain complex of $A_*(s)$ and the inclusion $A_*(t) \subset A_*(s)$ induces an isomorphism on H_0.
(d) if $s \in \mathcal{S}_0 \subset C_0 = Z_0$, then $H_0\varphi(s) \neq 0$ in $H_0(A_*(s))$.

Proposition 2.9.1 *Let φ and φ' be two morphisms of chain complexes from (C_*, ∂) to $(\bar{C}_*, \bar{\partial})$. Suppose that φ and φ' admit the same acyclic carrier A_* with respect to some basis $\mathcal{S}$ of C_*. Then $H_*\varphi = H_*\varphi'$.*

Proof The proof is similar to that of Proposition 2.5.9. By induction on m, we shall prove the following property:

Property $\mathcal{H}(m)$: there exists a linear map $D\colon C_m \to \bar{C}_{m+1}$ such that:

(i) $\bar{\partial}D(\alpha) + D(\partial\alpha) = \varphi(\alpha) + \varphi'(\alpha)$ for all $\alpha \in C_m$.
(ii) for each $s \in \mathcal{S}_m$, $D(s) \in A_{m+1}(s)$.

Property $\mathcal{H}(m)$ for all m implies that $H_*\varphi = H_*\varphi'$. Indeed, we then have a linear map $D\colon C_* \to \bar{C}_{*+1}$ satisfying

$$\varphi + \varphi' = \bar{\partial}\circ D + D\circ\partial\,. \tag{2.9.1}$$

Let $\beta \in Z_*$. By Eq. (2.9.1), one has $\varphi(\beta) + \varphi(\beta) = \bar{\partial}D(\beta)$ which implies that $H_*\varphi([\beta]) + H_*\varphi'([\beta])$ in $\bar{H}_*$.

Let us prove $\mathcal{H}(0)$. Let $s \in \mathcal{S}_0$. In $H_0(A_*(s)) = \mathbb{Z}_2$, one has $H_0\varphi(s) \neq 0$ and $H_0\varphi'(s) \neq 0$. Therefore $H_*\varphi(s) = H_*\varphi'(s)$ in $H_0(A_*(s))$. This implies that $\varphi(s) + \varphi'(s) = \bar{\partial}(\eta_s)$ for some $\eta_s \in A_1(s)$. We set $D(s) = \eta_s$. This procedure, for each $s \in \mathcal{S}_0$, provides a linear map $D\colon C_0 \to \bar{C}_1$, which, as $\partial C_0 = 0$, satisfies $\varphi(s) + \varphi'(s) = \bar{\partial}D(\alpha) + D(\partial(\alpha))$.

We now prove that $\mathcal{H}(m-1)$ implies $\mathcal{H}(m)$ for $m \geq 1$. Let $s \in \mathcal{S}_m$. The chain $D(\partial s)$ exists in $A_m(s)$ by $\mathcal{H}(m-1)$. Let $\zeta \in A_m(s)$ defined by

$$\zeta = \varphi(s) + \varphi'(s) + D(\partial s)$$

Using $\mathcal{H}(m-1)$, one checks that $\partial\zeta = 0$. Since $H_m(A_*(s)) = 0$, there exists $\nu \in A_{m+1}(s)$ such that $\zeta = \partial\nu$. Choose such an element ν and set $D(\sigma) = \nu$. This defines $D\colon C_m \to \bar{C}_{m+1}$ which satisfies (i) and (ii), proving $\mathcal{H}(m)$. □

2.10 Appendix B: Ordered Simplicial (Co)homology

This technical section may be skipped in a first reading. It shows that simplicial (co)homology may be defined using larger sets of (co)chains, based on ordered simplexes. This will be used for comparisons between simplicial and singular (co)homology (see § 17) and to define the cup and cap products in Chap. 4.

Let K be a simplicial complex. Define

$$\hat{\mathcal{S}}_m(K) = \{(v_0, \dots, v_m) \in V(K)^{m+1} \mid \{v_0, \dots, v_m\} \in \mathcal{S}(K)\}\,.$$

Observe that $\dim\{v_0, \dots, v_m\} \leq m$ and may be strictly smaller if there are repetitions amongst the v_i's. An element of $\hat{\mathcal{S}}_m(K)$ is an *ordered m-simplex* of K.

The definitions of ordered (co)chains and (co)homology are the same those for the simplicial case (see Sect. 2.2), replacing the simplexes by the ordered simplexes. We thus set

Definition 2.10.1 (*subset definitions*)

(a) An *ordered m-cochain* is a subset of $\hat{\mathcal{S}}_m(K)$.
(b) An *ordered m-chain* is a finite subset of $\hat{\mathcal{S}}_m(K)$.

The set of ordered m-cochains of K is denoted by $\hat{C}^m(K)$ and that of ordered m-chains by $\hat{C}_m(K)$. As in Sect. 2.2, Definition 2.10.1 are equivalent to

Definition 2.10.2 (*colouring definitions*)

(a) An *ordered m-cochain* is a function $a\colon \hat{\mathcal{S}}_m(K) \to \mathbb{Z}_2$.
(b) An *ordered m-chain* is a function $\alpha\colon \hat{\mathcal{S}}_m(K) \to \mathbb{Z}_2$ with finite support.

Definition 2.10.2 endow $\hat{C}^m(K)$ and $\hat{C}_m(K)$ with a structure of a $\mathbb{Z}_2$-vector space. The singletons provide a basis of $\hat{C}_m(K)$, in bijection with $\hat{\mathcal{S}}_m(K)$. Thus, Definition 2.10.2.b is equivalent to

Definition 2.10.3 $\hat{C}_m(K)$ is the $\mathbb{Z}_2$-vector space with basis $\hat{\mathcal{S}}_m(K)$:

$$\hat{C}_m(X) = \bigoplus_{\sigma\in\hat{\mathcal{S}}_m(X)} \mathbb{Z}_2\,\sigma\,.$$

We consider the graded $\mathbb{Z}_2$-vector spaces $\hat{C}_*(K) = \oplus_{m\in\mathbb{N}}\hat{C}_m(K)$ and $\hat{C}^*(K) = \oplus_{m\in\mathbb{N}}\hat{C}^m(K)$. The *Kronecker pairing* on ordered (co)chains

$$\hat{C}^m(K) \times \hat{C}_m(K) \xrightarrow{\langle\,,\,\rangle} \mathbb{Z}_2$$

is defined, using the various above definitions, by the equivalent formulae

$$\begin{aligned}\langle a, \alpha\rangle &= \sharp(a \cap \alpha) \pmod 2 && \text{using Definition 2.10.1a and b}\\ &= \textstyle\sum_{\sigma\in\alpha} a(\sigma) && \text{using Definitions 2.10.1a and 2.10.2b}\\ &= \textstyle\sum_{\sigma\in S_m(K)} a(\sigma)\alpha(\sigma) && \text{using Definitions 2.10.2a and b.}\end{aligned} \tag{2.10.1}$$

As in Lemma 2.2.4, we check that the map $\mathbf{k}\colon \hat{C}^m(K) \to \hat{C}_m(K)^\sharp$, given by $\mathbf{k}(a) = \langle a,\ \rangle$, is an isomorphism.

The *boundary operator* $\hat{\partial}\colon \hat{C}_m(K) \to \hat{C}_{m-1}(K)$ is the $\mathbb{Z}_2$-linear map defined, for $(v_0, \dots, v_m) \in \hat{\mathcal{S}}_m(K)$ by

$$\hat{\partial}(v_0, \dots, v_m) = \sum_{i=0}^{m} (v_0, \dots, \hat{v}_i, \dots, v_m)\,, \tag{2.10.2}$$

where $(v_0, \dots, \hat{v}_i, \dots, v_m) \in \hat{\mathcal{S}}_{m-1}$ is the m-tuple obtained by removing v_i. The *coboundary operator* $\hat{\delta} : C^m(K) \to C^{m+1}(K)$ is defined by the equation

$$\langle \hat{\delta} a, \alpha \rangle = \langle a, \hat{\partial} \alpha \rangle \, . \tag{2.10.3}$$

With these definition, $(\hat{C}_*(K), \hat{\partial}, \hat{C}^*(K), \hat{\delta}, \langle \, , \rangle)$ is a Kronecker pair. We define the vector spaces of *ordered cycles* $\hat{Z}_*(K)$, *ordered boundaries* $\hat{B}_*(K)$, *ordered cocycles* $\hat{Z}^*(K)$, *ordered coboundaries* $\hat{B}^*(K)$, *ordered homology* $\hat{H}_*(K)$ and *ordered cohomology* $\hat{H}^*(K)$ as in Sect. 2.3. By Proposition 2.3.5, the pairing on (co)chain descends to a pairing

$$H^m(K) \times H_m(K) \xrightarrow{\langle \, , \rangle} \mathbb{Z}_2$$

so that the map $\mathbf{k}$: $\hat{H}^m \to \hat{H}_m^\sharp$, given by $\mathbf{k}(a) = \langle a, \, \rangle$, is an isomorphism (*ordered Kronecker duality*).

Example 2.10.4 Let $K = pt$ be a point. Then, $\hat{\mathcal{S}}_m(pt)$ contains one element for each integer m, namely the $(m+1)$-tuple $(pt, \dots, pt)$. Then, $\hat{C}_m(pt) = \mathbb{Z}_2$ for all $m \in \mathbb{N}$ and the chain complex looks like

$$\cdots \xrightarrow{\approx} \hat{C}_{2k+1}(pt) \xrightarrow{0} \hat{C}_{2k}(pt) \xrightarrow{\approx} \hat{C}_{2k-1}(pt) \xrightarrow{0} \cdots \xrightarrow{\approx} \hat{C}_1(pt) \xrightarrow{0} \hat{C}_0(pt) \to 0 \, .$$

Therefore,

$$\hat{H}^*(pt) \approx \hat{H}_*(pt) \approx \begin{cases} 0 & \text{if } * > 0 \\ \mathbb{Z}_2 & \text{if } * = 0 \, . \end{cases}$$

One sees that, for a simplicial complex reduced to a point, the ordered (co)homology and the simplicial (co)homology are isomorphic.

Example 2.10.5 The *unit cochain* $\mathbf{1} \in \hat{C}^0(K)$ is defined as $\mathbf{1} = \hat{\mathcal{S}}_0(K)$. It is a cocycle and defines a class $\mathbf{1} = \hat{H}^0(K)$. If K is non-empty and connected, then $\hat{H}^0(K) \approx \mathbb{Z}_2$ generated by $\mathbf{1}$. Then $H_0(K) \approx \mathbb{Z}_2$ by Kronecker duality; one has $\hat{Z}_0(K) = \hat{C}_0(K)$ and $\alpha \in \hat{Z}_0(K)$ represents the non-zero element of $H_0(K)$ if and only if $\sharp\alpha$ is odd. The proofs are the same as for Proposition 2.4.1.

Example 2.10.6 Let L be a simplicial complex and CL be the cone on L. Then

$$\hat{H}^*(CL) \approx \hat{H}_*(CL) \approx \begin{cases} 0 & \text{if } * > 0 \\ \mathbb{Z}_2 & \text{if } * = 0 \, . \end{cases}$$

The proof is the same as for Proposition 2.4.6, even simpler, since D: $\hat{C}_m(CL) \to \hat{C}_{m+1}(CL)$ is defined, for $(v_0, \dots, v_m) \in \hat{\mathcal{S}}_m(CL)$ by the single line formula $D(v_0, \dots, v_m) = (\infty, v_0, \dots, v_m)$.

Let f: $L \to K$ be a simplicial map. We define $\hat{C}_* f$: $\hat{C}_*(L) \to \hat{C}_*(K)$ as the degree 0 linear map such that

$$\hat{C}_* f(v_0, \ldots, v_m) = (f(v_0), \ldots, f(v_m))$$

for all $(v_0, \ldots, v_m) \in \hat{\mathcal{S}}(L)$. The degree 0 linear map $\hat{C}^* f\colon \hat{C}^*(K) \to \hat{C}^*(L)$ is defined by

$$\langle \hat{C}^* f(a), \alpha \rangle = \langle a, \hat{C}_* f(\alpha) \rangle \, .$$

By Lemma 2.3.6, $(\hat{C}^* f, \hat{C}_* f)$ is a morphism of Kronecker pairs.

We now construct a functorial isomorphism between the ordered and non-ordered (co)homologies, its existence being suggested by the previous examples. Define ψ_*: $\hat{C}_*(K) \to C_*(K)$ by

$$\psi_*((v_0, \ldots, v_m)) = \begin{cases} \{v_0, \ldots, v_m\} & \text{if } v_i \neq v_j \text{ for all } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

We check that ψ is a morphism of chain complexes. We define $\psi^* : C^*(K) \to \hat{C}^*(K)$ by requiring that the equation $\langle \psi^*(a), \alpha \rangle = \langle a, \psi_*(\alpha) \rangle$ holds for all $a \in C^*(K)$ and all $\alpha \in \hat{C}_*(K)$. By Lemma 2.3.6, ψ^* is a morphism of cochain complexes and (ψ_*, ψ^*) is a morphism of Kronecker pairs between $(\hat{C}_*(K), \hat{C}^*(K))$ and $(C_*(K), C^*(K))$. It thus defines a morphism of Kronecker pairs $(H_*\psi, H^*\psi)$ between $(\hat{H}_*(K), \hat{H}^*(K))$ and $(H_*(K), H^*(K))$.

To define a morphism of Kronecker pairs in the other direction, choose a simplicial order $\leq$ on K (see 2.1.8). Define $\phi_{\leq *}\colon C_*(K) \to \hat{C}_*(K)$ as the unique linear map such that

$$\phi_{\leq *}(\{v_0, \ldots, v_m\}) = (v_0, \ldots, v_m) \, ,$$

where $v_0 \leq v_1 \leq \cdots \leq v_m$. We check that $\phi_{\leq *}$ is a morphism of chain complexes and define $\phi_{\leq}{}^*\colon \hat{C}^*(K) \to C^*(K)$ by requiring that the equation $\langle \phi_{\leq}{}^*(a), \alpha \rangle = \langle a, \phi_{\leq *}(\alpha) \rangle$ holds for all $a \in \hat{C}^*(K)$ and all $\alpha \in C_*(K)$. By Lemma 2.3.6, $(\phi_{\leq *}, \phi_{\leq}{}^*)$ is a morphism of Kronecker pairs between $(C_*(K), C^*(K))$ and $(\hat{C}_*(K), \hat{C}^*(K))$. It then defines a morphism of Kronecker pairs $(H_*\phi_{\leq}, H^*\phi_{\leq})$ between $(H_*(K), H^*(K))$ and $(\hat{H}_*(K), \hat{H}^*(K))$.

Proposition 2.10.7 *$H_*\psi \circ H_*\phi_{\leq} = \mathrm{id}_{H_*(K)}$ and $H_*\phi_{\leq} \circ H_*\psi = \mathrm{id}_{\hat{H}_*(K)}$.*

Proof As $\psi_* \circ \phi_{\leq *} = \mathrm{id}_{C_*(K)}$, the first equality follows from Lemma 2.3.7. For the second one, let $(v_0, \ldots, v_m) \in \hat{\mathcal{S}}_m(K)$. Let $\sigma = \{v_0, \ldots, v_m\} \in \mathcal{S}_k(K)$ with $k \leq m$. Clearly, $\phi_{\leq *} \circ \psi_*(v_0, \ldots, v_m) \in \hat{C}_*(\bar{\sigma})$. By what was seen in Examples 2.10.5 and 2.10.6, the correspondence $(v_0, \ldots, v_m) \mapsto \hat{C}_*(\overline{\{v_0, \ldots, v_m\}})$ is an acyclic carrier A_*, with respect to the basis $\hat{\mathcal{S}}_*(K)$, for both $\mathrm{id}_{\hat{C}(K)}$ and $\phi_{\leq *} \circ \psi_*$. Therefore, the equality $H_*\phi_{\leq} \circ H_*\psi = \mathrm{id}_{\hat{H}_*(K)}$ follows by Lemma 2.3.7 and Proposition 2.9.1.

Applying Kronecker duality to Proposition 2.10.7 gives the following

Corollary 2.10.8 $H^*\psi \circ H^*\phi_\leq = \mathrm{id}_{\hat{H}^*(K)}$ *and* $H^*\phi_\leq \circ H^*\psi = \mathrm{id}_{H^*(K)}$.

Corollary 2.10.9 $H_*\psi$ *and* $H^*\psi$ *are isomorphisms.*

Corollary 2.10.10 $H_*\phi_\leq$ *and* $H^*\phi_\leq$ *are isomorphisms which do not depend on the simplicial order* $\leq$.

Proof This follows from Proposition 2.10.7 and Corollary 2.10.8, since $H_*\psi$ and $H^*\psi$ do not depend on $\leq$. □

We shall see in Sect. 4.1 that $H^*\psi$ and $H^*\phi_\leq$ are isomorphisms of graded $\mathbb{Z}_2$-algebras. We now prove that they are also natural with respect to simplicial maps. Let $f: L \to K$ be a simplicial map. Let $\hat{C}_* f: \hat{C}_*(L) \to \hat{C}_*(K)$ be the unique linear map such that

$$\hat{C}_* f((v_0, \dots, v_m)) = (f(v_0), \dots, f(v_m))$$

for each $(v_0, \dots, v_m) \in \hat{\mathcal{S}}_m(K)$. Doing this for each $m \in \mathbb{N}$ produces a **GrV**-morphism $\hat{C}_* f: \hat{C}_*(L) \to \hat{C}_*(K)$. The formula $\hat{\partial} \circ \hat{C}_* f = \hat{C}_* f \circ \hat{\partial}$ is straightforward (much easier than that for non-ordered chains). Hence, we get a **GrV**-morphism $\hat{H}_* f: \hat{H}_*(L) \to \hat{H}_*(K)$. A **GrV**-morphism $\hat{C}^* f: \hat{C}^*(K) \to \hat{C}^*(L)$ is defined by the equation $\langle \hat{C}^* f(a), \alpha \rangle = \langle a, \hat{C}_* f(\alpha) \rangle$ required to hold for all $a \in \hat{C}^m(L)$, $\alpha \in \hat{C}_m(K)$ and all $m \in \mathbb{N}$. It is a cochain map and induces a **GrV**-morphism $\hat{H}^* f: \hat{H}^*(K) \to \hat{H}^*(L)$, Kronecker dual to $H_* f$.

Proposition 2.10.11 *Let* $f: L \to K$ *be a simplicial map. Let* $\leq$ *be a simplicial order on* K *and* $\leq'$ *be a simplicial order on* L. *Then the diagrams*

$$\begin{tikzcd}
\hat{H}_*(L) \arrow[r, "\hat{H}_* f"] \arrow[d, shift left, "H_*\psi"] & \hat{H}_*(K) \arrow[d, shift left, "H_*\psi"] \\
H_*(L) \arrow[r, "H_* f"] \arrow[u, shift left, bend left, "H_*\phi_{\leq'}"] & H_*(K) \arrow[u, shift left, bend left, "H_*\phi_{\leq}"]
\end{tikzcd}
\qquad \textit{and} \qquad
\begin{tikzcd}
\hat{H}^*(K) \arrow[r, "\hat{H}^* f"] \arrow[d, shift left, bend left, "H^*\phi^{\leq}"] & \hat{H}_*(L) \arrow[d, shift left, bend left, "H^*\phi^{\leq'}"] \\
H_*(K) \arrow[r, "H^* f"] \arrow[u, shift left, "H^*\psi"] & H_*(L) \arrow[u, shift left, "H^*\psi"]
\end{tikzcd}$$

are commutative.

Proof By Kronecker duality, only the homology statement requires a proof. It is enough to prove that $H_* f \circ H_*\psi = H_*\psi \circ \hat{H}_* f$ since the formula $\hat{H}_* f \circ H_*\phi_{\leq'} = H_*\phi_\leq \circ H_* f$ will follow by Corollary 2.10.8. Finally, the formula $\hat{C}_* f \circ C_*\phi_{\leq'} = C_*\phi_\leq \circ C_* f$ is straightforward. □

The above isomorphism results also work in relative ordered (co)homology. Let (K, L) be a simplicial pair. Denote by $i: L \hookrightarrow K$ the simplicial inclusion. We define the $\mathbb{Z}_2$-vector space of *relative ordered* (*co*)*chain* by

$$\hat{C}^m(K, L) = \ker \left(\hat{C}^m(K) \xrightarrow{\hat{C}^* i} \hat{C}^m(L) \right)$$

and

$$\hat{C}_m(K, L) = \operatorname{coker}\left(i_* : \hat{C}_m(L) \hookrightarrow \hat{C}_m(K)\right).$$

These inherit (co)boundaries $\hat{\delta} : \hat{C}^*(K, L) \to \hat{C}^*(K, L)$ and $\hat{\partial} = \hat{C}_*(K, L) \to \hat{C}_{*-1}(K, L)$ which give rise to the definition of *relative ordered (co)homology* $\hat{H}^*(k, L)$ and $\hat{H}_*(K, L)$. Connecting homomorphisms $\hat{\delta}_*$: $\hat{H}^*(L) \to \hat{H}^{*+1}(K, L)$ and $\hat{\partial}_*$: $\hat{H}_*(K, L) \to \hat{H}_{*-1}(L)$ are defined as in Sect. 2.7, giving rise to long exact sequences. Our homomorphisms ψ_*: $\hat{C}_*(K) \to C_*(K)$ and $\phi_{\leq *}$: $C_*(K) \to \hat{C}_*(K)$ satisfy $\psi_*(\hat{C}_*(L)) \subset C_*(L)$ and $\phi_{\leq *}(C_*(L) \subset \hat{C}_*(L)$, giving rise to homomorphisms on relative (co)chains and relative (co)homology $H_*\psi$: $\hat{H}_*(K, L) \to H - *(K, L)$, *etc*. Proposition 2.10.7 and Corollary 2.10.8 and their proofs hold in relative (co)homology. Hence, as for Corollaries 2.10.9 and 2.10.10, we get

Corollary 2.10.12 *$H_*\psi : \hat{H}_*(K, L) \to H_*(K, L)$ and $H^*\psi : H^*(K, L) \to \hat{H}^*(K, L)$ are isomorphisms.*

Corollary 2.10.13 *$H_*\phi_{\leq} : H_*(K, L) \to \hat{H}_*(K, L)$ and $H^*\phi_{\leq} : \hat{H}^*(K, L) \to \hat{H}^*(K, L)$ are isomorphisms which do not depend on the simplicial order $\leq$.*

2.11 Exercises for Chapter 2

2.1. Let $\mathcal{F}_n$ be the full complex on the set $\{0, 1, \ldots, n\}$ (see p. 24). What are the 2-simplexes of the barycentric subdivision $\mathcal{F}_2'$ of $\mathcal{F}_2$? How many n-simplexes does $\mathcal{F}_n'$ contain?

2.2. Compute the Euler characteristic and the Poincaré polynomial of the k-skeleton $\mathcal{F}_n^k$ of $\mathcal{F}_n$.

2.3. Let X be a metric space and let $\varepsilon > 0$. The *Vietoris-Rips complex* X_ε of X is the simplicial complex whose simplexes are the finite non-empty subset of X whose diameter is $< \varepsilon$ (the diameter of $A \subset X$ is the least upper bound of $d(x, y)$ for $x, y \in A$). In particular, $V(X_\varepsilon) = X$.

(a) Describe $|X_\varepsilon|$ for various ε when X is the set of vertices of a cube of edge 1 in $\mathbb{R}^3$. In particular, if $\sqrt{2} < \varepsilon \leq \sqrt{3}$, show that $|X_\varepsilon|$ is homeomorphic to S^3.

(b) Let X be the space n-th roots of unity, with the distance $d(x, y)$ being the minimal length of an arc of the unit circle joining x to y. Suppose that $4\pi/n < \varepsilon \leq 6\pi/n$.

(i) If n = 6, show that $|X_\varepsilon|$ is homeomorphic to S^2.
(ii) If $n \geq 7$ is odd, show that $|X_\varepsilon|$ is homeomorphic to a Möbius band.
(iii) If $n \geq 7$ is even, show that $|X_\varepsilon|$ is homeomorphic to $S^1 \times [0, 1]$.

Note: the complex X_ε was introduced by Vietoris in 1927 [201]. After its re-introduction by E. Rips for studying hyperbolic groups, it has been popularized under the name of *Rips complex*. For some developments and applications, see [84, 129] and Wikipedia's page "Vietoris-Rips complex".

2.4. Let $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{R}^n_{>0}$. A subset J of $\{1, \dots, n\}$ is called ℓ*-short* (or just *short*) if $\sum_{i \in J} \ell_i < \sum_{i \notin J} \ell_i$. Show that short subsets are the simplexes of a simplicial complex $\mathrm{Sh}(\ell)$ with $V(\mathrm{Sh}(\ell)) \subset J$ (used in Sect. 10.3). Describe $\mathrm{Sh}(1, 1, 1, 1, 3)$, $\mathrm{Sh}(1, 1, 3, 3, 3)$ and $\mathrm{Sh}(1, 1, 1, 1, 1)$. Compute their Euler characteristics and their Poincaré polynomials.

2.5. Let K be the simplicial complex with $V(K) = \mathbb{Z}$ and $\mathcal{S}_1(K) = \{\{r, r+1\} \mid r \in \mathbb{Z}\}$ ($|K| \approx \mathbb{R}$). Then $\mathcal{S}_1(K)$ is a 1-cocycle. Find all the cochains $a \in C^0(K)$ such that $\mathcal{S}_1(K) = \delta(a)$.

2.6. Find a simplicial pair (K, L) such that $|K|$ is homeomorphic to $S^1 \times I$ and $|L| = \mathrm{Bd}\,|K|$. In the spirit of Sect. 2.4.7, compute the simplicial cohomology of K and of (K, L) and find (co)cycles generating $H_*(K)$, $H_*(K, L)$, $H^*(K)$ and $H^*(K, L)$. Write completely the (co)homology sequence of (K, L).

2.7. Same exercise as before with $|K|$ the Möbius band and $|L| = \mathrm{Bd}\,|K|$.

2.8. Let $f\colon K \to L$ be a simplicial map between simplicial complexes. Suppose that L is connected and K is non-empty. Show that $H_0 f$ is surjective.

2.9. Let m, n, q be positive integers. If $m = nq$, the quotient map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ descends to a map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, giving rise to a simplicial map $f\colon \mathcal{P}_m \to \mathcal{P}_n$ between the simplicial polygons $\mathcal{P}_m$ and $\mathcal{P}_n$ (see Example 2.4.3). Compute $H^* f$.

2.10. Let M be an n-dimensional pseudomanifold. Let σ and σ' be two distinct n-simplexes of M. Find $a \in C^{n-1}(M)$ such that $\delta(a) = \{\sigma, \sigma'\}$.

2.11. Let M be a finite non-empty n-dimensional pseudomanifold. Let $\gamma \in Z_{n-1}(M)$ which is a boundary. Prove that γ is the boundary of exactly two n chains.

2.12. Let $f\colon M \to N$ be a simplicial map between finite n-dimensional pseudomanifolds. Show that the following two conditions are equivalent.

(a) $H_n f \neq 0$.
(b) There exists $\sigma \in \mathcal{S}(N)$ such that $\sharp f^{-1}(\{\sigma\})$ is odd.

2.13. Let $\{\pm 1\}$ be the 0-dimensional simplicial complex with vertices -1 and 1. Let K be a simplicial complex. The *simplicial suspension* ΣK is the join $K * \{\pm 1\}$.

(a) Let $\mathcal{P}_4$ be the polygon complex with 4-edges (see Example 2.4.3). Show that $\mathcal{P}_4 * K$ is isomorphic to the double suspension $\Sigma(\Sigma K)$. [Hint: show that the join operation is associative: $(K * L) * M \approx K * (L * M)$.]
(b) Prove that the suspension of a pseudomanifold is a pseudomanifold.
(c) Prove that the correspondence $K \mapsto \Sigma K$ gives a functor from **Simp** to itself.

2.14. Let A be a finite set. Show that $\dot{\mathcal{F}} A$ is a pseudomanifold.

2.15. Let M be an n-dimensional pseudomanifold which is infinite. What is $H_n(M)$?

2.16. Let (K, K_1, K_2) be a simplicial triad. Suppose that K_1 and K_2 are connected and that $K_1 \cap K_2$ is not empty. Show that K is connected.

2.17. Let (K, K_1, K_2) be a simplicial triad and let $K_0 = K_1 \cap K_2$.

(a) Prove that the homomorphism $H_*(K_1, K_0) \to H_*(K_, K_2)$ induced by the inclusion is an isomorphism (*simplicial excision*).
(b) Write the commutative diagram involving the homology sequences of (K_1, K_0) and $(K_, K_2)$. Using (a), construct out of this diagram the Mayer-Vietoris sequence for the triad (K, K_1, K_2).

2.18. Deduce the additivity formula for the Euler characteristic of Lemma 2.4.10 from the Mayer-Vietoris sequence.

2.19. Let M_1 and M_2 be two finite n-dimensional pseudomanifolds. Let $\sigma_i \in \mathcal{S}(M_i)$ and let $h\colon \sigma_1 \to \sigma_2$ be a bijection. The *simplicial connected sum* $M = M_1 \sharp M_2$ (using h) is the simplicial complex defined by

$$V(M) = V(M_1) \dot{\cup} V(M_2) \big/ \{v \sim h(v) \text{ for } v \in \sigma_1\}$$

and

$$\mathcal{S}(M) = \big(\mathcal{S}(M_1) - \{\sigma_1\}\big) \dot{\cup} \big(\mathcal{S}(M_2) - \{\sigma_2\}\big)\,.$$

Prove that M is a pseudomanifold. Compute $H_*(M)$ in terms of $H_*(M_1)$ and $H_*(M_2)$.

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